Available online at www.sciencedirect.com

ScienceDirect

Journal of Pure and Applied Algebra 212 (2008) 1229–1251

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

On the Alexander–Hirschowitz theorem

Maria Chiara Brambilla[a], Giorgio Ottaviani[b,*]

[a] *Dipartimento di Matematica e Applicazioni per l'Architettura, Università di Firenze, piazza Ghiberti 27, 50122 Firenze, Italy*
[b] *Dipartimento di Matematica U. Dini, Università di Firenze, viale Morgagni 67/A, 50134 Firenze, Italy*

Received 15 January 2007; received in revised form 31 July 2007; accepted 21 September 2007

Communicated by A.V. Geramita

Abstract

The Alexander–Hirschowitz theorem says that a general collection of k double points in $\mathbf{P}^n$ imposes independent conditions on homogeneous polynomials of degree d with a well-known list of exceptions. Alexander and Hirschowitz completed its proof in 1995, solving a long standing classical problem, connected with the Waring problem for polynomials. We expose a self-contained proof based mainly on the previous works by Terracini, Hirschowitz, Alexander and Chandler, with a few simplifications. We claim originality only in the case $d = 3$, where our proof is shorter. We end with an account of the history of the work on this problem.

MSC: 01-02; 14C20; 15A72; 14M17

1. Introduction

The aim of this paper is to expose a proof of the following theorem.

Theorem 1.1 (*Alexander–Hirschowitz*). *Let X be a general collection of k double points in $\mathbf{P}^n = \mathbf{P}(V)$ (over an algebraically closed field of characteristic zero) and let $S^d V^\vee$ be the space of homogeneous polynomials of degree d. Let $I_X(d) \subseteq S^d V^\vee$ be the subspace of polynomials through X, that is with all first partial derivatives vanishing at the points of X. Then the subspace $I_X(d)$ has the expected codimension* $\min\left((n+1)k, \binom{n+d}{n}\right)$ *except in the following cases*

- $d = 2, 2 \le k \le n$;
- $n = 2, d = 4, k = 5$;
- $n = 3, d = 4, k = 9$;
- $n = 4, d = 3, k = 7$;
- $n = 4, d = 4, k = 14$.

* Corresponding author.
E-mail addresses: brambilla@math.unifi.it (M.C. Brambilla), ottavian@math.unifi.it (G. Ottaviani).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.09.014

We remark that the case $n = 1$ is the only one where the assumption that X is general is not necessary.

More information on the exceptional cases is contained in Section 3.

This theorem has an equivalent formulation in terms of higher secant varieties. Given a projective variety Y, the k-secant variety $\sigma_k(Y)$ is the Zariski closure of the union of all the linear spans $\langle p_1, \dots, p_k\rangle$ where $p_i \in Y$ (see [34] or [39]). In particular $\sigma_1(Y)$ coincides with Y and $\sigma_2(Y)$ is the usual secant variety. Consider the Veronese embedding $V^{d,n} \subset \mathbf{P}^m$ of degree d of $\mathbf{P}^n$, that is the image of the linear system given by all homogeneous polynomials of degree d, where $m = \binom{n+d}{n} - 1$. It is easy to check that $\dim \sigma_k(V^{d,n}) \le \min((n+1)k - 1, m)$ and when the equality holds we say that $\sigma_k(V^{d,n})$ has the expected dimension.

Theorem 1.2 (*Equivalent formulation of* Theorem 1.1). *The higher secant variety $\sigma_k(V^{d,n})$ has the expected dimension with the same exceptions of Theorem 1.1.*

Theorem 1.2 still holds if the characteristic of the base field $\mathbf{K}$ is bigger than d and $d > 2$ [23, Corollary I.62], but the case $\mathrm{char}(\mathbf{K}) = d$ is open as far as we know. The equivalence between Theorems 1.1 and 1.2 holds if $\mathrm{char}(\mathbf{K}) = 0$, and since we want to switch freely between the two formulations we work with this assumption. Let us mention that in [5] Theorem 1.1 is stated with the weaker assumption that $\mathbf{K}$ is infinite.

Since the general element in $\sigma_k(V^{d,n})$ can be expressed as the sum of k dth powers of linear forms, a consequence of Theorem 1.2 is that the general homogeneous polynomial of degree d in $n+1$ variables can be expressed as the sum of $\left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil$ dth powers of linear forms with the same list of exceptions (this is called the Waring problem for polynomials, see [23]).

In the case $n = 1$, the Veronese embedding $V^{d,1}$ is the rational normal curve and there are no exceptions at all. The case $n = 2$ was proved by Campbell [9], Palatini [30] and Terracini [37], see the historical Section 7. In [30] Palatini stated Theorem 1.1 as a plausible conjecture. In [36] Terracini proved his famous two "lemmas", which turned out to be crucial keys to solve the general problem. In 1931 Bronowski claimed to have a proof of Theorem 1.1, but his proof was fallacious. Finally the proof was found in 1995 by Alexander and Hirschowitz along a series of brilliant papers, culminating with [5] so that Theorem 1.1 is now called the Alexander–Hirschowitz theorem. They introduced the so called differential Horace's method to attack the problem. The proof was simplified in [6]. In 2001 Chandler achieved a further simplification in [12,13]. The higher multiplicity case is still open and it is a subject of active research, due to a striking conjecture named after Segre–Gimigliano–Harbourne–Hirschowitz, see [15] for a survey.

In 2006 we ran a seminar in Firenze trying to understand this problem. This note is a result of that seminar, and reflects the historical path that we have chosen. We are able to present a self-contained and detailed proof of the Alexander–Hirschowitz theorem, starting from scratch, with several simplifications on the road tracked by Terracini, Hirschowitz, Alexander and Chandler.

The reader already accustomed to this topic can skip Section 4 which is added only to clarify the problem and jump directly to Sections 5 and 6, which contain our original contributions (especially Section 5 about cubics, while in Section 6 we supplied [12] with more details).

The Veronese varieties are one of the few classes of varieties where the dimension of the higher secant varieties is completely known. See [10,11,26,1] for a related work on Segre and Grassmann varieties.

We thank all the participants of the seminar for their criticism, especially Luca Chiantini. We thank also Ciro Ciliberto for his remarks concerning the historical Section 7 and Edoardo Ballico for the helpful comments on the previous version of our paper.

2. Notation and Terracini's two lemmas

For any real number x, $\lfloor x \rfloor$ is the greatest integer smaller than or equal to x, $\lceil x \rceil$ is the smallest integer greater than or equal to x. Let V be a vector space of dimension $n+1$ over an algebraically closed field $\mathbf{K}$ of characteristic zero. Let $\mathbf{P}^n = \mathbf{P}(V)$ be the projective space of lines in V. If $f \in V \setminus \{0\}$ we denote by $[f]$ the line spanned by f and also the corresponding point in $\mathbf{P}(V)$. Let $S = \oplus_d S^d V$ be the symmetric algebra of V and $S^\vee = \oplus_d S^d V^\vee$ its dual. We have the natural pairing $S^d V \otimes S^d V^\vee \to \mathbf{K}$ which we denote by $(,)$. Then $S^d V^\vee$ is the space of homogeneous polynomials over $\mathbf{P}(V)$ and a polynomial $h \in S^d V^\vee$ vanishes at $[f] \in \mathbf{P}(V)$ if and only if $(f^d, h) = 0$. The *Veronese* *variety* $V^{d,n}$ is the image of the embedding $[v] \mapsto [v^d]$ of $\mathbf{P}(V)$ in $\mathbf{P}(S^d V) = \mathbf{P}^m$, where $m = \binom{n+d}{n} - 1$. If $f \in V$, it is easy to check that the projective tangent space $T_{[f^d]}V^{d,n} \subseteq \mathbf{P}(S^d V)$ is equal to $\{[f^{d-1}g] | g \in V\}$ (to see this, compute the Taylor expansion of $(f + \epsilon g)^d$ at $\epsilon = 0$).

The maximal ideal corresponding to $f \in V$ is

$$m_{[f]} := \{h \in S^\vee | h(f) = 0\}.$$

It contains all the hypersurfaces which pass through $[f]$. Its power $m^2_{[f]}$ contains all the hypersurfaces which are singular at $[f]$, it defines a scheme which is denoted as $[f]^2$ and it is called a *double point*. Note that a hypersurface is singular at $[f]$ if and only if it contains $[f]^2$.

In order to state the relation between the higher secant varieties to the Veronese varieties and the double points of hypersurfaces we need the following proposition, well known to Palatini and Terracini, usually attributed to Lasker [25], the Hilbert's student who proved the primary decomposition for ideals in polynomial rings and is widely known as chess world champion at the beginning of XX century.

