Ammann Tilings in Symplectic Geometry

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Abstract. In this article we study Ammann tilings from the perspective of symplectic geometry. Ammann tilings are nonperiodic tilings that are related to quasicrystals with icosahedral symmetry. We associate to each Ammann tiling two explicitly constructed highly singular symplectic spaces and we show that they are diffeomorphic but not symplectomorphic. These spaces inherit from the tiling its very interesting symmetries.

Key words: symplectic quasifold; nonperiodic tiling; quasilattice

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1 Introduction

Our general aim is to study the connection between symplectic geometry and the nonperiodic tilings that are related to the geometry of quasicrystals[1]. Considering these tilings from the symplectic viewpoint provides a concrete way of obtaining new examples of highly singular symplectic spaces that are endowed with very rich symmetries. These examples effectively contribute to understanding the theoretical aspects of the geometry of this type of singular spaces. Moreover, we expect that symplectic geometry may be used to shed light on the study of the tilings.

We started this program with the study of two tilings of the plane: Penrose rhombus [2] and kite and dart [3] tilings.

In this article we focus our attention for the first time on three-dimensional tilings. We consider *Ammann tilings*, which are the three-dimensional analogues of Penrose rhombus tilings. They were introduced by Ammann in the 70's [10] and turned out to be related to quasicrystals with icosahedral symmetry [6]. As we will see later, the third dimension yields an initially unexpected richness and complexity.

The main idea underlying the connection between symplectic geometry and tilings is a generalization [8] of the Delzant construction [5], which we use to associate to each tile an explicitly constructed symplectic space. We recall that the Delzant construction associates a symplectic toric $2n$-manifold to each simple convex polytope in $(\mathbb{R}^n)^*$, which is rational with respect to a lattice and satisfies an additional integrality condition. The problem, in the setting of nonperiodic tilings related to quasicrystals, is that either the tiles are not individually rational, or they are not simultaneously rational with respect to the same lattice. In the generalized

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[1] Quasicrystals are special materials having discrete nonperiodic diffraction patterns that were experimentally discovered by Shechtman et al. [11] in 1982. They have atomic arrangements with symmetries that are not allowed in ordinary crystals. For a comprehensive review of this fascinating subject we refer the reader to the recent book by Steurer and Deloudi [12].

construction, however, the lattice is replaced by a *quasilattice* and rationality is replaced by the notion of *quasirationality*. The resulting space is a $2n$-dimensional *quasifold*, a generalization of manifolds and orbifolds that was first introduced by the second-named author in [8]; the group acting is no longer a torus but a *quasitorus* [8].

Ammann tilings are made of two kinds of tiles: an oblate rhombohedron and a prolate rhombohedron having same edge lengths. These rhombohedra, although separately rational, are not simultaneously rational with respect to the same lattice. However, the geometry of the tiling ensures that it is possible to choose a quasilattice F having the property that each rhombohedron of the tiling is quasirational with respect to F (Proposition 3.1). We then apply the generalized Delzant construction simultaneously to each rhombohedron and we show that there is one symplectic quasifold, M_b, associated to each of the oblate rhombohedra of the tiling and one symplectic quasifold, M_r, associated to each of the prolate rhombohedra (Theorem 4.1). Both quasifolds are globally the quotient of a manifold (the product of three 2-spheres) modulo the action of a discrete group. There is a unique diffeotype and two symplectotypes associated to the tiling. In fact, we show that M_b and M_r are diffeomorphic but not symplectomorphic, consistently with the fact that the two different types of tiles have different volumes (Theorem 6.1).

We remark that quasilattices are the fundamental structure underlying both nonperiodic tilings related to quasicrystals and the corresponding symplectic quasifolds. This is particularly evident for Ammann tilings, and the related physics of icosahedral quasicrystals. The novelty here, with respect to two-dimensional tilings, is that, in this context, there are actually *three* important quasilattices: the quasilattice F above, known as *face-centered lattice*, the *body-centered lattice*, I, and the *simple icosahedral lattice*, P. These are the only three quasilattices that have icosahedral symmetry [9]. The quasilattice P, up to a suitable rescaling, has the property of containing all of the vertices of the Ammann tiling. The quasilattices F and $\frac{1}{2}I$ are usually thought of in physics as the dual of each other, as they are projections of two 6-dimensional lattices that are dual to one another in the standard sense. Consistently, we show that, in our symplectic setting, F is the group quasilattice of the quasitorus $D^3 = R^3/F$ and that $\frac{1}{2}I$ can be thought of as its dual, the weight quasilattice (Section 5).

The paper is structured as follows: in Section 2 we recall the generalized Delzant construction; in Section 3 we introduce the quasilattices F, I and P, and we discuss their connection with the tiling; in Section 4 we construct the symplectic quasifolds M_b and M_r; in Section 5 we study their local geometry; finally, in Section 6 we show that they are diffeomorphic but not symplectomorphic.

2   The generalized Delzant construction

We now recall from [8] the generalized Delzant construction. For the notion of quasifold, of related geometrical objects and for a number of examples we refer the reader to the original article [8] and to [3], where some of the definitions were reformulated.

Let us recall what a *simple* convex polytope is.

Definition 2.1 (simple polytope)**.** A dimension n convex polytope $\Delta \subset \left(\mathbb{R}^n\right)^*$ is said to be *simple* if there are exactly n edges stemming from each vertex.

Let us next define the notion of *quasilattice*, introduced in [7]:

Definition 2.2 (quasilattice)**.** Let E be a real vector space. A *quasilattice* in E is the $\mathbb{Z}$-span of a set of $\mathbb{R}$-spanning vectors, $Y_1, \ldots, Y_d$, of E.

Notice that $\mathrm{Span}_{\mathbb{Z}}\{Y_1, \ldots, Y_d\}$ is a lattice if and only if it admits a set of generators which is a basis of E.

