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The Index of Isolated Critical Points and Solutions of Elliptic Equations in the Plane

G. ALESSANDRINI - R. MAGNANINI

1. - Introduction

The purpose of the present paper is to point out how statements about geometric quantities associated with solutions of elliptic equations can be derived from basic facts of differential topology like index calculus and the Gauss-Bonnet theorem. We will focus on two-dimensional problems.

The results we obtain are of two different kinds.

(1) Identities or estimates relating the number and character of critical points (i.e., zeroes of the gradient) of solutions of elliptic equations with the boundary data.

(2) Differential identities on the gradient length and the curvatures of the level curves and of the curves of steepest descent of an arbitrary smooth function with isolated critical points.

As a typical example of the kind (1), we shall demonstrate the following theorem.

THEOREM 1.1. *Let Ω be a bounded open set in the plane and let its boundary $\partial\Omega$ be composed of N simple closed curves $\Gamma_1, \ldots, \Gamma_N$, $N \geq 1$, of class $C^{1,\alpha}$. Consider the solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ of the Dirichlet problem:*

$$\Delta u = 0 \quad \textit{in } \Omega, \tag{1.1a}$$

$$u = a_j \quad \textit{on } \Gamma_j, \quad j = 1, \ldots, N, \tag{1.1b}$$

where $a_1, \ldots, a_N$ are given constants.

If $a_1, \ldots, a_N$ do not all coincide, then u has isolated critical points $z_1, \ldots, z_K$ in $\overline{\Omega}$, with finite multiplicities $m_1, \ldots, m_K$, respectively, and the

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following identity holds:

$$\sum_{z_k \in \Omega} m_k + \frac{1}{2} \sum_{z_k \in \partial\Omega} m_k = N - 2. \tag{1.2}$$

This theorem generalizes some results contained in [Wa, Section 8.1.3]. In [A1], a related result was proven in the case of non-constant Dirichlet data in simply connected domains, by completely different arguments. In Theorems 2.1, 2.2 we show how our present method can be used to obtain similar results when quite general oblique derivative boundary data are prescribed.

Moreover, in Section 4 we give a new proof of a result of Sakaguchi [Sa], concerning the number of critical points of the solution of an obstacle problem. We also prove identities, Theorem 3.3, 3.5, for the critical points of solutions of equations of the form $\Delta u = -f(u)$, which complement previous known results (see for instance [PM] and the references therein).

The main results in the spirit of (2) are summarized in the following Theorem 1.2, which needs the introduction of some preliminary notations.

It is convenient to use the complex variable $z = x + iy$ and to denote complex derivatives ∂_z, $\partial_{\bar{z}}$ as follows:

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Given a real-valued function u of class C^1 on an open set Ω in the plane, we define, locally, real-valued functions v, ω, and the complex-valued function ϕ by the relations:

$$e^{v - i\omega} = 2\partial_z u, \tag{1.3}$$

$$\phi = -2\partial_z e^{i\omega}. \tag{1.4}$$

It is clear that $e^v = \sqrt{u_x^2 + u_y^2}$ and that, away from the critical points of u, ω coincides with the angle formed by the gradient direction and the positive x-axis. Notice also that $h = \operatorname{Re}\phi$ and $k = \operatorname{Im}\phi$ are respectively the curvature of the level curves and of the curves of steepest descent of u, as defined in [T].

If $z_0 \in \Omega$ is an isolated critical point of u, we denote by $I(z_0)$ the index of the gradient vector field $\nabla u = (u_x, u_y)$ at z_0. As it is well-known, for every sufficiently small $r > 0$, we have that

$$I(z_0) = \frac{1}{2\pi} \int_{|z - z_0| = r} d\omega, \tag{1.5}$$

where the contour integral is understood in the counterclockwise orientation (see [Mi]).

THEOREM 1.2. *If $u \in C^2(\Omega)$ has only a finite number of critical points $z_1, \dots, z_K \in \Omega$, then the following identities hold in the sense of distributions:*

$$\Delta v - \operatorname{div}\left\{\frac{\Delta u}{u_x^2 + u_y^2}\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right\} = -2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k), \tag{1.6}$$

$$\Delta \omega - \operatorname{div}\left\{\frac{\Delta u}{u_x^2 + u_y^2}\begin{pmatrix} -u_y \\ u_x \end{pmatrix}\right\} = 0, \tag{1.7}$$

$$2e^{-i\omega}\partial_{\bar z}\phi - |\phi|^2 + e^{-v}\Delta u(\phi - \bar\phi) = -2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k). \tag{1.8}$$

The proof of this theorem will be carried out in Section 5. These formulas hold for any sufficiently smooth function. However, their usefulness is particularly apparent for solutions of elliptic equations. For instance, the above formulas take a much simpler form when u is harmonic. In such a case, if z_k is a critical point of u, then it has finite multiplicity m_k. More precisely, we have

$$\partial_z u = (z - z_k)^{m_k} g(z),$$

with $g(z)$ analytic and $g(z_k) \neq 0$; hence, by the argument principle for analytic functions, we can easily deduce that $I(z_k) = -m_k$. In Theorem 5.1 we shall give a generalized version of (1.6), (1.7) which is more suitable for solutions of elliptic equations with variable leading coefficients.

Formulas (1.6)–(1.8) unify, in a general setting, several identities which have been proven in different times, and used in different contexts. When the gradient never vanishes, (1.6) appears in Weatherburn [We], and (1.8) has been proven by Talenti, [T]. Pucci, [P], has obtained identities and inequalities of the type (1.7) in the treatment of solutions of elliptic equations in two or more variables. A formula like (1.6), in the presence of critical points, has been applied to the geometrical study of solutions of elliptic equations in [A1]; see also [A2] and the references therein.

All present results are based on the computation of contour integrals of the type $\int_{\partial G} \mathrm{d}\omega$ with suitable choices of the region $G \subseteq \Omega$. Index theory provides the appropriate tool, since

$$\int_{\partial G} \mathrm{d}\omega = 2\pi \sum_{z_k \in G} I(z_k). \tag{1.9}$$

For instance, the derivation of (1.8), in the case where u has no critical points, was carried out in [T], by writing $\mathrm{d}\omega$ in terms of ϕ, and then by taking into account that $\mathrm{d}\,\mathrm{d}\omega = 0$. On the other hand, if u has isolated critical points $z_1, \dots, z_K$, $\mathrm{d}\omega$ is a well defined closed form in $\Omega \backslash \{z_1, \dots, z_K\}$, while ω cannot be continuously defined as a single-valued function in $\Omega \backslash \{z_1, \dots, z_K\}$. By this remark and (1.9), we will show that $\mathrm{d}\,\mathrm{d}\omega$ defines a measure concentrated at $z_1, \dots, z_K$, and (1.8) will follow.

