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Matrix model for noncommutative gravity and gravitational instantons

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Abstract

We introduce a matrix model for noncommutative gravity, based on the gauge group $U(2) \otimes U(2)$. The vierbein is encoded in a matrix Y_μ, having values in the coset space $U(4)/(U(2) \otimes U(2))$, while the spin connection is encoded in a matrix X_μ, having values in $U(2) \otimes U(2)$. We show how to recover the Einstein equations from the $\theta \to 0$ limit of the matrix model equations of motion. We stress the necessity of a metric tensor, which is a covariant representation of the gauge group in order to set up a consistent second order formalism. We finally define noncommutative gravitational instantons as generated by $U(2) \otimes U(2)$ valued quasi-unitary operators acting on the background of the Matrix model. Some of these solutions have naturally self-dual or anti-self-dual spin connections.

1 Introduction

Unification of noncommutative geometry with gravity theories is a very challenging goal for a theoretical physicist [1]-[11]. Till now only gauge theories are proved to be consistent after deforming the ordinary product of fields into a noncommutative star product.

Gravity theories are usually constructed by requiring either diffeomorphism invariance or local Lorentz invariance. Since the Moyal star product breaks global Lorentz invariance in gauge theories, it also breaks diffeomorphism invariance in a gravity theory. Therefore in the noncommutative case it is possible to preserve only local Lorentz invariance, with the gauge group extended to $U(1, d-1)$ instead of $SO(1, d-1)$, in order that the gauge transformations multiplied with the star product are closed between them.

This approach has still many problems, mainly because the metric becomes complex, and the antisymmetric part of the metric may have nonphysical propagating modes [3]-[5]. It is however worth to explore all the consequences that such a new theory can give, before taking an opinion about it. In this paper we attempt to give a more solid construction of noncommutative gravity theory by introducing a new matrix model based on the $U(2) \otimes U(2)$ group (for Euclidean gravity) or $U(1,1) \otimes U(1,1)$ for the $U(2,2)$ case.

In this respect the first-order formalism, based on the vierbein and spin connection, turns out to be superior to the second order formalism, since it permits the definition of a matrix model, without the necessity of inverting the metric, which would be a rather difficult obstacle.

The vierbein is encoded in a matrix Y_μ, having values in the coset space $\frac{U(4)}{U(2)\otimes U(2)}$, while the spin connection is encoded in a matrix X_μ, having values in the gauge group $U(2) \otimes U(2)$. The action is a 4-form, thus preventing the use of any metric, and the corresponding equations of motion are proved to be a natural generalization of the Einstein equations, provided a certain separation between odd powers in θ and even powers in θ is made (as done in Ref. [9] and Ref. [12]).

We confirm the necessity of a complex metric tensor, but we point out a property which was not discussed before, i.e. the metric tensor cannot be defined to be invariant under local Lorentz invariance, but at most it can be a covariant representation of the gauge group. We believe that the introduction of a multiplet of metric tensors is necessary to set up a consistent second order formalism. Each component of this multiplet has no direct physical meaning, since it can be mixed with the other components by a gauge redefinition of the vierbein and spin connection.

In the last part of the article, we attack the problem of defining noncommutative gravitational instantons [13]-[17]. In this sense, our formalism based on the Matrix model approach turns out to be fruitful, since then the finite action solutions of the equations of motion are generated by quasi-unitary operators, as it has been successfully found in the Yang-Mills case [18]-[23]. The ultimate source of our quasi-unitary operator is a projector, with values in $U(2)_{\rm left} \otimes U(2)_{\rm right}$. We find that if the projector is restricted to be pure left or pure right, then the corresponding spin connection is self-dual or anti-self-dual, an important property which can make the bridge between our definition of gravitational instantons and the classical standard definition. However our class of solutions is more general than only self-dual ones. We finally outline how to construct examples of quasi-unitary operators giving rise to their sources, the $U(2) \otimes U(2)$ projectors. An important tool to construct these examples is the use of duality in noncommutative theory [24]-[25].

2 Matrix model for noncommutative gravity

We are going to introduce a gravity theory on a noncommutative plane defined by the commutators:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \qquad det|\theta_{ij}| \neq 0. \tag{2.1}$$

We remember that instead of using the commutation relations in full generality, it is possible to reduce them in a diagonal form as follows

$$\begin{aligned} [\hat{x}_1, \hat{x}_2] &= i\theta_1 \\ [\hat{x}_3, \hat{x}_4] &= i\theta_2, \end{aligned} \tag{2.2}$$

which is basically solved by two types of raising and lowering oscillator operators.

With the experience in the Yang-Mills case [23], we attack the problem of noncommutative gravity. Our proposal is based on gauging the local Lorentz symmetry, extended to $U(2) \otimes U(2)$ for consistency.

Firstly we define two types of matrices X_μ, Y_μ, where X_μ is, at least in the Euclidean case, a hermitian matrix with values in the group $U(2) \otimes U(2)$, while Y_μ is a hermitian matrix with values in the coset space $\frac{U(4)}{U(2)\otimes U(2)}$. Then we write the Einstein action for the noncommutative case as

$$S_E = \beta_E \, Tr[\gamma_5 \, \epsilon^{\mu\nu\rho\sigma} Y_\mu Y_\nu ([X_\rho, X_\sigma] - i\theta^{-1}_{\rho\sigma})]$$

$$Y_\mu^\dagger = Y_\mu \qquad X_\mu^\dagger = X_\mu \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{2.3}$$

This action can be derived as a part of a more symmetric action (inspired by Ref. [2] and Ref. [9]), based on two $U(4)$ fields $X_\mu^\pm$

$$X_\mu^\pm = X_\mu \pm Y_\mu, \tag{2.4}$$

given by the sum of two similar contributions:

$$\begin{aligned} S &= \beta\, Tr[\gamma_5\, \epsilon^{\mu\nu\rho\sigma}([X_\mu^+, X_\nu^+] - i\theta_{\mu\nu}^{-1})([X_\rho^+, X_\sigma^+] - i\theta_{\rho\sigma}^{-1}] \\ &+ \beta\, Tr[\gamma_5\, \epsilon^{\mu\nu\rho\sigma}([X_\mu^-, X_\nu^-] - i\theta_{\mu\nu}^{-1})([X_\rho^-, X_\sigma^-] - i\theta_{\rho\sigma}^{-1}]. \end{aligned} \tag{2.5}$$

With the trick of adding the second term, the odd powers in Y_μ all vanish and we are left with three terms; the term with zero powers of Y_μ is a pure topological term, representing noncommutative topological gravity, the term with two powers of Y_μ reproduces the action (2.3) which is the starting point of our article, and a term proportional to the square of the torsion $T_{\mu\nu}$, defined as

$$T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu], \tag{2.6}$$

while the term with four powers of Y_μ gives rise to the cosmological constant term.

Being the dependence from the torsion $T_{\mu\nu}$ quadratic, it is possible to set it equal to zero, because the variation of the other terms are consistent with this choice.

Neglecting the topological term and the cosmological constant term we can continue to discuss the action (2.3) as the basis for noncommutative gravity.

