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MITIGATION OF THE NON-UNIQUENESS PROBLEM FOR THE INDIRECT BOUNDARY ELEMENT METHOD

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ABSTRACT

Boundary Element Method fails to provide a unique solution for exterior problems at frequencies corresponding to internal resonances; this is the well-known non-uniqueness problem. This problem becomes even more critical at higher frequencies with the increasing modal density inside the cavity. Approaches have been developed to circumvent this limitation as the over determination points or the Burton-Miller formulation in Direct BEM. In Indirect BEM, the most commonly used approach consists in applying some absorption (Robin boundary conditions) inside the cavity to damp those internal resonances. Nevertheless, the approach has a large computational cost as it increases the number of unknowns.

In this paper we show that imposing Robin boundary condition only on certain percentage of elements allows to mitigate the non-uniqueness problem without drastically increase the computational cost. Numerical examples on industrial size problem using both standard IBEM and Fast Multipole BEM are presented to demonstrate the accuracy of the solution over the entire range of frequencies. Finally, equivalence with the Burton and Miller method for direct formulation is demonstrated.

1. INTRODUCTION

Nowadays, accuracy of numerical predictions is crucial in design applications and BEM is a well-established method for low-frequency acoustic problems. Its main advantage is that the dimensionality of the problem is decreased and its formulation is suitable for modeling infinite exterior problems. On the other hand, this methodology presents some disadvantages. To reach an accurate solution at higher frequencies becomes very demanding because at least 6 elements per wavelength are needed. Moreover, fully-populated matrices increase exponentially the size of the problem. To overcome this limitation, Fast Multipole BEM has been developed as an

improvement of the standard BEM technique. It allows to solve large acoustic problems quicker and using less computational resources [1, 2].

The other shortcoming is the fact that boundary integral formulations fail to provide a unique solution at certain frequencies, also called fictitious eigenfrequencies. The solution to an exterior problem is polluted by fictitious resonances corresponding to real eigenfrequencies of the interior problem. This problem is not physical but merely methematical and results from an ill-posed formulation. The non-uniqueness issue seriously degrades the solution over the whole frequency range, in particular at higher frequencies where the modal density increases. Moreover, the presence of a fictitious resonance slows down the convergence of the FMBEM iterative solver due to the resulting bad conditioning.

In order to overcome the issue several boundary integral formulations have been proposed. The family of formulations which assembly simultaneously the interior and exterior surface integrals, were first approached by Kupradze [3] and Copley [4], successively developed by Schenck [5] as the CHIEF method. Another type of improved formulation combines the original surface integral and its normal derivative. This methodology was first approached by Panich [6] and further developed by Kussumaul [7], Burton and Miller [8] and Zaman [9]. The last typology of improved formulations, which has been proposed by Ursell [10] consists in the usage of modified Green's functions.

All these are mainly direct improved formulations and they are limited either by the frequency range of application or by the computational request.

In this paper we propose an improvement of the method based on internal impedance boundary condition to mitigate the non-uniqueness issue for indirect BEM. Applying a prescribed value of internal impedance is proven to be mathematically equivalent to the well-known formulation proposed by Burton and Miller. Nevertheless, applying the condition over the entire boundary requires high computational efforts. The enhancement proposed consists in the application of the internal impedance boundary condition only over a certain percentage of elements of the model. This results in a practical method to avoid the non-uniqueness problem without highly increase the computational requirements of the problem.

In order to prove the accuracy provided by this methodology, two examples are shown. Different percentages of elements selected are used to investigate the accuracy of the solution vs computational cost.

2. BOUNDARY INTEGRAL FORMULATIONS

For solving exterior acoustic radiation problems two classes of boundary integral formulations exist, namely the direct and the indirect formulations [11].