2. - Critical points of $\mathcal{L}$-harmonic functions

We shall prove Theorem 1.1 in a greater generality than the one stated in the introduction. In fact, it is possible to replace (1.1a) with

$$\mathcal{L}u = 0 \qquad \text{in } \Omega, \tag{2.1}$$

where $\mathcal{L}$ is an elliptic operator of the form

$$\mathcal{L} = \partial_x(a\partial_x + b\partial_y) + \partial_y(b\partial_x + c\partial_y) + d\partial_x + e\partial_y, \tag{2.2}$$

where the variable coefficients, a, b, c are Lipschitz continuous and d, e are bounded measurable in $\overline{\Omega}$. Uniform ellipticity is assumed in the following form:

$$ac - b^2 = 1 \qquad \text{in } \Omega. \tag{2.3}$$

By the uniformization theorem (see [V]), we recall that we can choose a quasi-conformal mapping $\varsigma = \varsigma(z) = \xi(z) + i\eta(z)$ such that $u(\varsigma)$ satisfies the equation:

$$\Delta u + Pu_\xi + Qu_\eta = 0 \qquad \text{in } \varsigma(\Omega), \tag{2.4}$$

where

$$P, Q \in L^\infty(\varsigma(\Omega)). \tag{2.5}$$

It follows that $g = 2\partial_\varsigma u$ is a solution of

$$\partial_{\overline{\varsigma}} g + Rg + \overline{R}\,\overline{g} = 0 \qquad \text{in } \varsigma(\Omega), \tag{2.6}$$

where $R = \dfrac{1}{4}(P + iQ)$ is bounded on $\varsigma(\Omega)$.

By the well-known similarity principle (see [V]) there exists $s(\varsigma)$, Hölder continuous on the whole plane, and $G(\varsigma)$, analytic in $\varsigma(\Omega)$, such that

$$g(\varsigma) = e^{s(\varsigma)}G(\varsigma). \tag{2.7}$$

It follows, for instance, that every critical point $z_0 \in \Omega$ of u is isolated and $\partial_z u$ vanishes with finite multiplicity m_0 at z_0; moreover $I(z_0) = -m_0$.

Before proving Theorem 1.1, we shall state Theorems 2.1, 2.2. The following notations and definitions will be useful.

Let $\underline{\alpha}$ be a C^1 unitary vector field on $\partial\Omega$. We will denote by D the topological degree of $\underline{\alpha} : \partial\Omega \to S^1$, that is

$$D = \frac{1}{2\pi} \int_{\partial\Omega} \mathrm{d} \arg \underline{\alpha}. \tag{2.8}$$

DEFINITION 2.1. Let $\phi \in C^0(\partial\Omega)$.

(i) If $\partial\Omega$ is decomposed into two disjoint subsets J^+, J^- such that $\phi \geq 0$ on J^+ and $\phi \leq 0$ on J^-, we denote by $M(J^+)$ the number of connected components of J^+, which are proper subsets of a connected component of $\partial\Omega$. We denote by M the minimum of $M(J^+)$ among all such decompositions J^+, J^- of $\partial\Omega$.

(ii) If $I^+ = \{z \in \partial\Omega : \phi > 0\}$ and $I^- = \{z \in \partial\Omega : \phi < 0\}$, we denote by M^+ (respectively M^-) the number of connected components of I^+ (respectively I^-) which are proper subsets of a component of $\partial\Omega$.

REMARK. Notice that in (i), the definition of M would not change if we replace J^+ with J^-. Note also that $M \leq M^+, M^-$.

THEOREM 2.1. *Let Ω be as in Theorem* 1.1, *let $\mathcal{L}$ be as in* (2.2), (2.3), *and let $\underline{\alpha}$ be a C^1 unitary vector field on $\partial\Omega$ of degree D, as defined in* (2.8). *Suppose $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of* (2.1) *satisfying the oblique derivative boundary condition:*

$$\nabla u \cdot \underline{\alpha} = \phi \qquad \text{on } \partial\Omega, \tag{2.9}$$

where ϕ is a given continuous function on $\partial\Omega$. Let M be as in Definition 2.1 (i). *If M is finite and u has no critical points on $\partial\Omega$, then the interior critical points of u are finite in number and, letting $m_1, \dots, m_K$ be their multiplicities, we have:*

$$\sum_{k=1}^{K} m_k \leq M - D. \tag{2.10}$$

REMARK. It would be desirable to remove the hypothesis that u has no critical point on the boundary. In the following theorem we allow the presence of boundary critical points at the cost of requiring the more stringent assumption on the changes of sign of the boundary data ϕ, given by Definition 2.1 (ii).

THEOREM 2.2. *Let Ω, $\underline{\alpha}$, and ϕ be as in Theorem* 2.1 *and let M^+ and M^- be as in Definition* 2.1 (ii). *Suppose $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of* (2.1), (2.9). *If $M^+ + M^-$ is finite, then the interior critical points of u are finite in number and, letting $m_1, \dots, m_K$ be their multiplicities, we have:*

$$\sum_{k=1}^{K} m_k \leq \left[\frac{M^+ + M^-}{2}\right] - D, \tag{2.11}$$

Here $[x]$ = greatest integer $\leq x$.

PROOF OF THEOREM 1.1. Without loss of generality, we can suppose that $u = u(z)$ is the solution of (2.1), (1.1b) with $a = c = 1$, $b = 0$.

We start by analysing the character of a boundary critical point z_0. We choose a conformal mapping $\chi(z)$, regular up to the boundary, in a way that $\chi(\partial\Omega)$ is flat in a neighbourhood of $\chi(z_0)$. Since u is constant on $\chi(\partial\Omega)$, by a standard reflection argument (see [V]), we can continue u in a full neighbourhood $\mathcal{U}$ of $\chi(z_0)$ to a function u_* in such a way that $g_* = 2\partial_\chi u_*$ is a solution of

$$\partial_{\overline{\chi}} g_* + R_* g_* + \overline{R}_* \overline{g}_* = 0 \qquad \text{in } \mathcal{U},$$

where R_* is bounded in $\mathcal{U}$. This argument shows that, by the similarity principle (2.7), $\partial_z u$ vanishes at z_0 with positive order m_0; moreover, the level set $\{u = u(z_0)\}$ is made of $m_0 + 2$ distinct simple arcs crossing at z_0, two of which lie on $\partial\Omega$. Furthermore, a sort of an index can be computed at z_0, as well. Namely, by setting $\Sigma_\varepsilon = \{z \in \Omega : |z - z_0| = \varepsilon\}$ and

$$I(z_0) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\Sigma_\varepsilon} d\omega,$$

by the change of variable $\chi = \chi(z)$ and by (2.7), we readily obtain

$$I(z_0) = -\frac{1}{2} m_0. \tag{2.12}$$

Now, set $B_\varepsilon^k = \{z : |z - z_k| < \varepsilon\}$,

$$\mathcal{B}_\varepsilon = \bigcup_{z_k \in \partial\Omega} \overline{B_\varepsilon^k},$$

and consider the set $\Omega_\varepsilon = \Omega \setminus \mathcal{B}_\varepsilon$. Let us compute:

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} d\omega.$$

On one hand, if ε is small enough, all the interior critical points belong to Ω_ε, so that:

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} d\omega = -\lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} d \arg(\partial_z u) = -\sum_{z_k \in \Omega} m_k. \tag{2.13}$$

On the other hand, we have:

$$\frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} d\omega = -\sum_{z_k \in \partial\Omega} \frac{1}{2\pi} \int_{\partial B_\varepsilon^k \cap \Omega} d\omega + \frac{1}{2\pi} \int_{\partial\Omega \setminus \mathcal{B}_\varepsilon} d\omega.$$

By (2.12), if ε is small enough, we have:

$$-\sum_{z_k\in\partial\Omega}\frac{1}{2\pi}\int\limits_{\partial B_\varepsilon^k\cap\Omega} \mathrm{d}\omega = \frac{1}{2}\sum_{z_k\in\partial\Omega} m_k, \tag{2.14}$$

while

$$\frac{1}{2\pi}\int\limits_{\partial\Omega\setminus B_\varepsilon} \mathrm{d}\omega = \frac{1}{2\pi}\int\limits_{\Gamma_1\setminus B_\varepsilon} \mathrm{d}\omega - \sum_{j=2}^{N}\frac{1}{2\pi}\int\limits_{\Gamma_j\setminus B_\varepsilon} \mathrm{d}\omega, \tag{2.15}$$

where we assume that Γ_1 surrounds all the remaining Γ_j's and all such curves are oriented in the counterclockwise sense.