Proposition 2.1 (*Lasker*). *Given $T_{[f^d]}V^{d,n} \subseteq \mathbf{P}(S^d V)$, its (projectivized) orthogonal $\left(T_{[f^d]}V^{d,n}\right)^\perp \subseteq \mathbf{P}(S^d V^\vee)$ consists of all the hypersurfaces singular at $[f]$. More precisely, if we denote by $C(V^{d,n})$ the affine cone over $V^{d,n}$, then the following holds*

$$\left(T_{f^d} C(V^{d,n})\right)^\perp = \left(m^2_{[f]}\right)_d \subseteq S^d V^\vee.$$

Proof. Let $e_0, \ldots, e_n$ be a basis of V and $x_0, \ldots, x_n$ its dual basis. Due to the $GL(V)$-action it is enough to check the statement for $f = e_0$. Then $m_{[f]} = (x_1, \ldots, x_n)$, $m^2_{[f]} = (x_1^2, x_1 x_2, \ldots, x_n^2)$, so that $(m^2_{[f]})_d$ is generated by all monomials of degree d with the exception of $x_0^d, x_0^{d-1}x_1, \ldots, x_0^{d-1}x_n$. Since $T_{e_0^d} C(V^{d,n}) = \langle e_0^d, e_0^{d-1}e_1, \ldots, e_0^{d-1}e_n\rangle$ the thesis follows. $\square$

Lemma 2.2 (*First Terracini Lemma*). *Let $p_1, \ldots, p_k \in Y$ be general points and $z \in \langle p_1, \ldots, p_k\rangle$ a general point. Then*

$$T_z \sigma_k(Y) = \langle T_{p_1}Y, \ldots, T_{p_k}Y\rangle.$$

Proof. Let $Y(\tau) = Y(\tau_1, \ldots, \tau_n)$ be a local parametrization of Y. We denote by $Y_j(\tau)$ the partial derivative with respect to τ_j. Let p_i be the point corresponding to $\tau^i = (\tau^i_1, \ldots, \tau^i_n)$. The space $\langle T_{p_1}Y, \ldots, T_{p_k}Y\rangle$ is spanned by the $k(n+1)$ rows of the following matrix

$$\begin{matrix} \vdots \\ Y(\tau^i) \\ Y_1(\tau^i) \\ \vdots \\ Y_n(\tau^i) \\ \vdots \end{matrix}$$

(here we write only the ith block of rows, $i = 1, \ldots, k$).

We write also the local parametrization of $\sigma_k(Y)$ given by

$$\Phi(\tau^1, \ldots, \tau^k, \lambda_1, \ldots, \lambda_{k-1}) = \sum_{i=1}^{k-1} \lambda_i Y(\tau^i) + Y(\tau^k)$$

depending on kn parameters τ^i_j and $k-1$ parameters λ_i. The matrix whose rows are given by Φ and its $kn+k-1$ partial derivatives computed at z is

$$\begin{matrix} \sum_{i=1}^{k-1} \lambda_i Y(\tau^i) + Y(\tau^k) \\ \vdots \\ \lambda_i Y_1(\tau^i) \\ \vdots \\ \lambda_i Y_n(\tau^i) \\ \vdots \\ Y_1(\tau^k) \\ \vdots \\ Y_n(\tau^k) \\ Y(\tau^1) \\ \vdots \\ Y(\tau^{k-1}) \end{matrix}$$

and its rows span $T_z\sigma_k(Y)$. It is elementary to check that the two above matrices are obtained one from the other by performing elementary operations on rows, hence they have the same row space and the same rank. $\square$

The Proposition 2.1 and Lemma 2.2 allow to prove the equivalence between Theorems 1.1 and 1.2. Indeed let $X = \{p_1^2, \ldots, p_k^2\}$ be a collection of double points in $\mathbf{P}^n$ and choose some representatives $v_i \in V$ such that $[v_i] = p_i$ for $i = 1, \ldots, k$. The subspace

$$I_X(d) = \bigcap_{i=1}^{k} \left[m^2_{p_i}\right]_d$$

is equal by Proposition 2.1 to

$$\bigcap_{i=1}^{k} \left(T_{v_i^d} C(V^{d,n})\right)^{\perp} = \left(\langle T_{v_1^d} C(V^{d,n}), \ldots, T_{v_k^d} C(V^{d,n})\rangle\right)^{\perp} \subseteq S^d V^{\vee}$$

so that its codimension is equal to the dimension of

$$\langle T_{v_1^d} C(V^{d,n}), \ldots, T_{v_k^d} C(V^{d,n})\rangle \subseteq S^d V$$

which in turn is equal to

$$\dim\langle T_{[v_1^d]} V^{d,n}, \ldots, T_{[v_k^d]} V^{d,n}\rangle + 1,$$

where we consider now the projective dimension. Summing up, by using Lemma 2.2, the genericity assumption on the points and the fact that $\sigma_k(V^{d,n})$ is an irreducible variety, we get

$$\operatorname{codim} I_X(d) = \dim \sigma_k(V^{d,n}) + 1$$

and the equivalence between Theorems 1.1 and 1.2 is evident from this equality.

We say that a collection X of double points *imposes independent conditions on* $\mathcal{O}_{\mathbf{P}^n}(d)$ if the codimension of $I_X(d)$ in $S^d V^{\vee}$ is $\min\left\{\binom{n+d}{n}, k(n+1)\right\}$. It always holds $\operatorname{codim} I_X(d) \leq \min\left\{\binom{n+d}{n}, k(n+1)\right\}$. Moreover if $\operatorname{codim} I_X(d) = k(n+1)$ and $X' \subset X$ is a collection of k' double points then $\operatorname{codim} I_{X'}(d) = k'(n+1)$. On the other hand if $\operatorname{codim} I_X(d) = \binom{n+d}{n}$ and $X'' \supset X$ is a collection of k'' double points then $\operatorname{codim} I_{X''}(d) = \binom{n+d}{n}$.

Lemma 2.3 (*Second Terracini Lemma*). *Let X be a union of double points supported on p_i, $i = 1, \dots, k$. We identify the points p_i with their images on $V^{d,n}$ according to the Veronese embedding. Assume that X does not impose independent conditions on hypersurfaces of degree d. Then there is a positive dimensional variety $C \subseteq V^{d,n}$ through $p_1, \dots, p_k$ such that if $p \in C$ then $T_p V^{d,n} \subseteq \langle T_{p_1} V^{d,n}, \dots, T_{p_k} V^{d,n}\rangle$. In particular, by Proposition 2.1, every hypersurface of degree d which is singular at p_i is also singular along C.*

Proof. Let z be a general point in $\langle p_1, \dots, p_k\rangle$. By Lemma 2.2 we have

$$T_z\sigma_k(V^{d,n}) = \langle T_{p_1} V^{d,n}, \dots, T_{p_k} V^{d,n}\rangle.$$

The secant variety $\sigma_k(V^{d,n})$ is obtained by projecting on the last factor the abstract secant variety $\sigma^k(V^{d,n}) \subseteq V^{d,n} \times \cdots \times V^{d,n} \times \mathbf{P}^m$ which is defined as follows

$$\sigma^k(V^{d,n}) := \overline{\{(q_1, \dots, q_k, z) | z \in \langle q_1, \dots, q_k\rangle, \dim\langle q_1, \dots, q_k\rangle = k-1\}}$$

and has dimension $nk + (k-1)$.

By assumption the dimension of $\sigma_k(V^{d,n})$ is smaller than expected. Then the fibers Q_z of the above projection have positive dimension and are invariant under permutations of the first k factors. Note that $(p_1, \dots, p_k) \in Q_z$ and moreover $z \in \langle q_1, \dots, q_k\rangle$ for all $(q_1, \dots, q_k) \in Q_z$ such that $\dim\langle q_1, \dots, q_k\rangle = k-1$. In particular for any such q_1 we have that $T_{q_1} V^{d,n} \subseteq \langle T_{p_1} V^{d,n}, \dots, T_{p_k} V^{d,n}\rangle$.

The image of Q_z on the first (or any) component is the variety C we looked for. □

Remark. It should be mentioned that Terracini proved also a bound on the linear span of C, for details see [14]. The proofs of the two Lemmas that we have exposed are taken from [36].

The first application given by Terracini is the following version of Theorem 1.1 in the case $n = 2$ (see also the historical Section 7).

Theorem 2.4. *A general union of double points $X \subseteq \mathbf{P}^2$ imposes independent conditions on plane curves of degree d with the only two exceptions*

$d = 2$, *X given by two double points;*
$d = 4$, *X given by five double points.*

Proof. We first check the statement for small values of d. It is elementary for $d \le 2$. Now, every cubic with two double points contains the line through these two points (by Bézout theorem), hence every cubic with three double points is the union of three lines. It follows easily that the statement is true for $d = 3$. For $d = 4$ remind that any quartic with four double points contains a conic through these points (indeed impose to the conic to pass through a further point and apply the Bézout theorem). Hence there is a unique quartic through five double points, which is the double conic.

Assume that a general union X of k double points does not impose independent conditions on plane curves of degree d. If F is a plane curve of degree d through X, then by Lemma 2.3 F contains a double curve of degree $2l$ through X. Hence we have the inequalities

$$2l \le d \qquad \text{and} \qquad k \le \frac{l(l+3)}{2}.$$

We may also assume

$$\left\lfloor \frac{1}{3}\binom{d+2}{2} \right\rfloor \le k$$

because the left-hand side is the maximum expected number of double points imposing independent conditions on plane curves of degree d, so that we get the inequality

$$\left\lfloor \frac{(d+2)(d+1)}{6} \right\rfloor \le \frac{d}{4}\left(\frac{d}{2}+3\right)$$

which gives $d \le 4$ (already considered) or $d = 6$. So the theorem is proved for any $d \ne 6$. In the case $d = 6$ the last inequality is an equality which forces $k = 9$. It remains to prove that the unique sextic which is singular at 9 general points is the double cubic through these points, which follows again by Lemma 2.3. □

3. The exceptional cases

Two double points do not impose independent conditions to the linear system of quadrics. Indeed the system of quadrics singular at two points consists of cones having the vertex containing the line joining the two points, which has projective dimension $\binom{n}{2} > \binom{n+2}{2} - 2(n+1)$. The same argument works for k general points, $2 \le k \le n$. In the border case $k = n$, the only surviving quadric is the double hyperplane through the n given points.

In terms of secant varieties, the varieties $\sigma_k(V^{2,n})$ can be identified with the varieties of symmetric matrices of rank $\le k$ of order $(n+1) \times (n+1)$, which have codimension $\binom{n-k+2}{2}$.