In this paper we mainly discuss the $U(4)$ and $U(2,2)$ cases for simplicity, since we are mainly interested to introduce the noncommutative version of gravitational instantons, and we need to work with the Euclidean case. In the Euclidean case $U(4)$, the gamma matrices satisfy the hermitian condition

$$\gamma_a^\dagger = \gamma_a \qquad \gamma_5^\dagger = \gamma_5. \tag{2.7}$$

The matrix Y_μ can be developed in terms of basic $U(1)$ valued matrices :

$$Y_\mu = e_\mu^a \gamma_a + i f_\mu^a \gamma_a \gamma_5. \tag{2.8}$$

The hermitian condition of the Y_μ is reflected on a hermitian condition on the component vierbeins as follows:

$$
\begin{aligned}
(e_\mu^a)^\dagger &= e_\mu^a \\
(f_\mu^a)^\dagger &= f_\mu^a .
\end{aligned}
\tag{2.9}
$$

It is possible to define left and right combinations of the vierbeins as follows:

$$
e_\mu^{\pm a} = e_\mu^a \pm i f_\mu^a \qquad (e_\mu^{+a})^\dagger = e_\mu^{-a}, \tag{2.10}
$$

which are related by hermitian conjugation. These matrices have the nice property to be closed under the $U(2) \otimes U(2)$ gauge transformations. In fact a left-right decomposition can be made at the level of the Y_μ defining

$$
\begin{aligned}
Y_\mu &= Y_\mu^+ + Y_\mu^- \\
Y_\mu^\pm &= e_\mu^{\pm a} \gamma_a \left(\frac{1 \pm \gamma_5}{2} \right) .
\end{aligned}
\tag{2.11}
$$

In terms of $Y_\mu^\pm$ the action can be rewritten as

$$
S_E = -\beta_E \; Tr[\epsilon^{\mu\nu\rho\sigma}(Y_\mu^+ Y_\nu^- + Y_\mu^- Y_\nu^+)([X_\rho, X_\sigma] - i\theta_{\rho\sigma}^{-1})]. \tag{2.12}
$$

Note that since $Y_\mu^+ Y_\nu^+ = 0$, it is not possible to define an action containing only left combinations of the vierbein, but both left and right vierbeins are required to make S_E hermitian.

The matrix X_μ can be developed in terms of the basic matrices:

$$
X_\mu = \hat{p}_\mu + \omega_\mu^1 + \omega_\mu^5 \gamma^5 + i\omega_\mu^{ab} \gamma_{ab} \qquad \gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b], \tag{2.13}
$$

where the component matrices are all hermitian by construction:

$$
\hat{p}_\mu^\dagger = \hat{p}_\mu \quad \omega_\mu^{1\,\dagger} = \omega_\mu^1 \quad \omega_\mu^{5\,\dagger} = \omega_\mu^5 \quad \omega_\mu^{ab\,\dagger} = \omega_\mu^{ab}. \tag{2.14}
$$

We have introduced the distinction between $\hat{p}_\mu$ and ω_μ^1 since in a matrix model for non-commutative gravity it is necessary to separate the background from the fluctuations [23].

In general, the background for the matrix model of noncommutative gravity will be defined as

$$\begin{aligned} X_\mu &= \hat{p}_\mu \qquad \hat{p}_\mu = -\theta^{-1}_{\mu\nu}\hat{x}_\nu \\ Y_\mu &= \delta^a_\mu \gamma_a, \end{aligned} \tag{2.15}$$

and the fluctuations are generated by the matrices

$$\omega^1_\mu,\ \omega^5_\mu,\ \omega^{ab}_\mu,\ e^a_\mu - \delta^a_\mu,\ f^a_\mu. \tag{2.16}$$

The background is chosen to satisfy the equations of motion of the matrix model and to introduce the noncommutative coordinates as noncommutative analogues of the concepts of derivatives of the fields, as we normally do in the Yang-Mills case. Therefore the commutation relation , for example, of the background $\hat{p}_\mu$ with the matrix ω^1_ν is equivalent, at a level of the corresponding symbol, to a derivative action:

$$[\hat{p}_\mu, \omega^1_\nu] \rightarrow \partial_\mu \omega^1_\nu. \tag{2.17}$$

Moreover the operator products are transformed into star products of the corresponding symbols (through the Weyl map, see Ref. [24] for details).

In general, the matrix model is built on the gauge invariance

$$\begin{aligned} X_\mu &\rightarrow U^{-1} X_\mu U \\ Y_\mu &\rightarrow U^{-1} Y_\mu U, \end{aligned} \tag{2.18}$$

where U is the gauge transformation of $U(2) \otimes U(2)$; these types of transformations reproduce, in the commutative limit, the standard local Lorentz transformations for the vierbein and the spin connection. The gauge transformations of $U(2) \otimes U(2)$ are defined from the generators $1, \gamma_5, \gamma_{ab}$ and obey the unitary condition:

$$U^\dagger U = U U^\dagger = 1. \tag{2.19}$$

Therefore introducing the anti-hermitian matrix Λ

$$U = \exp[\Lambda] \qquad \Lambda^\dagger = -\Lambda \tag{2.20}$$

Λ can be developed in terms of basic gauge parameters:

$$\Lambda = i\Lambda_0 + i\Lambda_5\gamma_5 + \Lambda^{ab}\gamma_{ab}, \tag{2.21}$$

where the component parameters are all hermitian matrices:

$$\Lambda_0^\dagger = \Lambda_0 \qquad \Lambda_5^\dagger = \Lambda_5 \qquad \Lambda^{ab\,\dagger} = \Lambda^{ab}. \tag{2.22}$$

What does it change in the $U(2,2)$ scenario ? The action is always the same (2.3) with $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ (see Appendix). The matrix Y_μ is a matrix with values in the coset space $\frac{U(2,2)}{U(1,1)\otimes U(1,1)}$, and the matrix X_μ has values in $U(1,1)\otimes U(1,1)$.

To define the hermitian conjugation of these matrices, let us recall that $\Gamma_0 = \gamma_0\gamma_1$ is the hermitian conjugation matrix for the gamma matrices in the $U(2,2)$ case:

$$\begin{aligned} \gamma_0^2 &= \gamma_1^2 = -\gamma_2^2 = -\gamma_3^2 = 1 \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab} \quad (++--) \\ \gamma_a^\dagger &= \Gamma_0\gamma_a\Gamma_0 \qquad \gamma_5^\dagger = \gamma_5 \qquad \gamma_5\Gamma_0 = \Gamma_0\gamma_5. \end{aligned} \tag{2.23}$$

Then we define the hermitian conjugation for the matrices Y_μ, X_μ as

$$\begin{aligned} Y_\mu^\dagger &= \Gamma_0 Y_\mu \Gamma_0 \\ X_\mu^\dagger &= -\Gamma_0 X_\mu \Gamma_0. \end{aligned} \tag{2.24}$$

Again Y_μ can be developed in terms of the basic components as follows:

$$\begin{aligned} Y_\mu &= e_\mu^a\gamma_a + i f_\mu^a\gamma_a\gamma_5 \\ (e_\mu^a)^\dagger &= e_\mu^a \qquad (f_\mu^a)^\dagger = f_\mu^a, \end{aligned} \tag{2.25}$$

and X_μ

$$X_\mu = \hat{p}_\mu + \omega_\mu^1 + \omega_\mu^5\gamma_5 + i\omega_\mu^{ab}\gamma_{ab}, \tag{2.26}$$

where all the components are hermitian.

In general the $U(2,2)$ matrix model is built on the gauge invariance

$$
\begin{aligned}
X_\mu &\rightarrow U^{-1}X_\mu U \\
Y_\mu &\rightarrow U^{-1}Y_\mu U,
\end{aligned}
\tag{2.27}
$$

where U is a gauge transformation of $U(1,1)\otimes U(1,1)$. The gauge transformations of this group are again defined from the generators $1, \gamma_5, \gamma_{ab}$ and obey the condition

$$
\begin{aligned}
U^\dagger \Gamma_0 U = \Gamma_0 \\
U \Gamma_0 U^\dagger = \Gamma_0.
\end{aligned}
\tag{2.28}
$$

This reality condition assures that the matrix X_μ, once that it is gauge transformed, respects again the hermitian condition (2.24).

By defining $U = \exp[\Lambda]$ it follows that

$$
\Lambda^\dagger = \Gamma_0 \Lambda \Gamma_0. \tag{2.29}
$$

The matrix Λ can be developed in terms of the basic parameters:

$$
\Lambda = i\Lambda_0 + i\Lambda_5\gamma_5 + \Lambda^{ab}\gamma_{ab} \tag{2.30}
$$

with all hermitian components.

