A Harnack Inequality for Solutions of Doubly Nonlinear Parabolic Equations

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Abstract - We consider positive solutions of the doubly nonlinear parabolic equation

$$(|u|^{p-1})_t = \mathrm{div}(|Du|^{p-2}Du), \qquad p > 2.$$

We prove mean value inequalities for positive powers of nonnegative subsolutions and for negative powers of positive supersolutions using De Giorgi's methods. We combine them with Moser's logarithmic estimates to show that positive solutions satisfy a proper Harnack inequality.

AMS Subject Classification: 35K65, 35B05, 35B65

Key words: Nonlinear parabolic equation, mean value inequalities, Harnack estimate.

1. - Introduction

In the final Section of [16], without any explicit calculation, Trudinger states that Moser's method (see [11]) can be extended to prove the following

Theorem 1. *Let u be a positive solution of*

$$(|u|^{p-1})_t - \mathrm{div}(|Du|^{p-2}Du) = 0 \tag{1}$$

in Ω_T and suppose that the cylinder $[(x_0,t_0)+Q(2\rho,2^p\theta\rho^p)]\subset\Omega_T$. Then there exists a constant $C>1$ that depends only on the data and on θ s.t.

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} u \le C \inf_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} u$$

(we refer to the next Section for the notation). It is immediate to see that with the substitution $u^{p-1}=v$, (1) can be rewritten as

$$v_t - \left(\frac{1}{p-1}\right)^{p-1} \mathrm{div}(|v|^{2-p}|Dv|^{p-2}Dv) = 0$$

which is just a particular istance of the more general class of doubly nonlinear parabolic equations

$$v_t - \mathrm{div}(|v|^{m-1}|Dv|^{p-2}Dv) = 0 \tag{2}$$

where $p>1$ and $m+p>2$. This equation describes a lot of phenomena. Just to limit ourselves to the motion of fluids in media, when $p=2$ we obtain the well - known porous medium equation; if $m=1$ we have the parabolic p-laplacian, which describes the nonstationary flow in a porous medium of fluids with a power dependance of the tangential stress on the velocity of the displacement under elastic conditions; in the whole generality, that is when $p\neq 2$ and $m\neq 1$, (2) is a model for the polytropic case when we have dependance between stress and velocity of the displacement. However these are just few examples; the interested reader can find further applications in [1] or in [15].

Regularity issues for doubly nonlinear parabolic equations like (2) have been considered by a lot of authors and a complete bibliographic list cannot be given here: under this point of view, we refer to [4] (updated to 1993) and to [5] (just published). Let us just mention that, among others, continuity has been proved both in the degenerate ($p>2$ and $m>1$) and in the singular ($1<p<2$ or $0<m<1$) case in [8], [13] and [18]. Other interesting references can be found in [6], where the regularity in Sobolev spaces is considered.

Coming back to (1), the reason of Trudinger's statement basically lies in the p-homogeneity of the equation, that makes the proof of mean value inequalities for positive and negative powers of the solution as natural as in the case of the general parabolic equation with bounded and measurable coefficient a_{ij}

$$u_t - \mathrm{div}\left(\sum_{j=1}^N a_{ij}D_j u\right) = 0.$$

The Harnack inequality has indeed been proved with full details not only for (1), but more generally for (2) when $p>1$, $m+p>2$ and $m+p+\frac{p}{N}>3$ in [17] and the essential tools are the comparison principle, proper L^∞-estimates and the Hölder continuity of u. Now a natural question arises, namely if the particular link between m and p in (1) allows a different method, which does not require any previous knowledge of the regularity of u (not to mention the comparison principle).

The p-homogeneity of (1) naturally suggests an approach based on parabolic De Giorgi classes of order p (see [7]), but it is rather easy to see that they do not correspond to solutions of (1). Hence we have a twofold problem: understand what kind of De Giorgi classes are associated to positive solutions of (1) and verify if Trudinger's claim can be proved using De Giorgi's method, starting from the corresponding classes.

In this short note we characterize the classes associated to (1) and prove that proper mean value inequalities can indeed be obtained relying on them (see Sections 2 and 3). We then conclude in Section 4 with a proof of Theorem 1 based on logarithmic estimates first proved by Moser in [12].

As explained in Remark 1 (see the next Section for more details), our result applies to more general equations and under this point of view it can indeed be seen as a (limited!) extension of the Harnack inequality proved in [17].

When finishing this note, we learnt that T. Kuusi (see [9]) gave a full proof of Trudinger's statement using classical Moser's estimate.

