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# The Argument Principle for Quaternionic Slice Regular Functions

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## 1. Introduction

Let $\mathbb{H}$ be the skew field of Hamilton numbers. The elements of $\mathbb{H}$ are of the form $q = x_0 + e_1x_1 + e_2x_2 + e_3x_3$, where the $x_l$ are real and $e_1, e_2, e_3$ are imaginary units (i.e., their squares equal $-1$) such that $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$, and $e_3e_1 = -e_1e_3 = e_2$. We will also denote a generic element $w$ of $\mathbb{H}$ by $w = x_0 + \sum_{k=1}^{3} x_k e_k$ and define in a natural fashion the conjugate $\bar{w} = x_0 - \sum_{k=1}^{2} x_k e_k$ and the square norm $|w|^2 = w\bar{w} = \sum_{k=0}^{3} x_k^2$ of $w$. We will denote by $\mathbb{S}$ the (2-dimensional) sphere of imaginary units of $\mathbb{H}$—that is, the sphere $\mathbb{S} = \left\{I = \sum_{k=1}^{3} x_k e_k : \sum_{k=1}^{3} x_k^2 = 1\right\}$ whose elements $I$ are characterized by the property $I^2 = -1$. In particular, any $w \in \mathbb{H}$ uniquely defines two real numbers $x, y$ (with $y > 0$) and an imaginary unit $I_0$ such that $w = x + I_0 y$. If $w = x + I_0 y$, then we sometimes write $x = \operatorname{Re} w$ and $y = \operatorname{Im} w$; furthermore, we adopt the notation $w_J := x + Jy$ (with $J \in \mathbb{S}$) and $S_w = \{w_J : J \in \mathbb{S}\}$.

New theories of regular functions of a quaternionic (octonionic and hypercomplex) variable have been introduced (see [5; 7; 8; 9; 11]) that turn out to be interesting and rich. For a survey of recent results for these functions, we refer the interested reader to [2; 6] and the references therein.

According to these theories, a regular function in $\mathbb{H}$ is defined as follows.

Definition 1.1. Let $\Omega$ be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a real differentiable function with continuous real partial derivatives. Then $f$ is said to be *slice regular* if, for every $I \in \mathbb{S}$, its restriction $f_I$ to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and $I$ is holomorphic on $\Omega \cap L_I$.

Remark 1.2. The requirement for $f : \Omega \to \mathbb{H}$ to be slice regular is equivalent to saying that, for every $I$ in $\mathbb{S}$,

$$\bar{\partial}_I f(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) = 0$$

on $\Omega \cap L_I$.

From now on, we will always refer to slice regular functions and, for the sake of brevity, call them regular functions.

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If $\Omega$ is a domain in $\mathbb{H}$ and $f\colon \Omega \to \mathbb{H}$ is regular in $\Omega$, then the Cullen derivative of $f$ is (well) defined (see [8]) as follows:

$$\partial_C f(w) = f'(w)$$
$$= \begin{cases} \partial_I f(x+yI) \\ \qquad := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - I\dfrac{\partial}{\partial y}\right) f_I(x+yI) & \text{if } w = x+yI \text{ and } y \neq 0, \\ \dfrac{\partial}{\partial x} f_I(x+yI) & \text{if } w = x \in \mathbb{R}. \end{cases}$$

With this notation, any function that is regular in an open ball $B = B(0,R) = \{w \in \mathbb{H} : |w| < R\}$ centered at the origin has Cullen derivatives that are also regular functions in $B(0,R)$. Furthermore, $f$ is analytic (see [8; 11]); namely,

$$f(w) = \sum_{n=0}^{\infty} w^n \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0).$$

Since the pointwise multiplication of regular functions does not maintain regularity in general, we need to introduce the following definition (see also [3]).

DEFINITION 1.3. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be given quaternionic power series with radii of convergence greater than $R$. We define the *regular product* of $f$ and $g$ as the series $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$, whose coefficients $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ are obtained by discrete convolution from the coefficients of $f$ and $g$.

The regular product of $f$ and $g$, which we denote indifferently as $f * g$, $f * g(q)$, or $f(q) * g(q)$, has radius of convergence greater than $R$. It can be easily proven that the regular multiplication $*$ is an associative, noncommutative operation and that, when $f(q) \neq 0$,

$$f * g(q) = f(q)g(f(q)^{-1}qf(q)), \tag{1.1}$$

whereas $f * g(q) = 0$ if $f(q) = 0$.