Consider now a dimension n convex polytope $\Delta \subset (\mathbb{R}^n)^*$ having d facets. Then there exist elements $X_1, \ldots, X_d$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_d$ in $\mathbb{R}$ such that

$$\Delta = \bigcap_{j=1}^{d} \{\mu \in (\mathbb{R}^n)^* \mid \langle \mu, X_j \rangle \geq \lambda_j\}. \tag{2.1}$$

Definition 2.3 (quasirational polytope). Let Q be a quasilattice in $\mathbb{R}^n$. A convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be *quasirational* with respect to Q if the vectors $X_1, \ldots, X_d$ in (2.1) can be chosen in Q.

We remark that each polytope in $(\mathbb{R}^n)^*$ is quasirational with respect to some quasilattice Q: just take the quasilattice that is generated by the elements $X_1, \ldots, X_d$ in (2.1). Notice that if $X_1, \ldots, X_d$ can be chosen in such a way that they belong to a lattice, then the polytope is rational in the usual sense. Before we go on to describing the generalized Delzant construction we recall what a *quasitorus* is.

Definition 2.4 (quasitorus). Let $Q \subset \mathbb{R}^n$ be a quasilattice. We call *quasitorus* of dimension n the group and quasifold $D = \mathbb{R}^n/Q$.

For the definition of Hamiltonian action of a quasitorus on a symplectic quasifold we refer the reader to [8].

For the purposes of this article we will restrict our attention to the special case $n = 3$.

Theorem 2.1 (generalized Delzant construction [8]). *Let Q be a quasilattice in $\mathbb{R}^3$ and let $\Delta \subset (\mathbb{R}^3)^*$ be a simple convex polytope that is quasirational with respect to Q. Then there exists a 6-dimensional compact connected symplectic quasifold M and an effective Hamiltonian action of the quasitorus $D = \mathbb{R}^3/Q$ on M such that the image of the corresponding moment mapping is Δ.*

Proof. Let us consider the space $\mathbb{C}^d$ endowed with the standard symplectic form

$$\omega_0 = \frac{1}{2\pi i} \sum_{j=1}^{d} dz_j \wedge d\bar{z}_j$$

and the action of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ given by

$$\begin{aligned} \tau \colon\ T^d \times \mathbb{C}^d &\longrightarrow \mathbb{C}^d \\ \left(\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_d}\right), \underline{z}\right) &\longmapsto \left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_d} z_d\right). \end{aligned}$$

This is an effective Hamiltonian action with moment mapping given by

$$\begin{aligned} J \colon\ \mathbb{C}^d &\longrightarrow (\mathbb{R}^d)^* \\ \underline{z} &\longmapsto \sum_{j=1}^{d} |z_j|^2 e_j^* + \lambda, \qquad \lambda = \text{const} \in (\mathbb{R}^d)^*. \end{aligned}$$

The mapping J is proper and its image is given by the cone $\mathcal{C}_\lambda = \lambda + \mathcal{C}_0$, where $\mathcal{C}_0$ denotes the positive orthant of $(\mathbb{R}^d)^*$. Take now vectors $X_1, \ldots, X_d \in Q$ and real numbers $\lambda_1, \ldots, \lambda_d$ as in (2.1). Consider the surjective linear mapping

$$\begin{aligned} \pi \colon\ \mathbb{R}^d &\longrightarrow \mathbb{R}^3 \\ e_j &\longmapsto X_j. \end{aligned}$$
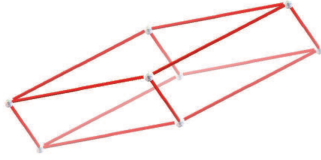
Figure 1. The oblate rhombohedron.
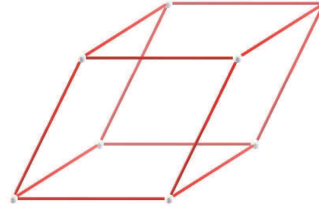
Figure 2. The prolate rhombohedron.

Consider the dimension 3 quasitorus $D = \mathbb{R}^3/Q$. Then the linear mapping π induces a quasitorus epimorphism $\Pi\colon T^d \longrightarrow D$. Define now N to be the kernel of the mapping Π and choose $\lambda = \sum_{j=1}^{d} \lambda_j e_j^*$. Denote by i the Lie algebra inclusion $\mathrm{Lie}(N) \to \mathbb{R}^d$ and notice that $\Psi = i^* \circ J$ is a moment mapping for the induced action of N on $\mathbb{C}^d$. Then the quasitorus T^d/N acts in a Hamiltonian fashion on the compact symplectic quasifold $M = \Psi^{-1}(0)/N$. If we identify the quasitori D and T^d/N via the epimorphism Π, we get a Hamiltonian action of the quasitorus D whose moment mapping Φ has image equal to $(\pi^*)^{-1}(\mathcal{C}_\lambda \cap \ker i^*) = (\pi^*)^{-1}(\mathcal{C}_\lambda \cap \mathrm{im}\,\pi^*) = (\pi^*)^{-1}(\pi^*(\Delta))$ which is exactly Δ. This action is effective since the level set $\Psi^{-1}(0)$ contains points of the form $\underline{z} \in \mathbb{C}^d$, $z_j \neq 0$, $j = 1, \ldots, d$, where the T^d-action is free. Notice finally that $\dim M = 2d - 2 \dim N = 2d - 2(d-3) = 6$. ∎

Remark 2.1. If we want to apply this construction to any simple convex polytope in $\left(\mathbb{R}^3\right)^*$, then there are two arbitrary choices involved. The first is the choice of a quasilattice Q with respect to which the polytope is quasirational, and the second is the choice of vectors $X_1, \ldots, X_d$ in Q that are orthogonal to the facets of Δ and inward-pointing as in (2.1).

3 Ammann tilings and quasilattices

The purpose of this section is to introduce three quasilattices P, F and I, that are relevant for our construction.

