EXISTENCE AND REGULARITY RESULTS FOR THE STEINER PROBLEM

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Abstract. Given a complete metric space X and a compact set $C \subset X$, the famous Steiner (or minimal connection) problem is that of finding a set S of minimum length (one-dimensional Hausdorff measure $\mathcal{H}^1$) among the class of sets

$$\mathcal{S}t(C) := \{S \subset X\colon S \cup C \text{ is connected}\}.$$

In this paper we provide conditions on existence of minimizers and study topological regularity results for solutions of this problem. We also study the relationships between several similar variants of the Steiner problem. At last, we provide some applications to locally minimal sets.

1. Introduction

In this paper we deal with the following problem. Let X be a complete metric space, $C \subset X$ be a compact set and define

$$\mathcal{S}t(C) := \{S \subset X\colon S \cup C \text{ is connected}\}.$$

We are interested to find an element of $\mathcal{S}t(C)$ with minimal length (i.e. 1-dimensional Hausdorff measure $\mathcal{H}^1$), namely,

$$\inf\{\mathcal{H}^1(S) \,:\, S \in \mathcal{S}t(C)\}. \tag{ST}$$

We introduce also the notation

$$\mathcal{M}(C) := \{S \in \mathcal{S}t(C)\colon \mathcal{H}^1(S) \le \mathcal{H}^1(S') \text{ for all } S' \in \mathcal{S}t(C)\}$$

for the set of solutions to this problem.

The above problem with C being a finite set of points in an Euclidean space is well-known under the name of *minimal connection* or *Steiner* problem. As it is generally recognized nowadays, the German-Swiss 19th century mathematician J. Steiner, whose name is attributed to this problem, has in fact little to do with its formulation. It appeared in fact more than a century after Steiner, in 1934, in the works of Czech mathematicians Jarnik and Kossler, but actually became famous only after having been cited in the book of Courant and Robbins. On the other hand, the first problem of this type (namely, with C being a triple of points in the plane) may be considered quite ancient: it was actually raised in 17th century by Fermat, the solution being found by Torricelli and further refined by Cavalieri (not to mention that, of course, the particular case of C being a couple of points, i.e. the *geodesic* problem, has even earlier history). By now this problem is subject of active study, and an extremely extensive literature on the subject exists (see, for instance, the book [6] and references therein).

It is worth mentioning that many other similar problem settings appear in the literature, usually under the same name of Steiner problem. In particular, one may

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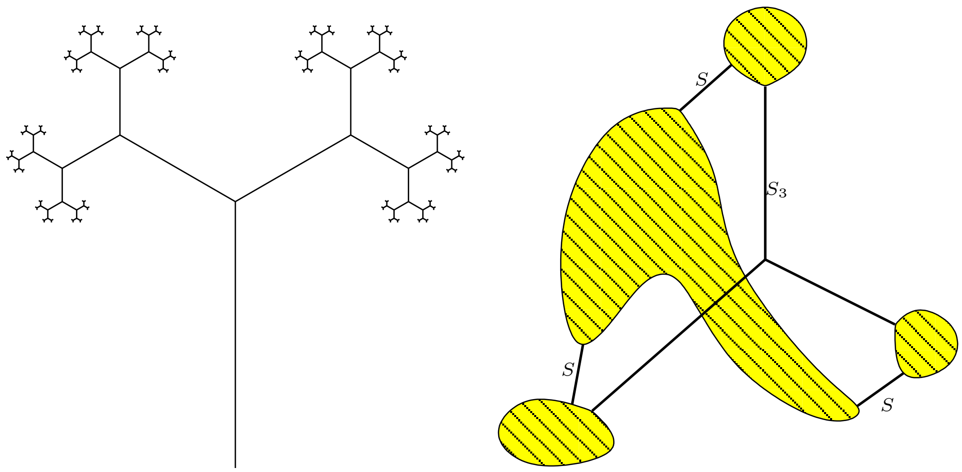

(A) A solution of a Steiner problem having a countable set of branching points.

(B) Comparison between length minimizers over $\mathcal{S}t(C)$ and over $\mathcal{S}t_3(C)$.

FIGURE 1

consider, instead of the class $\mathcal{S}t(C)$, the other natural classes of sets

$$\mathcal{S}t_1(C) := \{S \subset X \colon S \text{ is connected and } S \supset C\},$$
$$\mathcal{S}t_2(C) := \{S \subset X \colon S \cup C \text{ compact and connected}\},$$

and pose the problems of finding a set of minimal length (i.e. one-dimensional Hausdorff measure $\mathcal{H}^1$) among the class $\mathcal{S}t_i(C)$ for $i = 1, 2$. We will denote such problems (ST_i), $i = 1, 2$, and still refer to them as Steiner problems, the notation $\mathcal{M}_i(C)$, $i = 1, 2$ standing for the respective classes of minimizers of the length functional $\mathcal{H}^1$ (i.e. of solutions). For instance, in most papers one deals with problem (ST_1) which is easily reducible to problem (ST) and in fact is clearly less general than the latter (see section 6 below). Further, most results in the literature refer to the cases when the set C is finite and the ambient metric space X has some particular structure (e.g. is an Euclidean space or a Riemannian manifold). However, optimal networks connecting infinite sets are also quite important and may have highly complicated structure. For instance, in Figure 1(A) we have drawn a compact set S which is an infinite binary tree whose triple points (which are infinite) have equal angles of 120 degrees. We consider the set C composed by the endpoints of the tree S (one *root* and infinitely many *leafs*). The set C is not countable and totally disconnected (every connected component of C is a point). It is easy to verify that $S \setminus C$ is connected and that it is a locally minimal network (see Definition 8.1). However the proof that such a set S is actually a minimizer $S \in \mathcal{M}(C)$ is much more difficult than what one might think at first glance and will be provided in a forthcoming paper. We also mention that in the recent paper [4] another problem of this type was studied, namely, that of minimizing $\mathcal{H}^1$ over the set

$$\mathcal{S}t_3(C) := \{S \subset X \colon S \cup C \text{ and } S \text{ are both compact and connected}\}.$$

Though looking similar, this problem is a bit different and will not be studied here. In fact, in Figure 1(B) we present an example (with C being the union of four shaded regions in the plane) where the minimizer of length $\mathcal{H}^1$ over $\mathcal{S}t_3(C)$ (denoted by S_3 in the figure) is quite different from the solution S to (ST). In this example the minimizer S_3 is connected but it is not at all influenced by the presence of the biggest component of C.

In this paper we will study the general setting of the Steiner problem (ST), for generic ambient space X (the only result where we will need additional assumptions on X rather than just completeness, is the existence theorem) and for possibly infinite sets C. In particular, in Theorem 4.1 we will show the existence of solutions to this problem under additional requirement on the metric space X to be *proper* and connected. We will deduce this result from the analogous existence statement for the apparently easier problem (ST_2), for which the proof is by now quite standard and makes use of the metric space versions of the Blaschke theorem on compactness of Hausdorff topology and the Gołąb theorem on semicontinuity of the length $\mathcal{H}^1$ in the same way as in the proof of the existence of geodesics from [1]. Note however, that the proof of the Gołąb theorem used in the latter book, which is the only known version of this result for arbitrary complete metric spaces, contains a flaw, which we recover here giving the complete version of the respective proof. We will further study the topological properties of solutions to the Steiner problem. Namely, we will show, that without any extra assumption on the ambient space X every solution S to the problem (ST) having finite length $\mathcal{H}^1(S) < +\infty$ has the following quite natural properties (Theorems 5.1, 7.6, 7.4, 7.3):

- (i) $S \cup C$ is compact;
- (ii) $S \setminus C$ has at most countably many connected components, and each of the latter has strictly positive length;
- (iii) $\bar{S}$ contains no closed loops (homeomorphic images of $\mathbb{S}^1$);
- (iv) the closure of every connected component of S is a topological tree with endpoints on C (so that in particular it has at most countable number of branching points), and with at most one endpoint on each connected component of C and all the branching points having finite order (i.e. finite number of branches leaving them);
- (v) if C has a finite number of connected components, then $S \setminus C$ has finitely many connected components, the closure of each of which is a finite geodesic embedded graph with endpoints on C, and with at most one endpoint on each connected component of C;
- (vi) for every open set $U \subset X$ such that $C \subset U$ the set $\tilde{S} := S \setminus U$ is a subset of (the support of) a finite geodesic embedded graph. Moreover, for a.e. $\varepsilon > 0$ one has that for $U = \{x\colon \operatorname{dist}(x, C) < \varepsilon\}$ the set $\tilde{S}$ is a finite geodesic embedded graph (in particular, it has a finite number of connected components and a finite number of branching points).

It is worth mentioning that such results for a generic (i.e. not necessarily finite) compact set C are new even for the Euclidean space $X := \mathbb{R}^n$ (some results for a countable $C \subset \mathbb{R}^n$ were obtained in [8]).

We also will show that problems (ST_i), $i = 1, 2$, may be all naturally reduced to problem (ST). Finally, we will give some applications of the results proven to the study of locally minimal sets.

2. Notation and preliminaries

2.1. **Metric spaces.** The metric space X is said to be *proper* (or to possess the *Heine-Borel property*), if every closed ball of X is compact. In particular, this implies completeness, local compactness and σ-compactness of X.

For a set $S \subset X$ we denote by $\bar{S}$ its closure in X. The notation $B_r(x)$ stands for the open ball with radius r centered at $x \in X$. By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure, and, for the finite positive Borel measure μ over X the notation $\Theta^*_k(\mu, x)$ will stand for the k-dimensional upper density of μ at point $x \in X$ [1].

In the sequel we will need the following general result.

Theorem 2.1 (coarea inequality). *Let S be a non empty subset of a metric space X and let $f\colon X \to \mathbb{R}$ be any 1-Lipschitz function. Then*

$$\mathcal{H}^1(S) \geq \int_{\mathbb{R}} \mathcal{H}^0(S \cap f^{-1}(t))\, dt.$$

Proof. Let $\delta > 0$ be fixed. From the definition of Hausdorff measure, for all $\varepsilon > 0$ we can find a family of sets $\{B_i\}$ with $\operatorname{diam} B_i \leq \delta$, such that $\bigcup_i B_i \supset S$ and

$$\mathcal{H}^1_\delta(S) \geq \sum_i \operatorname{diam} B_i - \varepsilon.$$

Define $S_t := S \cap f^{-1}(t)$. For every t the family of sets B_i such that $t \in f(B_i)$ is a covering of S_t. The condition $t \in f(B_i)$ is equivalent to $1_{f(B_i)}(t) = 1$, where 1_E stands for the characteristic function of the set E, so we have

$$\mathcal{H}^0_\delta(S_t) \leq \sum_i 1_{f(B_i)}(t).$$

Notice now that given two points $\{x_i, y_i\} \subset B_i$ we have $|f(x_i) - f(y_i)| \leq \operatorname{diam} B_i$, hence $\sup f(B_i) - \inf f(B_i) \leq \operatorname{diam} B_i$. We conclude

$$\int_{\mathbb{R}} \mathcal{H}^0_\delta(S_t)\, dt \leq \sum_i \int_{\mathbb{R}} 1_{f(B_i)}(t)\, dt \leq \sum_i \int_{\inf f(B_i)}^{\sup f(B_i)} dt \leq \sum_i \operatorname{diam} B_i \leq \mathcal{H}^1_\delta + \varepsilon.$$

Letting $\varepsilon \to 0$ we get

$$\mathcal{H}^1_\delta(S) \geq \int_{\mathbb{R}} \mathcal{H}^0_\delta(S_t)\, dt$$

and letting $\delta \to 0$, we obtain the desired result by applying the monotone convergence theorem. □

2.2. **Connected spaces.** We recall that a topological space Σ is called

- *connected*, if it contains no subset except Σ and $\emptyset$ which is both closed an open,
- *continuum*, if it is connected and compact,
- *metric continuum*, if it is a metric space and a continuum in the topology induced by the metric,
- *locally connected*, if at every point there exists a fundamental system of open and connected neighborhoods.