THEOREM 1.4. *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence $R$, and let $\alpha \in B(0,R)$. Then $f(\alpha) = 0$ if and only if there exists a quaternionic power series $g$ with radius of convergence $R$ such that*

$$f(q) = (q-\alpha) * g(q). \tag{1.2}$$

This result (whose proof can be found in [3]) would of course be uninteresting if the other zeros of $f$ did not depend on the zeros of $g$. Fortunately, this is not the case: the zeros of a regular product $f * g$ are strongly related with those of $f$ and $g$, as shown by the following theorem (see [3]).

THEOREM 1.5 (Zeros of a regular product). *Let $f$ and $g$ be given quaternionic power series with radii greater than $R$ and let $\alpha \in B(0,R)$. Then $f * g(\alpha) = 0$ if and only if either $f(\alpha) = 0$ or $f(\alpha) \neq 0$ and $g(f(\alpha)^{-1}\alpha f(\alpha)) = 0$.*

In particular, if $f * g$ has a zero in $S = x + y\mathbb{S}$ then either $f$ or $g$ have a zero in $S$. However, the zeros of $g$ in $S$ need not be in one-to-one correspondence with the zeros of $f * g$ in $S$ that are not zeros of $f$.

EXAMPLE 1.6. Let $I \in \mathbb{S}$ be an imaginary unit. The regular product

$$(q - I) * (q + I) = q^2 + 1$$

has $\mathbb{S}$ as its zero set, whereas $q - I$ and $q + I$ vanish only at $I$ and $-I$, respectively.

EXAMPLE 1.7. Let $I, J \in \mathbb{S}$ be different imaginary units and suppose $I \neq -J$. The regular product

$$(q - I) * (q - J) = q^2 - q(I + J) + IJ$$

vanishes at $I$ but has no other zero in $\mathbb{S}$: given any $L \in \mathbb{S}$, we get

$$\begin{aligned} L^2 - L(I + J) + IJ = 0 &\iff L(I + J) = -1 + IJ \\ &\iff L(I + J) = I(I + J) \\ &\iff L = I, \end{aligned}$$

since $I + J \neq 0$.

The principal aim of this paper is to define a sort of logarithmic derivative of a regular function and apply it in order to “detect” the existence of zeros (and poles) of regular functions in the “symmetric regions” that naturally arise in this setting.

## 2. Symmetrization and Computation of the Zeros

In this section we summarize the results that characterize the zero set of $f$. This leads to the introduction of new power series related to $f$, as follows.

DEFINITION 2.1. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence $R$. We define the *regular conjugate* of $f$ as the series $f^c(q) = \sum_{n=0}^{+\infty} q^n \bar{a}_n$.

We remark that $f^c$ also has radius of convergence $R$ and that, in general, if $h = f * g$ then $h^c = g^c * f^c$. If we define the *symmetrized* of $f$ as $f^s = f * f^c = f^c * f$, then $f^s$ is analytic and has radius of convergence $R$. Notice furthermore that the coefficients of $f^s$ are all real and that, if the coefficients of $f$ are all real, then simply $f^s = f^2$. In particular (see [3]), we have the following statement.

PROPOSITION 2.2. *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ be a given quaternionic power series with radius of convergence $R$. If $\alpha = x_0 + I_0 y_0$ (with $x_0, y_0 \in \mathbb{R}$ and $I_0 \in \mathbb{S}$) is such that $f(\alpha) = f(x_0 + I_0 y_0) = 0$, then $f^s(x_0 + L y_0) = 0$ for all $L \in \mathbb{S}$.*

Actually, something more precise can be proven about the zeros of $f$, of $f^c$, and of $f^s$ (see again [3]).