Finally let us discuss the decomposition in $U(2)_{\text{ left}}$ and $U(2)_{\text{ right}}$ of the Euclidean $U(2)\otimes U(2)$ gauge transformations. This decomposition is obtained by requiring that the generators of $U(2)_{\text{ left}}$ are given by

$$
(1, \gamma_{ab})\left(\frac{1+\gamma_5}{2}\right). \tag{2.31}
$$

The application of the projector operator $\frac{1+\gamma_5}{2}$ on γ_{ab} reduces the number of generators from six to three. To see this property in detail we recall the identity:

$$
\gamma_5\gamma_{ab} = -\frac{1}{2}\epsilon_{abcd}\gamma_{cd}. \tag{2.32}
$$

Therefore the generators of $SU(2)_{\text{ left}}$ read

$$
M_{ab} = \frac{1}{2}[\gamma_{ab} - \frac{1}{2}\epsilon_{abcd}\gamma_{cd}]. \tag{2.33}
$$

Since $M_{ab} = -\frac{1}{2}\epsilon_{abcd}M_{cd}$ is anti-self-dual, the only independent generators are three

$$M_{0i} = \frac{1}{2}[\gamma_{0i} - \frac{1}{2}\epsilon_{ijk}\gamma_{jk}] = -i\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.34}$$

Analogously the generators of $U(2)_{\text{ right}}$ are given by

$$(1, \gamma_{ab})\left(\frac{1-\gamma_5}{2}\right), \tag{2.35}$$

where

$$N_{ab} = \frac{1-\gamma_5}{2}\gamma_{ab} = \frac{1}{2}[\gamma_{ab} + \frac{1}{2}\epsilon_{abcd}\gamma_{cd}] \tag{2.36}$$

is self-dual

$$N_{ab} = \frac{1}{2}\epsilon_{abcd}N_{cd}. \tag{2.37}$$

Therefore the only independent generators are

$$N_{0i} = \frac{1}{2}[\gamma_{oi} + \frac{1}{2}\epsilon_{ijk}\gamma_{jk}] = i\begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}. \tag{2.38}$$

What happens in the $U(1,1) \otimes U(1,1)$ case ?

The generators of $U(1,1)_{\text{ left}}$ are given by

$$(1, \gamma_{ab})\frac{1+\gamma_5}{2}, \tag{2.39}$$

where now the identity (2.32) reads $\gamma_5\gamma_{ab} = -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}$.

One can always define the generators of $U(1,1)_{\text{ left}}$ and $U(1,1)_{\text{ right}}$ as

$$\begin{aligned} M_{ab} &= \frac{1}{2}[\gamma_{ab} - \frac{1}{2}\epsilon_{abcd}\gamma^{cd}] \\ N_{ab} &= \frac{1}{2}[\gamma_{ab} + \frac{1}{2}\epsilon_{abcd}\gamma^{cd}], \end{aligned} \tag{2.40}$$

where now for $su(1,1)_{\text{ left}}$

$$
\begin{aligned}
M_{01} &= \frac{1}{2}[\gamma_{01} - \gamma_{23}] = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \\
M_{02} &= \frac{1}{2}[\gamma_{02} + \gamma_{31}] = \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \\
M_{03} &= \frac{1}{2}[\gamma_{03} + \gamma_{12}] = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
\tag{2.41}
$$

and for $su(1,1)_{\text{ right}}$

$$
\begin{aligned}
N_{01} &= \frac{1}{2}[\gamma_{01} + \gamma_{23}] = i \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix} \\
N_{02} &= \frac{1}{2}[\gamma_{02} - \gamma_{31}] = - \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \\
N_{03} &= \frac{1}{2}[\gamma_{03} - \gamma_{12}] = - \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}.
\end{aligned}
\tag{2.42}
$$

Let us define the chiral decomposition of the gauge transformation as

$$
U = \frac{1+\gamma_5}{2} U_L + \frac{1-\gamma_5}{2} U_R. \tag{2.43}
$$

The left part U_L must obey the hermitian condition

$$
\begin{aligned}
U_L^\dagger \frac{1+\gamma_5}{2} \Gamma_0 U_L &= \frac{1+\gamma_5}{2} \Gamma_0 \\
\frac{1+\gamma_5}{2} \Gamma_0 = i\sigma_1 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.44}
$$

Therefore reducing eq. (2.44) to the upper 2×2 subspace, we find

$$
U_L^\dagger \sigma_1 U_L = \sigma_1 \tag{2.45}
$$

and analogously for U_R

$$
U_R^\dagger \sigma_1 U_R = \sigma_1, \tag{2.46}
$$

which are completely equivalent to the usual $U(1,1)$ condition

$$U^{\dagger}\sigma_3 U = \sigma_3. \tag{2.47}$$

By defining

$$U_{L,R} = \exp[\Lambda_{L,R}] \qquad \Lambda^{\dagger}_{L,R} = -\sigma_1\Lambda_{L,R}\sigma_1, \tag{2.48}$$

it follows that Λ can be expanded as

$$\Lambda = i\Lambda_0 + i\Lambda_1\sigma_1 + \Lambda_2\sigma_2 + \Lambda_3\sigma_3, \tag{2.49}$$

where all the components are hermitian.

3 Equations of motions

Let us discuss the equations of motion of the matrix model. Since there are two independent matrices, we need to vary the action with respect to δX_μ and δY_μ independently, therefore obtaining two types of equations of motion.

The first one is due to the variation with respect to δX_μ

$$\delta S = Tr[\gamma_5 Y_\mu Y_\nu(\delta X_\rho X_\sigma + X_\rho \delta X_\sigma)\epsilon^{\mu\nu\rho\sigma}] \tag{3.1}$$

that is vanishing if the following tensor is null

$$T_{\mu\nu} = [X_\mu, Y_\nu] - [X_\nu, Y_\mu] = 0 \tag{3.2}$$

i.e. it is equivalent to the condition of null torsion.

The other equation of motion is obtained by varying with respect to δY_μ

$$\delta S = Tr[\gamma_5\epsilon^{\mu\nu\rho\sigma}(\delta Y_\mu Y_\nu + Y_\mu \delta Y_\nu)R_{\rho\sigma}] \tag{3.3}$$

where

$$R_{\mu\nu} = [X_\mu, X_\nu] - i\theta^{-1}_{\mu\nu} \tag{3.4}$$

and it is vanishing if the following condition is met

$$\epsilon^{\mu\nu\rho\sigma}\{Y_\nu, R_{\rho\sigma}\} = 0. \tag{3.5}$$

The two equations of motion (3.2) and (3.5) are not completely independent, since it exists, at the noncommutative level, an identity similar to the covariant conservation of the tensor $G_{\mu\nu}$

$$D_\mu G^{\mu\nu} = 0 \qquad G^{\mu\nu} = R_E^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_E \tag{3.6}$$

where we have distinguished $R_E^{\mu\nu}$, the Einstein Ricci tensor, from the antisymmetric matric $R_{\mu\nu}$ introduced before.

In fact, let us evaluate

$$\epsilon^{\mu\nu\rho\sigma}[X_\mu, \{Y_\nu, R_{\rho\sigma}\}] = 0. \tag{3.7}$$

We will prove that it corresponds to a trivial identity. It is enough to observe that (3.7) can be decomposed into a sum of terms which are zero

$$\epsilon^{\mu\nu\rho\sigma}[X_\mu, Y_\nu] = 0 \tag{3.8}$$

because of the null torsion condition, and

$$\epsilon^{\mu\nu\rho\sigma}[X_\mu, R_{\rho\sigma}] = 0 \tag{3.9}$$

because of the Jacobi identity.

Let us write the equations of motion (3.2) and (3.5) in components, to recognize the usual form of the Einstein equations in the commutative limit.

By introducing the parameterizations (2.8) and (2.13) of the Euclidean case we find that $T_{\mu\nu} = 0$ is equivalent to

$$\begin{aligned}
&[\hat{p}_\mu, e^a_\nu] + [\omega^1_\mu, e^a_\nu] - i\{\omega^5_\mu, f^a_\nu\} + 2i\{\omega^{ab}_\mu, e^b_\nu\} - \epsilon_{abcd}[\omega^{bc}_\mu, f^d_\nu] \\
&= (\mu \leftrightarrow \nu) \\
&[\hat{p}_\mu, f^a_\nu] + [\omega^1_\mu, f^a_\nu] + i\{\omega^5_\mu, e^a_\nu\} + 2i\{\omega^{ab}_\mu, f^b_\nu\} + \epsilon_{abcd}[\omega^{bc}_\mu, e^d_\nu] \\
&= (\mu \leftrightarrow \nu).
\end{aligned} \tag{3.10}$$

Therefore it is not possible to make the two vierbeins e^a_μ and f^a_μ proportional, because of the term proportional to ω^5_μ and ϵ_{abcd}.