A *path* (resp. *arc*) in Σ is defined as a continuous (resp. homeomorphic) image of an interval. Namely, we call a path (resp. arc) $[a, b] \subset \Sigma$ connecting the couple of points $\{a, b\} \subset \Sigma$ an image of some continuous (resp. continuous injective) map $\gamma\colon [0, L] \subset \mathbb{R} \to \Sigma$ satisfying $\gamma(0) = a$ and $\gamma(L) = b$ for some $L \in \mathbb{R}$. We also say in this case that the path (resp. arc) $[a, b] \subset \Sigma$ starts at a and ends at b. The map γ is called a parameterization of $[a, b]$. For the sake of brevity we will frequently abuse the notation identifying the path (resp. arc) with its parameterization, i.e. writing $\gamma = [a, b]$ instead of $\operatorname{Im} \gamma = [a, b]$. A path (resp. arc) is called *Lipschitz*, if it admits Lipschitz-continuous parameterization. At last, we let (a, b) stand for the path (resp. arc) without endpoints connecting a and b, i.e. $(a, b) := [a, b] \setminus \{a, b\}$.

A topological space Σ is called

- *arcwise connected*, if every couple of points of Σ is connected by some arc.
- *locally arcwise connected*, if at every point there exists a fundamental system of open and arcwise connected neighborhoods.

It is well-known that

(i) every arcwise connected (resp. locally arcwise connected) space is connected (resp. locally connected);
(ii) the reverse implication to the above is false;

(iii) nevertheless, a complete locally connected space is locally arcwise connected (Mazurkiewicz-Moore-Menger theorem II.1 from [7, § 50]);
(iv) a connected locally arcwise connected space is arcwise connected (theorem I.2 from [7, § 50]);
(v) a continuum is locally connected (hence also locally arcwise connected and arcwise connected by (iii) and (iv)), if an only if it is a path, i.e. a continuous image of an interval (Hahn-Mazurkiewicz-Sierpiński theorem II.2 from [7, § 50]).

Proposition 2.2. *Let $\Sigma \subset X$ be a closed connected set having $\mathcal{H}^1(\Sigma) < +\infty$. Then Σ is compact, arcwise connected and locally arcwise connected.*

Proof. By theorem 4.4.7 from [1], Σ is compact and arcwise connected (by Lipschitz continuous arcs). We now prove that Σ is locally connected (hence locally arcwise connected). Consider for this purpose an arbitrary $x_0 \in \Sigma$ and $\rho > 0$. It is enough to show the existence of a connected neighborhood of x_0 contained in $\bar{B}_\rho(x_0)$. Consider $\Sigma_\rho := \Sigma \cap \bar{B}_{2\rho}(x_0)$ and let Σ_i with $i \in I$ be the connected components of Σ_ρ. Recall that every connected component Σ_j is closed in Σ_ρ and hence is compact by theorem 4.4.7 from [1] since $\mathcal{H}^1(\Sigma_j) \leq \mathcal{H}^1(\Sigma) < +\infty$. Let Σ_0 be the connected component containing x_0. Consider the family Σ_j with $j \in J$ of such connected components Σ_j which intersect the smaller ball $\bar{B}_\rho(x_0)$. Every Σ_j is compact and does not contain x_0 so it has positive distance from x_0. If J is finite there would exist a minimal distance $\epsilon > 0$ such that $\Sigma \cap B_\epsilon(x_0) = \Sigma_0 \cap B_\epsilon(x_0)$ and hence Σ_0 would be a connected neighborhood of x_0. On the other hand suppose that J is infinite. Then we have infinitely many disjoint connected sets Σ_j such that $\Sigma_j \ni y_j$ with $y_j \in B_\rho(x_0)$. We also claim that every Σ_j has a point in $\partial B_{2\rho}(x_0)$. Otherwise we would conclude that $\Sigma_j \subset B_{2\rho}(x_0)$ which means that Σ_j is also open in Σ. This contradicts the connectedness assumption of Σ. So for every $j \in J$ we find points $y_j, z_j \in \Sigma_j$ such that $d(x_j, z_j) \geq \rho$ and hence $\mathcal{H}^1(\Sigma_j) \geq \rho$. Since J is infinite we would conclude that $\mathcal{H}^1(\Sigma) = +\infty$ which is in contradiction with the assumptions. □

We now introduce the following notions.

Definition 2.3. Let Σ be a connected space. Then $x \in \Sigma$ is called noncut point of Σ, if $\Sigma \setminus \{x\}$ is connected. Otherwise, x is called cut point of Σ.

It is worth mentioning that according to Moore theorem IV.5 from [7, § 47]) every continuum has at least two noncut points.

Another relevant notion is that of an order of a point and of branching points and endpoints.

Definition 2.4. Let Σ be a topological space. We will say that the order of the point $x \in \Sigma$ does not exceed $\mathfrak{n}$, writing

$$\mathrm{ord}_x \Sigma \leq \mathfrak{n},$$

where $\mathfrak{n}$ is a cardinal, if for every neighbourhood V of x there is an open subset $U \subset V$ such that $x \in U$ and $\#\partial U \leq \mathfrak{n}$, # standing for cardinality of a set.

The order of the point $x \in \Sigma$ is said to be equal $\mathfrak{n}$, written

$$\mathrm{ord}_x \Sigma = \mathfrak{n},$$

if $\mathfrak{n}$ is the least cardinal for which $\mathrm{ord}_x \Sigma \leq \mathfrak{n}$.

If $\mathrm{ord}_x \Sigma = \mathfrak{n}$, with $\mathfrak{n} \geq 3$, then x will be called branching point of Σ, while if $\mathrm{ord}_x \Sigma = 1$, then x will be called endpoint of Σ.

We recall that according to the theorem V.1 from [7, § 51], if $\mathrm{ord}_x \Sigma \leq 1$, then x is a noncut point of Σ. In particular, every endpoint of Σ is a noncut point.

A locally connected continuum not containing closed loops (homeomorphic images of $\mathbb{S}^1$) is called *topological tree*. It is well-known that a topological tree has at most countable number of branching points [7, theorem VI.7,§ 51].

We will also need the following easy lemmata of more or less folkloric character.

Lemma 2.5. Let C and S be two connected subsets of a metric space X. Then $C \cup S$ is connected, if and only if

$$\bar{C} \cap \bar{S} \cap (C \cup S) \neq \emptyset.$$

In particular, if C is closed, then $C \cup S$ is connected, if and only if $\bar{S}$ touches (i.e. is not disjoint from) C.

Proof. Suppose that $C \cup S$ is not connected. Then there exist A, B be two relatively open disjoint subsets of $C \cup S$ such that $A \cup B = C \cup S$. Since C and S are connected, we might suppose that $A \supset C$ and $B \supset S$. As a consequence $A = C$ and $B = S$ which means that C and S are relatively closed in $S \cup C$. In other words $\bar{S} \cap (S \cup C) = S$ and $\bar{C} \cap (S \cup C) = C$. Hence $\bar{C} \cap \bar{S} \cap (C \cup S) = C \cap S = A \cap B = \emptyset$. Hence we have proved one implication.

For the other implication suppose $S \cup C$ is connected. If $C \cap S \neq \emptyset$ then

$$\bar{C} \cap \bar{S} \cap (C \cup S) \supset C \cap S \neq \emptyset.$$

Otherwise S, C is a partition of $S \cup C$ and since the latter is connected, we known that either C or S is not relatively closed in $S \cup C$. Suppose that C is not relatively closed. Then there exists $x \in (\bar{C} \cap (S \cup C)) \setminus C$. Hence $x \in \bar{C} \cap S$ which means, in particular, that $x \in \bar{C} \cap \bar{S} \cap (C \cup S)$. □

Lemma 2.6. Let X be a metric space and $\Sigma \subset X$ be connected. Then $\mathcal{H}^1(\Sigma) = \mathcal{H}^1(\bar{\Sigma})$.

Proof. It suffices to prove the claim for the case $\mathcal{H}^1(\Sigma) < +\infty$. Let $\mu := \mathcal{H}^1 \llcorner \Sigma$. One has

$$\mu(\bar{B}_r(x)) = \mathcal{H}^1(\Sigma \cap \bar{B}_r(x)) \geq r$$

for all $x \in \Sigma$ and $r \leq \operatorname{diam} \Sigma/2$, in view of connectedness of Σ. Since by assumption the measure μ is finite, then the above estimate holds also for all $x \in \bar{\Sigma}$. In fact, if $x \in \bar{\Sigma}$ and $x_k \to x$, then $\bar{B}_r(x_k) \subset \bar{B}_{r+\varepsilon}(x)$ for all k such that $d(x, x_k) < \varepsilon$. Thus,

$$\mu(\bar{B}_{r+\varepsilon}(x)) \geq \limsup_k \mu(\bar{B}_r(x_k)),$$

and letting $\varepsilon \to 0^+$ in the above relationship, we get

$$\mu(\bar{B}_r(x)) \geq \limsup_k \mu(\bar{B}_r(x_k)).$$

One has now that the upper density Θ_1^* of μ with respect to $\mathcal{H}^1$ satisfies

$$\Theta_1^*(\mu, x) \geq 1/2 \text{ for all } x \in \bar{\Sigma},$$

which implies $\mu \geq \mathcal{H}^1 \llcorner \bar{\Sigma}/2$ by [1, theorem 2.4.1]. Therefore,

$$0 = \mu(\bar{\Sigma} \setminus \Sigma) \geq \frac{1}{2}\mathcal{H}^1 \llcorner \bar{\Sigma}(\bar{\Sigma} \setminus \Sigma) = \frac{1}{2}\mathcal{H}^1(\bar{\Sigma} \setminus \Sigma),$$

which concludes the proof. □

3. Generalized Gołąb theorem

In this section we prove the generalization to metric spaces of the classical Gołąb theorem on lower semicontinuity of one-dimensional Hausdorff measures on connected compact sets with respect to Hausdorff convergence. This claim was first proven in [1] (note however that there is a small flaw in the respective proof which we will recover here by using Lemma 3.2; apart from this point, our proof follows that of [1]). Our version of this theorem contains also the generalization of the semicontinuity result from [3] applicable to non-connected sets, which we will need to prove the existence result for the Steiner problem (ST) and which is absent in that of [1].