2. - Notation and Energy Inequalities

Let Ω be an open bounded domain in $\mathbf{R}^N$; for $T>0$ we denote by Ω_T the cylindrical domain $\Omega_T \equiv \Omega\times]0,T]$. In the following we will work with smooth solutions of the equation

$$(|u|^{p-1})_t - \mathrm{div}(|Du|^{p-2}Du) = 0 \qquad \text{in } \Omega_T \tag{3}$$

with $p>2$, but our estimates depend only on the data and not on the smoothness of the solutions, which is assumed just in order to simplify some calculations. Moreover we will deal with bounded nonnegative solutions, namely we assume that

$$\|u\|_{L^\infty(\Omega_T)} \le M, \qquad u(x,t) \ge 0 \quad \forall\,(x,t)\in\Omega_T$$

so that we can drop the modulus in the $|u|^{p-1}$ term. For $\rho>0$ denote by K_ρ the ball of radius ρ centered at the origin, i.e.

$$K_\rho \equiv \{x\in\mathbf{R}^N |\ |x|<\rho\}.$$

We let $[y+K_\rho]$ denote the ball centered at y and congruent to K_ρ, i.e.

$$[y+K_\rho] \equiv \{x\in\mathbf{R}^N |\ |x-y|<\rho\}.$$

For $\theta>0$ denote by $Q(\rho,\theta\rho^p)$ the cylinder of cross section K_ρ, height $\theta\rho^p$ and vertex at the origin, i.e.

$$Q(\rho,\theta\rho^p) \equiv K_\rho\times]-\theta\rho^p,0].$$

For a point $(y,s)\in\mathbf{R}^{N+1}$ we let $[(y,s)+Q(\rho,\theta\rho^p)]$ be the cylinder of vertex at (y,s) and congruent to $Q(\rho,\theta\rho^p)$, i.e.

$$[(y,s)+Q(\rho,\theta\rho^p)] \equiv [y+K_\rho]\times]s-\theta\rho^p,s].$$

The truncations $(u-k)_+$ and $(u-k)_-$ for $k\in\mathbf{R}$ are defined by

$$(u-k)_+ \equiv \max\{u-k,0\}; \qquad (u-k)_- \equiv \{k-u,0\}$$

and we set

$$A^\pm_{k,\rho}(\tau) \equiv \{x\in K_\rho :\ (u-k)_\pm(x,\tau)>0\}.$$

In the following with $|\Sigma|$ we denote the Lebesgue measure of a measurable set Σ.

Remark 1. *Even if we consider the p-Laplacian operator, all the following results still hold if we deal with a second order homogeneous operator*

$$\mathcal{L} = \mathrm{div}\,\mathbf{a}(x,t,u,Du)$$

where $\mathbf{a}:\Omega_T\times\mathbf{R}^{N+1}\to\mathbf{R}^N$ *is measurable and for a.e.* $(x,t)\in\Omega_T$ *satisfies*

$$\mathbf{a}(x,t,u,Du)\cdot Du \ge C_1|Du|^p,$$

$$|\mathbf{a}(x,t,u,Du)| \le C_2|Du|^{p-1}$$

for two given constant $0<C_1<C_2$*. The main point is that the lower order terms are zero* ∎.

Definition 1. *A measurable function* u *is a* <u>*local*</u> *weak sub (super) - solution of (3) if*

$$u\in C^0_{loc}(0,T;L^p_{loc}(\Omega))\cap L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega))$$

and for every compact subset $\mathcal{K}$ of Ω and for every subinterval $[t_1,t_2]$ of $]0,T]$ we have that

$$\int_{\mathcal{K}} u^{p-1}\zeta\,dx|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} [-u^{p-1}\zeta_t + \sum_{i=1}^{N} |Du|^{p-2} D_i u D_i \zeta]\,dxd\tau \le (\ge) 0 \tag{4}$$

for all testing function

$$\zeta \in W^{1,p}_{loc}(0,T;L^p(\mathcal{K})) \cap L^p_{loc}(0,T;W^{1,p}_0(\mathcal{K}))$$

with $\zeta \ge 0$. A function u that is both a local subsolution and a local supersolution is a local solution ■.

For general degenerate or singular parabolic equations of the type considered in [4], energy inequalities are proved both for $(u-k)_+$ and $(u-k)_-$ with $k \in \mathbf{R}$. Due to the presence of the $(u^{p-1})_t$ term, which gives rise to some difficulties when dealing with $(u-k)_-$, here we follow a different strategy in that we prove energy inequalities for $+$-truncations of u and $\frac{1}{u}$. The fact that we deal with $(\frac{1}{u}-k)_+$ instead of working with $(u-k)_-$ should not look so surprising as both are convex, monotone decreasing function of the argument u. For the sake of simplicity we state and prove the two energy inequalities indipendently from one another.

Proposition 1 (First Local Energy Estimate). *Let u be a locally bounded nonnegative weak subsolution of (3) in Ω_T. There exists a constant γ that can be determined a priori in terms of the data such that for every cylinder $[(y,s)+Q(\rho,\theta\rho^p)] \subset \Omega_T$ and for every $k \in \mathbf{R}_+$*

$$\frac{p-1}{p}\sup_{s-\theta\rho^p<t<s}\int_{[y+K_\rho]}(u-k)_+^p\varphi^p(x,t)\,dx + \iint_{[(y,s)+Q(\rho,\theta\rho^p)]}|D(u-k)_+|^p\varphi^p\,dxd\tau$$

$$\le \gamma\Bigg[\iint_{[(y,s)+Q(\rho,\theta\rho^p)]}(u-k)_+^p|D\varphi|^p\,dxd\tau +$$

$$+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k<k\}} p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau + \tag{5}$$

$$+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k>k\}} p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau +$$

$$+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{u-k=k\}}(p-1)2^{p-2}(u-k)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\Bigg]$$

for every $\varphi \in \mathcal{C}([(y,s)+Q(\rho,\theta\rho^p)])$ with $\varphi(\cdot,s-\theta\rho^p)=0$ where $\mathcal{C}(Q(\rho,\theta\rho^p))$ denotes the class of all piecewise smooth functions $\varphi : Q(\rho,\theta\rho^p) \to \mathbf{R}^+$ such that

1) $x \to \varphi(x,t) \in W^{1,\infty}_0(K_\rho) \quad \forall t \in]-\theta\rho^p,0]$;

2) $\varphi_t \ge 0$;

3) $|D\varphi| + \varphi_t \in L^\infty(Q(\rho,\theta\rho^p))$.