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. We will be using extensively the following fundamental identity

$$\phi = 1 + \frac{1}{\phi}. \tag{3.1}$$

Let σ be a positive real number and let us consider an Ammann tiling $\mathcal{T}$ with fixed edge length σ. Ammann tilings are nonperiodic tilings of three-dimensional space by so-called *golden rhombohedra*; rhombohedra are called golden when their facets are given by *golden rhombuses*, namely rhombuses with diagonals that are in the ratio of ϕ. There are two types of such rhombohedra which are called *oblate* and *prolate* (see Figs. 1 and 2)[2]. For a review of Ammann tilings we refer the reader to [10, 12].

Consider now the vectors in $\left(\mathbb{R}^3\right)^*$

$$\alpha_1 = \tfrac{1}{\sqrt{2}}(\phi - 1, 1, 0), \qquad \alpha_2 = \tfrac{1}{\sqrt{2}}(0, \phi - 1, 1), \qquad \alpha_3 = \tfrac{1}{\sqrt{2}}(1, 0, \phi - 1),$$
$$\alpha_4 = \tfrac{1}{\sqrt{2}}(1 - \phi, 1, 0), \qquad \alpha_5 = \tfrac{1}{\sqrt{2}}(0, 1 - \phi, 1), \qquad \alpha_6 = \tfrac{1}{\sqrt{2}}(1, 0, 1 - \phi).$$

These six vectors and their opposites point to the twelve vertices of an icosahedron that is inscribed in the sphere of radius $\sqrt{\frac{3-\phi}{2}}$ (see Figs. 3 and 4); they generate a quasilattice P that is known in physics as the *simple icosahedral lattice* [9].

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[2] All pictures were drawn using the ZomeCAD software.
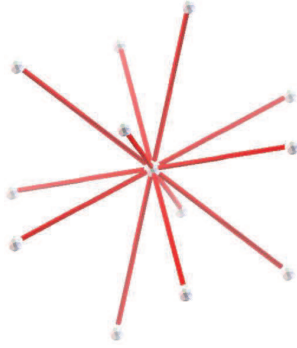
Figure 3. The vectors $\pm\alpha_1, \ldots, \pm\alpha_6$.
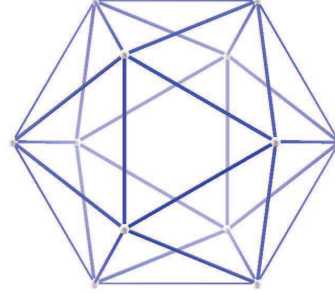
Figure 4. The icosahedron.

Let $\delta = \sqrt{\frac{2}{3-\phi}}\sigma$ and consider the two following golden rhombohedra: the oblate rhombohedron Δ_b^o, having nonparallel edges $\delta\alpha_4$, $\delta\alpha_5$, $\delta\alpha_6$, and the prolate rhombohedron Δ_r^o, having nonparallel edges $\delta\alpha_1$, $\delta\alpha_2$, $\delta\alpha_3$.

Denote by AB one edge of the tiling $\mathcal{T}$. From now on we will choose our coordinates so that $A = O$ and so that $B - A$ is parallel to α_1 with the same orientation.

Proposition 3.1. *Let $\mathcal{T}$ be an Ammann tiling with edges of length σ. Each vertex of the tiling lies in the quasilattice δP. Moreover, for each oblate rhombohedron Δ_b in $\mathcal{T}$ (respectively prolate rhombohedron Δ_r in $\mathcal{T}$) there is a rigid motion ρ, given by the composition of a translation with a transformation of the icosahedral group, such that $\rho(\Delta_b)$ is Δ_b^o (respectively $\rho(\Delta_r)$ is Δ_r^o).*

Proof. Consider first the icosahedron with its twenty pairwise parallel facets. To each pair of parallel facets there correspond two oblate rhombohedra, one the translate of the other, and two prolate rhombohedra, also one the translate of the other. Pick one representative for each such couple. This gives a total of ten oblate rhombohedra and ten prolate rhombohedra. Each of the ten oblate rhombohedra can be mapped to Δ_b^o via a transformation of the icosahedral group, and in the same way each of the ten prolate rhombohedra can be mapped to Δ_r^o.

Now, let C be a vertex of the tiling that is different from 0 and the above vertex B. We can join B to C with a broken line made of subsequent edges of the tiling. We denote the vertices of the broken line thus obtained by $T_0 = A, T_1 = B, \ldots, T_j, \ldots, T_m = C$. Since the tiles are oblate and prolate rhombohedra, each vector $Y_j = T_j - T_{j-1}$ is one of the vectors $\pm\alpha_k$, $k = 1, \ldots, 6$. Therefore we have that $C - A = T_m - T_0 = Y_m + \cdots + Y_1$. This implies that the vertex C lies in δP, that each oblate rhombohedron having C as vertex is the translate of one of the ten oblate rhombohedra described above and that each prolate rhombohedron having C as vertex is the translate of one of the ten prolate rhombohedra described above. We can therefore conclude that, for each oblate rhombohedron Δ_b having C as vertex, there exists a rigid motion ρ, given by the composition of a translation with a transformation of the icosahedral group, such that $\rho(\Delta_b) = \Delta_b^o$. The same is true for the prolate rhombohedra. ∎

We introduce now a quasilattice $F \subset \mathbb{R}^3$ with respect to which *all* of the rhombohedra of the tiling are quasirational (cf. Remark 2.1). This is necessary in order to apply the generalized Delzant procedure simultaneously to all of the rhombohedra in the tiling. We take F to be the quasilattice that is generated by the six vectors

$$U_1 = \tfrac{1}{\sqrt{2}}(1, \phi - 1, \phi), \qquad U_2 = \tfrac{1}{\sqrt{2}}(\phi, 1, \phi - 1), \qquad U_3 = \tfrac{1}{\sqrt{2}}(\phi - 1, \phi, 1),$$
$$U_4 = \tfrac{1}{\sqrt{2}}(-1, \phi - 1, \phi), \qquad U_5 = \tfrac{1}{\sqrt{2}}(\phi, -1, \phi - 1), \qquad U_6 = \tfrac{1}{\sqrt{2}}(\phi - 1, \phi, -1).$$

The quasilattice F is known in physics as the *face-centered lattice* [9].
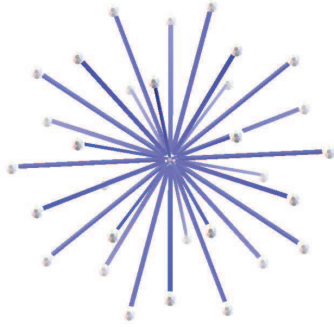
Figure 5. The star of norm $\sqrt{2}$ vectors of the quasilattice F.
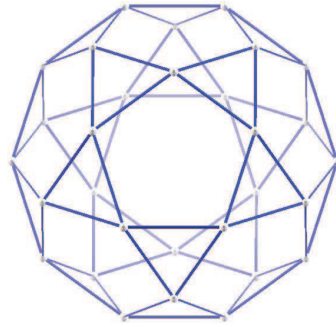
Figure 6. The icosidodecahedron.