We will need the following auxiliary assertions.

Lemma 3.1. Let C be an arcwise connected metric space containing three points $\{x, y, z\} \subset C$. Then

$$\mathcal{H}^1(C) \geq \frac{1}{2}(d(x,y) + d(x,z) + d(y,z)),$$

where d stands for the distance in C.

Proof. Consider an injective curve γ joining x with y. Then consider an injective curve η joining z with x. Let w be the first point of the curve η in common with γ. We obtain

$$\mathcal{H}^1(C) \geq d(x,w) + d(y,w) + d(z,w).$$

By the triangle inequality we have

$$\begin{aligned} d(x,y) &\leq d(x,w) + d(w,y) \\ d(x,z) &\leq d(x,w) + d(w,z) \\ d(y,z) &\leq d(y,w) + d(w,z), \end{aligned}$$

and summing up we obtain

$$d(x,y) + d(x,z) + d(y,z) \leq 2(d(x,w) + d(y,w) + d(z,w)) \leq 2\mathcal{H}^1(C).$$

$\square$

Lemma 3.2. Let X be a metric space, $x_0 \in X$, $r > 0$, $\varepsilon \in (0, r)$. Let $\gamma \colon [-r + \varepsilon, r - \varepsilon] \to X$ be a Lipschitz curve with $\gamma(0) = x_0$ and such that

$$|t - s| - \varepsilon \leq d(\gamma(t), \gamma(s)) \leq |t - s| + \varepsilon \qquad \text{for all } t, s \in [-r + \varepsilon, r - \varepsilon].$$

Let C be a compact subset of $\bar{B}_r(x_0)$ in X such that for every $t \in [-r+\varepsilon, r-\varepsilon]$ one has $d(\gamma(t), C) \leq \varepsilon$ and such that every connected component of C touches $\partial B_r(x_0)$. Then $\mathcal{H}^1(C) \geq 2r - 9\varepsilon$.

Proof. Let C_1 be a connected component of C such that $d(\gamma(0), C_1) \leq \varepsilon$. Let us define

$$T_1 := \{t \in [-r + \varepsilon, r - \varepsilon] \colon d(\gamma(t), C_1) \leq \varepsilon\}.$$

By the choice of C_1 we have $0 \in T_1$. Choose s_1, s_3, t_1, t_2 such that

$$-s_1 - \varepsilon < -s_3 < -s_1 \leq 0 \leq t_1 < t_2 < t_1 + \varepsilon$$

while $[-s_1, t_1] \subset T_1$ and $-s_3 \notin T_1$, $t_2 \notin T_1$.

Let now $x_1, y_1 \in C_1$ be such that $d(x_1, \gamma(t_1)) \leq \varepsilon$ and $d(y_1, \gamma(-s_1)) \leq \varepsilon$. Let $x_2, x_3 \in C$ be such that $d(x_2, \gamma(t_2)) \leq \varepsilon$ and $d(x_3, \gamma(-s_3)) \leq \varepsilon$ and let C_2, C_3 be connected components of C containing x_2 and x_3 respectively. In the case when $t_2 \geq r - \varepsilon$ we could not find x_2 as above, so we just take $C_2 := \emptyset$. Analogously let $C_3 := \emptyset$ if $s_1 \geq r - \varepsilon$.

By the choice of t_2 and s_3 we have that $x_2, x_3 \notin C_1$ hence $C_1 \neq C_2$ and $C_1 \neq C_3$. (but we might have $C_2 = C_3$). Let $z_i \in C_i \cap \partial B_r(x_0)$ for $i = 1, 2, 3$.

We have

$$\begin{aligned}
d(x_1,y_1) &\geq d(\gamma(t_1),\gamma(-s_1)) - 2\varepsilon \geq t_1+s_1-3\varepsilon,\\
d(x_2,x_3) &\geq d(\gamma(t_2),\gamma(-s_3)) - 2\varepsilon \geq t_2+s_3-3\varepsilon \geq t_1+s_1-5\varepsilon,\\
d(x_1,z_1) &\geq d(\gamma(t_1),z_1)-\varepsilon\\
&\geq d(z_1,x_0)-d(\gamma(t_1),x_0)-\varepsilon \geq r-t_1-2\varepsilon,\\
d(y_1,z_1) &\geq d(\gamma(-s_1),z_1)-\varepsilon\\
&\geq d(z_1,x_0)-d(\gamma(-s_1),x_0)-\varepsilon \geq r-s_1-2\varepsilon,\\
d(x_2,z_2) &\geq d(\gamma(t_2),z_2)-\varepsilon\\
&\geq d(z_2,x_0)-d(\gamma(t_2),x_0)-\varepsilon \geq r-t_2-2\varepsilon \geq r-t_1-3\varepsilon,\\
d(x_3,z_3) &\geq d(\gamma(-s_3),z_3)-\varepsilon\\
&\geq d(z_3,x_0)-d(\gamma(-s_3),x_0)-\varepsilon \geq r-s_3-2\varepsilon \geq r-s_1-3\varepsilon,\\
d(x_3,z_2) &\geq d(\gamma(-s_3),z_2)-\varepsilon\\
&\geq d(z_2,x_0)-d(\gamma(-s_3),x_0)-\varepsilon \geq r-s_3-2\varepsilon \geq r-s_1-3\varepsilon.
\end{aligned}$$

Suppose now that $C_2 \cap C_3 = \emptyset$. We now recall that if K is connected and $\{x,y\} \subset K$, then $\mathcal{H}^1(K) \geq d(x,y)$. Thus, applying the above estimates, we get

$$\begin{aligned}
\mathcal{H}^1(C_1) &\geq d(x_1,y_1) \geq t_1+s_1-3\varepsilon,\\
\mathcal{H}^1(C_2) &\geq d(x_2,z_2) \geq r-t_1-3\varepsilon,\\
\mathcal{H}^1(C_3) &\geq d(x_3,z_3) \geq r-s_1-3\varepsilon.
\end{aligned} \tag{1}$$

Note that the above inequalities hold even when $C_2 = \emptyset$ or $C_3 = \emptyset$. For instance, $C_2 = \emptyset$ when $t_2 > r-\varepsilon$ which implies $t_1 > r-2\varepsilon$, hence $r-t_1-3\varepsilon < 0$, and the case $C_3 = \emptyset$ is analogous. Summing the inequalities from (1), we get

$$\mathcal{H}^1(C) \geq \mathcal{H}^1(C_1)+\mathcal{H}^1(C_2)+\mathcal{H}^1(C_3) \geq 2r-9\varepsilon.$$

In the case $C_2 = C_3 \neq \emptyset$ we apply Lemma 3.1 getting

$$\begin{aligned}
\mathcal{H}^1(C_1) &\geq \frac{1}{2}(d(x_1,y_1)+d(x_1,z_1)+d(y_1,z_1)) \geq r-\frac{7}{2}\varepsilon,\\
\mathcal{H}^1(C_2) &\geq \frac{1}{2}(d(x_2,x_3)+d(x_2,z_2)+d(x_3,z_2)) \geq r-\frac{11}{2}\varepsilon,
\end{aligned}$$

so that

$$\mathcal{H}^1(C) \geq \mathcal{H}^1(C_1)+\mathcal{H}^1(C_2) \geq 2r-9\varepsilon,$$

concluding the proof. □

We may announce now the main result of this section.

Theorem 3.3 (Gołąb)**.** *Let C_ν be closed connected subsets of a complete metric space X. Suppose moreover that the sequence C_ν converges with respect to Hausdorff metric to a closed set $C \subset X$. Then C is connected and*

$$\mathcal{H}^1(C) \leq \liminf_\nu \mathcal{H}^1(C_\nu). \tag{2}$$

Moreover, if $\{K_\nu\}$ is a sequence of closed sets converging in Hausdorff distance to a closed set $K \subset X$, then

$$\mathcal{H}^1(C \setminus K) \leq \liminf_\nu \mathcal{H}^1(C_\nu \setminus K_\nu). \tag{3}$$

The claim (2) is the generalization of the classical Gołąb semicontinuity theorem which was originally formulated in the Euclidean space setting, i.e. for $X = \mathbb{R}^n$. The more general claim (3) is a generalization for metric spaces of the analogous Euclidean space result from [3].

Proof. We prove at once (3), since (2) is its particular case with $K = K_\nu = \emptyset$.

Without loss of generality we shall suppose that $L := \lim_\nu \mathcal{H}^1(C_\nu)$ exists and is finite, and that $\mathcal{H}^1(C_\nu) < L + 1$ for all n. Since Hausdorff convergence preserves connectedness, we know that C is connected. Also by Proposition 2.2 we have that every C_ν is compact. We are going to reduce to the case when X is compact. Consider the set

$$\tilde{X} := \overline{\bigcup_\nu C_\nu}.$$

Since X_ν is a Cauchy sequence with respect to Hausdorff convergence, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that all C_ν with $\nu > N$ are contained in the ε-neighborhood of C_N. Hence

$$\tilde{X} \subset \left(\bigcup_{\nu=1}^{N} C_\nu \right) \cup (C_N)_\varepsilon.$$

Since $\mathcal{H}^1(C_N) < +\infty$ we know that C_N is totally bounded and hence $\tilde{X}$ is totally bounded too. Being closed $\tilde{X}$ is compact and contains all C_ν and C. So, in the following, we shall suppose that X is compact, otherwise we could replace X with $\tilde{X}$.

Let $d_\nu := \operatorname{diam} C_\nu$, $d := \operatorname{diam} C$. We have $d_\nu \to d$. If $d = 0$ we get $\mathcal{H}^1(C) = 0$ and the proof is completed. Otherwise suppose that $d > 0$. Define the measures μ_ν by

$$\mu_\nu(B) := \mathcal{H}^1(B \cap C_\nu \setminus K_\nu), \quad n \in \mathbb{N}$$

for every Borel set $B \subset E$ and observe that μ_ν is a finite Borel measure. Up to a subsequence we can assume that $\mu_\nu \overset{*}{\rightharpoonup} \mu$ for some Borel measure μ. We recall that this implies

$$\mu(F) \geq \limsup_\nu \mu_\nu(F), \qquad \mu(G) \leq \liminf_\nu \mu_\nu(G)$$

whenever F is closed and G is open.