It should be noticed that on the boundary, the direction of the gradient of u, parallel to the normal $\underline{\nu}$, may switch from outward to inward or viceversa, when a critical point is crossed (precisely, it switches at points of odd multiplicity). At such points, ω has a jump of $\pm\pi$. However, there are as many positive jumps as the negative ones. Consequently, we obtain:

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int\limits_{\Gamma_j\setminus B_\varepsilon} \mathrm{d}\omega = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int\limits_{\Gamma_j\setminus B_\varepsilon} \mathrm{d}\arg\underline{\nu}, \qquad j=1,\dots,N.$$

By adding up over $j=1,\dots,N$ and taking into account (2.15), (2.13), and (2.14), we arrive at

$$-\sum_{z_k\in\Omega} m_k = \frac{1}{2}\sum_{z_k\in\partial\Omega} m_k + \frac{1}{2\pi}\int\limits_{\partial\Omega} \mathrm{d}\arg\underline{\nu}.$$

By the Gauss-Bonnet formula, we know that

$$\frac{1}{2\pi}\int\limits_{\partial\Omega} \mathrm{d}\arg\underline{\nu} = 2-N$$

(see [Mi]), thus, (1.2) follows.

PROOF OF THEOREM 2.1. Since u has only interior critical points, we have:

$$\sum_{k=1}^{K} m_k = -\frac{1}{2\pi}\int\limits_{\partial\Omega} \mathrm{d}\omega.$$

According to Definition 2.1 (i), let J^+, J^- be a decomposition of $\partial\Omega$ such that $M = M(J^+)$. Notice that

$$|\omega - \arg\underline{\alpha}| \le \frac{\pi}{2}, \qquad \text{on } J^+,$$
$$|\omega - \arg(-\underline{\alpha})| \le \frac{\pi}{2}, \qquad \text{on } J^-.$$

Let Γ_j be any connected component of $\partial\Omega$. If $\Gamma_j \subset J^+$, or $\Gamma_j \subset J^-$, then we have

$$\left| \frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d}\omega - \frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d} \arg \underline{\alpha} \right| \leq \frac{1}{2},$$

and, since the left-hand side is an integer, we get

$$\frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d}\omega = \frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d} \arg \underline{\alpha}.$$

If Γ_j contains points of both J^+ and J^-, then it contains as many components of J^+ as of J^-. Let M_j be the number of connected components of $J^+ \cap \Gamma_j$. Then we have $\sum M_j = M$.

If A and B are consecutive connected components of $J^+ \cap \Gamma_j$ and $J^- \cap \Gamma_j$ respectively, we obtain

$$\left| \frac{1}{2\pi} \int_{A\cup B} \mathrm{d}\omega - \frac{1}{2\pi} \int_{A\cup B} \mathrm{d} \arg \underline{\alpha} \right| \leq 1,$$

thus,

$$-\frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d}\omega \leq -\frac{1}{2\pi} \int_{\Gamma_j} \mathrm{d} \arg \underline{\alpha} + M_j.$$

Finally, by summing up over j, we have

$$\sum_{k=1}^{K} m_k = -\frac{1}{2\pi} \int_{\partial\Omega} \mathrm{d}\omega \leq -\frac{1}{2\pi} \int_{\partial\Omega} \mathrm{d} \arg \underline{\alpha} + M.$$

PROOF OF THEOREM 2.2. By the uniformization theorem, we can suppose that $u = u(z)$ is a solution of (2.1), (2.9) with $a = c = 1$, $b = 0$. In fact, a quasi-conformal transformation, regular up to the boundary, does not change the numbers D, M^+ and M^-.

The function $g = 2\partial_z u$ satisfies (2.6); we then apply the similarity principle (2.7) in the following way. As it is well-known (see [B]), fixed a component Γ_j of $\partial\Omega$, the Hölder continuous function s in (2.7) can be chosen to be real-valued on Γ_j. Let us denote by s_j this choice of s, and by G_j the corresponding analytic function in (2.7).

Now, fix $\varepsilon > 0$, $1 \leq j \leq N$, and consider the set $\{z \in \Omega: |G_j(z)| < \varepsilon\}$. This set is open, hence it is the union of countably many connected components: we select those components whose boundary contains *open portions* of Γ_j where $\phi \equiv 0$, and we name their union by $\mathcal{A}_j^\varepsilon$. Notice that, for sufficiently small ε, the $\mathcal{A}_j^\varepsilon$'s, $j = 1, \ldots, N$, are pairwise disjoint. Then, we set

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{j=1}^{N} \overline{\mathcal{A}_j^\varepsilon}.$$

Observe that $\partial\Omega_\varepsilon$ does not contain critical points of u, and furthermore, it can be decomposed as $\partial\Omega_\varepsilon = \Sigma \cup T \cup \Xi$ where Σ, T, and Ξ are disjoint subsets of $\partial\Omega_\varepsilon$, each one made of countably many arcs having the following properties:

(i) every component σ of Σ is a connected subset of $\partial\Omega$ where $\phi \not\equiv 0$, and each connected component of $\{z \in \partial\Omega : \phi(z) \not\equiv 0\}$ contains at most one arc $\sigma \subset \Sigma$;

(ii) every component τ of T is a connected subset of $\partial\Omega$ where $\phi \equiv 0$ and $|G_j| > \varepsilon$ on τ if $\tau \subset \Gamma_j$;

(iii) every component ξ of Ξ is an arc in Ω with endpoints on $\partial\Omega$, and we have $|G_j| = \varepsilon$ on ξ if this has endpoints on Γ_j.

Let us preserve on Σ, T, Ξ the positive orientation of $\partial\Omega_\varepsilon$. Notice that if $\phi \not\equiv 0$ on some component Γ_j of $\partial\Omega$, then, for sufficiently small ε, Γ_j is a component of Σ.

Denoting by $z_1, \ldots, z_k, \ldots$ the interior critical points of u with their respective multiplicities $m_1, \ldots, m_k, \ldots$, we have

$$\sum_{z_k \in \Omega} m_k = \lim_{\varepsilon \to 0^+} \sum_{z_k \in \Omega_\varepsilon} m_k = -\lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial\Omega_\varepsilon} \mathrm{d}\omega. \tag{2.16}$$

We observe that

$$\left| \int_\sigma \mathrm{d}\omega - \int_\sigma \mathrm{d}\arg\underline{\alpha} \right| \leq \pi, \qquad \text{for every } \sigma \subset \Sigma, \tag{2.17a}$$

and, in particular,

$$\int_{\Gamma_j} \mathrm{d}\omega = \int_{\Gamma_j} \mathrm{d}\arg\underline{\alpha}, \qquad \text{for every } \Gamma_j \subset \Sigma. \tag{2.17b}$$

Moreover, we have

$$\int_\tau \mathrm{d}\omega = \int_\tau \mathrm{d}\arg\underline{\alpha}, \qquad \text{for every } \tau \subset T. \tag{2.18}$$

Let now ξ be an arc of Ξ with endpoints on Γ_j and where $|G_j| = \varepsilon$. By our representation of $g = 2\partial_z u$, we have

$$\int_\xi \mathrm{d}\omega = \int_\xi \mathrm{d}\arg G_j + \int_\xi \mathrm{d}(\operatorname{Im} s_j) = \int_\xi \mathrm{d}\arg G_j,$$

since s_j is real-valued on Γ_j. Moreover, since $\arg G_j$ is the harmonic conjugate of $\log|G_j|$, and ξ is a level curve of $\log|G_j|$, we get

$$\int_\xi d\omega = \int_\xi d\arg G_j = \int_\xi |\nabla \log|G_j|| \geq 0, \tag{2.19}$$

where the last integral is unoriented and performed with respect to the arc lenght parameter.

