EXISTENCE OF SOLUTIONS FOR QUADRATIC INTEGRAL EQUATIONS ON UNBOUNDED INTERVALS

Giuseppe Anichini,
Facoltà di Ingegneria, Università di Firenze,
giuseppe.anichini@unifi.it

Giuseppe Conti,
Facoltà di Architettura, Università di Firenze,
gconti@unifi.it

March 1, 2011

Classification: 45G10, 47H09, 47H10, 47H30.

Abstract

In this paper, we investigate the existence of a solution on a semi-infinite interval for a quadratic integral equation of Urysohn or Volterra type in Fréchet spaces using a fixed point theorem due to Petryshyn and Fitzpatrick.

Keywords: Fixed point property; Fréchet space; cohomology; acyclic valued; quadratic integral equations.

1 Introduction and Notations

In this paper, we are going to study the solvability of a nonlinear quadratic integral equation of Urysohn or Volterra type. We will look for solutions of these equations in the Fréchet space of real functions being defined and continuous on the semi-infinite interval $J = \mathbb{R}^+ = [0, +\infty)$. Let us mention that the theory of quadratic integral equations has many useful applications in mathematical physics, in engineering, in economics, in biology, as well as in describing real world problems: it is worthwhile mentioning the applications of those equations, especially, the so-called quadratic integral equation of Chandrasekhar in the theory of radiative transfer, kinetic theory of gases, in the traffic theory and in the theory of neutron transport (see [11]), [13], [14]), [17]), [20], [21]). Of course integral equations of such a type are also often an object of mathematical investigations ([1], [2], [3], [5], [6], [7], [8], [9], [12]) mainly when the time t is allowed to vary on a bounded interval.

In this paper, we are going to study, in an abstract setting, the solvability of a nonlinear quadratic integral equation of the type

$$x(t) = f(t) + (Ax)(t) \int_0^t u(s, t, x(s))ds, \quad t \in J = [0, +\infty). \qquad (I)$$

Motivated by some previous papers considered here we try to extend the investigations to semi-infinite intervals. We will look for solutions of the equation (I) in the Fréchet space of continuous real functions being defined on J. The main tool used in achieving our main result is a fixed point theorem, for comapact acyclic multivalued maps of an appropriate operator on the Fréchet space.

Also more recently, a quadratic integral equation with linear modification of the argument is studied by many authors by means of a technique associated with measures of noncompactness, in order to prove the existence of solutions in a space like $C(J)$ (see, for istance, [9], [12]). The conditions imposed in these papers are a bit heavier than ours.

2 Preliminaries

In this paper we will denote by $F = (F, \{||\cdot||_n\}_{n\in\mathbb{N}})$ a Fréchet space, i.e. a locally convex space with the topology generated by a family of semi-norms $\{||\cdot||_n\}_{n\in\mathbb{N}}$, where $\mathbb{N}$ is the usual set of positive integer numbers. We assume that the family of semi-norms verifies: $||x||_1 \leq ||x||_2 \leq \cdots \leq ||x||_n \leq \cdots$, for every $x \in F$.

An example of Fréchet space is the space $F = C(J, \mathbb{R})$ of all continuous functions with the topology of the uniform convergence on compact subintervals of J. As a generating family of semi-norms for this topology one may consider $||x||_n = \max \{|x(t)| : t \in [0,n]\}$. We recall that the topology of F coincides with that of a complete metric space (F,d), where $d(x,y) = \sum_1^{+\infty} 2^{-n} \frac{||x-y||_n}{1+||x-y||_n}$, for $x,y \in F$.

A subset $M \subset F$ is said to be bounded if, for every $n \in \mathbb{N}$, there exists $k_n > 0$ such that $||x||_n \leq k_n$, for every $x \in M$.

If $U_n = \{x \in F : ||x||_n < 1,\ n \in \mathbb{N}, \}$ then $\epsilon U_n,\ \epsilon > 0,\ n \in \mathbb{N},$ is a basis of convex and symmetric neighbourhoods of the origin in F.

Let M be a subset of the Fréchet space F and let $T : M \longrightarrow F$ be a map. Let ϵ_n be a sequence of positive real numbers tending to zero. A sequence $\{T_n\}$ of maps $T_n : M \longrightarrow F$ is said to be an ϵ_n-approximation of F on M if $||T_n(x) - T(x)||_n \leq \epsilon_n$ for every $n \in \mathbb{N}$ and $x \in M$.