Now choose $x \in C \setminus K$, $r < \min\{d/2, \operatorname{dist}(x, K)/2\}$. We have

$$\mu(\bar{B}_r(x)) \geq \limsup_\nu \mu_\nu(\bar{B}_r(x)) = \limsup_\nu \mathcal{H}^1(C_\nu \cap \bar{B}_r(x)).$$

In the last equality we use the fact that for ν sufficiently large $r < \operatorname{dist}(x, K_\nu)$ because $\lim_\nu \operatorname{dist}(x, K_\nu) = \operatorname{dist}(x, K) > 2r$ and hence $K_\nu \cap \bar{B}_r(x) = \emptyset$. But on the other hand, letting $x_\nu \in C_\nu$ be such that $x_\nu \to x$ as $\nu \to \infty$, we get

$$\mathcal{H}^1(C_\nu \cap \bar{B}_r(x)) \geq \operatorname{dist}(x_\nu, \partial B_r(x)) \geq r - |x - x_\nu|.$$

Hence $\limsup_\nu \mathcal{H}^1(C_\nu \cap \bar{B}_r(x)) \geq r$ and we have thus

$$\mu(\bar{B}_r(x)) \geq r$$

for every $x \in C \setminus K$ and every r sufficiently small. Hence $\Theta_1^*(\mu, x) \geq \frac{1}{2}$ and by [1, theorem 2.4.1] we obtain

$$\mathcal{H}^1(C \setminus K) \leq 2\mu(X) \leq 2 \liminf_\nu \mu_\nu(X) = 2 \liminf_\nu \mathcal{H}^1(C_\nu) = 2L < +\infty.$$

Now we invoke the rectifiability theorem 4.4.5 from [1] which assures that $\mathcal{H}^1$-a.e. $x \in C \setminus K$ can be represented as $\gamma(0)$, where γ is a Lipschitz curve, with values in $C \setminus K$ defined in some open interval containing 0, and γ is metrically differentiable at 0. By reparameterization, we can assume that $|\gamma'|(0) = 1$. We can also suppose that γ is 1-Lipschitz and for every $\varepsilon > 0$ we can find $r > 0$ sufficiently small such that

$$|t - s| - r\varepsilon \leq d(\gamma(t), \gamma(s)) \leq |t - s| \qquad \text{for all } t, s \in (-r, r).$$

Applying Lemma 3.2 we conclude that

$$\mu_\nu(B_r(x)) = \mathcal{H}^1(C_\nu \setminus K \cap B_r(x)) \geq 2r - 9r\varepsilon$$

and letting $r \to 0$ we obtain $\bar\theta(\mu, x) \geq 1$ and hence

$$\mathcal{H}^1(C \setminus K) \leq \mu(X) \leq \liminf_\nu \mu_\nu(X) = \liminf_\nu \mathcal{H}^1(C_\nu \setminus K_\nu) = L,$$

concluding the proof. □

4. Existence of minimizers

The main result of this section is given by the following assertion.

Theorem 4.1 (existence of minimizers)**.** *If X is a proper connected metric space and $C \subset X$ is compact, then $\mathcal{M}(C)$ is not empty, i.e. problem (ST) admits a solution.*

Remark 4.2. The above existence result remains valid, if X is assumed to be just σ-compact and *uniformly locally compact* (i.e. such that there is a $\delta > 0$ such that any closed ball of radius δ is compact). In fact, by [9], in such a space one can introduce a new distance, topologically equivalent and locally uniformly identical to the original one, which implies that Hausdorff measures with respect to the new distance coincides with that with respect to the original one.

Notice that the existence result does not hold in every complete metric space. In particular, in [5, example 5.1] a complete metric space has been exhibited which does not contain geodesics between some couple of its points (i.e. in our terminology, the Steiner problem is not solvable even for the case when C consists of a couple of points).

Remark 4.3. According to Lemma 4.7 which is the core construction of the existence proof for problem (ST), one has that problem (ST) admit solutions whenever there is a minimizer of $\mathcal{H}^1$ in the smaller class $\mathcal{S}t_2(C)$. The latter may be true even in the case when X is not proper, for instance, it is true when X is a Hilbert space (in fact, the problems in this case may be reduced to problems in the compact $\overline{\mathrm{co}}\, C \subset X$ instead of the whole space X, because the projection operator to the closed convex set has unit norm and hence decreases Hausdorff measures).

Remark 4.4. It is important to emphasize that the above existence result does not guarantee that a minimizer $S \in \mathcal{M}(C)$ has finite length, i.e. $\mathcal{H}^1(C) < +\infty$.

For instance, consider the set $C \subset \mathbb{R}^2$ defined by

$$C := \{(1/k, 1/j)\colon k = 1, 2, \dots, \infty, \quad j = 1, 2, \dots, \infty\}$$

where we let $1/\infty := 0$. The set C is a countable, compact subset of $[0, 1]^2 \subset \mathbb{R}^2$. For every $S \in \mathcal{S}t(C)$, we claim that $\mathcal{H}^1(S) = \infty$. Consider the set $C_\nu \subset C$ defined by

$$C_\nu := \{(1/k, 1/j)\colon k = \nu, \nu+1, \dots, \nu^2, \quad j = \nu, \nu+1, \dots, \nu^2\}.$$

If we take $\rho_\nu := \frac{1}{2(\nu+1)^2}$ we easily notice that the balls $B_{\rho_\nu}(z)$ are pairwise disjoint for all $z \in C_\nu$. Hence we have

$$\mathcal{H}^1(S) \geq \sum_{z \in C_\nu} \mathcal{H}^1(S \cap B_{\rho_\nu}(z)) \geq (\nu^2 - \nu)^2 \rho_\nu \to +\infty \qquad \text{as } \nu \to \infty.$$

So $\mathcal{H}^1(S) = +\infty$ as claimed.

In the proof of the existence theorem we will use the following lemmata.

Lemma 4.5. Let $S \in \mathcal{S}t(C)$ with $\mathcal{H}^1(S) < +\infty$. If $x \in \bar S \setminus C$ then there is a connected component S_0 of S having $\mathcal{H}^1(S_0) > 0$ such that $x \in \bar S_0$.

Proof. Otherwise there is a sequence $\{x_\nu\} \subset S$ such that each x_ν belongs to a different connected component of S, while $x_\nu \to x$ as $\nu \to \infty$. But the closure of every connected component of S cannot be disjoint with C (otherwise $S \cup C$ would be disconnected by Lemma 2.5), and therefore, chosen a sufficiently small

$r > 0$ (such that $\bar{B}_r(x) \cap C = \emptyset$) one has that for all sufficiently large ν (such that $\operatorname{dist}(x, x_\nu) < r/2$) the closure of every connected component of S containing x_ν intersects $\partial B_r(x)$, and hence, has length exceeding $r/2$. By Lemma 2.6 the length of each connected component of S equals that of its closure, and since by assumption there are infinitely many such connected components, this would imply $\mathcal{H}^1(S) = +\infty$ contrary to the assumption. □

Lemma 4.6. Let S be a set such that $S \cup C$ is compact and connected (i.e. $S \in \mathcal{S}t_2(C)$). Then if S_0 is any connected component of S, we have

$$\inf\{d(x,y)\colon x \in S_0, y \in C\} = 0.$$

Proof. Reasoning by contradiction there would exist an $\varepsilon > 0$ such that the set

$$A := \{x\colon d(x, S_0) < \varepsilon\}$$

would be an open set containing S_0 and disjoint from C. Also, since S_0 is a connected component of S, there exists an open set B such that $S_0 = S \cap B$. Then we find that $S_0 = (S \cup C) \cap (A \cap B)$ which means that S_0 is relatively open in $S \cup C$. Clearly S_0 is also closed in $S \cup C$ because it is closed in S. Hence S_0 would be a connected component of $S \cup C$ which is a contradiction, since we assumed $S \cup C$ to be connected. □

Lemma 4.7. If problem (ST_2) is solvable, then so is problem (ST), i.e. $\mathcal{M}(C) \neq \emptyset$. Moreover, if S' is a solution to problem (ST_2), then $S' \in \mathcal{M}(C)$, i.e. S' is a solution also to problem (ST).

Proof. Let ℓ stand for the value of the minimum (which we suppose to be attained) of $\mathcal{H}^1$ over the smaller class $\mathcal{S}t_2(C) \subset \mathcal{S}t(C)$. It is worth emphasizing that the case $\ell = +\infty$ is not excluded. We will show the existence of solutions to problem (ST). Let $S \in \mathcal{S}t(C)$ be such that $\mathcal{H}^1(S) \leq \ell$. We will show that S is a solution to problem (ST). We may suppose that $\mathcal{H}^1(S) < +\infty$ (if no such S exists, then there is nothing to prove, since then automatically $\ell = +\infty$).

We first observe that by Lemma 4.5 if $x \in \bar{S}\backslash C$ then there a connected component S_0 of S having $\mathcal{H}^1(S_0) > 0$ such that $x \in \bar{S}_0$. Denote now by $\{S_\nu\}$ the at most countable set of connected components of S having positive length, i.e. $\mathcal{H}^1(S_\nu) > 0$ for all ν. The above proven claim implies that

$$\bar{S} \setminus C \subset \bigcup_\nu \bar{S}_\nu.$$

Hence,

$$\begin{aligned} \mathcal{H}^1(\bar{S} \setminus C) &\leq \mathcal{H}^1(\cup_\nu \bar{S}_\nu) \leq \sum_\nu \mathcal{H}^1(\bar{S}_\nu) \\ &= \sum_\nu \mathcal{H}^1(S_\nu) = \mathcal{H}^1(\cup_\nu S_\nu) \leq \mathcal{H}^1(S) \leq \ell, \end{aligned} \tag{4}$$

where we used the fact that $\mathcal{H}^1(S_\nu) = \mathcal{H}^1(\bar{S}_\nu)$ which is true by Lemma 2.6 since $\mathcal{H}^1(S_\nu) \leq \mathcal{H}^1(S) < +\infty$. Minding that $(\bar{S} \setminus C) \cup C = \bar{S} \cup C$ and the latter set is compact, we have $\bar{S} \setminus C \in \mathcal{S}t_0'(C)$, which implies $\mathcal{H}^1(\bar{S} \setminus C) \geq \ell$ by definition of ℓ. Together with the estimate (4) this gives

$$\mathcal{H}^1(\bar{S} \setminus C) = \mathcal{H}^1(S) = \ell,$$

and hence S solves problem (ST), i.e. $S \in \mathcal{M}(C)$ as claimed. In particular, if S is a solution to problem (ST_2), then $\mathcal{H}^1(S') = \ell$ and hence, $S' \in \mathcal{M}(C)$, which concludes the proof. □

Lemma 4.8. If X is a proper connected metric space, and $C \subset X$ is compact, then problem (ST_2) admits a solution.