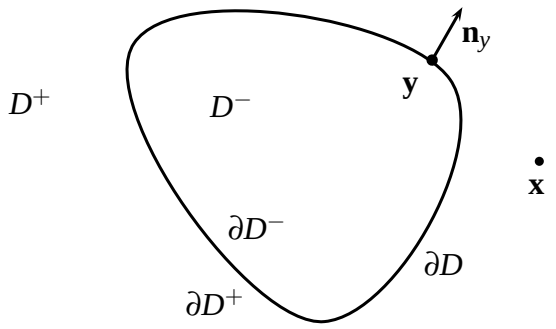

Figure 1: Domain definition

The Helmholtz equation evaluated at a point $\mathbf{x}$ of domain D (see Figure 1) is given by

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0, \tag{1}$$

where p is the acoustic pressure, $k = \omega/c$ is the wavenumber and c is the speed of sound. In order to obtain the direct formulation eq.(1) is integrated via Green's theorem using the free-space Green's function,

$$g(\mathbf{y},\mathbf{x}) = \frac{e^{-ikr}}{4\pi r}, \tag{2}$$

where $r = |\mathbf{x}-\mathbf{y}|$. This leads to the Helmholtz Integral Equation (HIE)

$$c_s(\mathbf{x})p(\mathbf{x}) = \int_{\partial D} p(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}\,ds - \int_{\partial D} g(\mathbf{y},\mathbf{x})\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}\,ds, \tag{3}$$

and the Normal Derivative Helmholtz Integral Equation (NDHIE)

$$c_s(\mathbf{x})\frac{\partial p(\mathbf{x})}{\partial n(\mathbf{x})} = \int_{\partial D} p(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})\partial n(\mathbf{x})}\,ds - \int_{\partial D} \frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}\,ds, \tag{4}$$

where $c_s(\mathbf{x})$ is a coefficient dependent on the position of the point $\mathbf{x}$. If $\mathbf{x}$ is inside the domain c_s is equal to 1, if it is outside the domain c_s is equal to 0 and if it is on the boundary c_s is equal to $\frac{1}{2}$ for smooth surface. These equations are first solved on the boundary and finally inside the domain. The direct formulation allows to evaluate the solution either on the exterior D^+ or on the interior domain D^-.

The indirect formulation expresses the acoustic pressure as the field in free-space due to monopoles and dipoles distributions over the boundary, respectively known as single layer potential, u_s, and double layer potential, u_d,

$$p(\mathbf{x}) = u_s(\mathbf{x}) + u_d(\mathbf{x}) = -\int_{\partial D} g(\mathbf{y},\mathbf{x})\sigma(\mathbf{y})\,ds + \int_{\partial D} \frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}\mu(\mathbf{y})\,ds. \tag{5}$$

The conditions are applied over both sides of the boundary, positive ∂D^+ and negative ∂D^-, and are directly connected to the potential densities. The derivative of the single layer potential presents a discontinuity approaching the boundary ∂D [12] and leads to a jump in terms of derivatives of pressure,

$$\begin{aligned} \frac{\partial p^+(\mathbf{y})}{\partial n} &= \frac{\partial p(\mathbf{y})}{\partial n} + \frac{\sigma(\mathbf{y})}{2} \\ \frac{\partial p^-(\mathbf{y})}{\partial n} &= \frac{\partial p(\mathbf{y})}{\partial n} - \frac{\sigma(\mathbf{y})}{2}, \end{aligned} \tag{6}$$

where $\frac{\partial p^+(\mathbf{y})}{\partial n}$ ($\frac{\partial p^-(\mathbf{y})}{\partial n}$) is the derivative of pressure on the positive (negative) side of the boundary ∂D^+ (∂D^-) and $\frac{\partial p(\mathbf{y})}{\partial n}$ is the derivative of pressure on the boundary ∂D. The double layer potential presents a discontinuity approaching the boundary ∂D and generates a jump in terms of pressure,

$$\begin{aligned} p^+(\mathbf{y}) &= p(\mathbf{y}) + \frac{\mu(\mathbf{y})}{2} \\ p^-(\mathbf{y}) &= p(\mathbf{y}) - \frac{\mu(\mathbf{y})}{2}, \end{aligned} \tag{7}$$

where $p^+(\mathbf{y})$ ($p^-(\mathbf{y})$) is the pressure on the positive (negative) side of the boundary ∂D^+ (∂D^-) and $p(\mathbf{y})$ is the pressure on the boundary ∂D. The solution of an indirect acoustic problem requests to find the potential densities, σ and μ, on ∂D and evaluate their contribution with eq.(5) at a point $\mathbf{x}$.

Three types of boundary conditions can be applied. The Dirichlet boundary condition defines the pressure while the Neumann boundary condition defines the derivatives of pressure. Finally, the Robin boundary condition, which relates the pressure and its derivative, can be written as follows,

$$\frac{\partial p}{\partial n} = -\frac{i\rho\omega}{Z}p, \tag{8}$$

where ρ is the density and Z is the impedance of the medium.