Finally, let C be a closed subset of a Fréchet space F; a map $T : C \longrightarrow F$ is said to be compact if T is continuous and $T(C)$ is a relatively compact subset of F.

Let W and V be metric spaces; W is a compact absolute retract (AR) if W is compact and, for every homeomorphism $f : W \longrightarrow V$, then $f(W) \subset V$ is a retract of V. It follows from the Dugundji extension theorem ([10]) that every compact and convex subset of a Fréchet space is a compact AR.

We say that $A \subset W$ is an R_δ –set in the space W if A is the intersection of countable decreasing sequence of absolute retracts contained in W. An R_δ –set is an acyclic set, i.e. it is acyclic with respect to any continuous theory of cohomology.

Let X be a subset of F; a multivalued map $S : X \longrightarrow \mathcal{P}(\mathcal{F})$, where $\mathcal{P}(\mathcal{F})$ is the family of all nonempty subsets of F, is said to be uppersemicontinuous (u.s.c.) if the graph of F is closed in $X \times F$.

The following results will be useful in the proof of our main theorem.

Proposition 1([15]): Let assume that the function $u : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following conditions:

i) $u(s,\cdot,\cdot) : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous for every $s \in J$;

ii) $u(\cdot,t,x)$ is Lebesgue measurable for every $(t,x) \in J \times \mathbb{R}$;

iii) there exist locally integrable functions $l,m : J \longrightarrow J$ such that $|u(s,t,x)| \leq l(s) + m(s)|x|$ for all $(s,t,x) \in J \times J \times \mathbb{R}$;

iv) there exists a locally integrable function $\alpha : J \longrightarrow J$ such that *supp* $u \subset \Omega_\alpha$ where $\Omega_\alpha = \{(s,t,x) \in J \times J \times \mathbb{R} \colon |x| \leq \alpha(s)\}$;

Then for every $\epsilon > 0$ and $a > 0$ there exists a map $\overline{u} : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies i), ii), iii), iv) and the following conditions:

v) there exists an integrable function $\phi : [0.a] \longrightarrow J$ such that $|\overline{u}(s,t,x) - u(s,t,x)| \leq \phi(s)$ for all $(s,t,x) \in [0,a] \times [0,a] \times \mathbb{R}$ and $\int_0^a \phi(s)ds < \epsilon$;

vi) there exists an integrable function $L : J \longrightarrow J$ such that $|\overline{u}(s,t,x) - \overline{u}(s,t,x')| \leq L(s)|x - x'|$ for all $(s,t) \in J \times J$ and $x, x' \in \mathbb{R}$.

Proposition 2([15]): Let F be a Frechét space and let $\overline{U}$ the closure of an open subset U of F and let $T : \overline{U} \longrightarrow F$ be a compact map. Let T_n be an ϵ_n-approximation of F on $\overline{U}$, where $T_n : \overline{U} \longrightarrow F$ are compact maps. If the equation $x - T_n(x) = y$ has at most one solution for every $n \in \mathbb{N}$ and $y \in \epsilon_n\overline{U_n}$, then the set of fixed points of T is a compact R_δ-set.

The following proposition can be deduced from Theorem 1 of ([22]).

Proposition 3: Let F be a Frechét space and let $S : F \longrightarrow F$ a compact, uppersemicontinuous multivalued map with acyclic and compact values; then S has a fixed point.

Proposition 4: (Gronwall inequality). Let $v, k, h : J \longrightarrow J$ be continuous functions such that $v(t) \leq k(t) + \int_0^t h(s)v(s)ds,\ t \in J$.

So we have $v(t) \leq k(t) \exp(\int_0^t h(s)ds),\ t \in J$.

3 Main result

Now we are ready to deal with the existence of solutions of an integral equation of the type:

$$x(t) = f(t) + (Ax)(t) \int_0^t u(s,t,x(s))ds, \quad t \in J. \tag{I}$$

.

Theorem 1: *Suppose that the following conditions are satisfied:*

1) The map $u : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function such that:

i) $u(s, \cdot, \cdot) : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous for every $s \in J$;

ii) $u(\cdot, t, x) : J \longrightarrow \mathbb{R}$ is Lebesgue measurable for every $(t, x) \in J \times \mathbb{R}$;

iii) $|u(s, t, x)| \leq \gamma(s) + \beta(s)|x|$ for all $(s, t, x) \in J \times J \times \mathbb{R}$, where $\gamma, \beta : J \longrightarrow J$ are locally integrable functions;

2) the map $f : J \longrightarrow \mathbb{R}$ is a continuous function.