PROPOSITION 2.3. *Let $f$ be a given quaternionic power series with radius of convergence $R$, and let $\alpha = x_0 + I_0 y_0$ (with $x_0, y_0 \in \mathbb{R}$ and $I_0 \in \mathbb{S}$) be such that $S_\alpha := x_0 + y_0\mathbb{S} \subseteq B(0, R)$. Then the zeros of $f$ in $S$ are in one-to-one correspondence with those of $f^c$.*

THEOREM 2.4 (Structure of the zero set). *Let $f : B(0, R) \to \mathbb{H}$ be a regular function and suppose that $f$ does not vanish identically. Then the zero set of $f$ consists of isolated points or isolated 2-spheres of the form $S = x + y\mathbb{S}$ for $x, y \in \mathbb{R}$.*

A nonreal zero $\alpha$ of a regular function $f$ is called *spherical* if $\bar{\alpha}$ is also a zero of $f$. Symmetrization allows us to transform any nonreal zero into a spherical zero, and these zeros cannot accumulate: if they did then zeros would accumulate in each complex line $L_I$, which is impossible for the identity principle of (holomorphic and) regular functions (see [8]) unless $f \equiv 0$. We recall that a domain $U \subseteq \mathbb{H}$ is called

(i) *an axially symmetric domain* if, for any $x + yI \in U$ with $y \neq 0$, the whole 2-sphere $x + y\mathbb{S}$ is contained in $U$;
(ii) *a slice domain* if $U \cap \mathbb{R}$ is nonempty and if $L_I \cap U$ is a domain in $L_I$ for all $I \in \mathbb{S}$.

Furthermore, we have the following theorem.

THEOREM 2.5. *Let $f$ be any given quaternionic power series with radius $R$. Then $f^s$ vanishes exactly on the 2-spheres (or singletons) $x + y\mathbb{S}$ containing a zero of $f$.*

The following Leibniz rule holds for the Cullen derivative of a $*$-product function (see [5]).

PROPOSITION 2.6. *Let $h = f * g$. Then*

$$h' = f' * g + f * g'.$$

Consider the derivative of $f^s = f * f^c = f^c * f$; we have, from Proposition 2.6, $(f^s)' = f' * f^c + f * (f^c)'$.

Assume that $f$ is a regular function and that $\alpha$ is a zero of $f$. Then we can write:

- $f(q) = (q - \alpha) * r(q)$ if $\alpha$ is not a spherical zero of $f$;
- $f(q) = (q^2 - 2q\,\mathrm{Re}(\alpha) + |\alpha|^2) * r(q)$ if $\alpha$ is a spherical zero of $f$.

In both cases we assume—for the moment—that $r(\alpha) \neq 0$. Thus we have:

- $f^s(q) = (q - \alpha) * r(q) * r^c(q) * (q - \bar{\alpha}) = (q - \alpha)^s * r^s(q)$ if $\alpha$ is not a spherical zero of $f$;
- $$\begin{aligned} f^s(q) &= (q^2 - 2q\,\mathrm{Re}(\alpha) + |\alpha|^2) * r(q) * r^c(q) * (q^2 - 2q\,\mathrm{Re}(\alpha) + |\alpha|^2)^c \\ &= (q^2 - 2q\,\mathrm{Re}(\alpha) + |\alpha|^2)^2 * r^s(q) \end{aligned}$$
  if $\alpha$ is a spherical zero of $f$.

Therefore,

- $(f^s)'(q) = ((q-\alpha)^s)' * r^s(q) + (q-\alpha)^s (r^s)'(q)$
  $= (2q - 2\operatorname{Re}\alpha) * r^s(q) + (q-\alpha)^s (r^s)'(q)$
  if $\alpha$ is not a spherical zero of $f$;
- $(f^s)'(q) = 2(q^2 - 2q\operatorname{Re}(\alpha) + |\alpha|^2) * (2q - 2\operatorname{Re}\alpha) * r^s(q)$
  $+ (q^2 - 2q\operatorname{Re}(\alpha) + |\alpha|^2)^2 * (r^s)'(q)$
  if $\alpha$ is a spherical zero of $f$.

Finally, we have the following definition.