The cases $d = 4, 2 \le n \le 4, k = \binom{n+2}{2} - 1$ are exceptional because there is a (unique and smooth) quadric through the points, and the double quadric is a quartic singular at the given points, while $\binom{n+4}{4} \le (n+1)\left[\binom{n+2}{2} - 1\right]$ exactly for $2 \le n \le 4$.

The corresponding defective secant varieties $\sigma_k(V^{4,n})$ $\left(\text{with } k = \binom{n+2}{2} - 1\right)$ are hypersurfaces whose equation can be described as follows.

For any $\phi \in S^4 V$, let $A_\phi\colon S^2 V^\vee \to S^2 V$ be the contraction operator. It is easy to check that if $\phi \in V^{4,n}$ then $rkA_\phi = 1$ (by identifying the Veronese variety with its affine cone). It follows that if $\phi \in \sigma_k(V^{4,n})$ then $rkA_\phi \le k$. When $k = \binom{n+2}{2} - 1$ also the converse holds and $\det A_\phi = 0$ is the equation of the corresponding secant variety $\sigma_k(V^{4,n})$. When $n = 2$ the quartics in $\sigma_5(V^{4,2})$ are sum of five 4-powers of linear forms and they are called Clebsch quartics [17].

The case $n = 4, d = 3, k = 7$ is more subtle. In this case, since $\binom{7}{3} = 7 \cdot 5$, it is expected that no cubics exist with seven given singular points. But indeed through seven points there is a rational normal curve C_4, which, in a convenient system of coordinates, has equation

$$rk \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix} \le 1.$$

Its secant variety is the cubic with equation

$$\det \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix} = 0,$$

which is singular along the whole C_4. This is the same J invariant which describes harmonic 4-ples on the projective line. The paper [16] contains a readable proof of the uniqueness of the cubic singular along C_4.

Let us mention that in [31] Reichstein gives an algorithm to find if $f \in S^3(\mathbf{C}^5)$ belongs to the hypersurface $\sigma_7(V^{3,4})$. For the invariant equation of this hypersurface, which has degree 15, see [28].

4. Terracini's inductive argument

Terracini in [38] considers a union X of double points on $\mathbf{P}^3$ and studies the dimension of the system of hypersurfaces through X by specializing some of the points to a plane $\mathbf{P}^2 \subseteq \mathbf{P}^3$. This is the core of an inductive procedure which has been considered by several authors since then. The appealing fact of the inductive procedure is that it covers almost all the cases with a very simple argument. This is the point that we want to explain in this section. The remaining cases, which are left out because they do not fit the arithmetic of the problem, have to be considered with a clever degeneration argument, which we postpone to Section 6.

Let X be a union of k double points of $\mathbf{P}^n$, let $\mathcal{I}_X$ be the corresponding ideal sheaf and fix a hyperplane $H \subset \mathbf{P}^n$. The trace of X with respect to H is the scheme $X \cap H$ and the residual of X is the scheme $\widetilde{X}$ with ideal sheaf $\mathcal{I}_X : \mathcal{O}_{\mathbf{P}^n}(-H)$. In particular if we specialize $u \le k$ points on the hyperplane H, the trace $X \cap H$ is given by u double points of $\mathbf{P}^{n-1}$, and the residual $\widetilde{X}$ is given by $k - u$ double points and by u simple points.

Theorem 4.1. *Let X be a union of k double points of $\mathbf{P}^n$ and fix a hyperplane $H \subset \mathbf{P}^n$ containing u of them. Assume that $X \cap H$ does impose independent conditions on $\mathcal{O}_H(d)$ and the residual $\widetilde{X}$ does impose independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d-1)$. Assume moreover one of the following pair of inequalities:*

(i) $un \le \binom{d+n-1}{n-1}$ $\qquad k(n+1) - un \le \binom{d+n-1}{n}$,

(ii) $un \ge \binom{d+n-1}{n-1}$ $\qquad k(n+1) - un \ge \binom{d+n-1}{n}$.

Then X does impose independent conditions on the system $\mathcal{O}_{\mathbf{P}^n}(d)$.

Proof. We want to prove that $I_X(d)$ has the expected dimension

$$\max\left(\binom{d+n}{n} - k(n+1), 0\right).$$

Taking the global sections of the restriction exact sequence

$$0 \longrightarrow \mathcal{I}_{\widetilde{X}}(d-1) \longrightarrow \mathcal{I}_X(d) \longrightarrow \mathcal{I}_{X\cap H}(d) \longrightarrow 0,$$

we obtain the so called Castelnuovo exact sequence

$$0 \longrightarrow I_{\widetilde{X}}(d-1) \longrightarrow I_X(d) \longrightarrow I_{X\cap H}(d) \tag{1}$$

from which we get the following inequality

$$\dim I_X(d) \le \dim I_{\widetilde{X}}(d-1) + \dim I_{X\cap H}(d).$$

Since $X \cap H$ imposes independent conditions on $\mathcal{O}_H(d)$ we know that $\dim I_{X\cap H}(d) = \max\left(\binom{d+n-1}{n-1} - un, 0\right)$; on the other hand, since $\widetilde{X}$ imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d-1)$ it follows that $\dim I_{\widetilde{X}}(d-1) = \max\left(\binom{d-1+n}{n} - (k-u)(n+1) - u, 0\right)$.

Then in case (i), we get $\dim I_X(d) \le \binom{d+n}{n} - k(n+1)$, while in case (ii), we get $\dim I_X(d) \le 0$. But since $\dim I_X(d)$ is always greater than or equal to the expected dimension, we conclude. $\square$

In many cases a standard application of the above theorem gives most of the cases of Theorem 1.1.

Let us see some examples in $\mathbf{P}^3$. It is easy to check directly that there are no cubic surfaces with five singular points (e.g. by choosing the five fundamental points in $\mathbf{P}^3$). This is the starting point of the induction.

Now consider $d = 4$ and a union X of 8 general double points. Setting $u = 4$ we check that the inequalities of case (i) of Theorem 4.1 are satisfied. Hence we specialize 4 points on a hyperplane H in such a way that they are general on H, then by Theorem 2.4 it follows that the trace $X \cap H$ imposes independent conditions on quartics. On the other hand we consider the residual $\widetilde{X}$, given by 4 double points outside H and 4 simple points on H. We know that the scheme $\widetilde{X}$ imposes independent conditions on cubics, since the previous step implies that 4 general double points do, and moreover we can add 4 simple points contained in a plane. This is possible because there exists no cubics which are unions of a plane and a quadric through 4 general double points. Theorem 4.1 applies and we conclude that 8 general double points impose independent conditions on $\mathcal{O}_{\mathbf{P}^3}(4)$. Notice that 9 double points (one of the exceptional cases in Theorem 1.1) do not impose independent conditions on quartic surfaces. Indeed if we apply the same argument we get as trace 5 double points on $\mathbf{P}^2$, which do not impose independent conditions on quartics by Theorem 2.4.

Consider now the case $d = 5$. To prove that a general union of 14 double points in $\mathbf{P}^3$ imposes independent conditions on quintics, it is enough to specialize $u = 7$ points on a plane in such a way that the trace is general and we apply the induction. On the other hand, also the residual imposes independent conditions on quartics by induction and since there is no quartics which are unions of a plane and a cubic through 7 general double points. Again Theorem 4.1 applies and we can conclude that any collection of general double points imposes independent conditions on $\mathcal{O}_{\mathbf{P}^3}(5)$.

For $d \ge 6$ we can apply this simple argument and by induction it is possible to prove that k double points impose independent conditions on surfaces of degree d with the following possible exceptions, for $6 \le d \le 30$:

$$(d, k) = (6, 21), (9, 55), (12, 114), (15, 204), (21, 506), (27, 1015), (30, 1364).$$

In particular if $d \neq 0 \bmod 3$, then it turns out that k double points impose independent conditions on surfaces of degree d. To extend the result to the case $d = 0 \bmod 3$ and the only possibly missing values of k (that is $k = \lceil \frac{(d+3)(d+2)(d+1)}{24} \rceil$) is much more difficult. We will do this job in full generality in Section 6.

5. The case of cubics

The inductive procedure of the previous section does not work with cubics ($d = 3$) because by restricting to a hyperplane we reduce to quadrics which have defective behavior. Nevertheless the case of cubics is the starting point of the induction, so it is crucial. Alexander and Hirschowitz solved this case in [5], by a subtle blowing up and by applying the differential Horace's method (see Section 6). Chandler solved this case with more elementary techniques in [13]. In this section we give a shorter (and still elementary) proof.

Given n, we denote $k_n = \lfloor \frac{(n+3)(n+2)}{6} \rfloor$ and $\delta_n = \binom{n+3}{3} - (n+1)k_n$. Notice that $k_n = \frac{(n+3)(n+2)}{6}$ for $n \neq 2 \bmod 3$. If $n = 3p+2$, we get $k_n = \frac{(n+3)(n+2)}{6} - \frac{1}{3} = \frac{(n+4)(n+1)}{6}$ and $\delta_n = p + 1 = \frac{n+1}{3}$.

This simple arithmetic remark shows that the restriction to codimension three linear subspaces has the advantage to avoid the arithmetic problems, and this is our new main idea. In this section we will prove the following theorem, which immediately implies the case $d = 3$ of Theorem 1.1.

Theorem 5.1. *Let $n \neq 2 \bmod 3$, $n \neq 4$. Then k_n double points impose independent conditions on cubics.*

Let $n = 3p + 2$, then k_n double points and a zero-dimensional scheme of length δ_n impose independent conditions on cubics.