Proof. Observe first that $\mathcal{S}t_2(C) \neq \emptyset$ because $X \in \mathcal{S}t_2(C)$ (since X is assumed to be connected). Let now $S_\nu \in \mathcal{S}t_2(C)$ be a minimizing sequence for the length functional $\mathcal{H}^1$, and define $\Sigma_\nu := S_\nu \cup C$. Assume also that $\mathcal{H}^1(S_\nu) \leq l$ for some $l < +\infty$ (otherwise the minimum for $\mathcal{H}^1$ is attained at every $S \in \mathcal{S}t_2(C)$ since every set in $\mathcal{S}t_2(C)$ has infinite length). Therefore, minding that C is compact, we may assume all Σ_ν are included in a unique compact set $\Omega \subset X$. In fact, for each fixed $\nu \in \mathbb{N}$, every connected component of S_ν has length, and hence diameter, not exceeding l, and therefore, by Lemma 4.6, S_ν belongs to the l-neighborhood of C. Since the space of non empty closed subsets of Ω endowed by the Hausdorff metric is a compact metric space according to the Blaschke theorem (theorem 4.4.6 of [1]), we get that up to a subsequence (not relabeled) $\Sigma_\nu \to \Sigma$ in the sense of Hausdorff convergence, while $\Sigma \subset X$ is still closed and connected. The generalized Gołąb theorem 3.3 gives

$$\mathcal{H}^1(\Sigma \setminus C) \leq \liminf_\nu \mathcal{H}^1(\Sigma_\nu \setminus C) = \liminf_\nu \mathcal{H}^1(S_\nu),$$

and hence $\Sigma \setminus C$ solves problem (ST_2) which proves the statement. □

Proof of Theorem 4.1. Combine Lemma 4.7 with Lemma 4.8. □

5. Basic topological properties

In this section we will prove the following theorem, which gives some finer topological properties of minimizers $S \in \mathcal{M}(C)$.

Theorem 5.1. *Let X be a metric space and let $C \subset X$ be compact. If $S \in \mathcal{M}(C)$, then $\bar{S} \setminus C \in \mathcal{M}(C)$, while if in addition $\mathcal{H}^1(S) < +\infty$, then*
- (a) *$S \cup C$ is compact;*
- (b) *$S \setminus C$ has at most countably many connected components, and each of the latter has strictly positive length;*
- (c) *$\bar{S}$ contains no closed loops (homeomorphic images of $\mathbb{S}^1$).*

The proposition below collects some even finer, though more technical, assertions on topological structure of connected components of minimizers $S \in \mathcal{M}(C)$, which are a by-product of the proof of Theorem 5.1.

Proposition 5.2. *Let X be a metric space and let $C \subset X$ be compact. If $S \in \mathcal{M}(C)$ with $\mathcal{H}^1(S) < +\infty$ and S_0 is a connected component of S, then*
- (c_1) *every $x \in S_0 \setminus C$ is a cut point of $\bar{S}_0$;*
- (c_2) *every $x \in \bar{S}_0 \cap C$ is an endpoint of $\bar{S}_0$.*

We emphasize that both Theorem 5.1 and Proposition 5.2 as well as all the results on topological structure of minimizers to the Steiner problem (ST), in contrast with the existence Theorem 4.1, hold in arbitrary (complete) metric spaces without any extra requirement on the ambient space (e.g. it should not necessarily be proper).

To prove Theorem 5.1 we use some auxiliary lemmata. First we will need the following lemma proven in [2].

Lemma 5.3. Let Σ be a locally connected metric continuum consisting of more than one point and $x \in \Sigma$ be a noncut point of Σ. Then there is a sequence of open sets $D_\nu \subset \Sigma$ satisfying
- (i) $x \in D_\nu$ for all ν;
- (ii) $\Sigma \setminus D_\nu$ are connected for all ν;
- (iii) $\operatorname{diam} D_\nu \searrow 0$ as $\nu \to \infty$;
- (iv) D_ν are connected for all ν.

The easy lemmata below are used in the proof of both Theorem 5.1 and Proposition 5.2.

Lemma 5.4. Let $\Sigma := S \cup C \subset X$ be a connected set of a metric space X, and S_0 be a connected component of S. If $S_0' \subset X$ is a connected set such that $\bar{S}_0 \cap C \subset \bar{S}_0' \cap C$, then the set

$$\Sigma' := (\Sigma \setminus S_0) \cup S_0'$$

is connected.

Proof. Let $C_0 := \bar{S}_0 \cap C$ and consider an arbitrary $y \in S_0' \setminus C_0$. We will show that for every $z \in \Sigma'$ there is a connected subset $\sigma \subset \Sigma'$ such that $\{y, z\} \subset \sigma$. Clearly it is so, if $z \in S_0'$ (since also $y \in S_0'$ and the latter set is connected). On the other hand, if $z \notin S_0'$, then consider a connected component Σ_z of $\Sigma \setminus S_0$ such that $z \in \Sigma_z$. One has clearly $\Sigma_z \cap \bar{S}_0 \neq \emptyset$ by Lemma 2.5, while $\Sigma_z \cap (\bar{S}_0 \setminus C) = \emptyset$ by definition of S_0. Hence, $\Sigma_z \cap C_0 \neq \emptyset$. Then clearly $\sigma := \Sigma_z \cup S_0'$ is connected, and $\{y, z\} \subset \sigma$. □

Lemma 5.5. If $S \in \mathcal{M}(C)$ with $\mathcal{H}^1(S) < +\infty$ and S_0 is one of its connected components and $x \in \bar{S}_0$ is a noncut point of $\bar{S}_0$, then $x \in C$.

Proof. Notice that by Lemma 2.6 we have $\mathcal{H}^1(\bar{S}_0) = \mathcal{H}^1(S_0) \leq \mathcal{H}^1(S) < +\infty$ hence, by Proposition 2.2 we know that S_0 is arcwise connected. By Lemma 5.3, if $x \in \bar{S}_0 \setminus C$ is a noncut point of $\bar{S}_0$, then there is a relatively open set $D \subset \bar{S}_0$, $x \in D$ such that $D \cap C = \emptyset$, $\mathcal{H}^1(D) > 0$ and $\bar{S}_0 \setminus D$ is connected. Then $S_0' := \bar{S}_0 \setminus D$ is closed (hence compact as a subset of a compact set $\bar{S}_0$), connected, and $\mathcal{H}^1(\bar{S}_0') < \mathcal{H}^1(\bar{S}_0) = \mathcal{H}^1(S_0)$. Consider

$$S' := S_0' \cup (S \setminus S_0).$$

Clearly, one has

$$\mathcal{H}^1(S') \leq \mathcal{H}^1(S_0') + \mathcal{H}^1(S \setminus S_0) < \mathcal{H}^1(S_0) + \mathcal{H}^1(S \setminus S_0) = \mathcal{H}^1(S)$$

Notice that since $D \cap C = \emptyset$, we have $\bar{S}_0' \cap C = S_0' \cap C = (\bar{S}_0' \setminus D) \cap C = \bar{S}_0 \cap C$. So we can apply Lemma 5.4 which states that $((S \cup C) \setminus S_0) \cup S_0'$ is connected. Then $S' \cup C = S_0' \cup (S \setminus S_0) \cup C = S_0' \cup ((S \cup C) \setminus S_0)$ is connected.

Therefore $S' \in \mathcal{S}t(C)$ and $\mathcal{H}^1(S') < \mathcal{H}^1(S)$ contrary to the assumption $S \in \mathcal{M}(C)$. □

Lemma 5.6. Let $\Sigma \subset X$ be a closed connected set satisfying $\mathcal{H}^1(\Sigma) < \infty$ which contains a simple closed curve Γ. Then $\mathcal{H}^1$-a.e. point $x \in \Gamma$ is a noncut point for Σ.

Proof. Let $x \in \Gamma$ be a cut point for Σ. Then $\Sigma \setminus \{x\}$ has at least two connected components, one of which containing the connected set $\Gamma \setminus \{x\}$. Let L_x be a connected component of $\Sigma \setminus \{x\}$ such that $L_x \cap \Gamma = \emptyset$. Notice that $\bar{L}_x \subset \Sigma$ is also connected, and since L_x is a maximal connected set in $\Sigma \setminus \{x\}$ we can state that $\bar{L}_x \subset L_x \cup \{x\}$.

We claim that $\bar{L}_x = L_x \cup \{x\}$. Otherwise we would have $L_x = \bar{L}_x$ which means that L_x is closed. Now notice that Σ is locally connected, by Proposition 2.2 and hence $\Sigma \setminus \{x\}$ is also locally connected. This means that L_x, which is a connected component of a locally connected space, is relatively open in $\Sigma \setminus \{x\}$ and then also in Σ. So L_x is open and closed in Σ which is a contradiction because Σ is connected. So the claim is proved.

As a consequence we notice that $\mathcal{H}^1(L_x) > 0$. In fact $\bar{L}_x$ is connected and it is not a single point, because L_x is not empty. Hence $\mathcal{H}^1(L_x) = \mathcal{H}^1(\bar{L}_x) > 0$.

Now we claim that if $x, y \in \Gamma$ are two different cutpoints of Σ then $L_x \cap L_y = \emptyset$. In fact suppose by contradiction that $L_x \cap L_y \neq \emptyset$. Then $L_x \cup L_y$ would be connected and disjoint from Γ. But since L_x is a maximal connected set in $\Sigma \setminus \{x\}$ we would have $L_x \supset L_x \cup L_y$ which means $L_x = L_y$. Then we notice that $L_y \cup \{x\} = \bar{L}_x$ is also connected and contained in $\Sigma \setminus \{y\}$ so $L_y \supset L_y \cup \{x\}$ i.e. $x \in L_y$. But this is a contradiction since $x \in \Gamma$ and $L_y \cap \Gamma = \emptyset$.

Hence for every noncut point $x \in \Gamma$ we are able to find disjoint sets L_x with $\mathcal{H}^1(L_x) > 0$. Since by assumption we have $\mathcal{H}^1(\Sigma) < +\infty$ then the set of noncut points of Σ in Γ is at most countable. $\square$

Lemma 5.7. Let $S \in \mathcal{S}t(C)$ with $\mathcal{H}^1(S) < +\infty$. Then $S \cup C$ is precompact.

Proof. Consider an arbitrary sequence $\{x_\nu\} \subset C \cup S$. We have to show that it admits a convergent subsequence. This is trivial if

(i) either it contains an (infinite) subsequence belonging to C (since C is compact by assumption), or
(ii) it contains an (infinite) subsequence belonging to some connected component S_0 of S (since $\mathcal{H}^1(S_0) \leq \mathcal{H}^1(S) < +\infty$ and thus $\mathcal{H}^1(\bar{S}_0) = \mathcal{H}^1(S) < +\infty$ by Lemma 2.6, which implies that $\bar{S}_0$ is compact, hence S_0 is precompact).