DEFINITION 2.7. Consider a regular function $f$ with a zero $\alpha$. Then we define

$$\mathcal{L}_f(q) := \frac{(f^s)'(q)}{f^s(q)}$$
$$= \begin{cases} 2\dfrac{q - \operatorname{Re}\alpha}{(q-\alpha)^s} + \dfrac{(r^s)'(q)}{r^s(q)} \\ \qquad = 2\dfrac{q - \operatorname{Re}\alpha}{(q-\alpha)^s} + \mathcal{L}_r(q) & \text{if } \alpha \text{ is not a spherical zero of } f, \\ 4\dfrac{q - \operatorname{Re}\alpha}{(q-\alpha)^s} + \dfrac{(r^s)'(q)}{r^s(q)} \\ \qquad = 4\dfrac{q - \operatorname{Re}\alpha}{(q-\alpha)^s} + \mathcal{L}_r(q) & \text{if } \alpha \text{ is a spherical zero of } f. \end{cases}$$

Notice that if, for a regular function $f$, we put

$$f^{-*} := \frac{f^c}{f^s}$$

(the position is justified by $f * f^{-*} = f^{-*} * f \equiv 1$), then we can rewrite the previous definition as

$$\mathcal{L}_f(q)$$
$$:= \frac{(f^s)'(q)}{f^s(q)}$$
$$= \begin{cases} (q-\bar{\alpha})^{-*} + (q-\alpha)^{-*} + \mathcal{L}_r(q) & \text{if } \alpha \text{ is not a spherical zero of } f, \\ 2(q-\bar{\alpha})^{-*} + 2(q-\alpha)^{-*} + \mathcal{L}_r(q) & \text{if } \alpha \text{ is a spherical zero of } f. \end{cases}$$

Recall that, on any $L_I$, the restriction $f_I$ of the regular function $f$ in $B(0, R)$ is holomorphic (and so is $r_I$, with $r$ as in the decomposition given previously). Therefore let $\alpha = x_0 + I_0 y_0$ and consider any point $\tilde{\alpha} = a^{-1}\alpha a$ with $a \in \mathbb{H}$ and $a \neq 0$; observe that $\tilde{\alpha} = x_0 + I_1 y_0$ (i.e., $\tilde{\alpha} = \alpha_{I_1}$ is on the sphere $S_\alpha$). In the complex plane $L_{I_1}$, take any disc $\Delta_l(\tilde{\alpha})$ centered at $\tilde{\alpha}$ and with radius $l > 0$ such that $|\tilde{\alpha} + le^{I_1\vartheta}| < R$ for $\vartheta \in [0, 2\pi)$. We can apply a well-known result in complex analysis (see e.g. [1; 12]) to obtain the following result.

PROPOSITION 2.8. *Let $f$ be a regular function in $B(0, R)$ not identically zero. Assume that $\alpha = x_0 + I_0 y_0$ is a zero of $f$. Given $I \in \mathbb{S}$, we consider $\Delta_l(\alpha_I) \subset B(0, R) \cap L_I$, which is any disc in $B(0, R) \cap L_I$ that is centered at $\alpha_I$ and such that $f^s$ never vanishes on $\partial\Delta_l(\alpha_I)$. Then*

$$\frac{1}{2\pi I}\int_{\partial\Delta_I(\alpha_I)}\mathcal{L}_f(z)\,\mathrm{d}z\neq 0.$$

*Conversely, if for a given disc* $\Delta_I(\alpha_I)\subset B(0,R)\cap L_I$ *we have*

$$\frac{1}{2\pi I}\int_{\partial\Delta_I(\alpha_I)}\mathcal{L}_f(z)\,\mathrm{d}z\neq 0,$$

*then the regular function* $f$ *must have a zero in the axially symmetric domain* $\Delta_I(\alpha_I)\times\mathbb{S}=\bigcup_{\eta\in\Delta_I(\alpha_I)}S_\eta$.

The regular function $\mathcal{L}_f(q)$ just introduced in $\mathbb{H}$ plays the role of the logarithmic derivative of $f$ in $\mathbb{C}$, and it can actually be used to replicate the notion of index for a zero of $f$. We recall that to each zero $\alpha$ of a regular polynomial $P$ we can associate a multiplicity $n_\alpha\in\mathbb{N}$ as in [10]; that is, we have the following result.