The vectors U_i have norm equal to $\sqrt{2}$. It can be easily seen that there are exactly 30 vectors in F having the same norm. These thirty vectors point to the vertices of an icosidodecahedron inscribed in the sphere of radius $\sqrt{2}$ (see Figs. 5 and 6).

Remark 3.1. Proposition 3.1 implies that, for each facet of the Ammann tiling, there is a pair of vectors $\{\alpha_i, \alpha_j\}$ such that the given facet is parallel to the plane Π_{ij} generated by $\{\alpha_i, \alpha_j\}$. We have 15 such possible pairs $\{\alpha_i, \alpha_j\}$, with $i, j = 1, \ldots, 6$, $i \neq j$. For each one of them, two of the 30 vectors above are orthogonal to the corresponding plane Π_{ij}. This ensures that all of the rhombohedra of the tiling are quasirational with respect to F.

Another quasilattice that will be useful in the sequel is the quasilattice $I \subset \left(\mathbb{R}^3\right)^*$ that is generated by the vectors $\{2\alpha_1, 2\alpha_2, 2\alpha_3, 2\phi\alpha_4, 2\phi\alpha_5, 2\phi\alpha_6\}$. The quasilattice I is known in physics as the *body centered lattice* [9].

Remark 3.2. The quasilattices P, F and I are invariant under icosahedral symmetries and are dense in their respective ambient spaces. One can show that, if we identify $\mathbb{R}^3$ with its dual using the standard inner product, we have the following proper inclusions:

$$I \subset F \subset P \subset \tfrac{1}{2}I. \tag{3.2}$$

Using the notation of Conway–Sloane [4], the lattices P, F and $\frac{1}{2}I$ can be obtained as the respective projections from the following lattices in $\mathbb{R}^6$: $\mathbb{Z}^6$,

$$D_6 = \mathrm{Span}\left\{\sum_{j=1}^{6} n_j e_j \,\Big|\, \sum_{j=1}^{6} n_j \text{ is even}\right\},$$

and

$$D_6^* = \mathrm{Span}\left\{\frac{1}{2}\sum_{j=1}^{6} n_j e_j \,\Big|\, n_j \equiv n_k \pmod{2}\right\},$$

the projection being given by

$$\mathbb{R}^6 \longrightarrow \mathbb{R}^3$$
$$e_j \longmapsto \alpha_j.$$

Coherently with (3.2) we have the following proper inclusions:

$$2D_6^* \subset D_6 \subset \mathbb{Z}^6 \subset D_6^*.$$

The lattice $\mathbb{Z}^6$ is self dual, whilst D_6 and D_6^* are the dual of one another. A notion of duality for the icosahedral quasilattices in dimension 3 is derived from the above relations of duality in $\mathbb{R}^6$. This is coherent with the symplectic setup. In fact, we will see that the quasilattice F is the group quasilattice (see Remark 4.1) and that the quasilattice $\frac{1}{2}I$ plays the role of its dual, the weight quasilattice (see Section 5).

4 The tiling from a symplectic viewpoint

In this section we perform the Delzant construction to obtain symplectic quasifolds that can be associated to the oblate and prolate rhombohedra of an Ammann tiling having edge length σ.

Let us consider the quasilattice F that we introduced in Section 3. As we have seen, all of the rhombohedra of our tiling are quasirational with respect to F.

We begin by considering the oblate rhombohedron Δ_b^o which has one of its vertices at the origin and is determined by the three non-parallel vectors $\delta\alpha_4$, $\delta\alpha_5$, $\delta\alpha_6$. This simple polytope has 6 facets. For our construction we choose the 6 vectors given by $X_1 = U_1$, $X_2 = U_2$, $X_3 = U_3$, $X_4 = -U_1$, $X_5 = -U_2$ and $X_6 = -U_3$. Then the corresponding coefficients are given by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = -\frac{\delta}{2\phi}$. Take now the surjective linear mapping defined by

$$\pi\colon \ \mathbb{R}^6 \to \mathbb{R}^3$$
$$e_i \ \mapsto X_i.$$

Its kernel, $\mathfrak{n}$, is the 3-dimensional subspace of $\mathbb{R}^6$ that is spanned by $e_1 + e_4$, $e_2 + e_5$ and $e_3 + e_6$. It is the Lie algebra of $N = \{\exp(X) \in T^6 \,|\, X \in \mathbb{R}^6, \pi(X) \in F\}$. If Ψ_b is the moment mapping of the induced N-action, then

$$\Psi_b\colon \ \mathbb{C}^6 \longrightarrow \left(\mathbb{R}^3\right)^*$$
$$\underline{z} \ \longmapsto \left(|z_1|^2 + |z_4|^2 - \frac{\delta}{2\phi}, |z_2|^2 + |z_5|^2 - \frac{\delta}{2\phi}, |z_3|^2 + |z_6|^2 - \frac{\delta}{2\phi}\right).$$

Therefore $\Psi_b^{-1}(0) = S_b^3 \times S_b^3 \times S_b^3$, where S_b^3 is the sphere in $\mathbb{R}^4$ centered at the origin with radius $b = \sqrt{\frac{\delta}{2\phi}}$. In order to compute the group N we need the following linear relations between the generators of the quasilattice F:

$$\begin{pmatrix} U_4 \\ U_5 \\ U_6 \end{pmatrix} = \begin{pmatrix} 1 & -\phi & 1 \\ 1 & 1 & -\phi \\ -\phi & 1 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}.$$