3. THE INTERNAL IMPEDANCE BOUNDARY CONDITION METHOD

The non-uniqueness problem is a mathematical shortcoming which directly influences the accuracy of the solution. Practically, the problem can be described as follows. Solving an exterior Neumann problem with direct formulation, we find that the solution is polluted by fictitious resonances at the corresponding internal Dirichlet problem eigenfrequencies. The issue is quite similar with the indirect formulation: the solution to an exterior Neumann (or Dirichlet) problem is polluted by the resonances in correspondence of the interior Neumann (or Dirichlet) problem.

Today the CHIEF and the Burton and Miller method seem to be the most popular in direct BEM. The CHIEF is easily implementable and assures a good accuracy of the solution but it is not so robust in the mid and high frequency range. The Burton and Miller method leads to a unique solution for all the wavenumbers.

Using BEM, the non-uniqueness problem can be avoided with a linear combination of HIE eq.(3) and NDHIE eq.(4),

$$\begin{aligned}\frac{1}{2}p(\mathbf{x}) + \frac{1}{2}\alpha\frac{\partial p(\mathbf{x})}{\partial n(\mathbf{x})} = \int_{\partial D} p(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds - \int_{\partial D} g(\mathbf{y},\mathbf{x})\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds + \\ +\alpha\int_{\partial D} p(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})\partial n(\mathbf{x})}ds - \alpha\int_{\partial D}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds,\end{aligned} \tag{9}$$

where the coefficient α usually is equal to i/k. Both the HIE and the NDHIE present a corresponding infinite set of fictitious eigenfrequencies. Since eq.(9)is a linear combination it can yield fictitious eigenfrequency if and only if both HIE and NDHIE have common eigenfrequencies, which is unexpected.

For indirect BEM, besides some improved indirect formulations, some practical approaches are used: they consist either in adding internal dummy surface with impedance or apply internal impedance boundary condition on the inner side of the boundary.

We will see that using an internal Robin boundary condition, with $Z = -\rho c$ the indirect formulation of the problem is mathematically equivalent to the direct one proposed by Burton and Miller to overcome the non-uniqueness problem on the whole frequency range. Furthermore a trick to reduce the high computational requirements will be illustrated.

3.1 Mathematical equivalence with the Burton and Miller method

Let suppose to solve an exterior direct problem using the boundary conditions defined in the exterior indirect problem. The linear combination between HIE and the NDHIE can be considered

as follows

$$\frac{1}{2}p^+(\mathbf{x})+\frac{1}{2}\alpha\frac{\partial p^+(\mathbf{x})}{\partial n(\mathbf{x})}=\int_{\partial D^+}p^+(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds-\int_{\partial D^+}g(\mathbf{y},\mathbf{x})\frac{\partial p^+(\mathbf{y})}{\partial n(\mathbf{y})}ds+$$
$$+\alpha\int_{\partial D^+}p^+(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}ds-\alpha\int_{\partial D^+}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p^+(\mathbf{y})}{\partial n(\mathbf{y})}ds. \tag{10}$$

Considering eq.(7) and eq.(6), we can compute the pressure jump,

$$p^+ + p^- = 2p, \tag{11}$$

and the derivative of pressure jump

$$\frac{\partial p^+}{\partial n}+\frac{\partial p^-}{\partial n}=2\frac{\partial p}{\partial n}. \tag{12}$$

Substituting p^+ from eq.(11) and $\frac{\partial p^+}{\partial n}$ from eq.(12) into eq.(10), and supposing that $\partial D^+ \approx \partial D^- \approx \partial D$ we obtain the sequent relation

$$p(\mathbf{x})+\frac{1}{2}\left[\alpha\frac{\partial p^+(\mathbf{x})}{\partial n(\mathbf{x})}-p^-(\mathbf{x})\right]=2\int_{\partial D}p(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds-2\int_{\partial D}g(\mathbf{y},\mathbf{x})\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds+$$
$$-2\int_{\partial D}g(\mathbf{y},\mathbf{x})\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds+2\alpha\int_{\partial D}p(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}ds-2\alpha\int_{\partial D}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds+$$
$$+\alpha\int_{\partial D}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p^-(\mathbf{y})}{\partial n(\mathbf{y})}ds+-\alpha\int_{\partial D}p^-(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}ds-\int_{\partial D}p^-(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds+$$
$$+\int_{\partial D}g(\mathbf{y},\mathbf{x})\frac{\partial p^-(\mathbf{y})}{\partial n(\mathbf{y})}ds. \tag{13}$$