Consider the remaining case, namely when for a subsequence of ν (not relabeled) one has $x_\nu \in S_\nu \setminus C$, where S_ν is some connected component of S having strictly positive length, and all S_ν are different. But then since

$$\sum_\nu \mathcal{H}^1(S_\nu) = \mathcal{H}^1(\cup S_\nu) \leq \mathcal{H}^1(S) < +\infty,$$

one has $\mathcal{H}^1(S_\nu) \to 0$ as $\nu \to +\infty$, and thus $\mathcal{H}^1(\bar{S}_\nu) \to 0$ because $\mathcal{H}^1(\bar{S}_\nu) = \mathcal{H}^1(S_\nu)$ by Lemma 2.6. Let $z_\nu \in \bar{S}_\nu \cap C$ (such a point exists in view of Lemma 2.5). Minding that $\mathcal{H}^1(\bar{S}_\nu) \geq d(x_\nu, z_\nu)$, we get $d(x_\nu, z_\nu) \to 0$ as $\nu \to +\infty$, and it suffices now to extract a convergent subsequence of $\{z_\nu\} \subset C$ to conclude the proof. $\square$

Proof of Theorem 5.1. Let $S \in \mathcal{M}(C)$ with $\mathcal{H}^1(S) < +\infty$. We will prove compactness of $S \cup C$ by showing that

$$\bar{S} \subset S \cup C. \tag{5}$$

In fact, the latter inclusion implies that $S \cup C$ is closed, hence compact by Lemma 5.7.

To prove (5), suppose the contrary, i.e. the existence of an $x \in \bar{S}$, with $x \notin S$ and $x \notin C$. Then minding Lemma 4.5, we conclude that there is a connected component S_0 of S with $\mathcal{H}^1(S_0) > 0$ satisfying $x \in \bar{S}_0$. Then x is a noncut point of $\bar{S}_0$. In fact, otherwise if $S_0 \setminus \{x\}$ were not connected, there would exist $y \in \bar{S}_0$ belonging to a connected component of $\bar{S}_0 \setminus \{x\}$ not containing S_0. Note that by Lemma 2.6 one has $\mathcal{H}^1(\bar{S}_0) = \mathcal{H}^1(S_0)$, hence in particular $\mathcal{H}^1(\bar{S}_0) \leq \mathcal{H}^1(S) < +\infty$, and therefore $\bar{S}_0$ is arcwise connected by Proposition 2.2. Consider then any arc $\sigma \subset \bar{S}_0$ connecting y to x. We have that $\sigma \cap S_0 = \emptyset$ (otherwise y would have been connected to a point of S_0 by an arc not containing x contrary to the assumption on y), hence $\sigma \subset \bar{S}_0 \setminus S_0$. But $\mathcal{H}^1(\sigma) > 0$ which contradicts the equality $\mathcal{H}^1(\bar{S}_0) = \mathcal{H}^1(S_0)$ thus concluding the proof of the fact that x is a noncut point of $\bar{S}_0$. We have now a contradiction with Lemma 5.5, which shows the validity of (5), and hence concludes the proof of compactness of $S \cup C$ which is statement (a) of the theorem.

To prove statement (b) note now that the closure of every connected component of $S \setminus C$ must touch C (otherwise $S \cup C$ would be disconnected by Lemma 2.5), and hence is not reduced to a point, so that in particular it must have strictly positive length. Therefore, there are only a countable number of such components.

For statement (c) suppose that Γ is a closed loop in $\bar{S}$. Then either $\Gamma \subset C$ or $\mathcal{H}^1(\Gamma \setminus C) > 0$. In the first case we get a contradiction because $\bar{S} \setminus \Gamma \in \mathcal{S}t(C)$ while $\mathcal{H}^1(\bar{S} \setminus \Gamma) < \mathcal{H}^1(\bar{S}) = \mathcal{H}^1(S)$. In the case $\mathcal{H}^1(\Gamma \setminus C) > 0$ by Lemma 5.6, there is a noncut point $z \in \Gamma \setminus C$ of some connected component of $\bar{S}$, in contradiction with Lemma 5.5, according to which one should have $z \in C$.

Finally, we show that $S \in \mathcal{M}(C)$ implies $\bar{S} \setminus C \in \mathcal{M}(C)$. In fact, in this case $(\bar{S} \setminus C) = \bar{S} \cup C = \overline{S \cup C}$ is connected since so is $S \cup C$. Hence it enough to prove

$\mathcal{H}^1(\bar{S} \setminus C) \leq \mathcal{H}^1(S)$. Assuming $\mathcal{H}^1(S) < +\infty$ (otherwise the assertion is trivial), we have by claim (a) that $\bar{S} \cup C = S \cup C$, hence

$$\bar{S} \setminus C = (\bar{S} \cup C) \setminus C = (S \cup C) \setminus C = S \setminus C,$$

which gives the desired statement. □

To prove Proposition 5.2 we will need in addition the following lemmata which will be also employed in the sequel.

Lemma 5.8. If $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$, then for every connected component S_0 of $S \setminus C$ one has $S_0 \in \mathcal{M}(C_0)$, where C_0 stands for the union of connected components of C touching $\bar{S}_0$.

Proof. Suppose the contrary, i.e. that there is an $S_0' \in \mathcal{S}t(C_0)$ such that $\mathcal{H}^1(S_0') < \mathcal{H}^1(S_0)$. Then for $S' := (S \setminus S_0) \cup S_0'$ one clearly has $\mathcal{H}^1(S') < \mathcal{H}^1(S)$. On the other hand,

$$S' \cup C = \Big((S \setminus S_0) \cup (S_0' \cup C_0) \Big) \cup C$$

and hence is connected by Lemma 5.4 (applied with the connected set $S_0' \cup C_0$ instead of S_0'), contrary to the optimality of S. □

Lemma 5.9. If $S \in \mathcal{M}(C)$, S is connected, $\mathcal{H}^1(S) < +\infty$, $S \cap C = \emptyset$ and $x \in \bar{S} \cap C$, then x is a noncut point of $\bar{S}$.

Proof. Suppose the contrary, i.e. that $x \in \bar{S} \cap C$ is a cut point and consider a $z \in S$, $z \neq x$. Combining Lemma 2.6 with the assumptions we get $\mathcal{H}^1(\bar{S}) = \mathcal{H}^1(S) < +\infty$, and hence in particular $\bar{S}$ is arcwise connected, and so is S (because $S = \bar{S} \setminus C$ and hence is relatively open in $\bar{S}$). For every $y \in \bar{S} \setminus C$, since $\bar{S} \setminus C = S$ by Theorem 5.1(a), there is an arc connecting y to z in S (hence, such a y belongs to the same connected component of $\bar{S} \setminus \{x\}$ as z). Therefore, we have that there is an $y \in \bar{S} \cap C$ such that every arc $[y, z]$ connecting y to z in $\bar{S}$ passes through x. But then for a subarc $[y, x] \subset [y, z]$ one has $[y, x] \subset C$, since otherwise, if there would be a point $u \in (y, x)$, $u \notin C$, then connecting u by some arc $[u, z] \subset S$ to z, we would get that the arc $[y, u] \circ [u, z]$ connects y to z without passing through x, contrary to the assumption. Letting now $S' := \bar{S} \setminus [y, x]$, we get $S' \cup C = S \cup C$ and hence is connected, while $\mathcal{H}^1(S') < \mathcal{H}^1(\bar{S}) = \mathcal{H}^1(S)$ contrary to the optimality of S. This concludes the proof. □

Proof of Proposition 5.2. It is obvious that (c_1) follows from Lemma 5.5. As for (c_2), let C_0 stand for the union of connected components of C touching $\bar{S}_0$. By Lemma 5.8 one has $S_0 \in \mathcal{M}(C_0)$, and hence every $x \in \bar{S}_0 \cap C = \bar{S}_0 \cap C_0$ is a cut point of $\bar{S}_0$ by Lemma 5.9 applied with S_0 and C_0, hence is an endpoint of $\bar{S}_0$, since $\bar{S}_0$ may not have closed loops by Theorem 5.1(c). □

6. Equivalence of different problem settings

We discuss now some of the consequences and by-products of Theorem 5.1 regarding the relationships between problem (ST) and problems (ST_i), $i = 1, 2$. First we note that problems (ST) and (ST_2) are in fact equivalent in the following sense.

Proposition 6.1. *If S solves problem (ST_2), then $S \in \mathcal{M}(C)$, i.e. S solves problem (ST). Conversely, every solution $S \in \mathcal{M}(C)$ to problem (ST) solves problem (ST_2). In particular, problems (ST_2) and (ST) are either both solvable or both not solvable, while in the former case*

$$\min\{\mathcal{H}^1(S') : S \in \mathcal{S}t(C)\} = \min\{\mathcal{H}^1(S') : S' \in \mathcal{S}t_2(C)\}. \tag{6}$$

Proof. The first claim is nothing else but Lemma 4.7, while the second one follows from Theorem 5.1. □

Consider now problem (ST_1). Clearly, there is plenty of situations when it has no solutions of finite length, while problem (ST) has such solutions (consider, for instance, the case of C being the union of two disjoint balls in $\mathbb{R}^n$). Nevertheless the following equivalence result holds.

Proposition 6.2. *Problems (ST) and (ST_1) are equivalent in the following sense.*

(i) *For every solution S to problem (ST) the set $S \cup C$ solves (ST_1);*
(ii) *if $\mathcal{H}^1(C) < +\infty$, then for every solution Σ to problem (ST_1) the set $\Sigma \setminus C$ solves (ST).*

In particular, for every solution Σ to problem (ST_1) its closure $\bar{\Sigma}$ also solves problem (ST_1), while if $\mathcal{H}^1(\Sigma) < +\infty$, then

(a') *Σ is compact,*
(b') *$\Sigma \setminus C$ has at most countably many connected components, and each of the latter has strictly positive length,*
(c') *$\overline{\Sigma \setminus C}$ contains no closed loops (homeomorphic images of $\mathbb{S}^1$).*

Proof. Let $S \in \mathcal{S}t(C)$ solve problem (ST). We show that $\Sigma := S \cup C$ solves (ST_1). In fact, suppose the contrary, i.e. the existence of a connected $\Sigma' \subset X$ such that $C \subset \Sigma'$ and $\mathcal{H}^1(\Sigma') < \mathcal{H}^1(\Sigma)$. Since $\mathcal{H}^1(\Sigma') < +\infty$, one has $\mathcal{H}^1(C) < +\infty$. Therefore,

$$\mathcal{H}^1(\Sigma' \setminus C) = \mathcal{H}^1(\Sigma') - \mathcal{H}^1(C) < \mathcal{H}^1(\Sigma) - \mathcal{H}^1(C) = \mathcal{H}^1(S \setminus C) \leq \mathcal{H}^1(S),$$

which contradicts the optimality of S for problem (ST), and thus shows the claim (i).