Proof - Since we assume u regular, we can rewrite (4) in a slightly different way, namely

$$\int_{t_1}^{t_2}\int_{\mathcal{K}}[(u^{p-1})_t\zeta + \sum_{i=1}^{N}|Du|^{p-2}D_iuD_i\zeta]\,dxd\tau \le 0. \tag{6}$$

After a translation we can assume $(y,s)\equiv(0,0)$ without loss of generality. Let us now fix $k \in \mathbf{R}_+$ and take $\zeta = (u-k)_+\varphi^p$ with $\varphi \in \mathcal{C}(Q(\rho,\theta\rho^p))$ and $\varphi(\cdot,-\theta\rho^p)=0$ as test function in (6) and integrate over $K_\rho\times]-\theta\rho^p,t]$ with $t \in]-\theta\rho^p,0]$. We obtain

$$\iint_{Q(\rho,\theta\rho^p+t)}(u^{p-1})_t(u-k)_+\varphi^p\,dxd\tau + \iint_{Q(\rho,\theta\rho^p+t)}|Du|^{p-2}Du\cdot D((u-k)_+\varphi^p)\,dxd\tau = 0$$

where $Q(\rho,\theta\rho^p+t)=K_\rho\times]-\theta\rho^p,t]\subseteq Q(\rho,\theta\rho^p)$. As usual

$$\iint_{Q(\rho,\theta\rho^p+t)}|Du|^{p-2}Du\cdot D((u-k)_+\varphi^p)\,dxd\tau=\iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(u-k)_+|^p\,dxd\tau+$$

$$+p\iint_{Q(\rho,\theta\rho^p+t)}|D(u-k)_+|^{p-2}\varphi^{p-1}(u-k)_+D(u-k)_+\cdot D\varphi\,dxd\tau$$

and for the estimate of the second term of the right - hand side we reason as usual. Let us now come to the estimate of $\iint_{Q(\rho,\theta\rho^p+t)}(u^{p-1})_t(u-k)_+\varphi^p\,dxd\tau$. Relying on the series expansion of $(1+z)^\alpha$ we have

$$(u^{p-1})_t=\begin{cases}(p-1)k^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{u-k}{k}\right)^n u_t & \text{if } 0<u-k<k\\ (p-1)(u-k)^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{k}{u-k}\right)^n u_t & \text{if } u-k>k>0\\ (p-1)2^{p-2}(u-k)^{p-2}u_t & \text{if } u-k=k>0\end{cases}\tag{7}$$

and hence

$$\iint_{Q(\rho,\theta\rho^p+t)}(u^{p-1})_t(u-k)_+\varphi^p\,dxd\tau=$$

$$=(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}k^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{u-k}{k}\right)^n u_t(u-k)_+\varphi^p\,dxd\tau+\tag{8}$$

$$+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}}(u-k)_+^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{k}{(u-k)_+}\right)^n u_t(u-k)_+\varphi^p\,dxd\tau+\tag{9}$$

$$+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}}2^{p-2}(u-k)_+^{p-1}u_t\varphi^p\,dxd\tau.\tag{10}$$

We clearly need to work distinctly on the previous three terms. Let us start from (8). We have

$$(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}k^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{u-k}{k}\right)^n u_t(u-k)_+\varphi^p\,dxd\tau=$$

$$=(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}(u-k)_+^{n+1}u_t\,\varphi^p\,dxd\tau=$$

$$=(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\left[\frac{(u-k)_+^{n+2}}{n+2}\right]_t\varphi^p\,dxd\tau=$$

$$=(p-1)\int_{K_\rho\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\,\varphi^p(x,t)\,dx+$$

$$-p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\,\varphi^{p-1}\varphi_t\,dxd\tau.$$

Let us now deal with (9). We obtain

$$(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}}(u-k)^{p-2}\sum_{n=0}^{\infty}\binom{p-2}{n}\left(\frac{k}{(u-k)_+}\right)^n u_t(u-k)_+\varphi^p\,dxd\tau=$$

$$= (p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n (u-k)_+^{p-1-n} u_t \varphi^p \, dx d\tau =$$

$$= (p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \left[\frac{(u-k)_+^{p-n}}{p-n}\right]_t \varphi^p \, dx d\tau =$$

$$= (p-1)\int_{K_\rho\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^p(x,t)\, dx +$$

$$-p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^{p-1}\varphi_t \, dx d\tau.$$

Finally, coming to (10) we get

$$(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2}(u-k)_+^{p-1} u_t \varphi^p \, dx d\tau = (p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2} \left[\frac{(u-k)_+^p}{p}\right]_t \varphi^p \, dx d\tau =$$

$$= \frac{p-1}{p}\int_{K_\rho\cap\{u-k=k\}} 2^{p-2}(u-k)_+^p \varphi^p(x,t)\, dx - (p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2}(u-k)_+^p \varphi^{p-1}\varphi_t \, dx d\tau.$$