Introducing the jumps (11) and (12) we transform the problem in an indirect one because we are using the single and double layer potentials. The fifth and the sixth terms on the right side of eq.(13), compared with eq.(4), referred to the negative side, yield to

$$\alpha\int_{\partial D}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p^-(\mathbf{y})}{\partial n(\mathbf{y})}ds-\alpha\int_{\partial D}p^-(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}ds=-\frac{1}{2}\alpha\frac{\partial p^-(\mathbf{x})}{\partial n(\mathbf{x})}. \tag{14}$$

The last two terms on the right side of eq.(13), compared with eq.(3), lead to

$$-\int_{\partial D}p^-(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds+\int_{\partial D}g(\mathbf{y},\mathbf{x})\frac{\partial p^-(\mathbf{y})}{\partial n(\mathbf{y})}ds=-\frac{1}{2}p^-(\mathbf{x}). \tag{15}$$

Eq.(13) can now be rewritten as

$$p(\mathbf{x})+\frac{1}{2}\left[\alpha\frac{\partial p^+(\mathbf{x})}{\partial n(\mathbf{x})}-p^-(\mathbf{x})\right]=2\int_{\partial D}p(\mathbf{y})\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{y})}ds-2\int_{\partial D}g(\mathbf{y},\mathbf{x})\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds+$$
$$+2\alpha\int_{\partial D}p(\mathbf{y})\frac{\partial^2 g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})\partial n(\mathbf{y})}ds-2\alpha\int_{\partial D}\frac{\partial g(\mathbf{y},\mathbf{x})}{\partial n(\mathbf{x})}\frac{\partial p(\mathbf{y})}{\partial n(\mathbf{y})}ds-\frac{1}{2}\left[\alpha\frac{\partial p^-(\mathbf{x})}{\partial n(\mathbf{x})}-p^-(\mathbf{x})\right]. \tag{16}$$

Terms one to four on the right side of eq.(16) are the same terms on the right side of the eq.(9). In order to prove the mathematical equivalence between eq.(16) and eq.(9) the last term has

to be zero and the second one on the left side has to be equal to $2\alpha\frac{\partial p}{\partial n}$. We need to find an impedance value Z that satisfy

$$\alpha\frac{\partial p^-}{\partial n} + p^- = 0 \tag{17}$$

$$\alpha\frac{\partial p^+}{\partial n} - p^- = 2\alpha\frac{\partial p}{\partial n}. \tag{18}$$

The only impedance value that satisfies both eq.(17) and eq.(18), with $\alpha = \frac{i}{k}$, is

$$Z = -\rho c. \tag{19}$$

We can conclude that solving an indirect problem with internal impedance boundary condition, $-\rho c$, applied over the entire boundary is mathematically equivalent to solve the Burton and Miller problem for the direct formulation. Using an impedance equal to $-\rho c$, the real internal resonances are completely damped, obtaining a solution not polluted by them. The negative sign is due to the outward normal of the model.

Note that the proposed method allows to avoid the fictitious resonances both on the Neumann and on the Dirichlet external problem in indirect BEM. The former is much more attractive for industrial applications because usually the velocity field over the boundary is given.

3.2 Elements selection to reduce computational efforts

Adding internal impedance increases the number of unknowns and is very demanding. It will be indeed shown in the next section that applying this method allows to completely damp fictitious resonances both at low and at high frequencies but, as expected, solving time is highly increased.

What is proposed here is a method to obtain a good compromise between reducing computational efforts and getting accurate results. This method consists in to applying the internal impedance boundary condition, $-\rho c$, only over a certain percentage of elements. In [13] are presented application cases with randomly selected stripes of elements, patches and randomly selected elements. Best results are reached with the random selection of the elements on the surface.

4. APPLICATION CASES

The method introduced is now applied to a real industrial case and to an academic one. Solutions are investigated both at low frequencies, using standard BEM, and at higher frequency, using FMBEM. The software used for both standard BEM and FMBEM is LMS Virtual.Lab Rev.9 [14]. Note that with FMBEM the accuracy of solution and the solving time are strictly related to the choice of the tolerance and the number of iterations for the iterative solver.