THEOREM 2.9. *Let* $P$ *be a regular polynomial of degree* $m$. *Then there exist* $p, m_1,\dots,m_p\in\mathbb{N}$ *and* $w_1,\dots,w_p\in\mathbb{H}$, *which are generators of the spherical roots of* $P$, *such that*

$$P(q)=(q^2-2q\,\mathrm{Re}(w_1)+|w_1|^2)^{m_1}\cdots(q^2-2q\,\mathrm{Re}(w_p)+|w_p|^2)^{m_p}Q(q),\quad (2.1)$$

*where* $\mathrm{Re}(w_i)$ *denotes the real part of* $w_i$ *and* $Q$ *is a regular polynomial with coefficients in* $\mathbb{H}$ *having only nonspherical zeros. Moreover, if* $n=m-2(m_1+\dots+m_p)$ *then there exist a constant* $c\in\mathbb{H}$, $t$ *distinct* 2*-spheres* $S_1=x_1+y_1\mathbb{S},\dots,S_t=x_t+y_t\mathbb{S}$, $t$ *integers* $n_1,\dots,n_t$ *with* $n_1+\dots+n_t=n$, *and (for any* $i=1,\dots,t$*)* $n_i$ *quaternions* $\alpha_{ij}\in S_i$, $j=1,\dots,n_i$, *such that*

$$Q(q)=\left[\mathop{\boxed{\ast}\!\prod}_{i=1}^{t}\mathop{\boxed{\ast}\!\prod}_{j=1}^{n_i}(q-\alpha_{ij})\right]c,\qquad (2.2)$$

*where* $\boxed{\ast}\!\prod$ *is the analogue of* $\prod$ *in the case of* $\ast$*-product.*

Following the ideas developed in [3; 4; 10], we can state the next definitions.

DEFINITION 2.10. Let $f: U\to\mathbb{H}$ be a regular function. If $x+Iy$ is a spherical zero of $f$, then its *spherical* multiplicity is defined as 2 times the largest integer $m$ for which it is possible to write $f(q)=(q^2-2qx+(x^2+y^2))^m g(q)$ for $g: U\to\mathbb{H}$ a regular function. Furthermore, we say that a zero $\alpha_1\in\mathbb{H}\setminus\mathbb{R}$ of $f$ has *isolated* multiplicity $k$ if $g$ can be written as

$$g(q)=(q-\alpha_1)\ast(q-\alpha_2)\ast\cdots\ast(q-\alpha_k)\ast h(q)$$

with $\alpha_j$ on the sphere $S_{\alpha_1}$ and such that $\alpha_j\neq\bar{\alpha}_{j+1}$ for $j=1,\dots,k-1$ and $h: U\to\mathbb{H}$ a regular function such that $h$ does not vanish on the sphere $S_{\alpha_1}$. Finally, for $x\in\mathbb{R}$ a zero of $f$, we say that $x$ has *isolated* multiplicity $n$ if we can write

$$g(q)=(q-x)^n h(q),$$

where $h: U\to\mathbb{H}$ is some regular function that does not vanish at $x$.

DEFINITION 2.11. Let $f$ be a regular function on a symmetric slice domain $\Omega$, and let $\alpha$ be a nonreal zero of $f$. If $S_\alpha = x + \mathbb{S}y \subset \Omega$, we define the *total* multiplicity of $S_\alpha$ to be the sum of the spherical multiplicity of $\alpha$ and all isolated multiplicities of points of $S_\alpha$. If instead $\alpha$ is real, then the total multiplicity of $x$ is defined as the isolated multiplicity of $f$.

We denote by $n_\alpha$ the total multiplicity of $S_\alpha$. After observing that in (2.2) if $\alpha_{ij}$ lies on the sphere $S_\alpha$ then $(q - \alpha_{ij})^s = (q - \alpha)^s$, from the previous calculations we conclude that, for sufficiently small $l > 0$,

$$\frac{1}{2\pi I} \int_{\partial \Delta_l(\alpha_I)} \mathcal{L}_f(z)\, \mathrm{d}z = \begin{cases} n_\alpha & \text{if } \alpha \text{ is not a real zero of } f, \\ 2n_\alpha & \text{if } \alpha \text{ is a real zero of } f. \end{cases}$$

As in the complex case, we are now interested in finding a notion of "total number of inverse images of a value $\beta \neq 0$ in a bounded region $V$" for a regular function $f$. Typically, one repeats for the regular function $q \mapsto f(q) - \beta$ the definition of $\mathcal{L}_{f-\beta}$ and then calculates the integrals as before. In the complex case, this leads one to evaluate