Then a straightforward computation gives that

$$N = \left\{\exp(X) \in T^6 \,|\, X = (r + \phi h, s + \phi k, t + \phi l, r, s, t), \ r, s, t \in \mathbb{R}, \ h, k, l \in \mathbb{Z}\right\}.$$

We can think of

$$S^1 \times S^1 \times S^1 = \left\{\, \exp(X) \in T^6 \,|\, X = (r, s, t, r, s, t), \ r, s, t \in \mathbb{R}\right\} \tag{4.1}$$

as being naturally embedded in N. The quotient group

$$\Gamma = \frac{N}{S^1 \times S^1 \times S^1}$$

is discrete. In conclusion, the symplectic quotient M_b is given by

$$M_b = \frac{\Psi_b^{-1}(0)}{N} = \frac{S_b^3 \times S_b^3 \times S_b^3}{N} = \frac{S_b^2 \times S_b^2 \times S_b^2}{\Gamma},$$

where S_b^2 is the sphere in $\mathbb{R}^3$ centered at the origin with radius b. The quasitorus $D^3 = \mathbb{R}^3/F$ acts on M_b in a Hamiltonian fashion, with image of the corresponding moment mapping given exactly by the oblate rhombohedron Δ_b^o.

Consider now the prolate rhombohedron Δ_r^o that has one vertex in the origin and is determined by the three nonparallel vectors $\delta\alpha_1$, $\delta\alpha_2$, $\delta\alpha_3$. We now choose the vectors given by $X_1 = U_4$, $X_2 = U_5$, $X_3 = U_6$, $X_4 = -U_4$, $X_5 = -U_5$ and $X_6 = -U_6$. Then the corresponding coefficients are given by $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = \lambda_5 = \lambda_6 = -\frac{\delta}{2}$. It is immediate to check that we obtain the same Lie algebra $\mathfrak{n}$ as in the case of the oblate rhombohedron. In order to see what happens to the corresponding group we need here the inverse relations:

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{\phi} \\ \frac{1}{\phi} & 1 & 1 \\ 1 & \frac{1}{\phi} & 1 \end{pmatrix} \begin{pmatrix} U_4 \\ U_5 \\ U_6 \end{pmatrix}.$$

To write the relations in this form we used the fundamental identity (3.1). This identity also implies that we obtain the same group N as in the case of the oblate rhombohedron.

The moment mapping Ψ_r is given by

$$\Psi_r \colon \ \mathbb{C}^6 \longrightarrow \left(\mathbb{R}^3\right)^*$$
$$\underline{z} \ \longmapsto \ \left(|z_1|^2 + |z_4|^2 - \tfrac{\delta}{2}, |z_2|^2 + |z_5|^2 - \tfrac{\delta}{2}, |z_3|^2 + |z_6|^2 - \tfrac{\delta}{2}\right).$$

Therefore

$$M_r = \frac{\Psi_r^{-1}(0)}{N} = \frac{S_r^3 \times S_r^3 \times S_r^3}{N} = \frac{S_r^2 \times S_r^2 \times S_r^2}{\Gamma},$$

where $S_r^2 \subset \mathbb{R}^3$ and $S_r^3 \subset \mathbb{R}^4$ are the spheres centered at the origin with radius $r = \sqrt{\frac{\delta}{2}}$. The quasifold M_r is acted on by the same quasitorus $D^3 = \mathbb{R}^3/F$ that we obtained for the oblate rhombohedron. This action is Hamiltonian and the image of the corresponding moment mapping is exactly the prolate rhombohedron Δ_r^o.

Remark 4.1. Let us remark that M_b and M_r are both global quotients and that this defines their quasifold structures. The quasilattice F can be viewed as the group quasilattice of the quasitorus D^3 acting on both.

Remark now that, by Proposition 3.1, *each* of the oblate and prolate rhombohedra in the tiling can be obtained from Δ_b^o and Δ_r^o respectively by a transformation of the icosahedral group composed with a translation. We can then prove the following

Theorem 4.1. *Consider an Ammann tiling having edge length σ. Then the compact connected symplectic quasifold corresponding to each oblate rhombohedron in the tiling is given by M_b, while the compact connected symplectic quasifold corresponding to each prolate rhombohedron is given by M_r.*

Proof. Observe that, for each oblate rhombohedron, there exists a transformation T in the icosahedral group that leaves the quasilattice F invariant, that sends the orthogonal vectors relative to the chosen oblate rhombohedron to the orthogonal vectors relative to Δ_b^o, and such that the dual transformation T^* sends Δ_b^o to a translate of the chosen oblate rhombohedron. The same reasoning applies to the prolate rhombohedra of the tiling. This implies that the reduced space corresponding to each oblate rhombohedron of the tiling, with the choice of orthogonal vectors and quasilattice specified above, is exactly M_b. This yields a unique symplectic quasifold, M_b, for all the oblate rhombohedra in the tiling. In the same way we prove that we obtain a unique symplectic quasifold, M_r, for all the prolate rhombohedra in the tiling. ∎

The quasifolds M_b and M_r can also be constructed as complex quotients and are Kähler [1].

5   Local geometry of the quasifolds M_b and M_r

In this section we study the equivariant geometry of the quasifolds M_b and M_r in a neighborhood of the D^3-fixed points.