Models are built choosing outward normal, imposing external velocity equal to 1 m/s over the whole boundary and applying internal impedance equal to -416.5 kg/m^{2}s over the entire boundary or only on a randomly selected group of elements.

4.1 Vibrating engine

The first case is a vibrating engine, whose acoustic response is evaluated at an external field point, Figure 2. The method, mesh composition and frequency range are indicated in Table 1.

In Figure 3, we can see that, using internal impedance applied over the entire boundary, fictitious resonances are completely eliminated both at low frequencies and at higher frequencies.

Solver	Nodes	Elements	Frequency Range	Frequency Step
Standard BEM	7464	7503	500-1000 Hz	10 Hz
FMBEM	7464	14946	1000-2500 Hz	10 Hz

Table 1: Engine. Solver, mesh composition and frequency range.

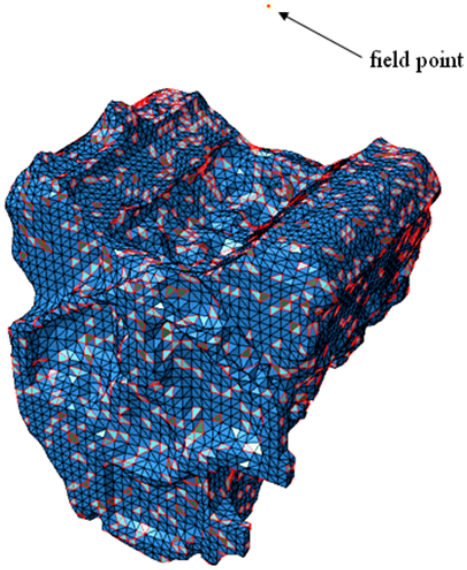

Figure 2: Vibrating engine: field point and 12% of selected elements with impedance boundary condition applied on.

For standard BEM, solving time is increased by about 5.5 when compared to the case without impedance applied.

In order to reduce the computational effort the impedance boundary condition is applied on a certain percentage of randomly selected elements, as shown in Figure 2. In Figure 4 and Figure 5 it can be seen that on the entire chosen frequency range a little percentage of elements allows to obtain an accurate solution. The curves are almost superposed and present the biggest errors in the highest part of the frequency range. Its amplitude is about 2 dB, negligible respect to the amplitude of the corresponding fictitious resonances.

Since FMBEM is an iterative solver, the solution time at each frequency is not constant. As a result, the evaluation of the computational requirements can be addressed only in terms of total CPU time for a set of frequencies. It is more straightforward and interesting to evaluate the advantages of the proposed method on a standard BEM solver, for which the solution time is constant at each frequency. For the engine model, the application of the impedance over the whole boundary surface increases the solution time with a factor 5.5. Applying the condition over 12% of the elements increases the solution time much less, about 150%, and applying it over 5% of the elements, the increase is about 50%. Both 12% and 5% of the elements allow to obtain an accurate solution.

4.2 Pulsating sphere

The second case presented is a very academic one: the pulsating sphere. The Neumann boundary condition yields strongly excited internal modes and damping fictitious resonances on the external solution is more difficult. In Table 2 are indicated the solver, the mesh composition and