The proof of Theorem 5.1 relies on the following description.

Proposition 5.2. *Let $n \geq 5$ and let $L, M, N \subset \mathbf{P}^n$ be general subspaces of codimension 3. Let l_i (resp. m_i, n_i) with $i = 1, 2, 3$ be three general points on L, (resp. M, N). Then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain $L \cup M \cup N$ and which are singular at the nine points l_i, m_i, n_i, with $i = 1, 2, 3$.*

Proof. For $n = 5$ it is an explicit computation, which can be easily performed with the help of a computer. Indeed in $\mathbf{P}^5$ it is easy to check that $I_{L \cup N \cup M, \mathbf{P}^5}(3)$ has dimension 26. Choosing three general points on each subspace and imposing them as singular points for the cubics, one can check that they impose 26 independent conditions.

For $n \geq 6$ the statement follows by induction on n. Indeed if $n \geq 6$ it is easy to check that there are no quadrics containing $L \cup M \cup N$. Then given a general hyperplane $H \subset \mathbf{P}^n$ the Castelnuovo sequence induces the isomorphism

$$0 \longrightarrow I_{L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{(L \cup M \cup N) \cap H, H}(3) \longrightarrow 0$$

hence specializing the nine points on the hyperplane H, since the space $I_{L \cup M \cup N, \mathbf{P}^n}(2)$ is empty, we get

$$0 \longrightarrow I_{X \cup L \cup M \cup N, \mathbf{P}^n}(3) \longrightarrow I_{(X \cup L \cup M \cup N) \cap H, H}(3),$$

where X denotes the union of the nine double points supported at l_i, m_i, n_i with $i = 1, 2, 3$. Then our statement immediately follows by induction. $\square$

Remark. Notice that Proposition 5.2 is false for $n = 4$. Indeed $I_{L \cup N \cup M, \mathbf{P}^4}(3)$ has dimension 23 and there is a unique cubic singular at the nine points l_i, m_i, n_i, $i = 1, 2, 3$. Also the following Propositions 5.3 and 5.4 are false for $n = 4$, indeed their statements reduce to the statement of Theorem 5.1, because a cubic singular at p and q must contain the line $\langle p, q \rangle$.

Proposition 5.3. *Let $n \geq 3$, $n \neq 4$ and let $L, M \subset \mathbf{P}^n$ be subspaces of codimension three. Let l_i (resp. m_i) with $i = 1, \ldots, n-2$ be general points on L (resp. M). Then there are no cubic hypersurfaces in $\mathbf{P}^n$ containing $L \cup M$ which are singular at the $2n-4$ points l_i, m_i with $i = 1, \ldots n-2$ and at three general points $p_i \in \mathbf{P}^n$, with $i = 1, 2, 3$.*

Proof. The case $n = 3$ is easy and it was checked in Section 4. For $n = 5, 7$ it is an explicit computation. Indeed it is easy to check that $\dim I_{L \cup M, \mathbf{P}^5}(3) = 36$ and that the union of three general points on L, three general points on M and three general points on $\mathbf{P}^5$ imposes 36 independent conditions on the system $I_{L \cup M, \mathbf{P}^5}(3)$. In the case $n = 7$ one can easily check that $\dim I_{L\cup M,\mathbf{P}^7}(3) = 54$, and that the union of five general points on L, five general points on M and three general points on $\mathbf{P}^5$ imposes 54 independent conditions.

For $n = 6$ or $n \geq 8$, the statement follows by induction from $n-3$ to n. Indeed given a third general codimension three subspace N, we get the exact sequence

$$0 \longrightarrow I_{L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{(L\cup M)\cap N,N}(3) \longrightarrow 0,$$

where the dimensions of the three spaces in the sequence are respectively 27, $9(n-1)$ and $9(n-4)$.

Let X denote the union of the double points supported at p_1, p_2, p_3, l_i and m_i with $i = 1,\ldots,n-2$. Let us specialize $n-5$ of the points l_i (lying on L) to $L\cap N$, $n-5$ of the points m_i (lying on M) to $M\cap N$ and the three points p_1, p_2, p_3 to N. Then we obtain a sequence

$$0 \longrightarrow I_{X\cup L\cup M\cup N,\mathbf{P}^n}(3) \longrightarrow I_{X\cup L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{(X\cup L\cup M)\cap N,N}(3),$$

where the trace $(X\cup L\cup M)\cap N$ satisfies the assumptions on $N = \mathbf{P}^{n-3}$ and we can apply the induction. Then we conclude, since the residual satisfies the hypotheses of Proposition 5.2. □

Proposition 5.4. *Let $n \geq 3$, $n \neq 4$ and let $L \subset \mathbf{P}^n$ be a subspace of codimension three.*

(i) *If* $n \neq 2 \bmod 3$ *then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain L and which are singular at $\frac{n(n-1)}{6}$ general points l_i on L and at $(n+1)$ general points $p_i \in \mathbf{P}^n$.*

(ii) *If* $n = 2 \bmod 3$ *then there are no cubic hypersurfaces in $\mathbf{P}^n$ which contain L, which are singular at $\frac{(n+1)(n-2)}{6}$ general points l_i on L and at $(n+1)$ general points $p_i \in \mathbf{P}^n$, and which contain a general scheme η supported at $q \in L$ such that* $\mathrm{length}(\eta) = \delta_n$ *and* $\mathrm{length}(\eta \cap L) = \delta_n - 1$.

Proof. The case $n = 3$ is easy and already checked in Section 4. For $n = 5$ let e_i for $i = 0,\ldots,5$ be a basis of V and choose L spanned by $p_i = [e_i]$ for $i = 0, 1, 2$. Consider the system of cubics with singular points at p_i for $i = 0,\ldots,5$, at $[e_0+\ldots+e_5]$ and at other two random points. Moreover impose that the cubics of the system contain a general scheme of length 2 supported at $[e_0+e_1+e_2]$. Note that such cubics contain L. A direct computation shows that this system is empty, as we wanted. For $n = 7$ the statement (i) can be checked, with the help of a computer, by computing the tangent spaces to $V^{3,7}$ at seven general points of L and at eight general points. The condition that the cubic contains L can be imposed by another simple point on L.

For $n = 6$ or $n \geq 8$ the statement follows by induction, and by the sequence

$$0 \longrightarrow I_{L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{L,\mathbf{P}^n}(3) \longrightarrow I_{L\cap M,M}(3) \longrightarrow 0,$$

where M is a general codimension three subspace. Denoting by X the union of the double points supported at the points l_i and p_i (and of the scheme η in case (ii)), we get

$$0 \longrightarrow I_{X\cup L\cup M,\mathbf{P}^n}(3) \longrightarrow I_{X\cup L,\mathbf{P}^n}(3) \longrightarrow I_{(X\cup L)\cap M,M}(3)$$

Assume now that $n \neq 2 \bmod 3$. We specialize $\frac{(n-3)(n-4)}{6}$ of the points l_i to $L\cap M$ and $n-2$ of the points p_i to M. Thus we have left $n-2$ points general on L and 3 points general on $\mathbf{P}^n$ and we can use Proposition 5.3 on the residual and the induction on the trace.

If $n = 2 \bmod 3$, we specialize $\frac{(n-2)(n-5)}{6}$ of the points l_i to $M\cap L$, and $n-2$ of the points p_i and the scheme η to M in such a way that $\eta \subset M$ and $\mathrm{length}(\eta\cap L) = \mathrm{length}(\eta\cap L\cap M) = \delta_n - 1$ (we can do this since $n \geq 8$) and we conclude analogously. □

Proof of Thorem 5.1. We fix a codimension three linear subspace $L \subset \mathbf{P}^n$ and we prove the statement by induction by using the exact sequence

$$0 \longrightarrow I_{L,\mathbf{P}^n}(3) \longrightarrow I_{\mathbf{P}^n}(3) \longrightarrow I_L(3)$$

Assume first $n \neq 2 \bmod 3$. We specialize to L as many points as possible in order that the trace with respect to L imposes independent conditions on the cubics of L. Precisely, we have $k_n = \frac{(n+3)(n+2)}{6}$ double points and we specialize $\frac{n(n-1)}{6}$ of them on L, leaving $(n+1)$ points outside. Then the result follows from Proposition 5.4 and by induction on n. The starting points of the induction are $n = 3$ (see Section 4) and $n = 7$ (in this case it is enough to check that 15 general tangent spaces to $V^{3,7}$ are independent; notice that for $n = 4$ the statement is false, see Section 3).

In the case $n = 2 \bmod 3$, we specialize $k_{n-3} = \lfloor\frac{n(n-1)}{6}\rfloor = \frac{(n+1)(n-2)}{6}$ double points on L and we leave $k_n - k_{n-3} = n + 1$ double points outside L. Moreover we specialize the scheme η on L in such a way that $\eta \cap L$ has length $\delta_n - 1 = \delta_{n-3}$. Thus Proposition 5.4 applies again and we conclude by induction. The starting point of the induction is $n = 2$ (see Theorem 2.4). $\square$

6. The degeneration argument: “la méthode d’Horace différentielle”

This section is devoted to the proof of Theorem 1.1 in the case $d \geq 4$.

In order to solve the arithmetic problems revealed in the Section 4, Alexander and Hirschowitz have introduced a clever degeneration argument, called the differential Horace’s method [3,4]. We follow in this section the simplified version of the method performed by Chandler in [12], trying to supply more details. For the convenience of the reader we describe first the case of sextics in $\mathbf{P}^3$ (see Proposition 6.2), which is enough to understand the main idea. In fact the pair $(6, 21)$ was the first gap we met at the end of Section 4. After this case we will provide the proof in full generality.