Thus, (2.16)–(2.19) imply that

$$\begin{aligned}\sum_{z_k\in\Omega} m_k &\leq \lim_{\varepsilon\to 0^+}\left\{-\frac{1}{2\pi}\int_\Sigma d\omega - \frac{1}{2\pi}\int_T d\omega\right\}\\ &\leq \frac{M^+ + M^-}{2} - \lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\partial\Omega\cap\partial\Omega_\varepsilon} d\arg\underline{\alpha}.\end{aligned}$$

Now, fixing $G = G_1$ and letting $\varepsilon \to 0$ the set $\partial\Omega\cap\partial\Omega_\varepsilon$ invades monotonically $\partial\Omega\backslash Z'$, where $Z' \subset Z = \{z \in \partial\Omega: |G(z)| = 0\}$. Now, Z, and hence Z', is a set of zero Lebesgue measure in $\partial\Omega$. In fact, for any $z_0 \in \Omega$, and for a sufficiently small r, we can find a conformal mapping $\chi: \Omega \cap B_r(z_0) \to B_1(0)$, the map χ^{-1} being piecewise C^1 on $\partial B_1(0)$. Since $G\circ\chi^{-1}$ is a bounded holomorphic function on the unit disk, it belongs to the Nevanlinna space, thus its zero set $\mathcal{W}$ on $\partial B_1(0)$ forms a set of zero Lebesgue measure (see [D]). The same holds for Z since $Z \cap B_r(z_0) \subset \chi^{-1}(\mathcal{W})$. By the dominated convergence theorem and (2.8), we obtain:

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\partial\Omega\cap\partial\Omega_\varepsilon} d\arg\underline{\alpha} = \frac{1}{2\pi}\int_{\partial\Omega} d\arg\underline{\alpha} = D,$$

and therefore

$$\sum_{z_k\in\Omega} m_k \leq \frac{M^+ + M^-}{2} - D,$$

that is the integer at the right-hand side is finite and (2.11) holds.

3. - Identities on the number of critical points of solutions of $\Delta u = -f(u)$

The following lemma gives a classification of isolated critical points of smooth functions in $\mathbb{R}^2$. Although results of the same kind, but in the more difficult case of higher dimensional spaces, which requires additional assumptions, are known (see [R]), the authors have not been able to find a detailed specific proof for the two dimensional case; therefore, an *ad hoc* proof is provided in the Appendix to this paper.

LEMMA 3.1. *Let u be a real-valued C^1 function in an open set Ω in the complex plane. Let $z_0 \in \Omega$ be an isolated critical point of u.*

Then, one of the following cases occurs.

(i) *There exists a neighbourhhod $\mathcal{U}$ of z_0 such that $\{z \in \mathcal{U}: u(z) = u(z_0)\}$ is exactly z_0, and we have $I(z_0) = 1$.*

(ii) *There exist a positive integer L and a neighbourhood $\mathcal{V}$ of z_0 such that the level set $\{z \in \mathcal{V}: u(z) = u(z_0)\}$ consists of L simple closed curves. If $L \geq 2$, each pair of such curves crosses at z_0 only. We have $I(z_0) = 1 - L$.*

REMARK. Observe that the statement of Lemma 3.1 can be summarized by saying that the index of an isolate zero of a conservative vector field never exceeds 1.

DEFINITION 3.2. If (i) holds, then z_0 is a local maximum or minimum point; in such a case we shall refer to it as an *extremal point*.

If (ii) holds with $L = 1$, we say that z_0 is a *trivial point*.

If (ii) holds with $L \geq 2$, we say that z_0 is a *saddle point* and if $L = 2$, z_0 is a *simple saddle point*. We will call *order* of a trivial or a saddle point the number $L - 1$.

We will consider now the following boundary value problem:

$$\Delta u = -f(u) \qquad \text{in } \Omega, \tag{3.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{3.2}$$

where f is a real-valued function with

$$f(t) > 0, \qquad \text{if } t > 0, \text{ or} \tag{3.3a}$$

$$t^{-1} f(t) > 0, \qquad \forall t \neq 0. \tag{3.3b}$$

The next theorem is in the same spirit as Morse theory, the condition (3.3a) takes the place of the non-degeneracy condition (see [Ms]).

THEOREM 3.3. *Let Ω be a bounded open domain in the plane and let its boundary $\partial\Omega$ be composed of N closed simple curves of class $C^{1,\alpha}$. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a non-negative solution of* (3.1), (3.2), *subject to* (3.3a). *If the critical points of u are isolated, then they only can be extremal, simple saddle, or trivial points. Moreover, if n_E and n_S denote respectively the number of extremal and saddle points, we obtain*

$$n_S - n_E = N - 2. \tag{3.4}$$

COROLLARY 3.4. *Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a non-negative solution of* (3.1), (3.2), *and suppose in addition that Ω is simply connected and f is real analytic and satisfies* (3.3a).

Then the critical points of u are isolated, and we have

$$n_S - n_E = -1. \tag{3.5}$$

REMARK. Notice that if Ω is not simply connected, then non-isolated critical points indeed occur. An example is provided by the first Dirichlet eigenfunction of the Laplace operator for a circular annulus.

THEOREM 3.5. *Let Ω be as in Theorem* 3.3, *and suppose f is a C^1 real-valued function satisfying* (3.3b). *If the critical points of a solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ are isolated, then they are of the following type:*

(i) *nodal critical points, that is u also vanishes at such points; indicate them by $z_1, \ldots, z_K$; then they have integral multiplicities $m_1, \ldots, m_K$, and $\partial_z u(z)$ is asymptotic to $c_k(z - z_k)^{m_k}$ as $z \to z_k$, with $c_k \neq 0$, $k = 1, \ldots, K$;*

(ii) *non-nodal critical points, which only can be extremal, simple saddle, or trivial points, as described in Definition* 3.2.

Finally, we have

$$\sum_{z_k \in \Omega} m_k + \frac{1}{2} \sum_{z_k \in \partial\Omega} m_k + n_S - n_E = N - 2, \tag{3.6}$$

where n_S, n_E were defined in Theorem 3.3.

PROOF OF THEOREM 3.3. By the maximum principle, $u > 0$ in Ω, hence $\Delta u < 0$ in Ω, by (3.3a). By the Hopf lemma, the normal derivative of u never vanishes on $\partial\Omega$, that is the critical points $z_1, \ldots, z_K$ of u are not on $\partial\Omega$. The Gauss-Bonnet theorem yields:

$$2 - N = \text{Euler characteristic of } \Omega = \sum_{k=1}^{K} I(z_k). \tag{3.7}$$

If the Hessian determinant of u is not zero at a critical point z_k, then it is readily seen that z_k is either a local maximum point or a simple saddle point, since $\Delta u(z_k) < 0$.