These two equations relate all the components of the spin connection $\omega^1_\mu, \omega^5_\mu, \omega^{ab}_\mu$ in terms of the generic vierbeins (e^a_μ, f^a_μ), treating them independently.

To help intuition, it is possible to restrict the general equations of motion (3.10) such that the symbol corresponding to the first operator equation contains only even powers of θ, while the symbol corresponding to the second equation contains only odd powers of θ (we refer to Ref. [9] for a detailed discussion on this point).

This reduction requires that the symbols of the operators e^a_μ, ω^{ab}_μ have an expansion in θ with only even powers, while the symbols of the operators $f^a_\mu, \omega^1_\mu, \omega^5_\mu$ have only odd powers. In this scenario, the usual free torsion condition for e^a_μ is recovered in the commutative limit since

$$[\omega^1_\mu, e^a_\nu] \sim \{\omega^5_\mu, f^a_\nu\} \sim [\omega^{bc}_\mu, f^a_\nu] \sim O(\theta^2), \tag{3.11}$$

taking into account that the commutator of operators, corresponding to the antisymmetric part of the star product, gives another odd contribution to the powers of θ, while the anticommutator is even.

Let us analyze in the same scenario the other equation of motion (3.5) which should give rise to the usual Einstein equations. Firstly let us compute the components of $R_{\mu\nu}$ as follows

$$\begin{aligned}
R_{\mu\nu} &= [X_\mu, X_\nu] - i\theta^{-1}_{\mu\nu} = \\
&= R^1_{\mu\nu} + R^5_{\mu\nu}\gamma^5 + R^{ab}_{\mu\nu}\gamma_{ab} \\
R^1_{\mu\nu} &= [\hat{p}_\mu, \omega^1_\nu] - [\hat{p}_\nu, \omega^1_\mu] + [\omega^1_\mu, \omega^1_\nu] + [\omega^5_\mu, \omega^5_\nu] \\
&= -2[\omega^{ab}_\mu, \omega^{ba}_\nu] \\
R^5_{\mu\nu} &= [\hat{p}_\mu, \omega^5_\nu] - [\hat{p}_\nu, \omega^5_\mu] + [\omega^1_\mu, \omega^5_\nu] - [\omega^1_\nu, \omega^5_\mu] \\
&= -\epsilon_{abcd}[\omega^{ab}_\mu, \omega^{cd}_\nu] \\
R^{ab}_{\mu\nu} &= i[\hat{p}_\mu, \omega^{ab}_\nu] - i[\hat{p}_\nu, \omega^{ab}_\mu] + i[\omega^1_\mu, \omega^{ab}_\nu] - i[\omega^1_\nu, \omega^{ab}_\mu] \\
&- \frac{i}{2}\epsilon_{abcd}([\omega^5_\mu, \omega^{cd}_\nu] - [\omega^5_\nu, \omega^{cd}_\mu]) + 2\{\omega^{bc}_\mu \omega^{ca}_\nu\} - 2\{\omega^{ac}_\mu \omega^{cb}_\nu\}.
\end{aligned} \tag{3.12}$$

Following the same reasoning done for the vierbein (from now on the distinction between odd and even powers of θ is always intended true for the symbols of the corresponding operators), it is possible to restrict $R_{\mu\nu}$ such that $R^1_{\mu\nu}, R^5_{\mu\nu}$ contain only odd powers of θ, while $R^{ab}_{\mu\nu}$ contains only even powers of θ, and therefore in the commutative limit only $R^{ab}_{\mu\nu}$ survives.

Then we are ready to compute the equations of motion (3.5):

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}[\{e^a_\nu, R^1_{\rho\sigma}\} + i[f^a_\nu, R^5_{\rho\sigma}] + 2[e^b_\nu, R^{ba}_{\rho\sigma}] + i\epsilon^{abcd}\{f^d_\nu, R^{bc}_{\rho\sigma}\}] = 0 \\ \epsilon^{\mu\nu\rho\sigma}[\{f^a_\nu, R^1_{\rho\sigma}\} - i[e^a_\nu, R^5_{\rho\sigma}] + 2[f^b_\nu, R^{ba}_{\rho\sigma}] - i\epsilon^{abcd}\{e^d_\nu, R^{bc}_{\rho\sigma}\}] = 0.\end{aligned} \tag{3.13}$$

Again it is not possible to make the two vierbeins proportional because of the odd terms $R^5_{\rho\sigma}$ and ϵ^{abcd}.

While the first equation of (3.13) can be restricted to odd powers of θ, the second one can contain only even powers of θ. Since

$$\{f^a_\nu, R^1_{\rho\sigma}\} \sim [e^a_\nu, R^5_{\rho\sigma}] \sim [f^b_\nu, R^{ba}_{\rho\sigma}] \sim O(\theta^2) \tag{3.14}$$

the only surviving term in the commutative limit is

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\{e^d_\nu, R^{bc}_{\rho\sigma}\} = 0 \tag{3.15}$$

which is in fact completely equivalent to the usual Einstein equations, where

$$R^{bc}_{\rho\sigma} \sim \partial_\rho\omega^{bc}_\sigma - \partial_\sigma\omega^{bc}_\rho + 2\{\omega^{cd}_\rho, \omega^{db}_\sigma\} - 2\{\omega^{bd}_\rho, \omega^{dc}_\sigma\} \tag{3.16}$$

is the usual antisymmetric classical tensor.

4 Gauge transformations of Y_μ and X_μ

Before analyzing the gauge transformations of Y_μ and X_μ, we recall that the distinction between odd and even powers of θ must be made also at a level of the gauge group $U(2)\otimes U(2)$, to be consistent.

Recalling the results of Ref. [12], it is possible to reduce the gauge symmetry from $U(2)\otimes U(2)$ to $SO(4)_*$, where the gauge parameter Λ

$$U = \exp[\Lambda] \qquad \Lambda = i\Lambda_0 + i\Lambda_5\gamma^5 + \Lambda^{ab}\gamma_{ab} \tag{4.1}$$

has the following property, i.e. Λ_0, Λ_5 contains only odd powers of θ, while Λ^{ab} contains only even powers of θ. This is a group property in the sense that the commutator of two gauge parameters has the same distinction.

We are now ready to analyze the gauge transformations of Y_μ and X_μ. The vierbein Y_μ transforms under $U(2) \otimes U(2)$ following the law

$$Y_\mu \to U^\dagger Y_\mu U. \tag{4.2}$$

At an infinitesimal level, it transforms as

$$\delta Y_\mu = [Y_\mu, \Lambda]. \tag{4.3}$$

By introducing the decomposition of Y_μ and Λ in terms of basic components we find

$$\begin{aligned} \delta e_\mu^a &= i[e_\mu^a, \Lambda_0] + 2\{e_\mu^b, \Lambda^{ba}\} - \{f_\mu^a, \Lambda_5\} + i\epsilon_{abcd}[f_\mu^d, \Lambda^{bc}] \\ \delta f_\mu^a &= i[f_\mu^a, \Lambda_0] + 2\{f_\mu^b, \Lambda^{ba}\} + \{e_\mu^a, \Lambda_5\} - i\epsilon_{abcd}[e_\mu^d, \Lambda^{bc}]. \end{aligned} \tag{4.4}$$

A rapid check shows that the first equation can be reduced to contain, in the case of $SO(4)_*$ transformations, only even powers of θ, while the second equation only odd powers of θ, taking always into account the additional odd contribution coming from the commutator of two operators.