If we now put everything together we obtain

$$(p-1)\int_{K_\rho\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} \frac{(u-k)_+^{n+2}}{n+2} \varphi^p(x,t)\, dx +$$

$$+(p-1)\int_{K_\rho\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^p(x,t)\, dx +$$

$$+\frac{p-1}{p}\int_{K_\rho\cap\{u-k=k\}} 2^{p-2}(u-k)_+^p \varphi^p(x,t)\, dx + \iint_{Q(\rho,\theta\rho^p+t)} \varphi^p |D(u-k)_+|^p \, dx d\tau \le$$

$$\le \gamma \left[\iint_{Q(\rho,\theta\rho^p+t)} (u-k)_+^p |D\varphi|^p \, dx d\tau + \right.$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^{p-2-n} \frac{(u-k)_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dx d\tau +$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}} \sum_{n=0}^{\infty} \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} \varphi^{p-1}\varphi_t \, dx d\tau +$$

$$\left. +(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}} 2^{p-2}(u-k)_+^p \varphi^{p-1}\varphi_t \, dx d\tau \right].$$

If $0 < u-k < k$ then

$$\sum_{n=0}^{\infty} (p-1) \binom{p-2}{n} k^{p-2-n} \frac{(u-k)_+^{n+2}}{n+2} =$$

$$= (p-1)\left[\binom{p-2}{0} k^{p-2} \frac{(u-k)_+^2}{2} + \binom{p-2}{1} k^{p-3} \frac{(u-k)_+^3}{3} + \dots\right] \ge \frac{p-1}{2}(u-k)_+^p$$

as $\binom{p-2}{1} > 0$. Analogously, if $u-k > k > 0$

$$\sum_{n=0}^{\infty} (p-1) \binom{p-2}{n} k^n \frac{(u-k)_+^{p-n}}{p-n} =$$

$$= (p-1)\left[\binom{p-2}{0}\frac{(u-k)_+^p}{p} + \binom{p-2}{1}\frac{(u-k)_+^{p-1}}{p-1}k + \dots\right] \geq \frac{p-1}{p}(u-k)_+^p.$$

We obtain

$$\frac{p-1}{p}\int_{K_\rho}(u-k)_+^p\varphi^p(x,t)\,dx + \iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(u-k)_+|^p\,dxd\tau \leq$$

$$\leq \gamma\left[\iint_{Q(\rho,\theta\rho^p+t)}(u-k)_+^p|D\varphi|^p\,dxd\tau + \right.$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k<k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{p-2-n}\frac{(u-k)_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau +$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k>k\}}\sum_{n=0}^{\infty}\binom{p-2}{n}k^{n}\frac{(u-k)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau +$$

$$\left. +(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{u-k=k\}}2^{p-2}(u-k)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\right]$$

and since $t\in]-\theta\rho^p,0]$ is arbitrary, we conclude ■.

Proposition 2 (Second Local Energy Estimate). *Let u be a locally bounded positive weak supersolution of (3) in Ω_T and let us set $v=\frac{1}{u}$. There exists a constant γ that can be determined a priori in terms of the data such that for every cylinder $[(y,s)+Q(\rho,\theta\rho^p)]\subset\Omega_T$ and for every $l\in\mathbf{R}_+$*

$$\frac{p-1}{p}\sup_{s-\theta\rho^p<t<s}\int_{[y+K_\rho]}(v-l)_+^p\varphi^p(x,t)\,dx + \iint_{[(y,s)+Q(\rho,\theta\rho^p)]}|D(v-l)_+|^p\varphi^p\,dxd\tau$$

$$\leq\gamma\left[\iint_{[(y,s)+Q(\rho,\theta\rho^p)]}(v-l)_+^p|D\varphi|^p\,dxd\tau + \right.$$

$$+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{v-l<l\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}l^{p-2-n}\frac{(v-l)_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau + \tag{11}$$

$$+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{v-l>l\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}l^{n}\frac{(v-l)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau +$$

$$\left.+\iint_{[(y,s)+Q(\rho,\theta\rho^p)]\cap\{v-l=l\}}(p-1)2^{p-2}(v-l)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\right]$$

for every $\varphi\in\mathcal{C}([(y,s)+Q(\rho,\theta\rho^p)])$ with $\varphi(\cdot,s-\theta\rho^p)=0$.

Proof - After a translation we can assume $(y,s)\equiv(0,0)$ without loss of generality. If we set $v=\frac{1}{u}$, (6) becomes

$$\int_{t_1}^{t_2}\int_{\mathcal{K}}[(p-1)\frac{1}{v^p}v_t\zeta + \sum_{i=1}^{N}\frac{|Dv|^{p-2}}{v^{2p-2}}D_ivD_i\zeta]\,dxd\tau\leq 0. \tag{12}$$