Let $X, Z \subseteq \mathbf{P}^n = \mathbf{P}(V)$ be zero-dimensional subschemes, $\mathcal{I}_X$ and $\mathcal{I}_Z$ the corresponding ideal sheaves and $\mathcal{D} = \mathcal{I}_Z(d)$ for some $d \in \mathbf{N}$. The space $\mathrm{H}^0(\mathcal{D})$ defines a linear system. The Hilbert function of X with respect to $\mathcal{D}$ is defined as follows:

$$h_{\mathbf{P}^n}(X, \mathcal{D}) := \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_X \otimes \mathcal{D}).$$

Notice that if $\mathcal{D} = \mathcal{O}_{\mathbf{P}^n}(d)$, then $\mathrm{H}^0(\mathcal{I}_X \otimes \mathcal{D}) = I_X(d) \subseteq S^d V^\vee$ and we get

$$h_{\mathbf{P}^n}(X, d) := h_{\mathbf{P}^n}(X, \mathcal{O}(d)) = \binom{d+n}{n} - \dim I_X(d).$$

In other words $h_{\mathbf{P}^n}(X, d)$ is the codimension of the subspace $I_X(d)$ in the space of homogeneous polynomials of degree d.

We say that X *imposes independent conditions on* $\mathcal{D}$ if

$$h_{\mathbf{P}^n}(X, \mathcal{D}) = \min\left(\deg X, h^0(\mathcal{D})\right)$$

This generalizes the definition given in Section 2 where $\mathcal{D} = \mathcal{O}(d)$.

In particular if $h_{\mathbf{P}^n}(X, \mathcal{D}) = \deg X$, we say that X is $\mathcal{D}$-independent, and in the case $\mathcal{D} = \mathcal{O}(d)$, we say d-independent. Notice that if $Y \subseteq X$, then if X is $\mathcal{D}$-independent, then so is Y. On the other hand if $h_{\mathbf{P}^n}(Y, d) = \binom{d+n}{n}$, then $h_{\mathbf{P}^n}(X, d) = \binom{d+n}{n}$.

A zero-dimensional scheme is called *curvilinear* if it is contained in a nonsingular curve. A curvilinear scheme contained in a union of k double points has degree smaller than or equal to $2k$.

The following crucial lemma is due to Chandler [12, Lemma 4].

Lemma 6.1 (*Curvilinear Lemma*). *Let $X \subseteq \mathbf{P}^n$ be a zero-dimensional scheme contained in a finite union of double points and $\mathcal{D}$ a linear system on $\mathbf{P}^n$. Then X is $\mathcal{D}$-independent if and only if every curvilinear subscheme of X is $\mathcal{D}$-independent.*

Proof. One implication is trivial. So let us assume that every curvilinear subscheme of X is $\mathcal{D}$-independent. Suppose first that X is supported at one point p. We prove the statement by induction on $\deg X$. If $\deg X = 2$, then X is curvilinear and the claim holds true.

Now suppose $\deg X > 2$ and let us prove that $h(X, \mathcal{D}) = \deg X$. Consider a subscheme $Y \subset X$ with $\deg Y = \deg X - 1$. We have

$$h(Y, \mathcal{D}) \leq h(X, \mathcal{D}) \leq h(Y, \mathcal{D}) + 1.$$

By induction $h(Y, \mathcal{D}) = \deg Y = \deg X - 1$. Then it is sufficient to construct a subscheme $Y \subset X$ with $\deg Y = \deg X - 1$ and $h(X, \mathcal{D}) = h(Y, \mathcal{D}) + 1$.

In order to do this, consider a curvilinear subscheme $\xi \subset X$, i.e. a degree 2 subscheme of a double point. By hypothesis we know that ξ is $\mathcal{D}$-independent, i.e. $h(\xi, \mathcal{D}) = 2$. Obviously we also have $h(p, \mathcal{D}) = 1$, where p

denotes the simple point. It follows that there exists a section s of $\mathcal{D}$ vanishing on p, and not on ξ. We define then $Y = X \cap Z$, where Z is the zero locus of s. Since X is contained in a union of double points, by imposing the condition $s = 0$ we obtain $\deg Y = \deg X - 1$. Moreover $h(X, \mathcal{D}) > h(Y, \mathcal{D})$ because s vanishes on Y and does not on X. Then we conclude that

$$h(X, \mathcal{D}) = h(Y, \mathcal{D}) + 1 = \deg Y + 1 = \deg X.$$

Now consider X supported at $p_1, \ldots, p_k$. Suppose by induction on k that the claim holds true for schemes supported at $k - 1$ points and we prove that $h(X, \mathcal{D}) = \deg X$. Let

$$A = X \cap p_k^2 \quad \text{and} \quad B = X \cap \{p_1, \ldots, p_{k-1}\}^2,$$

where $\{p_1, \ldots, p_{k-1}\}^2$ denotes the union of the double points p_i^2 and X is a disjoint union of A and B. Consider $\mathcal{D}' = \mathcal{I}_B \otimes \mathcal{D}$.

Let ζ be any curvilinear subscheme of A and $\mathcal{D}'' = \mathcal{D} \otimes \mathcal{I}_\zeta$. For every curvilinear $\eta \subset B$ we have

$$\begin{aligned} h(\eta, \mathcal{D}'') &= \dim \mathrm{H}^0(\mathcal{D} \otimes \mathcal{I}_\zeta) - \dim \mathrm{H}^0(I_{\zeta\cup\eta} \otimes \mathcal{D}) \\ &= \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_{\zeta\cup\eta} \otimes \mathcal{D}) - \dim \mathrm{H}^0(\mathcal{D}) + \dim \mathrm{H}^0(\mathcal{D} \otimes \mathcal{I}_\zeta) \\ &= h(\zeta \cup \eta, \mathcal{D}) - h(\zeta, \mathcal{D}) = (\deg \zeta + \deg \eta) - \deg \zeta = \deg \eta \end{aligned}$$

i.e. every curvilinear subscheme of B is $\mathcal{D}''$-independent. By induction it follows that B is $\mathcal{D}''$-independent, i.e. $h(B, \mathcal{D} \otimes \mathcal{I}_\zeta) = \deg B$.

Then we get in the same way

$$h(\zeta \cup B, \mathcal{D}) = h(\zeta, \mathcal{D}) + h(B, \mathcal{D}'') = \deg \zeta + \deg B,$$

and again

$$h(\zeta, \mathcal{D}') = h(\zeta \cup B, \mathcal{D}) - h(B, \mathcal{D})$$

hence putting together the last two equations and using the inductive assumption we get

$$h(\zeta, \mathcal{D}') = (\deg \zeta + \deg B) - \deg B = \deg \zeta.$$

We proved that every curvilinear subscheme of A is $\mathcal{D}'$-independent. Since A is supported at one single point, from the first part it follows that A is $\mathcal{D}'$-independent.

Obviously $\mathcal{I}_A \otimes \mathcal{D}' = \mathcal{I}_A \otimes \mathcal{I}_B \otimes \mathcal{D} = \mathcal{I}_X \otimes \mathcal{D}$. Then we conclude, by using induction on B, that

$$\begin{aligned} h(X, \mathcal{D}) &= \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_X \otimes \mathcal{D}) = \dim \mathrm{H}^0(\mathcal{D}) - \dim \mathrm{H}^0(\mathcal{I}_A \otimes \mathcal{D}') \\ &= h(B, \mathcal{D}) + h(A, \mathcal{D}') = \deg B + \deg A = \deg X. \quad \square \end{aligned}$$

Let us denote by $AH_{n,d}(k)$ the following statement: *there exists a collection of k double points in $\mathbf{P}^n$ which impose independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$.*

Before considering the general inductive argument, we analyze in detail the first interesting example. We ask how many conditions 21 double points impose on $\mathcal{O}_{\mathbf{P}^3}(6)$ and we will prove that $AH_{3,6}(21)$ holds true.

Proposition 6.2. *A collection of* 21 *general double points imposes independent conditions on* $\mathcal{O}_{\mathbf{P}^3}(6)$.

Proof. Notice that we cannot specialize u points in such a way that conditions either (i) or (ii) of Theorem 4.1 are satisfied. Then we choose u maximal such that $nu < k(n+1) - \binom{d+n-1}{n}$, that is $u = 9$.

By Theorem 2.4 and by Section 4 we know the following facts:

(i) $AH_{2,6}(9)$, and in particular 9 general double points in $\mathbf{P}^2$ are 6-independent,

(ii) $AH_{3,5}(12)$, and 12 general double points in $\mathbf{P}^3$ are 5-independent,

(iii) $AH_{3,4}(11)$, and there exist no quartic surfaces through 11 general double points.

Step 1: Fix a plane $\mathbf{P}^2 \subseteq \mathbf{P}^3$. Let $\gamma \in \mathbf{P}^2$ be a point and Σ a collection of 11 general points not contained in $\mathbf{P}^2$. By (ii), it follows that

$$h_{\mathbf{P}^3}(\{\gamma\}^2_{|\mathbf{P}^2} \cup \Sigma^2, 5) = \deg(\{\gamma\}^2_{|\mathbf{P}^2} \cup \Sigma^2) = 47.$$

Step 2: Now we want to add a collection of 9 points on $\mathbf{P}^2$ to the scheme $\{\gamma\}^2_{|\mathbf{P}^2} \cup \Sigma^2$. It is obvious that if we add 9 general simple points of $\mathbf{P}^3$ the resulting scheme would be 5-independent. But we want to add 9 points contained in the plane. In fact we obtain the same conclusion once we prove that there exists no quintic surface which is union of a plane and a quartic through Σ^2. Indeed by (iii) we know that $\dim I_{\Sigma^2}(4) = 0$, hence we can choose a collection Φ of 9 simple points in $\mathbf{P}^2$ in such a way that the scheme $\{\gamma\}^2_{|\mathbf{P}^2} \cup \Sigma^2 \cup \Phi$ is 5-independent.