In the commutative limit, since the terms

$$[e_\mu^a, \Lambda_0] \sim \{f_\mu^a, \Lambda_5\} \sim \epsilon_{abcd}[f_\mu^d, \Lambda^{bc}] \sim O(\theta^2), \tag{4.5}$$

the usual transformation of the vierbein under local Lorentz transformations is recovered

$$\delta e_\mu^a = 2\{e_\mu^b, \Lambda^{ba}\}. \tag{4.6}$$

The vierbein transformations are diagonal at a level of the combination $e_\mu^{\pm a}$, as already anticipated

$$\delta e_\mu^{\pm a} = i[e_\mu^{\pm a}, \Lambda_0] + 2\{e_\mu^{\pm b}, \Lambda^{ba}\} \pm i\{e_\mu^{\pm a}, \Lambda_5\} \pm i\epsilon_{abcd}[e_\mu^{\pm d}, \Lambda^{bc}]. \tag{4.7}$$

Analogously the spin connection transforms under $U(2) \otimes U(2)$ according to the law

$$X_\mu \to U^\dagger X_\mu U \tag{4.8}$$

and at an infinitesimal level

$$\delta X_\mu = [X_\mu, \Lambda]. \tag{4.9}$$

In components this equation reads

$$\begin{aligned}
\delta\omega^1_\mu &= i[\hat{p}_\mu, \Lambda_0] + i[\omega^1_\mu, \Lambda_0] + i[\omega^5_\mu, \Lambda_5] + 2i[\omega^{ab}_\mu, \Lambda^{ba}] \\
\delta\omega^5_\mu &= i[\hat{p}_\mu, \Lambda_5] + i[\omega^1_\mu, \Lambda_5] + i[\omega^5_\mu, \Lambda_0] + i\epsilon_{abcd}[\omega^{ab}_\mu, \Lambda^{cd}] \\
\delta\omega^{ab}_\mu &= -i[\hat{p}_\mu, \Lambda^{ab}] - i[\omega^1_\mu, \Lambda^{ab}] + \frac{i}{2}\epsilon_{abcd}[\omega^5_\mu, \Lambda^{cd}] + i[\omega^{ab}_\mu, \Lambda_0] \\
&- \frac{i}{2}\epsilon_{abcd}[\omega^{cd}_\mu, \Lambda_5] + 2\{\omega^{ac}_\mu, \Lambda^{cb}\} - 2\{\omega^{bc}_\mu, \Lambda^{ca}\}.
\end{aligned} \tag{4.10}$$

The same considerations made for the vierbein apply here, i.e. the spin connections ω^1_μ and ω^5_μ, restricted to odd powers of θ, and ω^{ab}_μ restricted to even powers of θ, maintain this property if the gauge parameters belong to $SO(4)_*$.

In the classical limit we notice that the only terms which survive are all the expected ones:

$$\delta\omega^{ab}_\mu = -i[\hat{p}_\mu, \Lambda^{ab}] + 2\{\omega^{ac}_\mu, \Lambda^{cb}\} - 2\{\omega^{bc}_\mu, \Lambda^{ca}\}. \tag{4.11}$$

These properties can also be made more clear and transparent by using the chiral decomposition for X_μ:

$$X_\mu = X^L_\mu \left(\frac{1+\gamma_5}{2}\right) + X^R_\mu \left(\frac{1-\gamma_5}{2}\right) \tag{4.12}$$

where

$$X^L_\mu = \hat{p}_\mu + (\omega^1_\mu + \omega^5_\mu) + i\omega^{ab}_\mu \gamma_{ab} \left(\frac{1+\gamma_5}{2}\right). \tag{4.13}$$

By using the property

$$\gamma_{ab}\left(\frac{1+\gamma_5}{2}\right) = \frac{1}{2}[\gamma_{ab} - \frac{1}{2}\epsilon_{abcd}\gamma_{cd}] \tag{4.14}$$

the $su(2)_{\text{ left}}$ part can be simplified to

$$i\omega_\mu^{ab}\gamma_{ab}\left(\frac{1+\gamma^5}{2}\right) = \frac{i}{2}(\omega_\mu^{ab} - \frac{1}{2}\omega_\mu^{cd}\epsilon_{abcd})\gamma_{ab}\left(\frac{1+\gamma^5}{2}\right)$$
$$= i\tilde{\omega}_\mu^{L\ ab}\gamma_{ab}\left(\frac{1+\gamma^5}{2}\right). \tag{4.15}$$

It is not difficult to show that $\tilde{\omega}_\mu^{L\ ab}$ is an anti-self-dual gauge connection:

$$\tilde{\omega}_\mu^{L\ ab} = \frac{1}{2}(\omega_\mu^{ab} - \frac{1}{2}\omega_\mu^{cd}\epsilon_{abcd})$$
$$\tilde{\omega}_\mu^{L\ ab} = -\frac{1}{2}\epsilon^{abcd}\tilde{\omega}^L_{\mu\ cd}. \tag{4.16}$$

Analogously X_μ^R is made by a self-dual gauge connection:

$$\begin{aligned} X_\mu^R &= \hat{p}_\mu + (\omega_\mu^1 - \omega_\mu^5) + i\tilde{\omega}_\mu^{R\ ab}\gamma_{ab}\left(\frac{1-\gamma_5}{2}\right) \\ \tilde{\omega}_\mu^{R\ ab} &= \frac{1}{2}(\omega_\mu^{ab} + \frac{1}{2}\omega_\mu^{cd}\epsilon_{abcd}) \\ \tilde{\omega}_\mu^{R\ ab} &= \frac{1}{2}\epsilon^{abcd}\tilde{\omega}^R_{\mu\ cd}. \end{aligned} \tag{4.17}$$

Reducing the gauge group to a pure left part by taking:

$$U = U_L\left(\frac{1+\gamma_5}{2}\right) + \left(\frac{1-\gamma_5}{2}\right) \qquad U_R = 1 \tag{4.18}$$

then only the left part of the spin connection changes according to

$$\begin{aligned} X_\mu^L &\rightarrow U_L^\dagger X_\mu^L U_L \\ X_\mu^R &\rightarrow X_\mu^R. \end{aligned} \tag{4.19}$$

The implications of this observation are interesting, since applying a pure left unitary transformation to the background connection $X_\mu = \hat{p}_\mu$ one constructs a (pure gauge) anti-self-dual spin connection, while X_μ^R remains pure background.

Decomposing the gauge parameter Λ into chiral components leads to:

$$\Lambda = \Lambda^L \left(\frac{1+\gamma_5}{2}\right) + \Lambda^R \left(\frac{1-\gamma_5}{2}\right) \tag{4.20}$$

where

$$\begin{aligned}
\Lambda^L &= i\Lambda_L^0 + \tilde{\Lambda}_L^{ab}\gamma_{ab}\left(\frac{1+\gamma_5}{2}\right) \qquad \tilde{\Lambda}_L^{ab} = \frac{1}{2}(\Lambda^{ab} - \frac{1}{2}\epsilon^{abcd}\Lambda_{cd}) \\
\Lambda^R &= i\Lambda_R^0 + \tilde{\Lambda}_R^{ab}\gamma_{ab}\left(\frac{1-\gamma_5}{2}\right) \qquad \tilde{\Lambda}_R^{ab} = \frac{1}{2}(\Lambda^{ab} + \frac{1}{2}\epsilon^{abcd}\Lambda_{cd}) \\
\Lambda_L^0 &= \Lambda_0 + \Lambda_5 \qquad \Lambda_R^0 = \Lambda_0 - \Lambda_5.
\end{aligned} \tag{4.21}$$

Defining $\omega_\mu^L = \omega_\mu^1 + \omega_\mu^5$, $\omega_\mu^R = \omega_\mu^1 - \omega_\mu^5$, X_μ^L can be rewritten as:

$$X_\mu^L = \hat{p}_\mu + \omega_\mu^L + i\tilde{\omega}_\mu^{ab}\gamma_{ab}\left(\frac{1+\gamma_5}{2}\right) \tag{4.22}$$

and the gauge property of X_μ^L now reads:

$$\begin{aligned}
\delta X_\mu^L &= [X_\mu^L, \Lambda^L] \\
\delta\omega_\mu^L &= i[\hat{p}_\mu, \Lambda_L^0] + i[\omega_\mu^L, \Lambda_L^0] + 4i[\tilde{\omega}_\mu^{L\ ab}, \tilde{\Lambda}_L^{ba}] \\
\delta\tilde{\omega}_\mu^{L\ ab} &= -i[\hat{p}_\mu, \tilde{\Lambda}^{L\ ab}] - i[\omega_\mu^L, \tilde{\Lambda}^{L\ ab}] + i[\tilde{\omega}_\mu^{L\ ab}, \Lambda_L^0] \\
&+ 2\{\tilde{\omega}_\mu^{L\ ac}, \tilde{\Lambda}_L^{cb}\} - 2\{\tilde{\omega}_\mu^{L\ bc}, \tilde{\Lambda}_L^{ca}\}.
\end{aligned} \tag{4.23}$$

In the classical limit the anti-self-dual spin connection $\tilde{\omega}_\mu^{L\ ab}$ transforms under the anti-self-dual gauge parameter $\tilde{\Lambda}_L^{ab}$.