Symmetrically, let $\mathcal{H}^1(C) < +\infty$, and assume that Σ be a solution to problem (ST_1). To show that $\Sigma \setminus C$ solves problem (ST), we suppose the contrary, i.e. the existence of an $S \subset X$ such that $S \cup C$ is connected and $\mathcal{H}^1(S) < \mathcal{H}^1(\Sigma \setminus C)$. Then

$$\mathcal{H}^1(S \cup C) \leq \mathcal{H}^1(S) + \mathcal{H}^1(C) < \mathcal{H}^1(\Sigma \setminus C) + \mathcal{H}^1(C) = \mathcal{H}^1(\Sigma),$$

contrary to the optimality of Σ for problem (ST). This shows (ii).

Claims (a'), (b'), (c') follow by combining Theorem 5.1 with the above claims (i) and (ii). □

7. Regularity of minimizers

The aim of this section is to prove that for every minimizer Σ, the set $\Sigma \setminus C$ is locally a finite embedded geodesic graph, as explained by the following definition.

Definition 7.1 (embedded graph)**.** Let $\Gamma = (V, A)$ be an abstract graph. The elements of V are the vertices of Γ while $A \subset V \times V$ identifies the arcs of Γ. An *embedding* of Γ into a metric space X is a couple of functions (f, g) with $f\colon A \times [0,1] \to X$ and $g\colon V \to X$ such that

(1) for each $a \in A$ the curve $f(a,\cdot)\colon [0,1] \to X$ is continuous;
(2) $f((v_0,v_1),0) = g(v_0)$, $f((v_0,v_1),1) = g(v_1)$;
(3) g is injective;
(4) given any $a, a' \in A$, $t, t' \in (0,1)$ if $t \neq t'$ or $a \neq a'$ then $f(a,t) \neq f(a',t')$ (i.e. f is injective if restricted to $A \times (0,1)$);

The couple $(\Gamma, (f,g))$ is called an embedded graph.

We say that the the embedded graph and the embedding is Lipschitz (resp. geodesic) if for every $a \in A$ the curve $t \mapsto f(a,t)$ is a Lipschitz curve (resp. a geodesic) in X.

We say that the embedded graph is finite/enumerable, if V is finite/enumerable.

The support of the embedded graph $(\Gamma, (f,g))$ is the set $[\Gamma] := f([0,1] \times A)$. If there is no confusion, we will always identify the graph with its support.

We will call each $g(v)$ for $v \in V$ a vertex of the graph, and $f((u,v),\cdot)$ an arc connecting vertices $g(u)$ with $g(v)$ for $(u,v) \in A$. Clearly the definition asserts that the arcs of an embedded graph do not intersect except at the vertices.

Theorem 7.2 (regularity for finite connections)**.** *If C is finite, then every minimizer $\Sigma \in \mathcal{M}_1(C)$ having $\mathcal{H}^1(S) < +\infty$, is a finite geodesic embedded graph.*

In particular, if X is a Riemannian manifold, every vertex of the graph is either a point of C or a triple joint.

Proof. Let $\Sigma \in \mathcal{M}_1(C)$. If C has only one element, the result is trivial, so suppose that C has at least two elements. In this case Σ is not finite and hence $\Sigma \setminus C \neq \emptyset$.

Fix a point $x_0 \in \Sigma \setminus C$ and let $C := \{c_1, \ldots, c_N\}$. Let $\Sigma_0 := \{x_0\}$. Then define the compact sets $\Sigma_1, \ldots, \Sigma_N$ and the continuous curves $\theta_1, \ldots, \theta_N \colon [0,1] \to X$ as follows. Recall that since $\mathcal{H}^1(\Sigma) < +\infty$, then Σ is arcwise connected. We set the curve θ_k to be the shortest arc in Σ joining c_k with a point $x_k \in \Sigma_{k-1}$ (such an arc exists since Σ is compact by Proposition 6.2(a')). Then define $\Sigma_k := \Sigma_{k-1} \cup \theta_k$. Every curve θ_k is injective and intersects the curves θ_j with $j < k$ only in the point x_k. Thus we have

$$\mathcal{H}^1(\Sigma_N) = \sum_{k=1}^{N} \mathcal{H}^1(\theta_k) \leq \mathcal{H}^1(\Sigma).$$

On the other hand Σ_N is compact, connected and contains C hence, by the minimality of Σ we get $\mathcal{H}^1(\Sigma_N) \geq \mathcal{H}^1(\Sigma)$. Therefore we conclude that $\mathcal{H}^1(\Sigma) = \mathcal{H}^1(\Sigma_N)$ and hence $\Sigma = \Sigma_N$.

For k fixed there might be points x_j with $j > k$ which lie on θ_k. By possibly splitting θ_k into pieces $\theta_k^1, \ldots, \theta_k^{n_k}$, we end up with a finite collection of injective curves which have pairwise disjoint interiors. Hence we have found an embedded graph with vertices on the set $C \cup \{x_0, x_1, \ldots, x_{N-1}\}$. □

We have then the following results.

Theorem 7.3 (pruning)**.** *Let $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$. Then for $\mathcal{L}^1$-a.e. $\varepsilon > 0$ one has that for $U_\varepsilon = \{x \in X \colon \operatorname{dist}(x, C) < \varepsilon\}$ the set $S_\varepsilon := S \setminus U$ is a finite geodesic embedded graph (in particular, it has a finite number of connected components and a finite number of branching points). Moreover, for every open set $U \subset X$ such that $C \subset U$ the set $\tilde{S} := S \setminus U$ is a subset of (the support of) a finite geodesic embedded graph, while for each connected component $\tilde{S}^0$ of $\tilde{S}$ one has $\tilde{S}^0 \in \mathcal{M}_1(\tilde{S}^0 \cap \partial U)$.*

Proof. The proof will be achieved in several steps.

Step 1. For every $\varepsilon > 0$ consider the open sets

$$U_\varepsilon := \{x \in X \colon \operatorname{dist}(x, C) < \varepsilon\},$$

We claim that for $\mathcal{L}^1$-a.e. $\varepsilon > 0$ the set $C_\varepsilon := S \cap \partial U_\varepsilon$ has finitely many points. This follows from the general coarea estimate of Lemma 2.1 (applied with $f(x) := \operatorname{dist}(x, C)$), namely,

$$\int_0^{+\infty} \mathcal{H}^0\left(S \cap \{x \in X \colon \operatorname{dist}(x, C) = t\}\right) dt \leq \mathcal{H}^1(S)$$

once we recall that $\mathcal{H}^0$ is just the counting measure, i.e. the cardinality of the set and that

$$\partial U_t = \{x \in X \colon \operatorname{dist}(x, C) = t\}.$$

Step 2. We will prove that for $\mathcal{L}^1$-a.e. $\varepsilon > 0$, the set $S_\varepsilon := S \setminus U_\varepsilon$ has finitely many connected components. To this aim we may suppose that C_ε has finitely many points. Let S_ε^0 be a connected component of S_ε. Define

$$C_\varepsilon^0 := S_\varepsilon^0 \cap \partial U_\varepsilon = S_\varepsilon^0 \cap C_\varepsilon.$$

Then $C_\varepsilon^0 \neq \emptyset$ for all sufficiently small $\varepsilon > 0$, unless C is connected (in which case one has $S \setminus C = \emptyset$ and hence there is nothing to prove). So to every connected component of S_ε we can associate the finite set C_ε^0. These sets are all disjoint subsets of C_ε because the connected components are disjoint. Since C_ε is finite we conclude that also the number of connected components of S_ε is finite.

Step 3. We prove now that for an arbitrary open set $U \subset X$ such that $C \subset U$, every connected component $\tilde{S}^0$ of the set $\tilde{S} := S \setminus U$ satisfies $\tilde{S}^0 \in \mathcal{M}_1(\tilde{S}^0 \cap \partial U)$. Clearly $\tilde{S}^0 \in \mathcal{S}t_1(\tilde{S}^0 \cap \partial U)$ by construction, and $\mathcal{H}^1(\tilde{S}^0) \leq \mathcal{H}^1(S) < +\infty$. Consider an arbitrary $\tilde{S}' \in \mathcal{S}t_1(\tilde{S}^0 \cap \partial U)$ and define

$$S' := (S \setminus \tilde{S}^0) \cup \tilde{S}'.$$

Obviously, if we suppose by contradiction that $\mathcal{H}^1(\tilde{S}') < \mathcal{H}^1(\tilde{S}^0)$ then also $\mathcal{H}^1(S') < \mathcal{H}^1(S)$. Further, clearly there is a connected component S^0 of S such that $\tilde{S}^0 \subset S^0$, and hence we have $S' = (S \setminus S^0) \cup ((S^0 \setminus \tilde{S}^0) \cup \tilde{S}')$, while $(S^0 \setminus \tilde{S}^0) \cup \tilde{S}'$ is connected. Hence by Lemma 5.4 the set S' is connected, which contradicts the optimality of S and proves the claim.

Step 4. Now we apply the result of Step 3 to the case $U = U_\varepsilon$. Then, with the notation of Step 1, we have that for $\mathcal{L}^1$-a.e. $\varepsilon > 0$, for every connected component S_ε^0 of S_ε we have $S_\varepsilon^0 \in \mathcal{M}_1(C_\varepsilon^0)$ where $C_\varepsilon^0 := S_\varepsilon^0 \cap \partial U_\varepsilon$. Hence by Theorem 7.2, the set S_ε^0 is a finite geodesic embedded graph. Since for every open $U \subset X$ such that $C \subset U$ there is a sufficiently small $\varepsilon > 0$ such that $U_\varepsilon \subset U$, then we have shown that every connected component of the set $\tilde{S} := S \cap \bar{U}$ is a subset of a finite geodesic embedded graph, which concludes the proof. □

In the case when the set C has finitely many connected components we can say more.

Theorem 7.4. *If $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$ with the set C having finitely many connected components, then $S \setminus C$ has finitely many connected components, the closure of each of which is a finite geodesic embedded graph with endpoints on C, and with at most one endpoint on each connected component of C.*

Proof. Let $\{C_i\}_{i=1}^k$ be connected components of C and consider an $\varepsilon > 0$ such that the sets

$$V_\varepsilon^i := \{x \in X \,:\, \operatorname{dist}(x, C_i) < \varepsilon\}$$

be pairwise disjoint. From Lemma 7.5 below we conclude that no connected component of $S \setminus C$ is entirely contained in some V_ε^i. But since by Theorem 7.3 the set $S \setminus \bigcup_i V_\varepsilon^i$ has only finitely many connected components, while $S \setminus \bigcup_i V_\varepsilon^i \subset S \setminus C$, then $S \setminus C$ has only a finite number of connected components.

Note that by Lemma 5.8 for every connected component S_0 of $S \setminus C$ one has $S_0 \in \mathcal{M}(C_0)$, where C_0 stands for the union of connected components of C touching $\bar{S}_0$. Then from Lemma 7.7 we get that $\bar{S}_0$ touches each connected component of C_0 in exactly one point, while by Proposition 5.2 the finite set $C_0' := C_0 \cap \bar{S}_0$ is the set of all endpoints of $\bar{S}_0$. Finally, it remains to observe that $\bar{S}_0 \in \mathcal{M}(C_0')$ and refer to Theorem 7.2. □

Lemma 7.5. If $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$, then for every connected component S_0 of $S \setminus C$ one has that $\bar{S}_0$ touches at least two different connected components of C.