Otherwise, $u_x = u_y = u_{xx}u_{yy} - u_{xy}^2 = 0$, at z_k, and, without loss of generality, we may assume $z_k = 0$, $u_{xx} = u_{xy} = 0$, at 0, and $u_{yy}(0) = -f(u(0)) < 0$. Let $\mathcal{U}$ be a neighbourhood of 0 where $u_{yy} < 0$. By reducing $\mathcal{U}$ if needed, Dini's theorem implies that the set $\{z \in \mathcal{U} : u_y(z) = 0\}$ is the graph of some smooth function $y = \psi(x)$ with $\psi(0) = 0$; we also have $u_y < 0$ if $y > \psi(x)$ and $u_y > 0$ if $y < \psi(x)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x, \psi(x)) = u_x(x, \psi(x)) + u_y(x, \psi(x))\psi'(x),$$

then $\frac{\mathrm{d}}{\mathrm{d}x}u(x,\psi(x))\neq 0$ if $x\neq 0$, because 0 is an isolated critical point. Therefore, $\{z\in \mathcal{U}: u(z)=u(0),\ u_y(z)=0\}=\{0\}$.

Fix $t\neq 0$; there exist at most two $y_1(t)$, $y_2(t)$, $y_1(t)<y_2(t)$, such that $u(t,y_1(t))=u(t,y_2(t))=0$, since $u_y(t,y)=0$ only at $y=\psi(t)$. Notice that $y_1(t)$ and $y_2(t)$ may not exist, but, if they do, they are distinct, because $u_{yy}<0$ on $\mathcal{U}$.

There are now three possibilities: a) $y_1(t)$, $y_2(t)$ do not exist for any $t\neq 0$, that is 0 is a maximal point; b) $y_1(t)$, $y_2(t)$ do exist for every $t\neq 0$, that is (ii) of Lemma 3.1 occurs with $L=2$; c) $y_1(t)$, $y_2(t)$ exist only for either $t>0$ or $t<0$, that is (i) of Lemma 3.1 is verified with $L=1$.

These remarks and formula (3.7) yield (3.3).

PROOF OF COROLLARY 3.4. As it is well-known (see for instance [Mo]), u is real analytic in Ω.

Suppose by contradiction that the set $Z=\{z\in\Omega: u_x(z)=u_y(z)=0\}$ is not discrete. As we observed in the previous proof, $Z\cap\partial\Omega=\emptyset$, by the Hopf lemma. Let $\{z_m\}\subseteq Z$, $z_0\in Z$, $z_m\neq z_0$, $z_m\to z_0$, as $m\to+\infty$. Since $\Delta u(z_0)<0$, we may suppose with no loss of generality that $u_{xx}(z_0)<0$; hence there exists a neighbourhood $\mathcal{U}$ of z_0 such that $\sigma=\{z\in\mathcal{U}: u_x(z)=0\}$ is a simple analytic arc. The fact that $u_y(z_m)=u_x(z_m)=0$, for infinitely many m's, implies that $u_y=0$ on σ by analytic continuation. This means that Z is composed by the union of a finite number of isolated points and analytic simple closed curves.

Let Σ be one of such curves and let $G\subset\Omega$ be the region such that $\partial G=\Sigma$. On ∂G, we have $u_x=u_y=0$, $u=c=$constant. We also know that $u>c$ on G, since u is superharmonic on Ω. Let $z_0\in\partial G$; if $\underline{\nu}$ and $\underline{\tau}$ are respectively the normal and tangential directions to ∂G at z_0, we have $u_{\nu\nu}(z_0)<0$, since $\Delta u(z_0)<0$, and $u_{\tau\tau}(z_0)=0$. This contradicts the fact that $u>u(z_0)$ in G, and that $u_x(z_0)=u_y(z_0)=0$.

PROOF OF THEOREM 3.5. The classification of non-nodal critical points goes as in Theorem 3.3. On the other hand, in a neighbourhood of an interior critical point, we have:

$$|\Delta u|\leq (\text{constant})|u|,$$

and the Hartman-Wintner theorem can be applied (see [Sch]), thus yielding (i). At a boundary critical point, a reflection argument, like the one used in the proof of Theorem 1.1, can be used, and again the Hartman-Wintner theorem applies, so that we obtain the asymptotic behaviour of u. Identity (3.5) follows, as before, by the computation of a limit of the type (2.13).

4. - Critical points of solutions of an obstacle problem

Here, we shall show how our method yields a simple proof of results of Sakaguchi [Sa] concerning the number of critical points in an obstacle problem. Let us begin with a description of the obstacle problem; for more details, the reader may consult [K-S].

We consider a simply connected bounded open set $\Omega \subset \mathbb{R}^2$, and a function $\psi \in C^2(\overline{\Omega})$ such that $\psi < 0$ on $\partial\Omega$ and $\max_{\overline{\Omega}} \psi > 0$. Suppose that a C^2 vector field $\mathbb{R}^2 \ni p \mapsto a = a(p) \in \mathbb{R}^2$ is given, which satisfies the following bounds:

$$\lambda|\xi|^2 \leq \sum_{j,k=1}^{2} \frac{\partial a_j}{\partial p_k}(p)\xi_j\xi_k \leq \Lambda|\xi|^2, \qquad \text{for every } \xi, \quad p \in \mathbb{R}^2, \tag{4.1}$$

and set

$$\mathcal{K} = \{v \in W_0^{1,2}(\Omega) : v \geq \psi\}. \tag{4.2}$$

The solution of the obstacle problem is defined as the unique function $u \in \mathcal{K}$ satisfying the variational inequality

$$\int_{\Omega} a(\nabla u) \cdot \nabla(v - u) \geq 0 \qquad \text{for every } v \in \mathcal{K},$$

where $\nabla u = (u_x, u_y)$.

An appropriate regularity theorem (see [K-S, IV, Theorem 6.3]) shows that $u \in W^{2,\infty}_{\mathrm{loc}}(\Omega)$. Moreover, denoting by

$$\mathcal{I} = \{z \in \Omega : u(z) = \psi(z)\} \tag{4.3}$$

the so-called coincidence set, we have

$$\begin{aligned} \operatorname{div}\{a(\nabla u)\} &= 0 \qquad \text{in } \Omega \backslash \mathcal{I}, \\ \operatorname{div}\{a(\nabla u)\} &\leq 0 \qquad \text{in } \Omega. \end{aligned}$$

It follows that $u > 0$ in Ω and also that, in the non-coincidence set $\Omega \backslash \mathcal{I}$, the Hartman-Wintner theorem applies to u, see again [Sch].

Now, we are in a position to state Sakaguchi's results (see [Sa, Theorems 1, 2]).

THEOREM 4.1. *Suppose that ψ has isolated critical points in Ω, and let N be the number of positive maximum points. Then the critical points of u in $\Omega \backslash \mathcal{I}$ are finite in number and, denoting by $m_1, \ldots, m_K$ their multiplicities, we have*

$$\sum_{k=1}^{K} m_k \leq N - 1. \tag{4.4}$$

If, in addition, all the critical points of ψ, *where* $\psi > 0$, *are of absolute maximum, then the equality holds in* (4.5), *that is*

$$\sum_{k=1}^{K} m_k = N - 1. \tag{4.5}$$

PROOF. The preliminary step of this proof consists in the following lemma.

LEMMA 4.2. *If the hypotheses of the Theorem* 4.1 *are satisfied, then the critical points of* u *in* $\Omega \backslash \mathcal{I}$ *are finite in number.*

A proof of this lemma can be obtained along the same lines of [A1, Lemma 1.1], and it is based on a combined use of the Hartman-Wintner theorem and the maximum principle (see Section 6).