5 Definition of the metric

It has been already pointed out in the literature that the metric is given by the star product of two vierbeins and it is no more symmetric, but these observations are not conclusive in my opinion. There is one more difficulty to set up a consistent second order formalism for noncommutative gravity, i.e. that the metric tensor is not even invariant under the gauge group $U(2) \otimes U(2)$, on which the model is defined.

The only way out to this further obstacle is to allow for a more general definition of metric, as a bilinear combination of the vierbein which is at least covariant under $U(2) \otimes U(2)$, i.e. a representation of the basic gauge group of the theory.

We have in fact at disposition the bilinear form

$$Y_\mu Y_\nu = G_{\mu\nu} + iB_{\mu\nu} \quad G_{\mu\nu} = \frac{1}{2}\{Y_\mu, Y_\nu\} \quad B_{\mu\nu} = -\frac{i}{2}[Y_\mu, Y_\nu] \tag{5.1}$$

which transforms in a covariant way under $U(2) \otimes U(2)$, having the same transformation properties of the spin connection X_μ (apart from the presence of two world indices instead of one).

Decomposing (5.1) into chiral parts one finds:

$$Y_\mu Y_\nu = Y_\mu^+ Y_\nu^- \left(\frac{1-\gamma_5}{2}\right) + Y_\mu^- Y_\nu^+ \left(\frac{1+\gamma_5}{2}\right). \tag{5.2}$$

Therefore we conclude that it is not possible to define a covariant metric tensor with only one vierbein e_μ^{+a}, but it is necessary the presence of both vierbeins.

Expanding the bilinear form (5.1) into components one finds:

$$Y_\mu Y_\nu = Y^0_{\mu\nu} + Y^5_{\mu\nu}\gamma_5 + Y^{ab}_{\mu\nu}\gamma_{ab} \tag{5.3}$$

with

$$\begin{aligned} Y^0_{\mu\nu} &= \eta_{ab}(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu) \\ Y^5_{\mu\nu} &= i\eta_{ab}(e^a_\mu f^b_\nu - f^a_\mu e^b_\nu) \\ Y^{ab}_{\mu\nu} &= \frac{1}{2}(e_\mu^{[\,a} e_\nu^{b\,]} - f_\mu^{[\,a} f_\nu^{b\,]}) - \frac{i}{2}\epsilon_{abcd}(e^c_\mu f^d_\nu - f^c_\mu e^d_\nu) \end{aligned} \tag{5.4}$$

where the symbols between parenthesis [ ] mean that we must take the antisymmetric combination of the indices.

The tensor $G_{\mu\nu}$ is symmetric and hermitian and its components are given by

$$\begin{aligned} G_{\mu\nu} &= g^0_{\mu\nu} + g^5_{\mu\nu}\gamma_5 + g^{ab}_{\mu\nu}\gamma_{ab} = \\ &= \frac{1}{2}[(\{e^a_\mu, e^b_\nu\} + \{f^a_\mu, f^b_\nu\}\eta_{ab} + i([e^a_\mu, f^b_\nu] + [e^a_\nu, f^b_\nu])\eta_{ab}\gamma_5 \\ &+ ([e^a_\mu, e^b_\nu] + [f^a_\mu, f^b_\nu] - \frac{i}{2}\epsilon^{abcd}(\{e^c_\mu, f^d_\nu\} + \{e^c_\nu f^d_\mu\}))\gamma_{ab}]. \end{aligned} \tag{5.5}$$

The $g^0_{\mu\nu}$ and $g^5_{\mu\nu}$ parts can be restricted to contain only even powers of θ, while $g^{ab}_{\mu\nu}$ contains only odd powers of θ, and since obviously

$$g^5_{\mu\nu} \sim O(\theta^2) \tag{5.6}$$

the only part which survives the commutative limit is

$$g^0_{\mu\nu} = \frac{1}{2}\{e^a_\mu, e^b_\nu\}\eta_{ab}. \tag{5.7}$$

Under a gauge transformation, $G_{\mu\nu}$ transforms as

$$G_{\mu\nu} \to U^\dagger G_{\mu\nu} U \tag{5.8}$$

or at an infinitesimal level

$$\delta G_{\mu\nu} = [G_{\mu\nu}, \Lambda]. \tag{5.9}$$

In components one finds

$$\begin{aligned}
\delta g^0_{\mu\nu} &= i[g^0_{\mu\nu}, \Lambda_0] + i[g^5_{\mu\nu}, \Lambda^5] + 2i[g^{ab}_{\mu\nu}, \Lambda^{ba}] \\
\delta g^5_{\mu\nu} &= i[g^0_{\mu\nu}, \Lambda^5] + i[g^5_{\mu\nu}, \Lambda^0] + i\epsilon_{abcd}[g^{ab}_{\mu\nu}, \Lambda^{cd}] \\
\delta g^{ab}_{\mu\nu} &= -i[g^0_{\mu\nu}, \Lambda^{ab}] + \frac{i}{2}\epsilon_{abcd}[g^5_{\mu\nu}, \Lambda^{cd}] + i[g^{ab}_{\mu\nu}, \Lambda_0] \\
&- \frac{i}{2}\epsilon_{abcd}[g^{cd}_{\mu\nu}, \Lambda_5] + 2\{g^{ac}_{\mu\nu}, \Lambda^{cb}\} - 2\{g^{bc}_{\mu\nu}, \Lambda^{ca}\}.
\end{aligned} \tag{5.10}$$

It is clear that $\delta g^0_{\mu\nu} \sim O(\theta^2)$, therefore one reobtains that $g^0_{\mu\nu}$ is gauge invariant in the classical limit.

For the antisymmetric and hermitian part $B_{\mu\nu}$ one finds analogous gauge transformations properties, and in this case the components $B^0_{\mu\nu}$, $B^5_{\mu\nu}$ can be restricted to contain only odd powers of θ, while $B^{ab}_{\mu\nu}$ only even powers of θ. In the classical limit there is one term which survives

$$B^{ab}_{\mu\nu} = -\frac{i}{4}(\{e^a_\mu, e^b_\nu\} - (a \leftrightarrow b)) \tag{5.11}$$

and it transforms in a covariant way as we can see from the formula (5.10). This antisymmetric part however decouples from the Einstein equations, and it can be neglected.

In conclusion, in order to setup a consistent second order formalism we believe that it is necessary to include all these components into the game. Some results contained in [11] confirm indirectly this picture.

6 Gravitational instantons

Given a solution of the equations of motion (3.2) and (3.5), it is possible to generate another one which is not trivially connected to the first one by introducing a quasi-unitary operator, which in the case of $U(2) \otimes U(2)$ is of the type

$$\begin{aligned} &UU^\dagger = 1 \qquad U^\dagger U = 1 - P_0 \\ &X_\mu \to U^\dagger X_\mu U \qquad Y_\mu \to U^\dagger Y_\mu U \end{aligned} \tag{6.1}$$

where P_0 is a projector with values in $U(2) \otimes U(2)$.

In particular we can start from the vacuum, which is defined by the background of the matrix model:

$$\begin{aligned} X_\mu &= \hat{p}_\mu \\ Y_\mu &= \delta_\mu^a \gamma_a \end{aligned} \tag{6.2}$$

and compute the following transformations

$$\begin{aligned} X_\mu &= U^\dagger \hat{p}_\mu U \\ Y_\mu &= U^\dagger \delta_\mu^a \gamma_a U. \end{aligned} \tag{6.3}$$

These are automatically solutions to the equations of motion of the matrix model (3.2) and (3.5), due to the property $UU^\dagger = 1$, and give a finite contribution to the matrix model action since then:

$$[X_\mu, X_\nu] - i\theta^{-1}_{\mu\nu} = i\theta^{-1}_{\mu\nu} P_0 \tag{6.4}$$

this commutator is a projector, and the trace defining the action is projected on a finite number of states. We call this generic solution a noncommutative gravitational instanton.