Let us now fix $l\in\mathbf{R}_+$ and take $\zeta=(v-l)_+v^{2p-2}\varphi^p$ with $\varphi\in\mathcal{C}(Q(\rho,\theta\rho^p))$ and $\varphi(\cdot,-\theta\rho^p)=0$ as test function in (12) and integrate over $K_\rho\times]-\theta\rho^p,t]$ with $t\in]-\theta\rho^p,0]$. With simple calculations we obtain

$$\iint_{Q(\rho,\theta\rho^p+t)}(p-1)(v-l)_+v^{p-2}v_t\varphi^p\,dxd\tau + \iint_{Q(\rho,\theta\rho^p+t)}|Dv|^{p-2}(Dv\cdot D(v-l)_+)\varphi^p\,dxd\tau +$$

$$+\iint_{Q(\rho,\theta\rho^p+t)}(2p-2)\frac{|Dv|^p}{v}(v-l)_+\varphi^p\,dxd\tau + \iint_{Q(\rho,\theta\rho^p+t)}p|Dv|^{p-2}(v-l)_+\varphi^{p-1}Dv\cdot D\varphi\,dxd\tau\leq 0$$

that is

$$\iint_{Q(\rho,\theta\rho^p+t)}(p-1)(v-l)_+v^{p-2}v_t\varphi^p\,dxd\tau+\iint_{Q(\rho,\theta\rho^p+t)}|D(v-l)_+|^p\varphi^p\,dxd\tau+$$

$$+\iint_{Q(\rho,\theta\rho^p+t)}(2p-2)|D(v-l)_+|^p\frac{(v-l)_+}{v}\varphi^p\,dxd\tau+$$

$$+\iint_{Q(\rho,\theta\rho^p+t)}p|D(v-l)_+|^{p-2}(v-l)_+\varphi^{p-1}D(v-l)_+\cdot D\varphi\,dxd\tau\leq 0.$$

We can then work as in the proof of the previous Proposition to conclude that

$$\frac{p-1}{p}\int_{K_\rho}(v-l)_+^p\varphi^p(x,t)\,dx+\iint_{Q(\rho,\theta\rho^p+t)}\varphi^p|D(v-l)_+|^p\,dxd\tau+$$

$$+(2p-2)\iint_{Q(\rho,\theta\rho^p+t)}\frac{(v-l)_+}{v}|D(v-l)_+|^p\varphi^p\,dxd\tau\leq\gamma\left[\iint_{Q(\rho,\theta\rho^p+t)}(v-l)_+^p|D\varphi|^p\,dxd\tau+\right.$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l<l\}}\sum_{n=0}^{\infty}\binom{p-2}{n}l^{p-2-n}\frac{(v-l)_+^{n+2}}{n+2}\,\varphi^{p-1}\varphi_t\;dxd\tau+$$

$$+p(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l>l\}}\sum_{n=0}^{\infty}\binom{p-2}{n}l^n\frac{(v-l)_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$\left.+(p-1)\iint_{Q(\rho,\theta\rho^p+t)\cap\{v-l=l\}}2^{p-2}(v-l)_+^p\varphi^{p-1}\varphi_t\,dxd\tau\right].$$

The third term on the left - hand side can be dropped since it is positive and relying on the arbitrarness of t we are finished ■.

3. - Mean Value Inequalities for Sub- and Supersolutions

As we discussed in the first Section, in [16] Trudinger states that it can be proved that positive solutions of (3) satisfy proper mean value inequalities relying on the method first developed in [11]. Here we show that the same results can be proved using De Giorgi's technique based on the energy inequalities for truncated subsolutions of the previous Section. We have

Proposition 3. *Let u be a nonnegative subsolution of (3) Then for all $\epsilon \in]0,p]$ there exists a positive constant C depending upon the data, θ and ϵ s. t. for all $[(x_0,t_0)+Q(\rho,\theta\rho^p)] \subset \Omega_T$ and for all $\sigma \in]0,1[$*

$$\sup_{[(x_0,t_0)+Q(\sigma\rho,\theta\sigma^p\rho^p)]} u \le \frac{C}{(1-\sigma)^{\frac{N+p}{\epsilon}}} \left(⨌_{[(x_0,t_0)+Q(\rho,\theta\rho^p)]} |u|^\epsilon \, dxd\tau \right)^{\frac{1}{\epsilon}} \quad \blacksquare. \tag{13}$$

Proposition 4. *Let u be a positive supersolution of (3). Then for all $\epsilon \in]0,p]$ there exists a positive constant D depending upon the data, θ and ϵ s. t. for all $[(x_0,t_0)+Q(\rho,\theta\rho^p)] \subset \Omega_T$ and for all $\sigma \in]0,1[$ the function $v = \frac{1}{u}$ satisfies*

$$\sup_{[(x_0,t_0)+Q(\sigma\rho,\theta\sigma^p\rho^p)]} v \le \frac{D}{(1-\sigma)^{\frac{(N+p)}{\epsilon}}} \left(⨌_{[(x_0,t_0)+Q(\rho,\theta\rho^p)]} |v|^\epsilon \, dxd\tau \right)^{\frac{1}{\epsilon}} \quad \blacksquare. \tag{14}$$

Proof - Due to the same structure of (5) and (11), we limit ourselves to the proof of (13). We assume $k > 0$ and set

$$k_j = k(1 - \frac{1}{2^j}).$$

As usual we can suppose that $(x_0,t_0) = (0,0)$. Let us now consider the second term on the right - hand side of (5) with respect to level k_{j+1}.