3) A is a continuous operator from the Fréchet space $F = C(J, \mathbb{R})$ into itself such that there exists a real number $h > 0$ for which $|(Ax)(t)| \leq h$ for every $t \in J$.

Then the integral equation (I) has at least one solution in $F = C(J, \mathbb{R})$.

Proof: Let q be an element of $F = C(J, \mathbb{R})$; consider the equation

$$x(t) = f(t) + (Aq)(t) \int_0^t u(s, t, x(s))ds, \quad t \in J. \qquad (I_q)$$

Let S the (multivalued) map from F into itself which associates to any $q \in F$ the solutions of (I_q). Clearly, the fixed points of the map S are the solutions of equation (I).

Let $q \in F$ be fixed; let x be a solution of the equation (I_q): in order to prove our theorem, we need the conditions of Proposition 3 be satisfied.

To that aim the following steps in the proof have to be established:

- the set $S(F)$ is (equi)bounded and equicontinous;
- the map S is uppersemicontinous;
- The set $S(q)$ is an acyclic set, for every $q \in F$.

We have from 1) and 3):

$|x(t)| \leq |f(t)| + |(Aq)(t)| \int_0^t |u(s, t, x(s))|ds \leq$

$|f(t)| + h \int_0^t (\gamma(s) + \beta(s)|x(s)|)ds, \quad t \in J.$

By Gronwall inequality we obtain:

$|x(t)| \le (|f(t)| + h\int_0^t \gamma(s)ds)\exp(h\int_0^t \beta(s)ds) = \alpha(t), \quad t \in J.$

If we put $\alpha_n = max\{|\alpha(t)|; t \in [0,n]\}$, then we have $||x||_n \le \alpha_n$, so that the set $S(F)$ is bounded.

Let $\psi : [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$. For every $q \in F$, define $\overline{u}_q : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\overline{u}_q(s,t,x) = \psi(\frac{x}{1+\alpha(t)})(Aq)(t)u(s,t,x).$$

The map $\overline{u}_q$ satisfies conditions i), ii), iii) of our Proposition 1 with $l(s) = h\gamma(s)$ and $m(s) = h\beta(s)$.

We have $\overline{u}_q(s,t,x) = (Aq)(t)u(s,t,x)$ when $|x| \le \alpha(s)$; hence the set of solutions of equation (I_q) coincides with the set of solutions of the equation

$$x(t) = f(t) + \int_0^t \overline{u}_q(s,t,x(s))ds, \quad t \in J. \qquad (U_q)$$

Moreover supp $\overline{u}_q \subset \Omega_{\overline{\alpha}}$ where $\overline{\alpha}(s) = 2h(\alpha(s)+1)$.

Now we prove that the set $S(F)$ is a set of equicontinuous maps.

Let $x \in S(F)$; by recalling the conditions established in Proposition 1, for $n \in \mathbb{N}$ and arbitrarily chosen $\epsilon > 0$, there exists $\delta_1 > 0$ such that for every measurable $B \subset [0,n]$ with meas $(B) < \delta_1$, $\int_B (l(s) + \overline{\alpha}(s)\, m(s))\, ds \le \frac{\epsilon}{5}$.

For $n_1 \in \mathbb{N}$ we put $B_{n_1} = \{s \in [0,n] : \forall w \in \mathbb{R}, \forall t, t_1 \in [0,n],$ with $|t - t_1| < \frac{1}{n_1}$ implies $|\overline{u}_q(s,t,w) - \overline{u}_q(s,t_1,w)| < \frac{\epsilon}{5n}.\}$

So there exists $\overline{n} \in \mathbb{N}$ such that meas $([0,n] \backslash B_{\overline{n}}) < \delta_1$.