Step 3: By (i), it follows that the scheme $(\Phi^2_{|\mathbf{P}^2} \cup \gamma) \subseteq \mathbf{P}^2$ has Hilbert function

$$h_{\mathbf{P}^2}(\Phi^2_{|\mathbf{P}^2} \cup \gamma, 6) = 28$$

i.e. it is 6-independent.

Now for $t \in \mathbf{K}$, let us choose a flat family of general points $\delta_t \subseteq \mathbf{P}^3$ and a family of planes $\{H_t\}$ such that

- $\delta_t \in H_t$ for any t,
- $\delta_t \notin \mathbf{P}^2$ for any $t \neq 0$,
- $H_0 = \mathbf{P}^2$ and $\delta_0 = \gamma \in \mathbf{P}^2$.

Now consider the following schemes: $\{\delta_t\}^2$, Φ^2, where Φ is the collection of 9 points introduced in Step 2 and Σ^2, the collection of 11 double points introduced in Step 1. Then in order to prove that $AH_{3,6}(21)$ holds, it is enough to prove the following claim.

Claim: There exists $t \neq 0$ such that the scheme $\{\delta_t\}^2$ is independent with respect to the system $I_{\Phi^2 \cup \Sigma^2}(6)$.

Proof of the claim. Assume by contradiction that the claim is false. Then by Lemma 6.1 for all t there exist pairs (δ_t, η_t) with η_t a curvilinear scheme supported in δ_t and contained in $\{\delta_t\}^2$ such that

$$h_{\mathbf{P}^3}(\Phi^2 \cup \Sigma^2 \cup \eta_t, 6) < 82.$$

Let η_0 be the limit of η_t.

By the semicontinuity of the Hilbert function and by the previous inequality we get

$$h_{\mathbf{P}^3}(\Phi^2 \cup \Sigma^2 \cup \eta_0, 6) \leq h_{\mathbf{P}^3}(\Phi^2 \cup \Sigma^2 \cup \eta_t, 6) < 82. \tag{2}$$

Consider the following two possibilities

(1) $\eta_0 \not\subset \mathbf{P}^2$.
By applying the Castelnuovo exact sequence to $\Sigma^2 \cup \Phi^2 \cup \eta_0$, and by using Step 2 and Step 3, we obtain

$$h_{\mathbf{P}^3}(\Sigma^2 \cup \Phi^2 \cup \eta_0, 6) \geq h_{\mathbf{P}^3}(\Sigma^2 \cup \Phi \cup \widetilde{\eta}_0, 5) + h_{\mathbf{P}^2}((\Phi^2_{|\mathbf{P}^2} \cup \gamma), 6) = 54 + 28 = 82,$$

a contradiction with (2).

(2) $\eta_0 \subset \mathbf{P}^2$.
By the semicontinuity of the Hilbert function there exists an open neighborhood O of 0 such that for any $t \in O$

$$h_{\mathbf{P}^3}(\Phi \cup \Sigma^2 \cup \{\delta_t\}^2_{|H_t}, 5) \geq h_{\mathbf{P}^3}(\Phi \cup \Sigma^2 \cup \{\gamma\}^2_{|\mathbf{P}^2}, 5) = 9 + 44 + 3 = 56$$

and the equality holds. In particular the subscheme $\Phi \cup \Sigma^2 \cup \eta_0 \subset \Phi \cup \Sigma^2 \cup \{\gamma\}^2_{|\mathbf{P}^2}$ is 5-independent, then $h_{\mathbf{P}^3}(\Phi \cup \Sigma^2 \cup \eta_t, 5) = 9 + 44 + 2 = 55$ for all $t \in O$.
Hence for any $t \in O$, by applying again the Castelnuovo exact sequence, we get

$$h_{\mathbf{P}^3}(\Phi^2 \cup \Sigma^2 \cup \eta_t, 6) \geq h_{\mathbf{P}^3}(\Phi \cup \Sigma^2 \cup \eta_t, 5) + h_{\mathbf{P}^2}(\Phi^2_{|\mathbf{P}^2}, 6) = 55 + 27 = 82$$

contradicting again the inequality (2) above.

This completes the proof of the proposition. □

Remark. We want to comment “why” the proof of Proposition 6.2 works. A double point in $\mathbf{P}^3$ has length 4; specializing it on a plane we get a trace of length 3 and a residual of length 1. Among the 21 points, 9 points are specialized on the plane $\mathbf{P}^2$, and 11 remain outside. After this process has been performed, the trace defines a subspace of codimension 27 in $\mathrm{H}^0(\mathcal{O}_{\mathbf{P}^2}(6)) \simeq \mathbf{K}^{28}$ and there is no more room in the trace to specialize the last point on $\mathbf{P}^2$, nor there is room in the residual to keep it outside. Thanks to the degeneration argument, called the differential Horace’s method, the last point $\{\gamma\}^2$ “counts like” a point of length 1 in the trace, and there is room for it. This single point in the trace, which allows to solve the problem, reminds us of the Roman legend of the Horaces.

In Theorem 6.4 below we describe the general inductive argument. It could not be enough to specialize only one point γ, in general we need to specialize ϵ points, with $0 \le \epsilon < n$ to be chosen. We need the following easy numerical lemma, proved by Chandler [12] in a slightly different form.

Lemma 6.3. *Fix the integers $2 \le n, 4 \le d, 0 \le k \le \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$ and let $u \in \mathbf{Z}$, $0 \le \epsilon < n$ such that $nu + \epsilon = k(n+1) - \binom{n+d-1}{n}$. Then we have*

(i) $n\epsilon + u \le \binom{n+d-2}{n-1}$;
(ii) $\binom{n+d-2}{n} \le (k-u-\epsilon)(n+1)$;
(iii) $k-u-\epsilon \ge n+1$, *for $d=4$ and $n \ge 10$.*

Proof. We have

$$u \le \frac{1}{n}\left(\binom{n+d}{n} + (n+1) - \binom{n+d-1}{n}\right) = \frac{1}{n}\binom{n+d-1}{n-1} + \frac{n+1}{n}$$

hence

$$n\epsilon + u \le n(n-1) + \frac{1}{n}\binom{n+d-1}{n-1} + \frac{n+1}{n}$$

and the right-hand side is smaller than or equal to $\binom{n+d-2}{n-1}$ except for $(n,d) = (3,4), (4,4), (5,4)$. In these cases the inequality (i) can be checked directly.

The inequality (ii) follows from (i) and from the definition of u and ϵ.

In order to prove (iii) let us remark that by definition of u we get

$$k-u-\epsilon = -\frac{k}{n} + \frac{1}{n}\binom{n+3}{n} - \frac{(n-1)\epsilon}{n} \ge \frac{1}{n}\left(-\frac{1}{n+1}\binom{n+4}{n} - 1 + \binom{n+3}{n} - (n-1)^2\right)$$

and the right-hand side is greater than or equal to $n+1$ for $n \ge 10$. $\square$

Theorem 6.4. *Fix the integers $2 \le n, 4 \le d, \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor \le k \le \left\lceil \frac{1}{n+1}\binom{n+d}{n} \right\rceil$ and let $u \in \mathbf{Z}$, $0 \le \epsilon < n$ such that $nu + \epsilon = k(n+1) - \binom{n+d-1}{n}$. Assume that $AH_{n-1,d}(u) AH_{n,d-1}(k-u)$, $AH_{n,d-2}(k-u-\epsilon)$, hold. Then $AH_{n,d}(k)$ follows.*

Proof. We will construct a scheme $\Phi^2 \cup \Sigma^2 \cup \Delta_t^2$ of k double points which imposes independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$.

Step 1: Choose a hyperplane $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$. Let $\Gamma = \{\gamma^1, \ldots, \gamma^\epsilon\}$ be a collection of ϵ general points contained in $\mathbf{P}^{n-1}$ and Σ a collection of $k-u-\epsilon$ points not contained in $\mathbf{P}^{n-1}$. By induction we know that $AH_{n,d-1}(k-u)$ holds, then it follows

$$h_{\mathbf{P}^n}(\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2, d-1) = \min\left((n+1)(k-u) - \epsilon, \binom{n+d-1}{n}\right).$$

From the definition of ϵ it follows that $\binom{n+d-1}{n} = (n+1)(k-u) - \epsilon + u$ and since $u \ge 0$, we obtain

$$h_{\mathbf{P}^n}(\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2, d-1) = (n+1)(k-u) - \epsilon.$$

Step 2: Now we want to add a collection of u simple points in $\mathbf{P}^{n-1}$ to the scheme $\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2$ and we want to obtain a $(d-1)$-independent scheme. Notice that from Step 1 it follows that $\dim I_{\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2}(d-1) = u$. Thus it is enough to prove that there exist no hypersurfaces of degree $d-1$ which are unions of $\mathbf{P}^{n-1}$ and of a hypersurface of degree $d-2$ through Σ^2. In fact by induction we know that $\dim I_{\Sigma^2}(d-2) = \max\left(0, \binom{n+d-2}{n} - (k-u-\epsilon)(n+1)\right)$ and this dimension vanishes by (ii) of Lemma 6.3.