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau =$$

$$= \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau +$$

$$+ \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} p(p-1) \sum_{n=[p-2]+1}^{\infty} \binom{p-2}{n} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau.$$

Notice that $\forall \, s > 0$

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_j)_+^s \, dxd\tau = \iint_{Q(\rho,\theta\rho^p)\cap\{k_j<u<2k_{j+1}\}} (u-k_j)_+^s \, dxd\tau \ge$$

$$\ge \iint_{Q(\rho,\theta\rho^p)\cap\{k_{j+1}<u<2k_{j+1}\}} (u-k_j)_+^s \, dxd\tau \ge (k_{j+1}-k_j)^s |A_{j+1}| = \frac{k^s}{2^{(j+1)s}} |A_{j+1}| \tag{15}$$

where $A_{j+1} = \{k_{j+1} < u < 2k_{j+1}\}$. Then

$$p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} k_{j+1}^{p-2-n} \frac{(u-k_{j+1})_+^{n+2}}{n+2} \varphi^{p-1}\varphi_t \, dxd\tau$$

$$\le p(p-1) \sum_{n=0}^{[p-2]} \binom{p-2}{n} \frac{k_{j+1}^{p-2-n}}{n+2} \left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}} (u-k_{j+1})_+^p (\varphi^{p-1}\varphi_t)^{\frac{p}{n+2}} \, dxd\tau \right)^{\frac{n+2}{p}} |A_{j+1}|^{1-\frac{n+2}{p}}$$

$$\leq C_\varphi\, p(p-1)\sum_{n=0}^{[p-2]}\binom{p-2}{n}\frac{k_{j+1}^{p-2-n}}{n+2}\left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau\right)^{\frac{n+2}{p}}\cdot$$

$$\cdot\left(\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau\right)^{1-\frac{n+2}{p}}\frac{2^{(j+1)(p-(n+2))}}{k^{p-(n+2)}}$$

where we have taken (15) into account and $C_\varphi := \sup\,\varphi^{p-1}\varphi_t$,

$$\leq C_\varphi\, p(p-1)\sum_{n=0}^{[p-2]}\binom{p-2}{n}\frac{2^{(j+1)(p-(n+2))}}{n+2}\left(1-\frac{1}{2^{j+1}}\right)^{p-(n+2)}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau$$

$$\leq C_\varphi\, p(p-1)\sum_{n=0}^{[p-2]}\binom{p-2}{n}\frac{2^{(j+1)(p-(n+2))}}{n+2}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau$$

$$\leq \gamma(p)\,2^{jp}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau.$$

Then we have

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^{p-2-n}\frac{(u-k_{j+1})_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau\leq$$

$$\leq \gamma(p)\,2^{jp}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau+$$

$$+C_\varphi p(p-1)\binom{p-2}{[p-2]+1}k_{j+1}^{p-3-[p-2]}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}\frac{(u-k_{j+1})_+^{[p-2]+3}}{[p-2]+3}\,dxd\tau.$$

Moreover

$$0<(u-k_{j+1})_+<k_{j+1}\quad\Rightarrow\quad\frac{1}{k_{j+1}^{[p-2]-(p-3)}}<\frac{1}{(u-k_{j+1})_+^{[p-2]-(p-3)}}$$

$$\Rightarrow\quad\frac{(u-k_{j+1})_+^{[p-2]+3}}{[p-2]+3}k_{j+1}^{p-3-[p-2]}<\frac{1}{[p-2]+3}(u-k_{j+1})_+^p=\gamma(p)\,(u-k_{j+1})_+^p\leq\gamma(p)\,(u-k_j)_+^p$$

and we can then conclude that

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^{p-2-n}\frac{(u-k_{j+1})_+^{n+2}}{n+2}\varphi^{p-1}\varphi_t\,dxd\tau \tag{16}$$

$$\leq\gamma(p)\,2^{jp}\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau+C_\varphi\gamma(p)\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}<k_{j+1}\}}(u-k_j)_+^p\,dxd\tau.$$

We can now consider the third term on the right - hand side of (5) with respect to level k_{j+1}. We have

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}}p(p-1)\sum_{n=0}^{\infty}\binom{p-2}{n}k_{j+1}^{n}\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau=$$

$$=\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}}p(p-1)\sum_{n=0}^{[p-2]+1}\binom{p-2}{n}k_{j+1}^{n}\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau+$$

$$+\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}}p(p-1)\sum_{n=[p-2]+2}^{\infty}\binom{p-2}{n}k_{j+1}^{n}\frac{(u-k_{j+1})_+^{p-n}}{p-n}\varphi^{p-1}\varphi_t\,dxd\tau$$

$$\leq \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1) \sum_{n=0}^{[p-2]+1} \binom{p-2}{n} k_{j+1}^n \frac{(u-k_{j+1})_+^{p-n}}{p-n} \varphi^{p-1}\varphi_t \, dxd\tau.$$

Moreover

$$k_{j+1} < (u-k_{j+1})_+ \quad \Rightarrow \quad k_{j+1}^n < (u-k_{j+1})_+^n,$$

and we obtain

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1) \sum_{n=0}^{\infty} \binom{p-2}{n} k_{j+1}^n \frac{(u-k_{j+1})_+^{p-n}}{p-n} \varphi^{p-1}\varphi_t \, dxd\tau$$

$$\leq C_\varphi \gamma(p) \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)(u-k_{j+1})_+^p \, dxd\tau \tag{17}$$

$$\leq C_\varphi \gamma(p) \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}>k_{j+1}\}} p(p-1)(u-k_j)_+^p \, dxd\tau.$$