Now we put $\delta = \min(\delta_1, \frac{1}{\overline{n}})$; then, $\forall t, t_1 \in [0,n]$, with $0 < t - t_1 < \delta$ and $x \in S(F)$, we have

$|x(t) - x(t_1)| \le \int_0^t |\overline{u}_q(s,t,x(s)) - \overline{u}_q(s,t_1,x(s))|ds +$

$\int_{t_1}^t |\overline{u}_q(s,t_1,x(s))|\, ds \le \int_{B_{\overline{n}}} \frac{\epsilon}{5n}\, ds + \displaystyle\int_{[0,n]\backslash B_{\overline{n}}} 2h(\gamma(s) + \overline{\alpha}(s)\beta(s))\, ds +$

$\displaystyle\int_{t_1}^t 2h(\gamma(s) + \overline{\alpha}(s)\beta(s))\, ds < \frac{\epsilon}{5} + 2\frac{\epsilon}{5} + 2\frac{\epsilon}{5} = \epsilon.$

Now we prove that the map S is uppersemicontinuous.

Let $q_n, q_0 \in F$ and let $q_n \longrightarrow q_0$; let $x_n \in S(q_n)$ and $x_n \longrightarrow x_0 \in F$. We need to show that $x_0 \in S(q_0)$.

From the continuity of operator A, it follows that $\lim_{n\to+\infty} A(q_n) = A(q_0)$; from the continuity of the function $u(s,\cdot,\cdot)$ for every $s \in J$, from 1)- iii) and the Dominated Lebesgue Convergence Theorem it follows:

$$\lim_{n\to+\infty} \int_0^t u(s,t,x_n(s))\, ds = \int_0^t \lim_{n\to+\infty} u(s,t,x_n(s))\, ds = \int_0^t u(s,t,x_0(s))\, ds.$$

Hence we have:

$$\lim_{n\to+\infty} x_n(t) = \lim_{n\to+\infty} (f(t) + (Aq_n)(t) \int_0^t u(s,t,x_n(s))\, ds =$$

$$f(t) + (Aq_0)(t) \int_0^t u(s,t,x_0(s))\, ds.$$

The latter means that $\lim_{n\to+\infty} x_n(t) = x_0(t) \in S(q_0)$, so that the map S is uppersemicontinuous.

Now it remain to show that the set $S(q)$ is an acyclic set for every $q \in F$.

Let $n \in \mathbb{N}$ be arbitrarily chosen and let $a = n$, $\epsilon = \dfrac{1}{n}$. From Proposition 1 there exists a map $u_n : J \times J \times \mathbb{R} \longrightarrow \mathbb{R}$ and a function $\overline{u}$, with appropriate functions L_n, ϕ_n, l_n, m_n and α_n satisfying conditions i), ii), iii), iv),v), vi).

Now, for every fixed $q \in F$, let us define the operators $T, T_n : F \longrightarrow F$ as follows:

$$T(x)(t) = f(t) + (Aq)(t) \int_0^t \overline{u}(s,t,x(s))\, ds, \quad t \in J$$

$$T_n(x)(t) = f(t) + (Aq)(t) \int_0^t u_n(s,t,x(s))\, ds, \quad t \in J$$

With a proof similar to previous one, we can show that T and T_n are compact operators.

It is easy to see that $\{T_n\}$ is an ϵ_n-approximation of T on F, where $\epsilon_n = \frac{h}{n}$. In fact, for $t \in [0, n]$, we have:

$|T_n(x)(t) - T(x)(t)| = |(Aq)(t) \int_0^t u_n(s, t, x(s))\, ds - (Aq)(t) \int_0^t \overline{u}(s, t, x(s))\, ds \leq$

$h \int_0^t |u_n(s, t, x(s)) - \overline{u}(s, t, x(s))|\, ds < \frac{h}{n}.$

Hence we obtain $||T_n(x)(t) - T(x)(t)||_n \leq \epsilon_n$.

Finally we have only to show that, for every $n \in \mathbb{N}$ and $y \in F$, the equation $x - T_n(x) = y$ has at most one solution.

Suppose that x_1 and x_2 are solutions of the previous equation; we have:

$|x_1(t) - x_2(t)| \leq |(Aq)(t) \int_0^t u_n(s, t, x_1(s))\, ds - (Aq)(t) \int_0^t u_n(s, t, x_2(s))\, ds| \leq$

$h \int_0^t |u_n(s, t, x_1(s)) - u_n(s, t, x_2(s))|\, ds \leq h \int_0^t L_n(s)|x_1(s) - x_2(s)|\, ds.$

By the Gronwall inequality we obtain $|x_1(t) - x_2(t)| = 0$ for every $t \in J$.