Then it follows that we can choose a collection Φ of u simple points in $\mathbf{P}^{n-1}$ in such a way that the scheme $\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2 \cup \Phi$ is $(d-1)$-independent, i.e.

$$h_{\mathbf{P}^n}(\Gamma^2_{|\mathbf{P}^{n-1}} \cup \Sigma^2 \cup \Phi, d-1) = (n+1)(k-u) - \epsilon + u = \binom{n+d-1}{n}.$$

Now we split the proof in two cases.

First case: $k(n+1) \leq \binom{d+n}{n}$.

Step 3: The assumption $k(n+1) \leq \binom{d+n}{n}$ implies that $k = \left\lfloor \frac{1}{n+1}\binom{n+d}{n} \right\rfloor$ and $nu + \epsilon \leq \binom{d+n-1}{n-1}$.

By induction we know that $AH_{n-1,d}(u)$ holds, hence the scheme $(\Phi^2_{|\mathbf{P}^{n-1}} \cup \Gamma) \subseteq \mathbf{P}^{n-1}$ has Hilbert function

$$h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup \Gamma, d) = \min\left(nu + \epsilon, \binom{d+n-1}{n-1}\right) = nu + \epsilon,$$

that is the scheme is d-independent.

Now for $(t_1, \ldots, t_\epsilon) \in \mathbf{K}^\epsilon$, let us choose a flat family of general points $\{\delta^1_{t_1}, \ldots, \delta^\epsilon_{t_\epsilon}\} \subseteq \mathbf{P}^n$ and a family of hyperplanes $\{H_{t_1}, \ldots, H_{t_\epsilon}\}$ such that

- $\delta^i_{t_i} \in H_{t_i}$ for any $i = 1, \ldots, \epsilon$ and for any t_i,
- $\delta^i_{t_i} \notin \mathbf{P}^{n-1}$ for any $t_i \neq 0$ and for any $i = 1, \ldots, \epsilon$,
- $H_0 = \mathbf{P}^{n-1}$ and $\delta^i_0 = \gamma^i \in \mathbf{P}^{n-1}$ for any $i = 1, \ldots, \epsilon$.

Now let us consider the following schemes:

- $\Delta^2_{(t_1,\ldots,t_\epsilon)} = \{\delta^1_{t_1}, \ldots, \delta^\epsilon_{t_\epsilon}\}^2$, notice that $\Delta^2_{(0,\ldots,0)} = \Gamma^2$;
- Φ^2, where Φ is the collection of u points introduced in Step 2;
- Σ^2, the collection of $k-u-\epsilon$ double points introduced in Step 1.

In order to prove that there exists a collection of k points in $\mathbf{P}^n$ which impose independent conditions on $\mathcal{O}_{\mathbf{P}^n}(d)$, it is enough to prove the following claim.

Claim: There exists $(t_1, \ldots, t_\epsilon)$ such that the scheme $\Delta^2_{(t_1,\ldots,t_\epsilon)}$ is independent with respect to the system $I_{\Phi^2 \cup \Sigma^2}(d)$.

Proof of the claim. Assume by contradiction that the claim is false. Then by Lemma 6.1 for all $(t_1, \ldots, t_\epsilon)$ there exist pairs $(\delta^i_{t_i}, \eta^i_{t_i})$ for $i = 1, \ldots, \epsilon$, with $\eta^i_{t_i}$ a curvilinear scheme supported in $\delta^i_{t_i}$ and contained in $\Delta^2_{(t_1,\ldots,t_\epsilon)}$ such that

$$h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup \eta^1_{t_1} \cup \ldots, \eta^\epsilon_{t_\epsilon}, d) < (n+1)(k-\epsilon) + 2\epsilon. \tag{3}$$

Let η^i_0 be the limit of $\eta^i_{t_i}$, for $i = 1, \ldots, \epsilon$.

Suppose that $\eta^i_0 \not\subset \mathbf{P}^{n-1}$ for $i \in F \subseteq \{1, \ldots, \epsilon\}$ and $\eta^i_0 \subset \mathbf{P}^{n-1}$ for $i \in G = \{1, \ldots, \epsilon\} \setminus F$.

Given $t \in \mathbf{K}$, let us denote $Z^F_t = \cup_{i \in F}(\eta^i_t)$ and $Z^G_t = \cup_{i \in G}(\eta^i_t)$. Denote by $\widetilde{\eta^i_0}$ the residual of η^i_0 with respect to $\mathbf{P}^{n-1}$ and by f and g the cardinalities respectively of F and G.

By the semicontinuity of the Hilbert function and by (3) we get

$$h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup Z^F_0 \cup Z^G_t, d) \leq h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup Z^F_t \cup Z^G_t, d) < (n+1)(k-\epsilon) + 2\epsilon. \tag{4}$$

On the other hand, by the semicontinuity of the Hilbert function there exists an open neighborhood O of 0 such that for any $t \in O$

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma^2 \cup (\cup_{i \in F} \widetilde{\eta^i_0}) \cup Z^G_t, d-1) \geq h_{\mathbf{P}^n}(\Phi \cup \Sigma^2 \cup (\cup_{i \in F} \widetilde{\eta^i_0}) \cup Z^G_0, d-1).$$

Since $\Phi \cup \Sigma^2 \cup (\cup_{i \in F} \widetilde{\eta^i_0}) \cup Z^G_0 \subseteq \Phi \cup \Sigma^2 \cup \Gamma^2_{|\mathbf{P}^{n-1}}$, by Step 2 we compute

$$h_{\mathbf{P}^n}(\Phi \cup \Sigma^2 \cup (\cup_{i \in F} \widetilde{\eta^i_0}) \cup Z^G_0, d-1) = u + (n+1)(k-u-\epsilon) + f + 2g.$$

Since $\Phi^2_{|\mathbf{P}^{n-1}} \cup (\cup_{i \in F} \gamma_i)$ is a subscheme of $\Phi^2_{|\mathbf{P}^{n-1}} \cup \Gamma$, by Step 3 it follows that

$$h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup (\cup_{i \in F} \gamma_i), d) \geq nu + f.$$

Hence for any $t \in O$, by applying the Castelnuovo exact sequence to the scheme $\widetilde{\Phi} \cup \Sigma \cup Z_0^F \cup Z_t^G$, we get,

$$\begin{aligned} h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup Z_0^F \cup Z_t^G, d) &\geq h_{\mathbf{P}^n}(\Phi \cup \Sigma^2 \cup (\cup_{i \in F} \widetilde{\eta_0^i}) \cup Z_t^G, d-1) + h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup (\cup_{i \in F} \gamma_i), d) \\ &\geq (u + (n+1)(k-u-\epsilon) + f + 2g) + (nu + f) = (n+1)(k-\epsilon) + 2\epsilon, \end{aligned}$$

contradicting the inequality (4) above. This completes the proof of the claim and that of the first case.

Second case: $k(n+1) > \binom{d+n}{n}$.

It follows that $k = \lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$ and $nu + \epsilon > \binom{d+n-1}{n-1}$.

If $\binom{d+n-1}{n-1} - nu < 0$ then we are in the easy case (ii) of Theorem 4.1 (indeed the second inequality of (ii) is equivalent to $\epsilon \geq 0$). Then $AH_{n,d}(k)$ holds by applying Theorem 4.1. Indeed the assumptions of Theorem 4.1 are satisfied: in particular the assumption on the trace follows from $AH_{n-1,d}(u)$, while the assumption on the residual follows from $AH_{n,d-1}(k-u)$, and $AH_{n,d-2}(k-u-\epsilon)$, which in particular implies $AH_{n,d-2}(k-u)$ by Step 2.

So we may assume that $0 \leq \nu := \binom{d+n-1}{n-1} - nu < \epsilon$.

Step 3: Differently from the first case, now we obtain

$$h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup \Gamma, d) = \binom{d+n-1}{n-1} < nu + \epsilon.$$

Note that if we substitute to Γ its subset $\overline{\Gamma} = \{\gamma_1, \ldots, \gamma_\nu\}$ we get

$$h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup \overline{\Gamma}, d) = \binom{d+n-1}{n-1} = nu + \nu$$

and the advantage of this formulation is that now we can apply Lemma 6.1 to the scheme $\Phi^2_{|\mathbf{P}^{n-1}} \cup \overline{\Gamma}$.

Now choose a flat family of general points $\{\delta^1_{t_1}, \ldots, \delta^\epsilon_{t_\epsilon}\} \subseteq \mathbf{P}^n$ and a family of hyperplanes $\{H_{t_1}, \ldots, H_{t_\epsilon}\}$ with the same properties as above.

Let us denote

$$\overline{\Delta}_{(t_1,\ldots,t_\epsilon)} = \{\delta^1_{t_1}, \ldots, \delta^\nu_{t_\nu}\}^2 \cup \{\delta^{\nu+1}_{t_{(\nu+1)}}\}^2_{|H_{t_{(\nu+1)}}} \cup \ldots \{\delta^\epsilon_{t_\epsilon}\}^2_{|H_{t_\epsilon}}.$$

Since obviously we have

$$h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup \Delta^2_{(t_1,\ldots,t_\epsilon)}, d) \geq h_{\mathbf{P}^n}(\Phi^2 \cup \Sigma^2 \cup \overline{\Delta}_{(t_1,\ldots,t_\epsilon)}, d),$$

in order to conclude it is enough to prove the following claim.

Claim: There exists $(t_1, \ldots, t_\epsilon)$ such that the scheme $\overline{\Delta}_{(t_1,\ldots,t_\epsilon)}$ is independent with respect to the system $I_{\Phi^2 \cup \Sigma^2}(d)$.