As for the last term on the right - hand side of (5), it is easy to check that

$$\iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}=k_{j+1}\}} (u-k_{j+1})_+^p \varphi^{p-1}\varphi_t \, dxd\tau \leq \iint_{Q(\rho,\theta\rho^p)\cap\{u-k_{j+1}=k_{j+1}\}} (u-k_j)_+^p \varphi^{p-1}\varphi_t \, dxd\tau. \tag{18}$$

Relying on (5) and (16) - (18) we conclude that

$$\frac{p-1}{p} \sup_{-\theta\rho^p<t<0} \int_{K_\rho} (u-k_{j+1})_+^p \varphi^p(x,t)\, dx + \iint_{Q(\rho,\theta\rho^p)} |D(u-k_{j+1})_+|^p \varphi^p \, dxd\tau$$

$$\leq \gamma \left[\iint_{Q(\rho,\theta\rho^p)} (u-k_j)_+^p |D\varphi|^p \, dxd\tau + C_\varphi 2^{jp} \iint_{Q(\rho,\theta\rho^p)} (u-k_j)_+^p \, dxd\tau \right].$$

When $p = 2$, this last inequality is the standard starting point to prove boundedness of u, as shown in [10], Chapter II, Section 6. In our case, even if we are dealing with a general $p > 2$, it is not difficult to see that the same calculations still hold, due to the p-homogeneity of both sides.

Similar estimates are developed in [4], Chapter V, to obtain boundedness estimates for solutions of degenerate parabolic equations, like the parabolic p-laplacian ∎.

4. - A Harnack Inequality

We can now come to the proof of Theorem 1. First of all let us recall the main Lemma of [12], which is actually a suitable adaptation to the parabolic setting of an idea introduced in [2] for the elliptic setting. We denote by $Q(\rho)$, $\rho > 0$ any family of domains satisfying $Q(\rho) \subset Q(r)$ for $0 < \rho < r$. We have

Proposition 5. *Let m, μ, c_0, δ be positive constants and let $w > 0$ be a measurable function defined in a neighborhood of $Q(1)$ and such that*

$$\sup_{Q(\rho)} w^p < \frac{c_0}{(r-\rho)^m} \iint_{Q(r)} w^p \, dx \tag{19}$$

for all ρ, r, p satisfying

$$\frac{1}{2} \le \rho < r \le 1, \qquad 0 < p < \mu^{-1}.$$

Moreover, let

$$|\{x \in Q(1) : \ \ln w > s\}| < \frac{c_0 \mu}{s^\delta} \tag{20}$$

for all $s > 0$. Then there exists a constant $\gamma = \gamma(\mu, m, c_0, \delta)$ such that

$$\sup_{Q(\frac{1}{2})} w < \gamma. \tag{21}$$

It is worth to notice that in [12] the parameter δ is taken equal to one, but as it is remarked in [3] any positive δ can do. In [12], Proposition 5 is the key point in proving the Harnack inequality for parabolic equations with bounded and measurable coefficients. Here we follow the same strategy and therefore we need the equivalent of (20) in our setting. We have

Proposition 6. *Fix $\theta > 0$ and $\sigma \in]0,1[$. If u is a positive solution of (3) in $[(x_0,t_0) + Q(\rho, \theta\rho^p)]$, there exists a constant $c = c(u, \sigma)$ such that, for all $s > 0$*

$$|\{(x,t) \in {Q_{\sigma\rho}}^+ : \log u < -s - c\}| \le \frac{C}{s^{p-1}} |Q(\rho, \theta\rho^p)| \tag{22}$$

and

$$|\{(x,t) \in {Q_{\sigma\rho}}^- : \log u > s - c\}| \le \frac{C}{s^{p-1}} |Q(\rho, \theta\rho^p)| \tag{23}$$

where ${Q_{\sigma\rho}}^+ = [x_0 + K_{\sigma\rho}] \times]t_0 - \theta\sigma^p\rho^p, t_0]$ and ${Q_{\sigma\rho}}^- = [x_0 + K_{\sigma\rho}] \times]t_0 - \theta\rho^p, t_0 - \theta\sigma^p\rho^p]$. Here the constant C is independent of s, u, (x_0,t_0) and K_ρ.

Proof - Things are very much the same as in the proof of the analogous proposition of [12]. Here we closely follow the exposition given in Lemma 5.4.1 of [14]. First of all we set $(x_0,t_0) = (0,0)$ as always and take $\theta = 1$. Let K'_ρ be any concentric ball larger than K_ρ. For any nonnegative $\zeta \in C_0^\infty(K'_\rho)$ we consider the test function $\varphi = \dfrac{\zeta^p}{u^{p-1}}$. If we insert it in (3), relying on the regularity of u we obtain

$$\int_{K'_\rho} \left[(u^{p-1})_t \frac{\zeta^p}{u^{p-1}} + |Du|^{p-2} Du \cdot D(\frac{\zeta^p}{u^{p-1}}) \right] dx = 0$$

and also

$$(p-1)\frac{\partial}{\partial t} \int_{K'_\rho} (\zeta^p \ln u) \, dx + (p-1) \int_{K'_\rho} \zeta^p \frac{1}{u^p} |Du|^p \, dx + p \int_{K'_\rho} |Du|^{p-2} \frac{\zeta^{p-1}}{u^{p-1}} Du \cdot D\zeta \, dx = 0.$$