Let us begin by describing an atlas for the quasifold M_b. The charts of this atlas are indexed by the vertices of the polytope: in our case we find an atlas given by eight charts, each of which corresponds to a vertex of the oblate rhombohedron. Consider for example the origin: it is given by the intersection of the facets whose orthogonal vectors are X_1, X_2 and X_3. Let B_b be the ball in $\mathbb{C}$ of radius b, namely

$$B_b = \{z \in \mathbb{C} \,|\, |z| < b\}.$$

Consider the following mapping, which gives a slice of $\Psi_b^{-1}(0)$ transversal to the N-orbits

$$B_b \times B_b \times B_b \xrightarrow{t_1} \{\underline{z} \in \Psi_b^{-1}(0) \,|\, z_4 \neq 0,\ z_5 \neq 0,\ z_6 \neq 0\}$$
$$(z_1, z_2, z_3) \longmapsto \left(z_1, z_2, z_3, \sqrt{b^2 - |z_1|^2}, \sqrt{b^2 - |z_2|^2}, \sqrt{b^2 - |z_3|^2}\right).$$

This induces the homeomorphism

$$(B_b \times B_b \times B_b)/\Gamma_1 \xrightarrow{\tau_1} U_1$$
$$[\underline{z}] \longmapsto [t_1(\underline{z})],$$

where the open subset U_1 of M_b is the quotient

$$\{\underline{z} \in \Psi_b^{-1}(0) \,|\, z_4 \neq 0,\ z_5 \neq 0,\ z_6 \neq 0\}/N$$

and the discrete group Γ_1 is given by $\Gamma_1 \simeq N \cap (S^1 \times S^1 \times S^1 \times \{1\} \times \{1\} \times \{1\})$, hence

$$\Gamma_1 = \exp\left\{(\phi h, \phi k, \phi l) \,|\, h, k, l \in \mathbb{Z}\right\}. \tag{5.1}$$

The triple $(U_1, \tau_1, (B_b \times B_b \times B_b)/\Gamma_1)$ is a chart of M_b. Analogously, we can construct seven other charts, corresponding to the remaining vertices of the oblate rhombohedron, each of which is characterized by a different combination of the variables. One can show that these eight charts are compatible and give an atlas of M_b.

One can check that the moment map, locally, on the first chart is given by

$$\Phi([z_1 : z_2 : z_3]) = \frac{\phi\alpha_5}{\langle \phi\alpha_5, U_1 \rangle}|z_1|^2 + \frac{\phi\alpha_6}{\langle \phi\alpha_6, U_2 \rangle}|z_2|^2 + \frac{\phi\alpha_4}{\langle \phi\alpha_4, U_3 \rangle}|z_3|^2$$
$$= \phi\alpha_5|z_1|^2 + \phi\alpha_6|z_2|^2 + \phi\alpha_4|z_3|^2,$$

while the isotropy action of D^3 on $\mathbb{C}^3/\Gamma_1$ is given by

$$\left(D^3, \mathbb{C}^3/\Gamma_1\right) \longrightarrow \mathbb{C}^3/\Gamma_1$$
$$([X], (z_1, z_2, z_3)) \longmapsto \left(e^{2\pi i \phi\alpha_5(X)}z_1, e^{2\pi i \phi\alpha_6(X)}z_2, e^{2\pi i \phi\alpha_4(X)}z_3\right). \tag{5.2}$$

To obtain the local expression of the moment mapping on the other seven charts it suffices to replace $\phi\alpha_5$, $\phi\alpha_6$, $\phi\alpha_4$ in (5.2) with all the possible combinations of $\pm\phi\alpha_5$, $\pm\phi\alpha_6$, $\pm\phi\alpha_4$ respectively. Notice that the vectors U_1, U_2, U_3 are three of the six generators of F, while $\phi\alpha_4$, $\phi\alpha_5$, $\phi\alpha_6$ are three of the six generators of $\frac{1}{2}I$.

An atlas for the the quasifold M_r can be constructed in the same way. It can be shown that the moment mapping for the prolate rhombohedron, is given, locally on the chart corresponding to the origin, by

$$\Phi([z_1 : z_2 : z_3]) = \frac{\alpha_2}{\langle \alpha_2, U_4 \rangle}|z_1|^2 + \frac{\alpha_3}{\langle \alpha_3, U_5 \rangle}|z_2|^2 + \frac{\alpha_1}{\langle \alpha_1, U_6 \rangle}|z_3|^2 = \alpha_2|z_1|^2 + \alpha_3|z_2|^2 + \alpha_1|z_3|^2,$$

while the isotropy action of D^3 on $\mathbb{C}^3/\Gamma_1$ is given by

$$
\begin{aligned}
\left(D^3, \mathbb{C}^3/\Gamma_1\right) &\longrightarrow \mathbb{C}^3/\Gamma_1 \\
([X], (z_1, z_2, z_3)) &\longmapsto \left(e^{2\pi i \alpha_2(X)} z_1, e^{2\pi i \alpha_3(X)} z_2, e^{2\pi i \alpha_1(X)} z_3\right).
\end{aligned}
$$

Again, notice that the vectors U_4, U_5, U_6 are the three remaining generators of F, while α_1, α_2, α_3 are the three remaining generators of $\frac{1}{2}I$. In conclusion, the weights of the isotropy action of the quasitorus D^3 on a neighboorhood of the D^3-fixed points for both M_b and M_r generate the quasilattice $\frac{1}{2}I$. Therefore $\frac{1}{2}I$ can be thought of, in this setting, as the weight quasilattice of D^3. This is consistent with the fact that $\frac{1}{2}I$ is dual to the group quasilattice F (cf. Remark 3.2).

Remark 5.1. Remark that, since $\alpha_i(X)$, $\phi\alpha_i(X)$ lie in $\mathbb{Z} + \phi\mathbb{Z}$ whenever $X \in F$, and since the local group in each chart of M_b and M_r is equal to Γ_1, the above actions are well defined.

Remark 5.2. If we choose as group lattice tF instead, $t \in \mathbb{R}$, then the corresponding weight lattice would have to be $\frac{1}{2t}I$. But this would not be consistent with the inclusion and projection schemes in Remark 3.2. This is the main reason underlying our choice of the norm of the vectors α_j, $j = 1, \ldots, 6$.