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - \beta}\, \mathrm{d}z,$$

where the boundary of the region $V$ is a (simple) closed curve $\gamma\colon [t_0, t_1] \to \mathbb{C}$ such that $f(\gamma(t)) \neq \beta$ for all $t \in [t_0, t_1]$. To make the definition consistent, in the case of a regular function $f$ in $\mathbb{H}$ we must consider

$$\frac{1}{2\pi I} \int_{\partial \Delta_l(\alpha_I)} \frac{[(f(z) - \beta)^s]'}{(f(z) - \beta)^s}\, \mathrm{d}z \tag{2.3}$$

with $\alpha_I$ and $l > 0$ small enough and such that $f(\alpha_I + le^{It}) \neq \beta$ for all $t \in [0, 2\pi)$. Notice that $[(f(q) - \beta)^s]' \neq (f^s)'(q) - \beta$ and $[(f(q) - \beta)^s]' \neq (f^s)'$. Furthermore, the value obtained in (2.3) is *independent* of the choice of the slice $L_I$ because the functions involved in the definition have all real coefficients. Therefore, if $f(\alpha) = \beta$ then we define $g^s_\beta(q) := (f(q) - \beta)^s$; clearly, $g^s_\beta(\alpha) = 0$ and $g^s_\beta(L_I) \subseteq L_I$ for every $I \in \mathbb{S}$. Moreover, $g^s_\beta|_{L_I}$ is holomorphic for every $I \in \mathbb{S}$.

We write $\beta = f(\alpha)$ and observe that the equality

$$\frac{1}{2\pi I} \int_{\partial \Delta_l(\alpha_I)} \frac{[(f(z) - \beta)^s]'}{(f(z) - \beta)^s}\, \mathrm{d}z = \frac{1}{2\pi I} \int_{\partial \Delta_l(\alpha_I)} \frac{(g^s_\beta(z))'}{g^s_\beta(z)}\, \mathrm{d}z$$

does not depend on the choice of $I$, since $g^s_\beta$ maps any $L_I$ into itself for any $I \in \mathbb{S}$; furthermore, we observe that the integrand function in

$$\frac{1}{2\pi I_\beta} \int_{\partial \Delta_l(\alpha_{I_\beta})} \frac{[(f(z) - \beta)^s]'}{(f(z) - \beta)^s}\, \mathrm{d}z$$

is (locally) continuous in $\beta$ where it is defined. This means that the value of

$$\frac{1}{2\pi I_\beta} \int_{\partial \Delta_l(\alpha_{I_\beta})} \frac{[(f(z) - \beta)^s]'}{(f(z) - \beta)^s}\, \mathrm{d}z$$

remains constant for any choice of $\beta'$ in an open neighborhood of $\beta$, and since $\beta = f(\alpha)$ we have

$$\frac{1}{2\pi I_\beta}\int_{\partial\Delta_I(\alpha_{I_\beta})}\frac{[(f(z)-\beta)^s]'}{(f(z)-\beta)^s}\,\mathrm{d}z = \frac{1}{2\pi I_{\beta'}}\int_{\partial\Delta_I(\alpha_{I_{\beta'}})}\frac{[(f(z)-\beta')^s]'}{(f(z)-\beta')^s}\,\mathrm{d}z \neq 0.$$

Hence, by Proposition 2.8 we obtain a version of the open mapping theorem as presented in [4] for axially symmetric domains.

THEOREM 2.12. *Let $f\colon B(0,R)\to\mathbb{H}$ be a regular function. If $U$ is an axially symmetric open subset of $B(0,R)$, then $f(U)$ is open. In particular, $f(B(0,R))$ is an open set.*

## 3. The Argument Principle

Given the peculiar properties of the zeros of regular functions, it is natural to look for singularities resembling the poles of holomorphic complex functions. This question has the following complete answer, as given in [13].

PROPOSITION 3.1. *Consider a quaternionic Laurent series $f(q) = \sum_{n\in\mathbb{Z}} q^n a_n$ with quaternionic coefficients $a_n \in \mathbb{H}$. There exists a spherical shell*

$$A = A(0,R_1,R_2) = \{q\in\mathbb{H} : R_1 < |q| < R_2\}$$

*such that*: (i) *the series $f^+(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $f^-(q) = \sum_{n=1}^{+\infty} q^{-n}a_{-n}$ both converge absolutely and uniformly on the compact subsets of $A$*; (ii) *$f^+(q)$ diverges for $|q| > R_2$*; (iii) *$f^-(q)$ diverges for $|q| < R_1$. If $A$ is not empty (i.e., if $0 \le R_1 < R_2$), then the function $f\colon A\to\mathbb{H}$ defined by $f(q) = \sum_{n\in\mathbb{Z}} q^n a_n = f^+(q) + f^-(q)$ is regular.*

This proposition ensures the existence of functions that are regular on a punctured ball $B(0,R)\setminus\{0\}$ and have a singularity at 0. Moreover, any function that is regular on a spherical shell $A(0,R_1,R_2)$ admits a Laurent series expansion centered at 0. The latter statement is a special case of the following result.