If we set $w = -\log u$, we can rewrite the previous inequality as

$$\frac{\partial}{\partial t} \int_{K'_\rho} \zeta^p w \, dx = -\int_{K'_\rho} \zeta^p |Dw|^p \, dx + \frac{p}{p-1} \int_{K'_\rho} |Dw|^{p-2} \zeta^{p-1} Dw \cdot D\zeta \, dx$$

from which we obtain in a standard way

$$\frac{\partial}{\partial t}\int_{K'_\rho}\zeta^p w\,dx + C_1\int_{K'_\rho}\zeta^p|Dw|^p\,dx \le C_2(\sup_{K'_\rho}|D\zeta|^p)\,|K'_\rho|. \tag{24}$$

Let us now choose $\zeta(x) = (1-\frac{|x|}{\rho})_+$: ζ is not smooth, but it can easily be approximated by nonnegative C_0^∞ functions in K'_ρ. Then the weighted Poincaré inequality of Theorem 5.3.4 of [14] becomes

$$\int_{K_\rho}|w-W|^p\zeta^p\,dx \le A_0\rho^p\int_{K_\rho}|Dw|^p\zeta^p\,dx \tag{25}$$

with

$$W = \frac{\int_{K_\rho} w\zeta^p\,dx}{\int_{K_\rho}\zeta^p\,dx}. \tag{26}$$

If we divide (24) by $\int_{K_\rho}\zeta^p\,dx$ and take into account (25) and (26), we obtain

$$\frac{\partial W}{\partial t} + \frac{1}{A_1\rho^{N+p}}\int_{K_{\sigma\rho}}|w-W|^p\,dx \le \frac{A_2}{\rho^p}$$

for some constants A_1, $A_2 > 0$. We can rewrite this inequality as

$$\frac{\partial \bar W}{\partial t} + \frac{1}{A_1\rho^{N+p}}\int_{K_{\sigma\rho}}|\bar w-\bar W|^p\,dx \le 0$$

where $\bar w(x,t) = w(x,t) - A_2\rho^{-p}(t-t')$, $\bar W(t) = W(t) - A_2\rho^{-p}(t-t')$ with $t' = -\sigma^p\rho^p$. We now set $c(u) = \bar W(t')$ and

$$K_t^+(s) = \{x \in K_{\sigma\rho}:\ \bar w(x,t) > c+s\},$$
$$K_t^-(s) = \{x \in K_{\sigma\rho}:\ \bar w(x,t) < c-s\}$$

and we can finish exactly as in Lemma 5.4.1 of [14], with the only difference that the exponent for s is $p-1$ instead of 1 ■.

We can now conclude with the

Proof of Theorem 1 - As always we assume $(x_0,t_0) = (0,0)$. Fix $\theta > 0$ and let u be a positive solution of (3) in $K_{2\rho}\times]-2^p\theta\rho^p,0]$. By Proposition 6 with $\sigma = \frac{1}{2}$ we have

$$|\{(x,t)\in K_\rho\times]-2\theta\rho^p,-\theta\rho^p] : \log u > s-c\}| \le$$

$$\le |\{(x,t)\in K_\rho\times]-2^p\theta\rho^p,-\theta\rho^p] : \log u > s-c\}| \le \frac{C_1}{s^{p-1}}|Q(\rho,\theta\rho^p)|$$

and by Proposition 3

$$\sup_{[(0,-\theta\rho^p)+Q(\sigma\rho,\theta\sigma^p\rho^p)]} u^\epsilon \le \frac{C_2}{(1-\sigma)^{(N+p)}}\left(\iint_{[(0,-\theta\rho^p)+Q(\rho,\theta\rho^p)]} u^\epsilon\,dxd\tau\right).$$

We can then apply Proposition 5 and conclude that

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} e^c\,u \le C_3. \tag{27}$$

Analogously, by Proposition 6

$$|\{(x,t)\in K_\rho\times]-\theta\rho^p,0] : \log u < -s-c\}| \le \frac{C_1}{s^{p-1}}|Q(\rho,\theta\rho^p)|$$

and by Proposition 4

$$\sup_{Q(\sigma\rho,\theta\sigma^p\rho^p)} \left(\frac{1}{u}\right)^\epsilon \leq \frac{C_2}{(1-\sigma)^{(N+p)}} \left(\fint\!\!\fint_{Q(\rho,\theta\rho^p)} \left(\frac{1}{u}\right)^\epsilon dx d\tau \right).$$

We can then apply Proposition 5 and conclude that

$$\sup_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} e^{-c}\, u^{-1} \leq C_4. \tag{28}$$

We can now multiply (27) by (28) and conclude that

$$\sup_{[(0,-\theta\rho^p)+Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)]} u \leq C_5 \inf_{Q(\frac{\rho}{2},\frac{\theta}{2^p}\rho^p)} u \tag{29}$$

with $C_5 = C_3 C_4$ ∎.

5. - Acknowledgements

This work was partially supported by F.A.R. funds of the University of Pavia, by grants of M.U.R.S.T. (cofin 1999-9901107579, cofin2000-MM01151559) and of I.M.A.T.I. - C.N.R., Pavia.

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