6 Diffeotype and symplectotype of the tiles

The purpose of this section is to prove the following

Theorem 6.1. *The quasifolds M_b and M_r are diffeomorphic but not symplectomorphic.*

Before proceeding with the proof of this theorem we need a few more facts on the local geometry of the quasifold M_b. Let us denote by p_b the projection

$$
S_b^2 \times S_b^2 \times S_b^2 \to M_b.
$$

Denote by V_n the open subset of S_b^2 given by S_b^2 minus the south pole and by V_s the open subset of S_b^2 given by S_b^2 minus the north pole. Then, on $\Psi_b^{-1}(0)$, consider the action of $S^1 \times S^1 \times S^1$ given by (4.1). We obtain

$$
V_n \times V_n \times V_n = \left\{ \underline{z} \in \Psi_b^{-1}(0) \mid z_4 \neq 0,\, z_5 \neq 0,\, z_6 \neq 0 \right\}/\left(S^1 \times S^1 \times S^1\right)
$$

and

$$
U_1 = (V_n \times V_n \times V_n)/\Gamma.
$$

We have the following commutative diagram:

$$
\begin{array}{ccc}
B_b \times B_b \times B_b & \xrightarrow{\ t_1\ } & \left\{ \underline{z} \in \Psi_b^{-1}(0) \mid z_4 \neq 0,\, z_5 \neq 0,\, z_6 \neq 0 \right\} \\
\downarrow & & \downarrow \\
B_b \times B_b \times B_b & \xrightarrow{\ \tilde{\tau}_1\ } & V_n \times V_n \times V_n \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_b} \\
(B_b \times B_b \times B_b)/\Gamma_1 & \xrightarrow{\ \tau_1\ } & U_1.
\end{array} \tag{6.1}
$$

The mapping $\tilde{\tau}_1$ is induced by the diagram and can be written as $\tau_n \times \tau_n \times \tau_n$, with $\tau_n\colon B_b \to V_n$. Observe that the mapping

$$
\begin{aligned}
\mathbb{C} &\longrightarrow V_n \\
w &\longmapsto \left[\tau_n\big(bw/\sqrt{1+|w|^2}\big)\right]
\end{aligned}
$$

is just the stereographic projection from the north pole. We denote by τ_s the analogous mapping $\tau_s\colon B_b \longrightarrow V_s$. The two charts (B_b, τ_n, V_n) and (B_b, τ_s, V_s) give a symplectic atlas of S_b^2, whose standard symplectic structure is induced by the standard symplectic structure on B_b. Analogously, at a local level, the symplectic structure of the quotient M_b is induced by the standard symplectic structure on $B_b \times B_b \times B_b$.

We have already seen that the quasifold M_b is a global quotient of a product of three 2-spheres by the discrete group Γ. We remark that the atlas above is the quotient by Γ of the atlas of the product of three spheres, given by the eight triples $V_n \times V_n \times V_n$, $V_n \times V_n \times V_s$, $V_n \times V_s \times V_n$, $V_s \times V_n \times V_n$, $V_n \times V_s \times V_s$, $V_s \times V_n \times V_s$, $V_s \times V_s \times V_n$, $V_s \times V_s \times V_s$.

We are now ready to prove Theorem 6.1:

Proof. Let us begin by showing that M_b and M_r are diffeomorphic. Let us denote by p_r the projection

$$S_r^2 \times S_r^2 \times S_r^2 \to M_r.$$

The natural Γ-equivariant diffeomorphism $f^\dagger\colon S_b^2 \times S_b^2 \times S_b^2 \to S_r^2 \times S_r^2 \times S_r^2$ induces a homeomorphism $f\colon M_b \to M_r$; in general, a homeomorphism between two global quotients that is induced by an equivariant diffeomorphism of the manifolds turns out to be a quasifold diffeomorphism [3, Definition A.2].

Let us now show that M_b and M_r are not symplectomorphic. Denote by ω_b and ω_r the symplectic forms of M_b and M_r respectively. Suppose that there is a symplectomorphism $h\colon M_b \longrightarrow M_r$, namely a diffeomorphism h such that $h^*(\omega_r) = \omega_b$. We prove that this implies that the homeomorphism $h\colon M_b \to M_r$ lifts to a symplectomorphism $\tilde{h}\colon S_b^2 \times S_b^2 \times S_b^2 \to S_r^2 \times S_r^2 \times S_r^2$, leading thus to a contradiction: such symplectomorphism cannot exists, since the two manifolds have different symplectic volumes. To start with recall from [3, Remark 2.9] that, to each point $m \in M_b$, one can associate the groups Γ_m and $\Gamma_{h(m)}$. The definition of diffeomorphism implies that these two groups are isomorphic. Let $n_b \in S_b^2$ be the north pole and take $m_0 = p_b(n_b \times n_b \times n_b)$. Then, since $\Gamma_m \simeq \Gamma_{h(m)}$, without loss of generality the point $h(m_0)$ can be taken to be $p_r(n_r \times n_r \times n_r)$, where $n_r \in S_r^2$ is the north pole. Consider the chart U_1 that we constructed above. Then, by definition of quasifold diffeomorphism [3, Definition A.23] and [3, Remark A.24], there exists an open subset $U \subset U_1$ such that $m_0 \in U$ and $h \circ \tau_1^{-1}\colon \tau_1^{-1}(U) \to h(U)$ is a diffeomorphism of the universal covering models induced by $\tau_1^{-1}(U) \subset B_b \times B_b \times B_b/\Gamma_1$ and $h(U) \subset M_r$ respectively. Moreover, by [3, Proposition A.9], any open subset $W \subset U$ enjoys the same property. We can choose $W_0 \subset U_1$ such that $\tilde{W}_0 = (\tau_1 \circ p_1)^{-1}(W_0))$ is a product of three balls. In particular, $\tilde{W}_0$ is simply connected. Denote now by $\tilde{W}_{r,0} = (p_r)^{-1}(h(W)))$; this is an open subset of $S_r^2 \times S_r^2 \times S_r^2$, which is also connected, due to the action of Γ on $S_r^2 \times S_r^2 \times S_r^2$. Denote by $W_{r,0}^\sharp$ its universal covering. Now consider a point $\underline{z}^1 \in B_b \times B_b \times B_b$ such that $z_1^1 \neq 0$, $z_2^1 \neq 0$ and $z_3^1 \neq 0$ and let $m = (\tau_1 \circ p_1)(\underline{z}^1)$. For the sequel it is crucial to remark that, because of the action of Γ_1 given in (5.1), any Γ_1-invariant open subset of $B_b \times B_b \times B_b$ that contains the point $\underline{z}^1$, contains also the product of circles $\{(z_1, z_2, z_3) \in B_b \times B_b \times B_b \,|\, |z_1| = |z_1^1|, |z_2| = |z_1^2|, |z_3| = |z_1^3|\}$. Hence, for each point $(\tau_1 \circ p_1)(t\underline{z}^1)$ with $t \in [0, 1]$, we can find an open subset $W_t \subset U_1$, containing that point, such that the homeomorphism $\tau_1^{-1} \circ h$, restricted to $\tau_1^{-1}(W_t)$, is a diffeomorphism, and $(\tau_1 \circ p_1)^{-1}(W_t)$ is the product of three open annuli. We can cover the curve by a finite number of these W_t's: $W_0, W_1, \ldots, W_s$, with $W_j \cap W_{j+1} \neq \varnothing$. Notice that $(\tau_1 \circ p_1)^{-1}(W_j \cap W_{j+1})$, $j = 0, \ldots, s-1$, is itself a product of three open annuli. The subsets $\tilde{W}_j = (\tau_1 \circ p_1)^{-1}(W_j)$ and $\tilde{W}_{r,j} = (p_r)^{-1}(h(W_j))$ are open and connected.