Proof. Suppose the contrary, i.e. that for some connected component S_0 of $S \setminus C$ one has that $\bar{S}_0$ touches just a single connected component C_0 of C. Let then $S' := S \setminus (\bar{S}_0 \setminus C)$. Then $S' \cup C$ is connected by Lemma 5.4 (applied with C_0 in place of S_0') since

$$S' \cup C = \Big((S \setminus S_0) \cup C_0\Big) \cup C.$$

On the other hand, since clearly $\mathcal{H}^1(S_0) = \mathcal{H}^1(\bar{S}_0) > 0$, then $\mathcal{H}^1(S') < \mathcal{H}^1(S)$, contrary to the optimality of S. □

We can prove also the following general result.

Theorem 7.6. *If $S \in \mathcal{M}(C)$ and $\mathcal{H}^1(S) < +\infty$, then the closure of every connected component of S is a topological tree with endpoints on C, and with at most one endpoint on each connected component of C. Each branching point of this tree has finite order and the number of branching points is at most countable.*

Proof. The closure $\bar{S}_0$ of every connected component S_0 of S is a locally connected continuum by Proposition 2.2. The fact that it is a topological tree follows then from Theorem 5.1(c). Its endpoints belong to C by Proposition 5.2(c_2) and there is at most one endpoint on each connected component of C by Lemma 7.7 below. Mind that every topological tree has at most countable number of branching points [7, theorem VI.7,§ 51]. Further, every branching point x of $\bar{S}_0$ must belong to $X \setminus C$ by Proposition 5.2(c_1), hence to $X \setminus C_\varepsilon$ for all sufficiently small $\varepsilon > 0$ (depending of course on x. Minding that $\bar{S}_0 \setminus C_\varepsilon$ is a finite geodesic embedded graph for an appropriate $\varepsilon > 0$ by Theorem 7.3, we have $\mathrm{ord}_{\bar{S}_0} x < +\infty$. □

Lemma 7.7. Let $S \in \mathcal{M}(C)$ be connected, $S \cap C = \emptyset$, and $\mathcal{H}^1(S) < +\infty$. Then the set $\bar{S} \cap C^0$ is a singleton for every connected component C^0 of C.

Proof. Suppose the contrary, i.e. that $\bar{S} \cap C^0$ contains at least two points, $\{x, y\} \subset \bar{S} \cap C^0$. Since $\bar{S}$ is a topological tree by Theorem 5.1(c), there is a unique arc $[x, y] \subset \bar{S}$. Clearly $\mathcal{H}^1(\bar{S} \cap C) = 0$, since otherwise for $\tilde{S} := \bar{S} \setminus C$ one would have

$$\mathcal{H}^1(\tilde{S}) = \mathcal{H}^1(\bar{S}) - \mathcal{H}^1(\bar{S} \cap C) < \mathcal{H}^1(\bar{S}) = \mathcal{H}^1(S)$$

(minding Lemma 2.6 in the last equality), which would contradict the optimality of S since $\tilde{S} \in \mathcal{S}t(C)$. Therefore, $\mathcal{H}^1$-a.e. point of $[x, y]$ is outside of C. Consider then an open $U \subset X$ with $C \subset U$ such that $\mathcal{H}^1([x, y] \setminus U) > 0$. Since every connected component of $[x, y] \setminus U$ belongs to some connected component of $\bar{S} \setminus U$ which, by Theorem 7.2 is a finite graph, then there is a $z \in [x, y] \setminus U$ such that $\mathrm{ord}_{\bar{S}} z = 2$. This means in particular that there exists an open $V \subset X$ such that $z \in V$, $\bar{V} \cap C = \emptyset$ and $\#(\partial V \cap \bar{S}) = 2$. Therefore there exists an $x' \in \partial V \cap \bar{S}$ such that $[x, x'] \subset [x, y]$ is outside of V, and, symmetrically, an $y' \in \partial V \cap \bar{S}$ such that $[y', y] \subset [x, y]$ is outside of V. We prove now that for $S' := S \setminus V$ one has that $S' \cup C$ is connected. This is enough to arrive at a contradiction since $\mathcal{H}^1(S') < \mathcal{H}^1(S)$ against the optimality of S.

Therefore, it remains to prove that $S' \cup C$ is connected, which will be achieved by Lemma 5.4 applied with $S_0 := S$ and $S_0' := (S \setminus V) \cup C^0$ once we show that S_0' is connected. Proving the latter claim amounts to showing that for every $u \in S_0'$ there is a connected set $\sigma \subset S_0'$ containing both x and u. This is clearly the case when $u \in C^0$ (then just take $\sigma := C^0$). Otherwise, if $u \in S \setminus V$, then consider the arc $[u, x] \subset \bar{S}$. If $[u, x] \cap V = \emptyset$, then we may just take $\sigma := [u, x]$, while if $[u, x] \cap V \neq \emptyset$, then $[u, y'] \cap V = \emptyset$ and hence we take $\sigma := ([u, y'] \circ [y', y]) \cup C^0$. □

Example 7.8. It is worth emphasizing that Theorem 7.6 does not assert that every connected component of C is touched by the closure of some connected component of S. In fact, the latter assertion is, generally speaking, false. Consider for instance $X = \mathbb{R}$, $C \subset [0, 1]$ be the Cantor set. Then a set S is optimal for problem (ST), if and only if

$$[0, 1] \setminus C \subset S \subset [0, 1]$$

In particular, $S := [0, 1] \setminus C \in \mathcal{M}(C)$. This set is open and hence is a countable union of intervals (which are hence connected components of S). Therefore, the closures of connected components of S touch only a countable number of connected components of C (recall that the latter form an uncountable set since C is totally

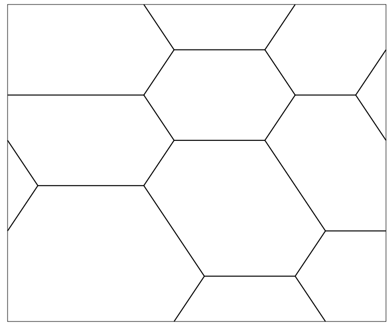

Figure 2. A locally minimal network.

disconnected). In other words, for a more than countable number of connected components of C it is true that they are not touched by the closures of any single connected component of S (rather, for each such component C^0 of C there is a sequence of components of S getting arbitrarily close to C^0).

Corollary 7.9. If $S \in \mathcal{M}(C)$, then the closure of every connected component of S has at most countable number of branching points.

Proof. Since $\mathcal{H}^1(S\cap C) = 0$, one has that the connected components of S belonging to C are singletons for which the statement being proven is automatically valid. Thus one has to consider only such connected components S_0 of S that $S_0 \setminus C \neq \emptyset$. By Theorem 7.6, one has that $\bar{S}_0$ is a topological tree and hence it has at most countable number of branching points, and it remains to observe that the set of such connected components is at most countable by Theorem 5.1(b) (equivalently, one may have referred to Theorem 7.3). □

8. A remark on local minima

In this section we study the following concept.

Definition 8.1. Let X be a metric space. A set $\Sigma \subset X$ is called *locally minimal network*, if for every $x \in X$ there is an open $U \subset X$ such that $x \in U$, $\Sigma \cap \partial U \neq \emptyset$ and

$$\Sigma \cap \bar{U} \in \mathcal{M}(\overline{\Sigma \cap U} \cap \partial U).$$

In Figure 2 we present an example of a locally minimal network in the plane. Notice that such sets can be unbounded and may have loops.

We can immediately list now some of the basic properties of locally minimal networks. In fact, every locally minimal network is

- *locally connected* (by definition),
- *closed* (in view of Theorem 5.1(a)).

However, the most important property of locally minimal networks is the following.

Theorem 8.2. *Let X be a complete metric space. Every locally minimal network $\Sigma \subset X$ is locally a finite geodesic graph, in the sense that for every compact set $K \subset X$ there is a closed set $S \supset K$ such that $\Sigma \cap S$ is the (support of a) finite geodesic embedded graph. In particular, $\Sigma \cap S$ has a finite number of connected components and a finite number of branching points (hence also $\Sigma \cap K$ has a finite number of branching points).*

Proof. Let $\tilde{K} := K \cap \Sigma$. The set $\tilde{K}$ is compact since Σ is closed. Now, for every $x \in \tilde{K}$ choose an open $U \subset X$ such that $x \in U$, $\Sigma \cap \partial U \neq \emptyset$ and

$$\Sigma \cap \bar{U} \in \mathcal{M}(\overline{\Sigma \cap U} \cap \partial U).$$

By Theorem 7.3 combined with the coarea inequality (Theorem 2.1) there is an open $V \subset U$ with $x \in V$, such that $\#(\Sigma \cap \partial V) < +\infty$ and $\Sigma \cap \bar{V}$ is a finite graph (it is enough to take $V := \{y \in U \ : \ \mathrm{dist}\,(y, \Sigma \cap \partial U) > \varepsilon\}$ for a sufficiently small $\varepsilon > 0$). By compactness the set $\tilde{K}$ may be covered by a finite family of the latter sets $\{V_i\}_{i=1}^l$. Let $\tilde{S} := \bigcup_i \bar{V}_i$. We show that $\Sigma \cap \tilde{S}$ is the support of some finite geodesic embedded graph. For this purpose we assume without loss of generality that for all $i = 1, \ldots, l$ and all $j = 1, \ldots, l$ all the points in $\Sigma \cap \bar{V}_i \cap \partial V_j$ (there is only a finite number of them by construction) are vertices of a finite geodesic embedded graph with support $\Sigma \cap \bar{V}_i$ (if it is not so, just add the as new vertices and split the respective arcs accordingly, so that the support of the graph will not change). Let us prove the claim by induction. In fact, $\Sigma \cap \bar{V}_1$ is the support of a finite geodesic embedded graph. Suppose that $\Sigma^k := \Sigma \cap (\bar{V}_1 \cup \ldots \cup \bar{V}_k)$ is so for some $k < l$. Notice that every arc σ of the graph $\Sigma \cap \bar{V}_{k+1}$ is either an arc of Σ^k or may touch Σ^k only at the vertices of the latter (since every such arc by construction meets every ∂V_i only at the vertex). Hence by adding to Σ^k all the arcs σ of the graph $\Sigma \cap \bar{V}_{k+1}$ which do not belong to Σ^k, we get a finite graph with support Σ^{k+1}. Thus $\Sigma \cap \tilde{S} = \Sigma^{k+1}$ is the support of a finite geodesic embedded graph as claimed.

To conclude the proof it suffices to take as a set S any closed $S \supset K$ satisfying $S \cap \Sigma = \tilde{S} \cap \Sigma$ (e.g. one can take $S := K \cup \tilde{S}$). ☐

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