Now, since the critical points of u in $\mathcal{I}$ are also critical points of ψ, we have that the interior critical points of u in Ω are finite in number, and we denote them by $z_1, \dots, z_L$. Since Ω is simply connected, we have:

$$\sum_{\ell=1}^{L} I(z_\ell) = 1,$$

and also $I(z_\ell) = -m_\ell$ when $z_\ell \in \Omega \backslash \mathcal{I}$. Moreover, since u has no interior minima, by Lemma 3.1, we get

$$\sum_{z_\ell \in I} I(z_\ell) \leq \sum_{\substack{z_\ell \in I \\ z_\ell \text{ max point}}} I(z_\ell) \leq N,$$

so that

$$1 = \sum_{z_\ell \in I} I(z_\ell) + \sum_{z_\ell \in \Omega \backslash I} I(z_\ell) \leq N - \sum_{k=1}^{K} m_k,$$

and (4.4) follows.

Observe now that the points of absolute maximum of ψ are in the set $\mathcal{I}$, hence, if all the critical points of ψ, where $\psi > 0$, are points of absolute maximum, then the critical points of u in $\mathcal{I}$ are nothing else than such N points of absolute maximum, and consequently

$$\sum_{z_\ell \in I} I(z_\ell) = N,$$

thus (4.5) follows.

5. - Identities relating the functions v, ω, h, and k

In this section we will prove Theorem 1.2. We wish to remark that formulas (1.6)–(1.8) can be adapted to suit the needs of treating solutions of elliptic equations with general principal part. The next Theorem, which is in fact a corollary of Theorem 1.2, provides such an adaptation for formulas (1.6), (1.7).

Let

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

be a positive definite symmetric matrix with Lipschitz continuous entries satisfying the normalization condition (2.3), that is

$$\det A = ac - b^2 = 1.$$

We shall consider an elliptic operator in divergence form:

$$\mathcal{L} = \operatorname{div}(A\nabla \cdot\,) = \partial_x(a\partial_x + b\partial_y) + \partial_y(b\partial_x + c\partial_y). \tag{5.1}$$

Given $u \in C^2(\Omega)$, we introduce in place of v, ω, the following functions which are more strictly related to the metric intrinsic to $\mathcal{L}$:

$$\tilde{v} = \log|\sqrt{A}\nabla u|, \tag{5.2a}$$

$$\tilde{\omega} = \arg(\sqrt{A}\nabla u); \tag{5.2b}$$

here $\sqrt{A}$ denotes the positive definite symmetric square root of A. It is useful to introduce the following expression

$$q = -\frac{a - c + 2ib}{a + c + 2}.$$

THEOREM 5.1. *If $u \in C^2(\Omega)$ has only a finite number of critical points $z_1, \dots, z_K \in \Omega$ of indices $I(z_1), \dots, I(z_K)$, respectively, then the following identities hold in the sense of distributions:*

$$\mathcal{L}\tilde{v} - \operatorname{div}\left(\frac{\mathcal{L}u}{|\sqrt{A}\nabla u|^2} A\nabla u\right) + \mathcal{A}(q) = -2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k), \tag{5.3}$$

$$\mathcal{L}\tilde{\omega} - \operatorname{div}\left(\frac{\mathcal{L}u}{|\sqrt{A}\nabla u|^2} (\nabla u)^{\perp}\right) + \mathcal{B}(q) = 0; \tag{5.4}$$

here $(\nabla u)^{\perp} = (-u_y, u_x)$, and

$$\mathcal{A}(q) + i\mathcal{B}(q) = \frac{1}{2}\,\mathcal{L}\left(\log\frac{1}{1 - |q|^2}\right) - 4\partial_z\left(\frac{q_z}{1 - |q|^2}\right) + 4\partial_{\bar{z}}\left(\frac{\bar{q}q_z}{1 - |q|^2}\right).$$

The proof of Theorems 1.2 and 5.1 is based on the following lemma.

LEMMA 5.2. *Suppose* $u \in C^2(\Omega)$ *has isolated critical points* $z_1, \dots, z_K \in \Omega$ *of indices* $I(z_1), \dots, I(z_K)$, *respectively. Let* v, ω *be given by* (1.3). *Then the following identities hold in the sense of distributions*

$$\mathrm{d}\,\mathrm{d}\omega = 2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k)\,\mathrm{d}x\,\mathrm{d}y, \tag{5.5}$$

$$\mathrm{d}\,\mathrm{d}v = 0. \tag{5.6}$$

Here, $\delta(\cdot - z_k)\,\mathrm{d}x\,\mathrm{d}y$ *denotes the Dirac measure with pole at* z_k.

PROOF. Let $\psi \in C_0^\infty(\Omega)$. By means of a partition of unity, we can assume, without loss of generality, that the support of ψ contains only one critical point of u, say z_k.

If $\langle \cdot, \cdot \rangle$ denotes the L^2-based duality between distributional 2-forms and functions, for any integrable 1-form η, we have

$$\langle \mathrm{d}\eta, \psi \rangle = -\int \mathrm{d}\psi \wedge \eta.$$

By the use of local changes of coordinates for which ψ becomes an independent variable, we can write

$$-\int \mathrm{d}\psi \wedge \eta = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{+\partial\{\psi>t\}} \eta.$$

Notice that this formula can be viewed as a special case of Federer's coarea formula (see [F], Theorem 3.2.12). Note also that, by Sard's lemma, $\partial\{\psi > t\}$ is C^1-smooth for almost every t. Now, setting $\eta = \mathrm{d}\omega$ yields

$$\langle \mathrm{d}\,\mathrm{d}\omega, \psi \rangle = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{+\partial\{\psi>t\}} \mathrm{d}\omega.$$

By changing ψ with $-\psi$ if needed, we can suppose that $\psi(z_k) \geq 0$. Let t be a regular value of ψ. If $t > \psi(z_k)$, then $z_k \notin \{\psi > t\}$, while $+\partial\{\psi > t\} = -\partial\{\psi < t\}$ and $\{\psi < t\}$ is bounded, when $t \leq 0$. In both cases, we get

$$\int_{+\partial\{\psi>t\}} \mathrm{d}\omega = 0.$$

On the other hand, when t is regular and $0 < t < \psi(z_k)$, then $z_k \in \{\psi > t\}$, and consequently

$$\int_{+\partial\{\psi>t\}} \mathrm{d}\omega = 2\pi I(z_k).$$

Finally, we have

$$\langle \mathrm{d}\,\mathrm{d}\omega, \psi \rangle = \int_{0}^{\psi(z_k)} \mathrm{d}t \int_{+\partial\{\psi>t\}} \mathrm{d}\omega = 2\pi I(z_k)\psi(z_k),$$

and (5.5) follows. Analogously, we obtain $\langle \mathrm{d}\,\mathrm{d}v, \psi \rangle = 0$, since $\int_\gamma \mathrm{d}v = 0$ for every smooth closed curve γ, and (5.6) follows.