Obviously one can introduce more structure into the game, by requiring that the noncommutative solutions have a smooth θ-limit and coincide in the commutative limit with some known and classified solution [13]-[17]. It is not the purpose of the present paper, however we believe that our definition can be adjusted to achieve all these goals. For the moment we limit ourself to indicate some general property of our finite action solutions of the equations of motion.

Firstly the projector P_0 can be decomposed into chiral parts

$$P_0 = P_0^L \left(\frac{1+\gamma_5}{2}\right) + P_0^R \left(\frac{1-\gamma_5}{2}\right) \tag{6.5}$$

where P_0^L and P_0^R are two independent $U(2)$-valued projectors. Let us note that in the particular case in which one of these two projectors is null, the corresponding quasi-unitary operator is of the form:

$$U = U_L \left(\frac{1+\gamma_5}{2}\right) + \left(\frac{1-\gamma_5}{2}\right) \quad \text{or} \quad U = \left(\frac{1+\gamma_5}{2}\right) + U_R \left(\frac{1-\gamma_5}{2}\right) \tag{6.6}$$

and it produces a nontrivial spin connection only for the left or right sector:

$$X_\mu = U_L^\dagger \hat{p}_\mu U_L \left(\frac{1+\gamma_5}{2}\right) + \hat{p}_\mu \left(\frac{1-\gamma_5}{2}\right) \tag{6.7}$$

or

$$X_\mu = \hat{p}_\mu \left(\frac{1+\gamma_5}{2}\right) + U_R^\dagger \hat{p}_\mu U_R \left(\frac{1-\gamma_5}{2}\right). \tag{6.8}$$

The corresponding spin connection is automatically anti-self-dual or self-dual, according to the choice $P_0^R = 0$ or $P_0^L = 0$.

Therefore solutions with self-dual or anti-self-dual spin connections are naturally implemented, although the class of solution defined by (6.5) is more general.

What is the generic form of the projector P_0^L with values in $U(2)_{\text{left}}$? We don't have the general proof but we believe that the more general solution is of the form

$$P_0^L = P_0^+ \left(\frac{1+\sigma_3}{2}\right) + P_0^- \left(\frac{1-\sigma_3}{2}\right) \tag{6.9}$$

where $P_0^\pm$ are two independent projectors of $U(1)$, apart from eventual isomorphisms of the Hilbert space, which are implemented by unitary transformations of the type

$$P_0 \to U^\dagger P_0 U \qquad U^\dagger U = UU^\dagger = 1. \tag{6.10}$$

We can therefore reduce the general problem of projectors with values in $U(2) \otimes U(2)$ to the $U(1)$ case.

This is also clear from the duality between $U(2)$ and $U(1)$ gauge groups on the noncommutative plane which can be described as follows.

Consider the Hilbert space created by the commutation rules of the coordinates and label it with a quantum number n. The correspondence between $U(2)$ and $U(1)$ is obtained by, firstly enlarging the one-oscillator basis to the $U(2)$ basis

$$|n; a> \ \ \forall n \in N \quad a = 0, 1 \tag{6.11}$$

and noting the isomorphism between Hilbert spaces:

$$\mathcal{H} \rightarrow \mathcal{H} \qquad |n : a> \rightarrow |2n + a> \qquad a = 0, 1. \tag{6.12}$$

Therefore we can relabel the tensorial product of the Hilbert space of one oscillator and the gauge group $U(2)$ with a new quantum number $n' = 2n + a$ that describes only oscillator states with $U(1)$ gauge group [24].

Once that the general $U(2) \otimes U(2)$ projector is reduced to a $U(1)$ projector, we need to build quasi-unitary operators with gauge group $U(1)$ in four dimensions.

In four dimensions the Hilbert space on which the quasi-unitary operator acts is (see Ref. [24]) generally the tensorial product of two Hilbert spaces of one oscillator

$$\mathcal{H} \times \mathcal{H} \quad |n_1, n_2> \quad \forall\, n_1, n_2 \in N. \tag{6.13}$$

We can therefore introduce a duality between four dimensions and two dimensions, by observing that a couple of numbers can be made isomorphic to a number, for example

$$(n_1, n_2) \rightarrow \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_2 \qquad \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \tag{6.14}$$

and therefore we can relabel the Hilbert space with only one quantum number.

Explicitly the construction of a general class of quasi-unitary operators with values in $U(1)$ in four dimensions follows these steps. Let us define two new quantum numbers

$$n = n_1 + n_2 \qquad k = n_2 \tag{6.15}$$

and let us introduce a short notation for the state:

$$|n_1, n_2 >\equiv |\frac{n(n+1)}{2} + k >= |n;k > . \qquad (6.16)$$

A basis of the two-dimensional Hilbert space is determined by the states

$$|n;k > \qquad 0 \leq k \leq n \qquad \forall n \in N. \qquad (6.17)$$

We must allow the continuation of the notation (6.17) to states with $k \geq n$ keeping in mind the following equivalence relation:

$$|n;k >= |n+1; k-n-1 > . \qquad (6.18)$$

In the two-dimensional basis, the generic finite projector operator P_0 can be represented in the following form, apart from an isomorphism of the Hilbert space,

$$P_0 = \sum_{i=0}^{m-1} |i >< i| \qquad (6.19)$$

that represents a configuration with instanton number m.

In the two-dimensional basis it is easy to derive the quasi-unitary operator U that produces the projector operator P_0:

$$\begin{aligned} UU^\dagger &= 1 \qquad U^\dagger U = 1 - P_0 \\ U &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} |n;k >< n; k+m| \\ U^\dagger &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} |n;k+m >< n;k|. \end{aligned} \qquad (6.20)$$

To derive the quasi-unitary operator in the equivalent basis $|n_1, n_2 >$ we must pullback the duality from the four-dimensional plane and the two-dimensional one. The problem is complicated in general, and it is simple to perform it only in the simplest case, with a configuration with instanton number m. Then the quasi-unitary operator can be reexpressed as:

$$U = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1+1, n_2 >< n_1, n_2+1|$$

$$
\begin{aligned}
&+ \sum_{n_1=n_2=0}^{\infty} |0, n_2 >< n_1 + 1, 0| \\
U^\dagger &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} |n_1, n_2 + 1 >< n_1 + 1, n_2| \\
&+ \sum_{n_1=n_2=0}^{\infty} |n_1 + 1, 0 >< 0, n_2|. \qquad (6.21)
\end{aligned}
$$

In summary, the construction of gravitational instantons can be derived as in the Yang-Mills case by introducing quasi-unitary operators with values in $U(2) \otimes U(2)$. Self-dual or anti-self-dual solutions can be achieved restricting the gauge group to its left part or right part only, being the general solution defined by eqs. (6.5) without any particular symmetry. The question of the smoothness of the classical limit remains to be investigated, as well as the (noncommutative) characterization of these solutions in terms of topological invariants.

7 Conclusions

In this paper we have attempted to define noncommutative gravity theory with a Matrix model approach. The noncommutative plane is taken as a background solution of the Matrix model, and the fluctuations are the vierbein and spin connections.

Two types of vierbein are needed to make the formalism consistent at the noncommutative level. It is not possible to make them proportional, but it is possible to restrict the equations of motion in such a way that one type of vierbein has only even powers of θ, and the other one only odd powers of θ, recovering in the classical limit the usual gravity theory with only one vierbein.

The spin connection has other two $U(1)$ parts, which however can be restricted to contain only odd powers of θ and therefore are negligible in the $\theta \to 0$ limit.

These properties can be respected by gauge transformations, if the gauge parameters are restricted to the $SO(4)_*$, the smallest subgroup of $U(2) \otimes U(2)$, consistent with the star product. The distinction between odd and even powers of θ is fruitful also in the discussion of the $\theta \to 0$ limit of the equations of motion, in which we recover the Einstein equations.