4 Esempio

Let us consider the following quadratic integral equation of the Volterra type:

$$x(t) = f(t) + \frac{x(t)}{1 + (x(t))^2} \int_0^t \frac{s^3}{1 + s^2} \frac{t}{1 + t^2}\, x(s)\, ds, \quad t \in [0, +\infty) \qquad (Q)$$

where $f : [0, +\infty) \longrightarrow \mathbb{R}$ is a continuous function.

Clearly we have $|(Ax)(t)| \leq 1$ and $|\frac{s^3}{1 + s^2} \frac{t}{1 + t^2}\, x| \leq s|x|, \quad s, t \in [0, +\infty)$.

The hypotheses of our Theorem 1 are verified, so that the equation (Q) has at least one solution.

References

[1] G. Anichini - G. Conti *Boundary value problem for implicit ODE's in a singular case,* Differential Equations and Dynamical Systems, Vol. 7 no. 4, (1999), pag. 437 - 456.

[2] G. Anichini - G. Conti, *Some properties of the solution set for integral non compact equations*, Far East Journal of Mathematical Sciences (FJMS), 24, no.3, (2007), pag. 415, 423.

[3] G. Anichini, G. Conti, *Existence of solutions of some quadratic integral equations,* Opuscola mathematica, 4, (2008), pag. 433 -440.

[4] J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics, Vol. 95, Camb. Univ. Press, Cambridge, 1990.

[5] I.K. Argyros, *On a class of quadratic integral equations with perturbation*, Funct. Approx. Comment. Math. 20, (1992), pag. 51–63.

[6] J. Banas, M. Lecko, W.G. El-Sayed, *Existence theorems for some quadratic integral equations*, J. Math. Anal. Appl. 222, (1998), pag. 276 - 285.

[7] Banas, J., Rocha Martin, J., Sadarangani, *K. On solutions of a quadratic integral equation of Hammerstein type.* Math. Comput. Modelling 43, no. 1-2, (2006), pag. 97–104.

[8] J. Banas, K. Sadarangani, *Solvability of Volterra-Stieltjes operator-integral equations and their applications*, Computers Math. Applic. 41 (12),(2001), pag. 1535-1544.

[9] M. Benchohra, M. Abdalla Darwish, *On unique solvability of quadratic integral equations with linear modification of the argument*, Miskolc Math. Notes, vol $10-1$, (2009), pag. $3-10$.

[10] K. Borsuk, *Theory of retracts*, Polish Scientific Publishers, Warszawa, 1967.

[11] L.W. Busbridge, *The Mathematics of Radiative Transfer*, Cambridge Univ. Press, Cambridge, England, 1960.

[12] Caballero, J., López, B.. Sadarangani, K., *Existence of nondecreasing and continuous solutions of an integral equation with linear modification of the argument*, Acta Math. Sin. (Engl. Ser.) 23, no. 9, (2007), pag. 1719–1728.

[13] K.M. Case, P.F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading, MA, (1967).

[14] M. Crum, *On an integral equation of Chandrasekhar*, Quart. J. Math. Oxford Ser. 2. no.18, (1947), pag. 244 – 252.

[15] K. Czarnowski, T. Pruszko, *On the structure of fixed point sets of compact maps in B_0 spaces with applications to integral and differential equations in unbounded domain*, J. of Math. Anal. Appl., 154, no. 1, (1991), pag. 151 – 163.

[16] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[17] P. Edström, *A fast and stable solution method for the radiative transfer problem*. SIAM Rev. 47, no. 3, (2005), pag. 447–468

[18] M. Frigon, A. Granas, *Résultats de type Leray - Schauder pour des contractions sur des espaces de Fréchet*, Ann. Sci. Math. Québec, vol. 22, no. 2, (1998), pag. 161 – 168.

[19] G. Gabor, *On the acyclicity of fixed point sets of multivalued maps*, Topological Methods in Nonlinear Analysis, no. 14, (1999), pag. 327 343.

[20] S. Hu, M. Khavanin, W. Zhuang, *Integral equations arising in the kinetic theory of gases*, Appl. Anal. no. 34, (1989), pag. 281 - 286.

[21] L. K. Nowosad, B. R. Saltzberg, *On the solution of a quadratic integral equation arising in signal design*, Journal of the Franklin Institute, 281, no. 6, (1966), pag. 437 - 454.

[22] W. V. Petryshyn, P. M. Fitzpatrick, *Degree theory for noncompact multivalued vector fields*, Bull. Amer. Math. Soc., no. 79, (1973), pag. 609 - 613.