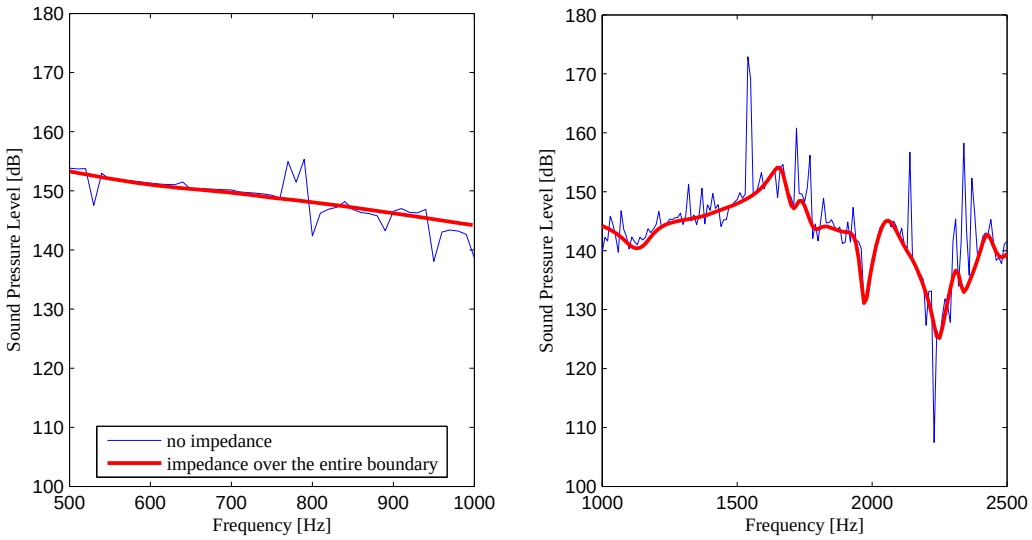

Figure 3: Engine. Solution to the external Neumann problem with standard BEM (left) and Fast Multipole BEM (right).

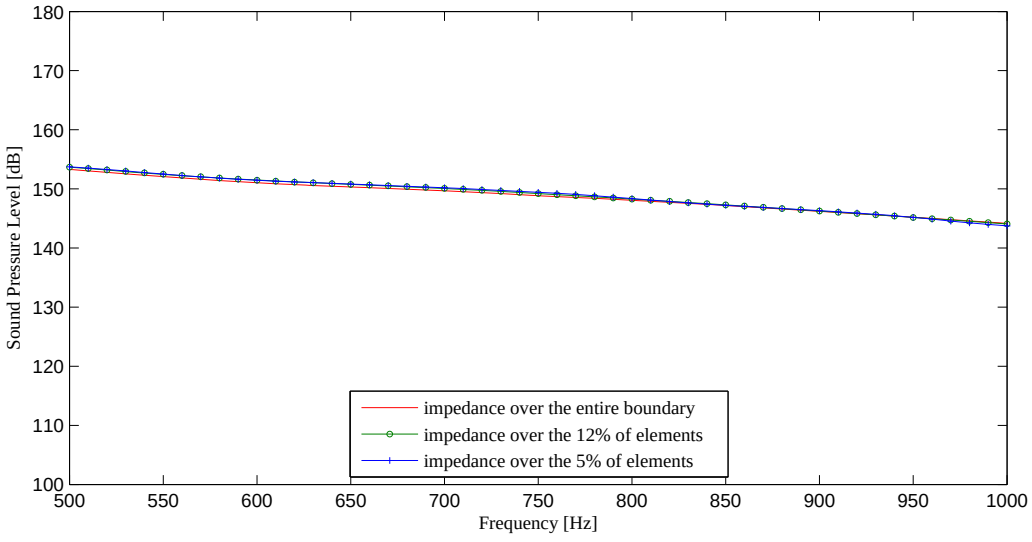

Figure 4: Engine, standard BEM. Solution to the external Neumann problem with impedance boundary condition applied over a certain percentage of randomly selected elements.

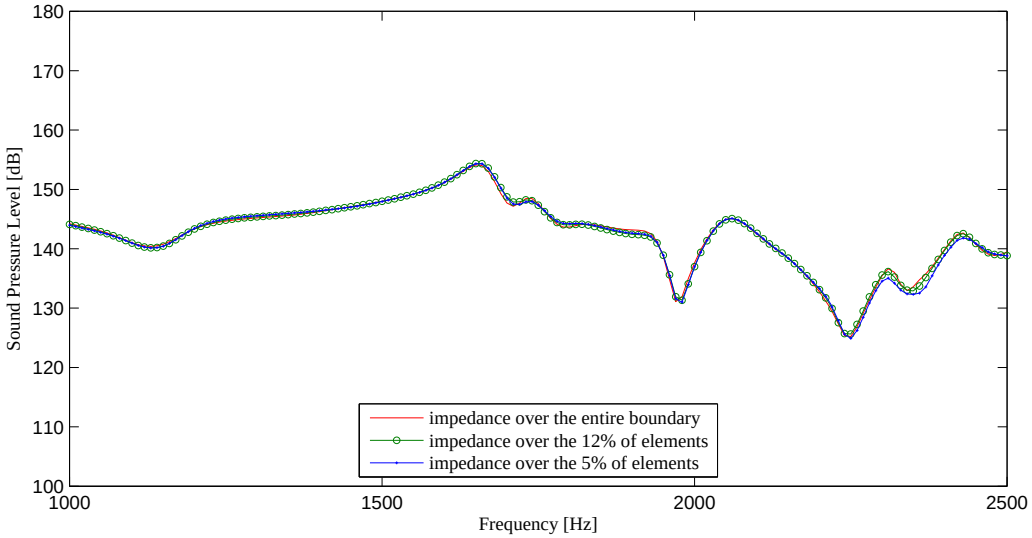

Figure 5: Engine, FMBEM. Solution to the external Neumann problem with impedance boundary condition applied over a certain percentage of randomly selected elements.