We have then attacked the problem of defining a consistent second order formalism. The metric tensor which is bilinear in the vierbein cannot be defined to be invariant under $U(2) \otimes U(2)$ or even $SO(4)_*$ in the noncommutative case, but only covariant. The need of a multiplet of metric tensors, each one having no direct physical meaning, is confirmed indirectly by the computations of ref. [11].

Finally we have attempted to give a definition of noncommutative gravitational instantons. We have introduced $U(2) \otimes U(2)$ valued quasi-unitary operators which generate, once that are applied to the background, nontrivial solutions of the equations of motion.

In the case of a pure left or right quasi-unitary operators, the corresponding solution has a self-dual or anti-self-dual spin connection, a property which defines the commutative gravitational instantons. Our class of finite action solutions is more general. We have then constructed explicit examples of quasi-unitary operators based on the concept of duality of Hilbert spaces, which is typical of the noncommutative plane.

Finally let us briefly mention the problems left; firstly the construction of a consistent second-order formalism, with the use of the multiplet of metric tensors outlined in this paper, secondly the careful analysis of the physical degrees of freedom of the metric tensor, and the cancellation of the unphysical ones and thirdly a more careful analysis of the noncommutative gravitational instantons and of their link with the commutative case. The last project will require to take control of the $\theta \to 0$ limit of the nonperturbative solutions constructed in this article, and the introduction of topological invariants for the noncommutative case. In any case we believe that the language of matrices, introduced in this paper, which is more familiar to a physicist than the star product formalism, will help in making progress in this research field.

A Appendix

The gamma matrices are known to generate the $U(4)$ and $U(2,2)$ algebra. In this appendix we recall some basic properties, like the basic representations and commutation properties which are used during this paper.

Firstly we repeat the $U(4)$ (Euclidean) case. We must solve the anticommutation relations

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab} \tag{A.1}$$

with the constraints $\gamma_a^\dagger = \gamma_a$. The solution to this requirements is (the so-called chiral representation)

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \tag{A.2}$$

where σ_i are the Pauli matrices. The properties

$$\gamma_5^2 = 1 \quad \gamma_5^\dagger = \gamma_5 \tag{A.3}$$

identify γ_5 as the combination

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

In the case of $U(2,2)$, we have to solve the anticommutation relations:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \qquad (++--) \tag{A.5}$$

with the constraints

$$\gamma_0^\dagger = \gamma_0 \quad \gamma_1^\dagger = \gamma_1 \quad \gamma_2^\dagger = -\gamma_2 \quad \gamma_3^\dagger = -\gamma_3. \tag{A.6}$$

A possible choice is , in the chiral representation

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{A.7}$$

Again γ_5 identified with the properties (A.3) can be chosen as in the Euclidean case (A.4).

The hermitian conjugation property can be encoded in the following property

$$\gamma_a^\dagger = \Gamma_0\gamma_a\Gamma_0 \tag{A.8}$$

where $\Gamma_0 = \gamma_0\gamma_1$.

The composition properties of gamma matrices can be summarized as follows:

$$\begin{aligned}
&[\gamma_a, \gamma_{bc}] = 2\eta_{ab}\gamma_c - 2\eta_{ac}\gamma_b \\
&\{\gamma_a, \gamma_{bc}\} = 2\epsilon_{abcd}\gamma_5\gamma^d \\
&[\gamma_{ab}, \gamma_{cd}] = 2(\eta_{ad}\gamma_{bc} + \eta_{bc}\gamma_{ad} - \eta_{ac}\gamma_{bd} - \eta_{bd}\gamma_{ac}) \\
&\{\gamma_{ab}, \gamma_{cd}\} = 2(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}) + 2\epsilon_{abcd}\gamma_5 \\
&\gamma_5\gamma_{ab} = -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}
\end{aligned} \tag{A.9}$$

where we raise the indices with the tensor η^{ab}. Of course in the Euclidean case the distinction between upper and lower indices is superfluous.

References

[1] A. H. Chamseddine, "Noncommutative gravity", hep-th/0301112.

[2] A. H. Chamseddine, "Invariant actions for noncommutative gravity", hep-th/0202137.

[3] A. H. Chamseddine, "Complex gravity and noncommutative geometry", hep-th/0010268, Int. J. Mod. Phys. **A16**, 759 (2001).

[4] A. H. Chamseddine, "Deforming Einstein's gravity", hep-th/0009153, Phys. Lett. **504**, 33 (2001).

[5] A. H. Chamseddine, "Complexified gravity in noncommutative spaces", hep-th/0005222, Comm. Math. Phys. **218**, 283 (2001).

[6] H. Garcia-Compean, O. Obregon, C. Ramirez, M. Sabido, "Noncommutative self-dual gravity", hep-th/0302180.

[7] H. Garcia-Compean, O. Obregon, C. Ramirez, M. Sabido, "Noncommutative topological theories of gravity", hep-th/0210203.

[8] S. Cacciatori, A. H. Chamseddine, D. Klemm, L. Martucci, W. Sabra, D. Zanon, "Noncommutative gravity in two dimensions", hep-th/0203038, Class. Quant. Grav. **19**, 4029 (2002).

[9] M.A. Cardella, D. Zanon, "Noncommutative deformation of four-dimensional Einstein gravity", hep-th/0212071.

[10] S. Cacciatori, D. Klemm, L.Martucci, W. Sabra, D. Zanon, "Noncommutative Einstein AdS Gravity in three dimensions", hep-th/0201103, Phys. Lett. **B356**, 101 (2002).

[11] M. Banados, O. Chandia, N. Grandi, F.A. Schaposnik and G.A. Silva, "Three-dimensional noncommutative gravity", hep-th/0104264, Phys. Rev.**D64**, 084012 (2001).

[12] L. Bonora, M. Schnabl, M.M. Sheikh-Jabbari and A. Tomasiello, " Noncommutative SO(N) and SP(N) gauge theories ", hep-th/0006091, Nucl. Phys. **B589**,461 (2000).

[13] T. Eguchi, P. B. Gilkey, A. J. Hanson, " Gravitation, gauge theories and differential geometry ", Phys. Reports **66** , 213 (1980).

[14] T. Eguchi, A. J. Hanson, " Asymptotically flat self-dual solutions to Euclidean gravity ", Phys. lett. **B74**, 249 (1978).

[15] T. Eguchi, A. J. Hanson, " Self-dual solutions to Euclidean gravity ", Ann. Phys. (N.Y.) **120**, 82 (1979).

[16] G. W. Gibbons and S. W. Hawking, " Gravitational Multi-instantons ', Phys. Lett. **B78**, 430 (1978).

[17] G. W. Gibbons and S. W. Hawking, " Classification of gravitational instantons symmetries ", Comm. Math. Phys. **66**, 291 (1979).

[18] N. Nekrasov, A. Schwarz, " Instantons on noncommutative R^4 and (2,0) superconformal six dimensional theory ", Comm. Math. Phys. **198** (1998), 689.

[19] A. Kapustin, A. Kutznetsov, D. Arlov, " Noncommutative instantons and twistor transform ", Comm. Math. Phys. **221** (2001) 385, hep-th/0002193.

[20] M. Hamanaka, " ADHM/Nahm construction of localized solitons in noncommutative gauge theories ", Phys. Rev. **D65** (2002) 085022, hep-th/0109070.

[21] K. Furuuchi, " Instantons on noncommutative R^4 and projection operators ", Prog. Theor. Phys. **103** (2000) 1043, hep-th/9912047.

[22] P. Krauss and M. Shigemori, " Noncommutative instantons and the Seiberg-Witten map ", JHEP **0206** (2002), 034, hep-th/0110035.

[23] P. Valtancoli, "Noncommutative instantons on $D = 2N$ planes from Matrix models", hep-th/0209118, to be published in Int. J. Mod. Phys. A.

[24] E. Kiritsis, C. Sochichiu, " Duality in noncommutative gauge theories as a nonperturbative Seiberg-Witten map ", hep-th/0202065.

[25] C. Sochichiu, " On the equivalence of noncommutative models in various dimensions ",hep-th/0007127, JHEP **0008**, 048 (2000).