THEOREM 3.2. *Let $f$ be a regular function on a domain $\Omega$, let $p = x + Iy$ for $p\in\mathbb{H}$, and let $L_I = \mathbb{R} + I\mathbb{R}$ be the complex line through $p$. If $\Omega$ contains an annulus $A_I = A(p,R_1,R_2)\cap L_I$, then there exist $\{a_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{H}$ such that $f_I(z) = \sum_{n\in\mathbb{Z}}(z-p)^n a_n$ for all $z\in A_I$. If, moreover, $p\in\mathbb{R}$, then $f(q) = \sum_{n\in\mathbb{Z}}(q-p)^n a_n$ for all $q\in A(p,R_1,R_2)\cap\Omega$.*

DEFINITION 3.3. Let $f$, $p$, and $\{a_n\}_{n\in\mathbb{Z}}$ be as in Theorem 3.2. The point $p$ is called a *pole* if there exists an $n\in\mathbb{N}$ such that $a_{-m} = 0$ for all $m > n$; the minimum of such an $n\in\mathbb{N}$ is called the *order* of the pole and is denoted $\mathrm{ord}_f(p)$. If $p$ is not a pole for $f$ then we call it an *essential singularity* for $f$.

Notice that, by the final statement of Theorem 3.2, real singularities are completely analogous to singularities of holomorphic functions of one complex variable but bear no resemblance to the case of several complex variables.

We are now able to introduce an analogue of the concept of a meromorphic function (see [13]).

DEFINITION 3.4. Let $\Omega$ be a domain in $\mathbb{H}$, let $\mathcal{S} \subseteq \Omega$, and suppose the intersection $\mathcal{S}_I = \mathcal{S} \cap L_I$ is a discrete subset of $\Omega_I = \Omega \cap L_I$ for all $I \in \mathbb{S}$. A regular function $f \colon \Omega \setminus \mathcal{S} \to \mathbb{H}$ is said to be *semiregular* on $\Omega$ if $\mathcal{S}$ does not contain essential singularities for $f$.

Notice that a function $f$ is semiregular if the restriction $f_I$ is meromorphic for all $I \in \mathbb{S}$. As proven in [13], $f$ is semiregular on $B(0,R)$ if and only if $f|_{B(0,R_1)}$ is a left regular quotient for all $R_1 < R$. This enables the definition of a multiplication operation $*$ on the set of semiregular functions on a ball as well as the proof of the following result.

THEOREM 3.5 (Structure of the poles). *If $f$ is a semiregular function on $B = B(0,R)$, then $f$ extends to a regular function on $B$ minus a union of isolated real points or isolated 2-spheres of the type $x + y\mathbb{S} = \{x + yI : I \in \mathbb{S}\}$ for $x, y \in \mathbb{R}$ and $y \neq 0$. All the poles on each 2-sphere $x + y\mathbb{S}$ have the same order, with the possible exception of one pole that must have lesser order.*

Let us now consider a semiregular function $f$ that can be locally written as $f = h^{-*} * g = h^{-s}h^c * g$, where $h$ and $g$ are regular functions. Then we deduce from the properties of the regular conjugate and $*$ product that

$$\begin{aligned} f^c &= g^c * h * h^{-s}, \\ f^s &= f * f^c = h^{-s}h^c * g * g^c * h * h^{-s} \\ &= h^{-s} * h^c * g^s * h * h^{-s} \\ &= h^{-s}g^s h^c * h * h^{-s} \\ &= h^{-s}g^s = (h^s)^{-1}g^s. \end{aligned}$$

Given any pair $G, H$ of quaternionic power series with real coefficients, from the properties of derivation that are similar to the analogue in $\mathbb{C}$ it follows that

$$(H^{-1}G)' = -H^{-2}H'G + H^{-1}G'.$$

Therefore we conclude that

$$(f^s)' = -(h^s)^{-2}(h^s)'g^s + (h^s)^{-1}(g^s)'$$

and so

$$\mathcal{L}_f = (f^s)^{-1}(f^s)' = (g^s)^{-1}(g^s)' - (h^s)^{-1}(h^s)' = \mathcal{L}_g - \mathcal{L}_h.$$

Hence, by recalling the results of the previous section, we obtain the following analogue for semiregular functions of the argument principle for meromorphic functions in $\mathbb{C}$.