Solver	Nodes	Elements	Frequency Range	Frequency Step
Standard BEM	2402	2400	500-1000 Hz	10 Hz
FMBEM	15002	30000	2000-3000 Hz	10 Hz

Table 2: Sphere. Solver, mesh composition and frequency range.

the frequency range. The solution has been evaluated at an external point.

Using impedance boundary condition applied over the entire boundary, fictitious resonances are mitigated both with standard BEM and with FMBEM, Figure 6.

In Figure 7, the solution obtained using standard BEM is proposed. Selecting 20% and 40% of elements we get similar and very accurate results. Using 10% of elements allows to damp the fictitious resonance but it still pollutes the curve. In Figure 8, results computed with FMBEM are shown. Even in this case, solution obtained selecting 10% of elements does not allow to mitigate the problem, instead, applying the condition over the 20% and 40% of elements assures good accuracy.

We can conclude that in this case 20% of elements are sufficient to get accurate results over the whole examined frequency range. The application of the condition over the whole surface in standard BEM causes an increase of the solving time with a factor 4. Applying the absorption over the 40%, 20% and 10% of the elements increases the solution time respectively by 250%, 150% and 100%.

5. CONCLUSIONS

The non-uniqueness is one potential shortcoming with BEM technique and can seriously pollute the solution and drastically impact the efficiency of FMBEM solvers. In the past several improved boundary integral formulations have been proposed by many authors to overcome the issue. What we have proposed here is a practical method to obtain accurate results reducing computational efforts.

In this paper we have shown that using the internal impedance $-\rho c$ allows to mitigate the fictitous resonances on the entire frequency range but the computational cost drastically increases.

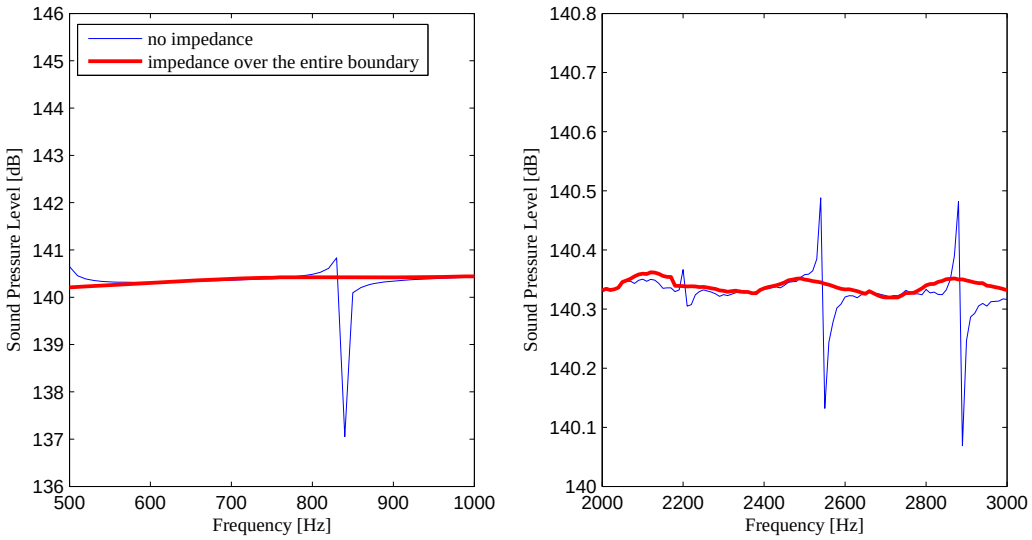

Figure 6: Sphere. Solution to the external Neumann problem with standard BEM (left) and Fast Multipole BEM (right).