PROOF OF THEOREM 1.2. By (1.4), we have:

$$\partial_z \omega = \frac{1}{2} i\phi e^{-i\omega}. \tag{5.7}$$

On the other hand, we get

$$\Delta u = 4\partial_z \partial_{\overline{z}} u = 2\partial_z e^{v+i\omega} = 2e^{v+i\omega}\partial_z v - e^v \phi,$$

hence

$$\partial_z v = \frac{1}{2} \phi e^{-i\omega} + \frac{1}{2} \Delta u e^{-v-i\omega}. \tag{5.8}$$

Since v and ω are real-valued, we can write:

$$\begin{aligned} \mathrm{d}v &= \frac{1}{2} [\phi e^{-i\omega}\, \mathrm{d}z + \overline{\phi} e^{i\omega}\, \mathrm{d}\overline{z}] + \frac{1}{2} \Delta u [e^{-v-i\omega}\, \mathrm{d}z + e^{-v+i\omega}\, \mathrm{d}\overline{z}], \\ \mathrm{d}\omega &= \frac{1}{2} i[\phi e^{-i\omega}\, \mathrm{d}z - \overline{\phi} e^{i\omega}\, \mathrm{d}\overline{z}]. \end{aligned} \tag{5.9}$$

By applying Lemma 5.2 to (5.9), we get:

$$2\partial_{\overline{z}}(\phi e^{-i\omega}) = \partial_z(\Delta u e^{-v+i\omega}) - \partial_{\overline{z}}(\Delta u e^{-v-i\omega}) - 2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k). \tag{5.10}$$

On the other hand, (5.7) and (5.8) give also

$$\Delta v - i\Delta\omega = 2\partial_{\overline{z}}(\phi e^{-i\omega}) + \partial_z(\Delta u e^{-v+i\omega}) + \partial_{\overline{z}}(\Delta u e^{-v-i\omega}),$$

and, by (5.10), we have

$$\Delta v - i\Delta\omega = 2\partial_z(\Delta u e^{-v+i\omega}) - 2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k). \tag{5.11}$$

It is worthwhile to notice that this last identity may be re-written as follows:

$$\Delta \log(\partial_z u) - \partial_z\left(\frac{\Delta u}{\partial_z u}\right) = -2\pi \sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k). \tag{5.12}$$

By taking the real and the imaginary parts in (5.11), we obtain (1.6) and (1.7). Formula (1.8) follows by simply differentiating in (5.10), and by using (1.4).

PROOF OF THEOREM 5.1. By the uniformization theorem (see [V]), we may find a change of coordinates in Ω, $\varsigma = \varsigma(z) = \xi(z) + i\eta(z)$, such that $\varsigma(z)$ satisfies the Beltrami equation:

$$\varsigma_{\bar z} = q\varsigma_z, \tag{5.13}$$

and we have

$$\mathcal{L} = J\Delta_\varsigma, \tag{5.14}$$

where $\Delta_\varsigma = 4\partial_\varsigma\partial_{\bar\varsigma}$ and

$$J = |\varsigma_z|^2 - |\varsigma_{\bar z}|^2 = (1 - |q|^2)|\varsigma_z|^2$$

is the jacobian determinant of the mapping $\varsigma(z)$.

Now, we write the identity (5.12) in the ς coordinate and, by a simple calculation, we obtain in the z coordinate:

$$\begin{aligned}\mathcal{L}\left(\log\frac{\mathcal{M}u}{1-|q|^2}\right) - \mathcal{M}\left(\frac{\mathcal{L}u}{\mathcal{M}u}\right) - \mathcal{L}\left(\log\varsigma_z\right) - \frac{\mathcal{L}u}{\mathcal{M}u}\bar\varsigma_{\bar z}\mathcal{M}\left(\frac{1}{\bar\varsigma_{\bar z}}\right)\\ = -2\pi\sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k),\end{aligned}$$

where $\mathcal{M} = \partial_z - \bar q\partial_{\bar z}$. We compute the terms depending on ς_z by using equations (5.13) and (5.14), and we obtain:

$$\begin{aligned}\mathcal{L}\left(\log\frac{\mathcal{M}u}{\sqrt{1-|q|^2}}\right) - \mathcal{M}\left(\frac{\mathcal{L}u}{\mathcal{M}u}\right) + \bar q_{\bar z}\frac{\mathcal{L}u}{\mathcal{M}u} + \frac{1}{2}\mathcal{L}\left(\log\frac{1}{1-|q|^2}\right)\\ - 4\partial_z\left(\frac{q_z}{1-|q|^2}\right) + 4\partial_{\bar z}\left(\frac{\bar q q_z}{1-|q|^2}\right)\\ = -2\pi\sum_{k=1}^{K} I(z_k)\delta(\cdot - z_k).\end{aligned} \tag{5.15}$$

Identities (5.3) and (5.4) follow directly from (5.15), by taking the real and imaginary parts.

6. - Appendix

PROOF OF LEMMA 3.1. With no loss of generality, we may assume that $u(z_0) = 0$, $z_0 = 0$. If (i) does not hold, there exists $r_0 > 0$ such that the sets

$$B_r^+ = \{z \in \overline{B_r(0)}: u(z) > 0\}, \qquad B_r^- = \{z \in \overline{B_r(0)}: u(z) < 0\}$$

are both non-empty for every $r < r_0$. Moreover, being z_0 an isolated critical point, no connected component of B_r^+ or B_r^- is contained in the interior of $B_r(0)$. Such connected components are finite in number; in fact, we can argue as follows. By the Dini's theorem, for any $\tilde{z} \in \partial B_r$, there is a disk $B_\varrho(\tilde{z})$ such that $\{z \in B_\varrho(\tilde{z}): u(z) = 0\}$ is either empty or a simple curve that crosses transversally $\partial B_\varrho(\tilde{z})$. By compactness, there is a finite covering $B_{\varrho_1}(z_1), \dots, B_{\varrho_M}(z_M)$ of $\partial B_r(0)$. Therefore, if $\mathcal{A} = \bigcup_{m=1}^{M} B_{\varrho_m}(z_m)$, then the set $\{z \in \mathcal{A}: u(z) = 0\}$ is composed by a finite number of non-intersecting simple curves. Thus, $\{z \in \mathcal{A}: u(z) > 0\}$ has finitely many components and, since each component of B_r^+ contains points in $\mathcal{A}$, also such components are finite in number. The same argument applies to B_r^-.

Let $\mathcal{U} = B_r(0) \cup \mathcal{A}$; $\mathcal{U}$ is a simply connected neighbourhod of 0, with piecewise smooth boundary, since it may have angular points at the intersections of the circles $\partial B_{\varrho_1}(z_1), \dots, \partial B_{\varrho_M}(z_M)$. Let $p_1, \dots, p_N$ be such angular points; we can suppose $u(p_n) \neq 0$, $n = 1, \dots, N$, otherwise we can add to $\mathcal{U}$ a small disk $\mathcal{D}$ centered at p_n in such a way that $\mathcal{U}_0 = \{z \in \mathcal{U} \setminus \{0\}: u(z) = 0\}$ and $\mathcal{U}_0 \cup \mathcal{D}$ have the same number of components. If now $u(p_n) \neq 0$, $n = 1, \dots, N$, we may smooth $\partial\mathcal{U}$ near $p_1, \dots; p_N$ by enlarging $\mathcal{U}$ in such a way that the number of connected components of $\mathcal{U}_0$ does not change. Note in addition that each component of this new version of $\mathcal{U}_0$ crosses $\partial\mathcal{U}$ transversally, by a similar continuity argument.

Let $\mathcal{E}_1^+, \dots, \mathcal{E}_K^+$ (respectively $\mathcal{E}_1^-, \dots, \mathcal{E}_L^-$) be the connected components of $\mathcal{U}^+ = \{z \in \mathcal{U} \setminus \{0\}: u(z) > 0\}$ (respectively $\mathcal{U}^- = \{z \in \mathcal{U} \setminus \{0\}: u(z) < 0\}$) which contain 0 in their boundary. Each of the sets $\mathcal{E} = \mathcal{E}_k^+$, or $\mathcal{E}_\ell^-$ is simply connected and its boundary can be decomposed as follows:

(i) two simple arcs σ_1, σ_2 each having one endpoint at 0 and the other one on $\partial\mathcal{U}$,

(ii) an arc τ which is composed by finitely many portions of $\partial\mathcal{U}$ and arcs where $u = 0$ but which do not reach 0.