We divide the remaining part of the proof in subsequent steps:

Step 1: consider first W_0. Since the isotropy of Γ_1 at 0 is the whole Γ_1, we can apply [3, Lemma 6.2]. We find that $\tilde{W}_{r,0}$ is itself simply connected and that the homeomorphism $h \circ \tau_1$ lifts to a diffeomorphism $\tilde{h}_0 : \tilde{W}_0 \to \tilde{W}_{r,0}$.

Step 2: consider the homeomorphism $h_1 = h \circ \tau_1$ defined on $\tau_1^{-1}(W_1)$. By construction h_1 is a diffeomorphism of the universal covering models of the induced models. We find the following diagram:

$$
\begin{array}{ccc}
W_1^\sharp & \xrightarrow{\ h_1^\sharp\ } & W_{r,1}^\sharp \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho_1} \\
\tilde{W}_1 & & \tilde{W}_{r,1} \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q_1} \\
\tau_1^{-1}(W_1) & \xrightarrow{\ h_1\ } & h(W_1).
\end{array}
$$

Consider the restriction of h_1 to $\tau_1^{-1}(W_0 \cap W_1)$. This restriction admits a lift, given by the restriction of $h_1^\sharp$ to $(\pi_1 \circ p_1)^{-1}(\tau_1^{-1}(W_0 \cap W_1))$. Furthermore, by Step 1, the restriction of h_1 admits another lift, defined on $p_1^{-1}(\tau_1^{-1}(W_0 \cap W_1))$, which is the restriction of $\tilde{h}_0$. Therefore, by [3, Lemma 6.3], the restriction of $\rho_1 \circ h_1^\sharp$ to $(\pi_1 \circ p_1)^{-1}(\tau_1^{-1}(W_0 \cap W_1))$ descends to a diffeomorphism defined on $p_1^{-1}(\tau_1^{-1}(W_0 \cap W_1))$.

Step 3: we consider $W_0 \cap W_1 \subset W_1$ and we apply [3, Lemma 6.5] to the homeomorphism $h \circ \tau_1$ defined on $\tau_1^{-1}(W_1)$. We deduce that $h \circ \tau_1$ is a diffeomorphism of the model $(\tau_1 \circ p_1)^{-1}(W_1)/\Gamma_1$ with the model induced by $h(W_1) \subset M_r$.

Step 4: we apply Step 3 to the other successive intersections. We find that $h \circ \tau_1$ is a diffeomorphism of the model $(\tau_1 \circ p_1)^{-1}(\underset{i=1}{\overset{k}{\cup}} W_i)/\Gamma_1$ with the model induced by $h(\underset{i=1}{\overset{k}{\cup}} W_i) \subset M_r$. Remark now that a slight modification of the above argument applies to any choice of point $\underline{z}^1 \in B_b \times B_b \times B_b$, $\underline{z}_1 \neq 0$.

Let $\epsilon > 0$ be arbitrarily small. Consider the product of closed balls $\overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon}$. This, by Step 4, can be covered by a finite number of connected open subsets of the kind $(\tau_1 \circ p_1)^{-1}(\underset{i=1}{\overset{k}{\cup}} W_i)/\Gamma_1$, whose intersection is a product of three balls centered at the origin. Now [3, Lemma A.3], which guarantees the uniqueness of the lift up to the action of Γ, implies that the homeomorphism h admits a lift to $\tilde{\tau}_1(\overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon} \times \overline{B}_{b-\epsilon})$. This in turn implies that $h\colon U_1 \to h(U_1)$ admits a lift

$$\tilde{h}_1\colon \ V_n \times V_n \times V_n \to p_r^{-1}(h(U_1)).$$

We apply the same argument to the other eight charts. These charts intersect on the dense connected open subset where the action of the quasitorus D^3 is free. By the uniqueness of the lift [3, Lemma A.3], we obtain a global lift $\tilde{h} : S_b^2 \times S_b^2 \times S_b^2 \to S_r^2 \times S_r^2 \times S_r^2$. Moreover, since diagram (6.1) preserves the symplectic structures, we have that $\tilde{h}$ is a symplectomorphism between $S_b^2 \times S_b^2 \times S_b^2$ to $S_r^2 \times S_r^2 \times S_r^2$, which is impossible. ∎

In conclusion, similarly to what happens in dimension two for Penrose rhombus tilings [2], there is a *unique* quasifold structure that is naturally associated to any Ammann tiling with fixed edge length, and two distinct symplectic structures that distinguish the oblate and the prolate rhombohedra.

Acknowledgements

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