THEOREM 3.6. *Given a semiregular function $f \in B(0,R)$ and $I \in \mathbb{S}$, consider $B_I(0,r) := B(0,r) \cap L_I$, $r < R$, which is any disc in $B(0,R) \cap L_I$ such that $\partial B_I(0,r)$ does not pass through any of the zeros or poles of $f$. Then*

$$\frac{1}{4\pi I}\int_{\partial B_I(0,rf)} \mathcal{L}_f(z)\,\mathrm{d}z$$

*counts the difference between the sum of* (*isolated and spherical*) *multiplicities of zeros of* $f$ *and the sum of orders of poles of* $f$ *that lie in* $B(0,r)$.

As a consequence of Theorem 2.5 and Theorem 3.6, we obtain the following analogue of the Rouché theorem.

COROLLARY 3.7. *Let* $h$ *and* $g$ *be two regular functions in* $B(0,R)$, *and assume that* $|h^s - g^s| < |h^s|$ *on* $\partial B(0,r)$, $r < R$. *Then* $h$ *and* $g$ *have the same number of zeros* (*counted with their multiplicities*) *in* $B(0,r)$.

*Proof.* First observe that if $|h^s - g^s| < |h^s|$ on $\partial B(0,r)$ then $h^s$ and $g^s$ are zero-free on $\partial B(0,r)$. Moreover, this implies that the function $(h^s)^{-1}g^s$ is the "symmetrized" (see paragraph following Definition 2.1) of the semiregular function $f = h^{-*} * g = (h^s)^{-1}h^c * g$ and is such that $|f^s - 1| < 1$ on $\partial B(0,r)$. Hence, given any $I \in \mathbb{S}$, we have

$$\frac{1}{4\pi I}\int_{\partial(B(0,r)\cap L_I)} \mathcal{L}_f(z)\,\mathrm{d}z = 0$$

and so the assertion follows from Theorem 2.5 and Theorem 3.6. □

We conclude this paper with a version of the Hurwitz theorem.

THEOREM 3.8. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of regular functions in* $B(0,R)$, *and assume that* $f_n \to f$ *as* $n \to +\infty$ *uniformly on each compact subset of* $B(0,R)$. *Then either* $f \equiv 0$ *or every zero of* $f^s$ *is a limit of a sequence of spheres of zeros of* $(f_n)^s$ *for* $n > n_0$.

*Proof.* Observe that the limit function $f$ is regular; furthermore, we recall (see [14]) that the convergence of $\{f_n\}_{n\in\mathbb{N}}$ to $f$ uniformly on each compact subset of $B(0,R)$ implies the uniform convergence (on compacta of $B(0,R)$) of $\{(f_n)^s\}_{n\in\mathbb{N}}$ to $f^s$.

Assume that $f(q_0) = 0$ for $q_0 = x_0 + I_0 y_0$ but that $f \neq 0$ in a neighborhood $U(q_0) = L_{I_0} \cap B(q_0,r)$ in $B(0,R)$ for sufficiently small $r > 0$. This implies that $f^s(q_0) = 0$ but $f^s \neq 0$ in $U(q_0)$. Assume that $r$ is chosen such that $f^s \neq 0$ in $\partial U(q_0) := L_{I_0} \cap \overline{B(q_0,r)}$. Let $m$ be the minimum of $|f^s|$ on $\partial U(q_0)$. Then, for all $n > n_0$ and $q \in \partial U(q_0)$, we have

$$|(f_n)^s(q) - f^s(q)| < m < |f^s(q)|.$$

From Corollary 3.7 it now follows that $(f_n)^s$ has the same number of zeros of $f^s$ inside $\partial U(q_0)$; in other words, $(f_n)^s$ must vanish at least once inside $\partial U(q_0)$ for all $n > n_0$. □

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