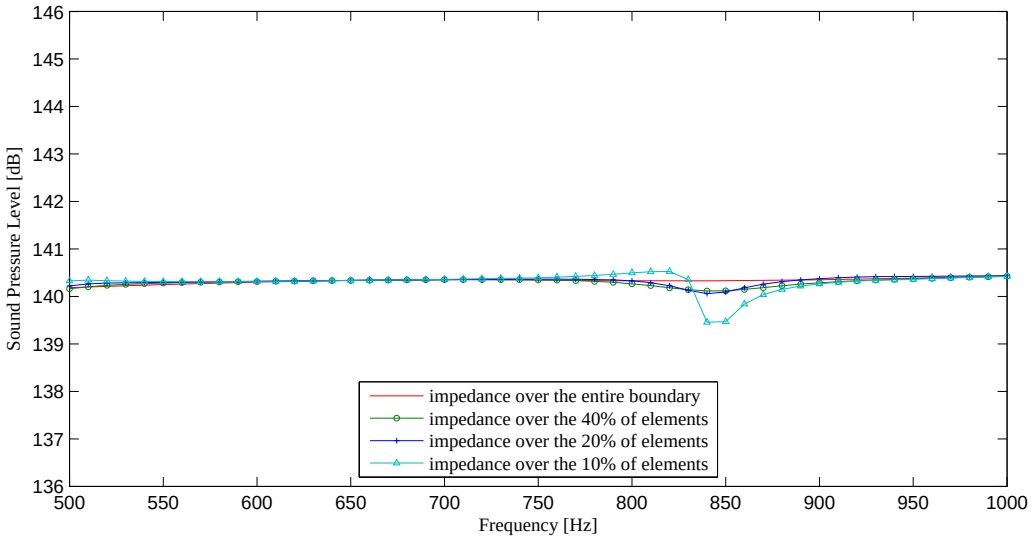

Figure 7: Sphere, standard BEM. Solution to the external Neumann problem with impedance boundary condition applied over a certain percentage of randomly selected elements.

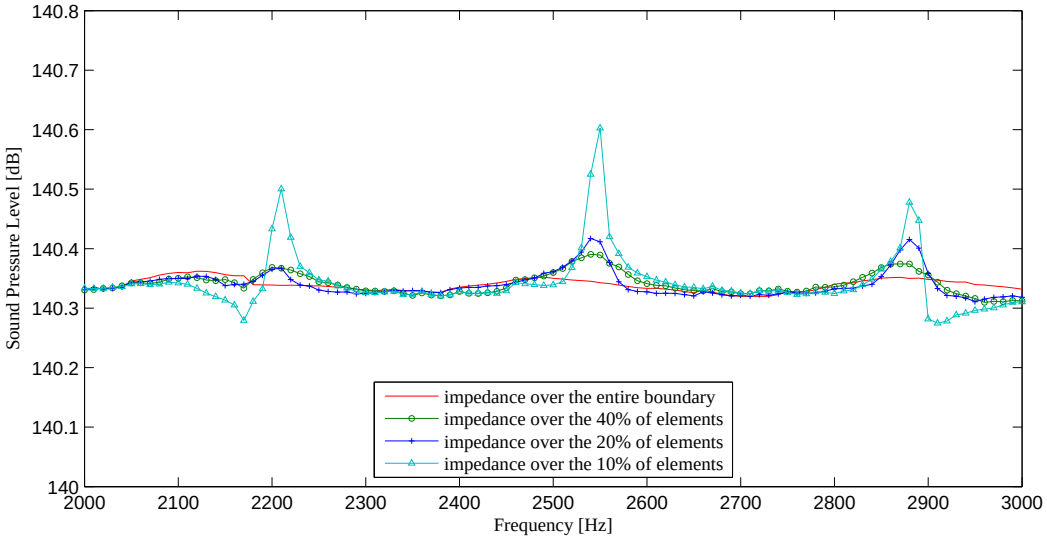

Figure 8: Sphere, FMBEM. Solution to the external Neumann problem with impedance boundary condition applied over a certain percentage of randomly selected elements.

We have illustrated that we can reach an accurate and less demanding solution applying the condition over a certain percentage of elements. This value is strictly connected to the internal problem: the stronger are the internal resonances, the heavier is the pollution and the larger is the percentage of elements needed.

Further improvement can be done in order to provide some practical guidelines to avoid the issue as function of the application, getting accurate results with the lowest computational cost.

6. ACKNOLEDGMENTS

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