Let P_1, P_2 be the endpoints on $\partial\mathcal{U}$ of σ_1, σ_2, respectively. By the transversality condition mentioned above, we may join P_1 to P_2 with an arc $\tilde{\tau}$ in $\mathcal{E}$ having the following properties: (a) $\tilde{\tau}$ is tangent to $\partial\mathcal{U}$ at P_1 and P_2; (b) $u \neq 0$ on $\tilde{\tau} \setminus \{P_1, P_2\}$.

Let $\tilde{\mathcal{E}} \subset \mathcal{E}$ be the simply connected region surrounded by $\sigma_1 \cup \sigma_2 \cup \tilde{\tau}$. Moving along $\partial\mathcal{U}$, the sign of u changes at each P_j. Hence, we have $L = K$. Moreover, by replacing $\mathcal{E} = \mathcal{E}_k^+$, $\mathcal{E}_\ell^-$ with $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_k^+$, $\tilde{\mathcal{E}}_\ell^-$, for every k, $\ell = 1, \dots, L$, and $\mathcal{U}$ with $\mathcal{V}$, the domain surrounded by the union of the $\tilde{\tau}_k^+$, $\tilde{\tau}_\ell^-$ arcs, we have that the following holds: $\mathcal{V}$ is a simply connected neighbourhood of 0 with C^1 boundary; the level set $\{z \in \mathcal{V}\backslash\{0\}: u(z) = 0\}$ is composed by $2L$ disjoint simple arcs each having one endpoint at 0 and the other on $\partial\mathcal{V}$; each of such arcs reaches $\partial\mathcal{V}$ transversally at distinct points, and separates points where $u > 0$ from points where $u < 0$. If we glue two by two such arcs at 0, we get case (ii).

Now we proceed by computing the index. If $z_0 = 0$ is an extremal point, say a minimum point, for $\varepsilon > 0$ small enough, there is a bounded simply connected component of $\{z: u(z) < \varepsilon\}$ which contains 0 and whose boundary is C^1. Hence,

$$I(z_0) = \frac{1}{2\pi} \int\limits_{\partial\{u<\varepsilon\}} \mathrm{d}\arg \underline{\nu} = 1,$$

by the Gauss-Bonnet formula.

Let now suppose that $z_0 = 0$ is either a trivial or a saddle point of order L. Let $\tilde{\mathcal{E}}$ be one of the sets $\tilde{\mathcal{E}}_k^+$, $\tilde{\mathcal{E}}_\ell^-$ defined above. We have

$$I(z_0) = \frac{1}{2\pi} \sum_{\ell=1}^{L} \left(\int\limits_{\tilde{\tau}_\ell^+} \mathrm{d}\omega + \int\limits_{\tilde{\tau}_\ell^-} \mathrm{d}\omega \right).$$

We want to compute $\int_{\tilde{\tau}_\ell^+} \mathrm{d}\omega$, $\int_{\tilde{\tau}_\ell^-} \mathrm{d}\omega$.

Consider the region $\mathcal{E}_\varepsilon = \{z \in \mathcal{E}: |u(z)| < \varepsilon\}$; for $\varepsilon > 0$ sufficiently small, $\partial\mathcal{E}_\varepsilon$ is formed by 5 arcs σ_1, σ_2, γ, α_1, and α_2, where σ_1, σ_2 are the arcs defined above, α_1, α_2 are contained in $\partial\mathcal{V}$, and $\gamma \subset \mathcal{E}$ and is such that $|u| = \varepsilon$ on γ. Clearly,

$$\int\limits_{\tilde{\tau}} \mathrm{d}\omega = \int\limits_{\alpha_1} \mathrm{d}\omega + \int\limits_{\gamma} \mathrm{d}\omega + \int\limits_{\alpha_2} \mathrm{d}\omega,$$

and we have $\int_{\alpha_1} \mathrm{d}\omega$, $\int_{\alpha_2} \mathrm{d}\omega \to 0$, as $\varepsilon \to 0$. On the other hand, we can compute $\int_\gamma \mathrm{d}\omega$ as follows: let $\underline{v}$ be the unitary vector field, tangent to the lines $|u| = \varepsilon$ at the points of $\partial\mathcal{V}$ near $p_1, \dots, p_{2L}$, where $u = 0$. We may orientate and continue $\underline{v}$ in such a way that, on all $\partial\mathcal{V}$, it forms an acute angle with the exterior normal $\underline{\nu}$ to $\partial\mathcal{V}$.

Let Q_1, Q_2 be the endpoints of γ ordered along $\tilde{\tau}$ with respect to the counterclockwise orientation on $\partial\mathcal{V}$. We have then

$$\int\limits_{\gamma} \mathrm{d}\omega = \arg \underline{v}(Q_1) - \arg \underline{v}(Q_2) - \pi,$$

hence

$$\int_{\gamma} d\omega \to \int_{\tilde{\gamma}} d \arg \underline{v} - \pi,$$

as $\varepsilon \to 0$.

Finally,

$$\frac{1}{2\pi} \int_{\partial\Omega} d\omega = \frac{1}{2\pi} \int_{\partial\Omega} d \arg \underline{v} - L = 1 - L,$$

by the Gauss-Bonnet formula.

PROOF OF LEMMA 4.2. By contradiction, let us suppose that there are infinitely many critical points of u in $\Omega \backslash \mathcal{I}$. We distinguish two cases:

(i) there exist $t > 0$ and a connected component Γ of $\{z \in \Omega : u(z) = t\}$ which contains infinitely many critical points;

(ii) there exist infinitely many connected components of level lines of u in Ω, each of which contains at least one critical point.

Case (i). We show that the set $\{z \in \Omega : u(z) > t\}$ has infinitely many connected components, thus contradicting the assumption that ψ has finitely many maximum points.

If α, β are simple closed curves in Γ, then none of the two surrounds the other. Otherwise, in the region between them, we would have $u < t$, which is impossible by the maximum principle. Therefore, there exists a simple closed curve $\chi \subset \Gamma$ which contains infinitely many critical points of u, $\{z_k\}_{k=1}^{\infty}$, $z_k \in \Omega \backslash \mathcal{I}$, $k = 1, 2, \ldots$. Near each z_k, the level line Γ is composed of $2(m_k + 1)$ $(m_k > 0)$ branches meeting at z_k. Two of such branches belong to χ, while all the others point to the exterior of χ and cannot reach another z_j, $j \neq k$.

Therefore, by induction, for each z_k, we find at least one loop γ_k in $\{u = t\}$ passing through z_k and lying in the exterior of χ, and of $\gamma_1, \ldots, \gamma_{k-1}$. Each γ_k surrounds a connected component of $\{u > t\}$. This is a contradiction.

Case (ii). There are infinitely many Jordan curves $\mathcal{G} = \{\alpha_j\}_{j=1}^{\infty}$, $\alpha_j \cap \alpha_k = \emptyset$, $j \neq k$, each of which contains a critical point and it belongs to a different component of a level line of u. If each curve in $\mathcal{G}$ surrounds a finite number of curves in $\mathcal{G}$, then we are finished.

Conversely, suppose that α_{j+1} is surrounded by α_j for every j. If $z_j \in \alpha_j$ is a critical point, then, by the same reasoning as above, in the exterior of α_j there is at least a Jordan curve β_j, containing z_j and on which u is constant. Thus, the interiors of the β_j's are all disjoint and each of them contains a point of relative maximum of u. This yields a contradiction.

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