We can prove the claim exactly as in the first case. Indeed note that $\{\nu+1, \ldots, \epsilon\} \subseteq G$. Then $\Phi^2_{|\mathbf{P}^{n-1}} \cup (\cup_{i \in F} \gamma_i)$ is a subscheme of $\Phi^2_{|\mathbf{P}^{n-1}} \cup \overline{\Gamma}$, hence by Lemma 6.1 it follows again that

$$h_{\mathbf{P}^{n-1}}(\Phi^2_{|\mathbf{P}^{n-1}} \cup (\cup_{i \in F} \gamma_i), d) \geq nu + f.$$

So the above proof of the claim works smoothly. This completes the proof of the second case. □

Theorem 6.4 allows us to prove Theorem 1.1, once we have checked the initial steps of the induction. Thanks to Theorem 5.1, the only problems occurring in the initial steps depend on quadrics and on the exceptional cases. It is easy to see that the only cases we have to study explicitly are $\mathcal{O}_{\mathbf{P}^n}(4)$ for $5 \leq n \leq 9$. Indeed for $n \geq 10$ we can apply (iii) of Lemma 6.3 and the easy fact that $AH_{n,2}(k)$ holds if $k \geq n+1$, because there are no quadrics with $n+1$ general double points.

Even for $n = 9$ we have $k = 71$ (or respectively 72), $(u, \epsilon) = (54, 4)$, (respectively $(55, 5)$) and still $k - u - \epsilon \geq n + 1$ so that $AH_{n,2}(k-u-\epsilon)$ holds, moreover we need $AH_{8,4}(54)$ (respectively $AH_{8,4}(55)$) and $AH_{9,3}(17)$ that will turn out to hold by the induction procedure. The same argument applies for $n = 6, 8$.

For $n = 7$, we have to consider $k = 41$ or 42. For $k = 41$ it applies Theorem 4.1(i) with $u = 30$, while for $k = 42$ it applies Theorem 4.1(ii) again with $u = 30$.

In the remaining case $n = 5$ we have $k = 21$ and neither Theorem 4.1 nor Theorem 6.4 apply because we always need $AH_{4,4}(14)$ which does not hold and indeed it is the last exceptional case of Theorem 1.1. This case can be checked explicitly, by verifying that 21 general tangent spaces to $V^{4,5}$ are independent, with the help of a computer, or by an ad hoc argument, either as in [3] or as in the last paragraph of [12].

This completes the proof of Theorem 1.1.

Remark. Alexander and Hirschowitz called the assumption $AH_{n-1,d}(u)$ in Theorem 6.4 the *dime* (lower dimension) and the other assumptions the *degue* (lower degree).

7. Historical remarks

7.1. The one-dimensional case and the Sylvester Theorem

In the case $n = 1$ the Veronese variety $V^{d,1}$ is the rational normal curve C_d. It is easy to check that the higher secant variety $\sigma_k(C_d)$ has always the expected dimension (moreover this is true for arbitrary curves, see [39, Example V.1.6]). In the setting of Theorem 1.1 this follows from the fact that the space of one variable polynomials, with given roots of fixed multiplicities, has always the expected dimension. Indeed there are well-known explicit interpolation formulas to handle this problem which go back to Newton and Lagrange.

The equations of the higher secant varieties to the rational normal curves C_d were computed by Sylvester in 1851. In modern notation, given a vector space U of dimension two and $\phi \in S^{2m}U$ it is defined the contraction operator $A_\phi\colon S^m U^\vee \longrightarrow S^m U$ and we have that $\phi \in \sigma_k(C_{2m})$ if and only if $rkA_\phi \le k$, while in the odd case we have $\phi \in S^{2m+1}U$, the contraction operator $A_\phi\colon S^m U^\vee \longrightarrow S^{m+1}U$ and again we have that $\phi \in \sigma_k(C_{2m+1})$ if and only if $rkA_\phi \le k$. It turns out that the equations of the higher secant varieties of the rational normal curve are given by the minors of A_ϕ. The matrices representing A_ϕ were called catalecticant by Sylvester [35]. In 1886 Gundelfinger [20] treated the same problem from a different point of view by finding the covariants defining $\sigma_k(C_d)$ in the setting of classical invariant theory. In [35] Sylvester also found the canonical form of a general $\phi \in S^{2m+1}U$ as sum of $m + 1$ uniquely determined powers of linear forms. This is the first case of the Waring problem for polynomials.

Making precise the statement of Sylvester, we denote

$$f_{p,q} = \frac{\partial^{p+q} f}{\partial x^p \partial y^q}$$

and we get the following

Theorem 7.1 (*Sylvester*). *Let $f(x, y)$ be a binary form of degree $2m + 1$ over the complex numbers. Consider the $(m + 1) \times (m + 1)$ matrix F whose (i, j) entry is $f_{2m-i-j,i+j}$ for $0 \le i, j \le m$ and denote $g(x, y) = \det F$.*

(i) *If $g(x, y)$ does vanish identically then $f \in \sigma_m(C_{2m+1})$, and the converse holds.*

(ii) *If $g(x, y)$ does not vanish identically then factorize*

$$g(x, y) = \prod_{i=1}^{m+1} (p_i x + q_i y).$$

There are uniquely determined constants c_i such that

$$f(x, y) = \sum_{i=1}^{m+1} c_i (p_i x + q_i y)^{2m+1}$$

if and only if $g(x, y)$ has distinct roots. (A convenient choice of p_i, q_i allows of course to take $c_i = 1$.)

It is worth to rewrite and prove Sylvester theorem in the first nontrivial case, which is the case of quintics, as Sylvester himself did. The general case is analogous. Let

$$f = a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5.$$

We have that $f \in \sigma_k(C_5)$ if and only if

$$rk \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{bmatrix} \leq k.$$

We have the formula

$$\frac{1}{5!} \begin{bmatrix} f_{4,0} & f_{3,1} & f_{2,2} \\ f_{3,1} & f_{2,2} & f_{1,3} \\ f_{2,2} & f_{1,3} & f_{0,4} \end{bmatrix} = \begin{bmatrix} a_0x + a_1y & a_1x + a_2y & a_2x + a_3y \\ a_1x + a_2y & a_2x + a_3y & a_3x + a_4y \\ a_2x + a_3y & a_3x + a_4y & a_4x + a_5y \end{bmatrix}$$

moreover Sylvester found the following equality between determinants

$$\begin{vmatrix} a_0x + a_1y & a_1x + a_2y & a_2x + a_3y \\ a_1x + a_2y & a_2x + a_3y & a_3x + a_4y \\ a_2x + a_3y & a_3x + a_4y & a_4x + a_5y \end{vmatrix} = \begin{vmatrix} y^3 & -x^2y & x^2y & -x^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix}$$

and Cayley pointed out to him [35] that it follows from

$$\begin{bmatrix} y^3 & -x^2y & x^2y & -x^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & y & x & 0 \\ 0 & 0 & y & x \\ 0 & 0 & 0 & y \end{bmatrix} = \begin{bmatrix} y^3 & 0 & 0 & 0 \\ a_0 & a_0x + a_1y & a_1x + a_2y & a_2x + a_3y \\ a_1 & a_1x + a_2y & a_2x + a_3y & a_3x + a_4y \\ a_2 & a_2x + a_3y & a_3x + a_4y & a_4x + a_5y \end{bmatrix}.$$

We get that $f \in \sigma_2(C_5)$ if and only if

$$\begin{vmatrix} f_{4,0} & f_{3,1} & f_{2,2} \\ f_{3,1} & f_{2,2} & f_{1,3} \\ f_{2,2} & f_{1,3} & f_{0,4} \end{vmatrix} \equiv 0$$

(this is one of Gundelfinger's covariants) and this proves (i).

In case (ii) we have the factorization

$$\begin{vmatrix} f_{4,0} & f_{3,1} & f_{2,2} \\ f_{3,1} & f_{2,2} & f_{1,3} \\ f_{2,2} & f_{1,3} & f_{0,4} \end{vmatrix} = (p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$$

and Sylvester proves in [35] the "remarkable discovery" that there are constants c_i such that

$$f = c_1(p_1x + q_1y)^5 + c_2(p_2x + q_2y)^5 + c_3(p_3x + q_3y)^5$$

if and only if the three roots are distinct.

In particular the three linear forms $p_ix + q_iy$ are uniquely determined, so that we get generically a canonical form as a sum of three 5th powers. The proof goes as follows. Consider the covariant

$$g(a, x, y) = \begin{vmatrix} y^3 & -x^2y & xy^2 & -x^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix}$$

which is called apolar to f (we do not need this concept). To any catalecticant matrix

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

such that $rkA = 1$, it is associated a unique $(x, y) \in \mathbf{P}^1$ such that

$$rk \begin{bmatrix} y^3 & -x^2y & xy^2 & -x^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{bmatrix} = 1$$

(it is easy to see this by looking at the parametric equations of the rational normal curve).

Assume now that the general catalecticant matrix A is the sum of three catalecticant matrices of the same shape A^i of rank 1. We may write $a = a^1 + a^2 + a^3$. Let $(x_i, y_i) \in \mathbf{P}^1$ be the point associated to a^i. Now compute $g(a^1 + a^2 + a^3, x_1, y_1)$. By linearity on rows, the determinant splits in 27 summands, among them there are 19 which contain a row in A^1, which vanish because any row of A^1 is dependent with $(y_1^3, -x_1^2y_1, x_1y_1^2, -x_1^3)$, and other 8 which vanish because by the pigeon-hole principle they contain at least two rows from A^2 or from A^3. It follows that $g(a^1 + a^2 + a^3, x_1, y_1) = 0$, then (x_1, y_1) is a root of the covariant $g(a, x, y)$. Since the same argument works also for (x_i, y_i) with $i = 2, 3$, this ends the proof of the uniqueness in Sylvester theorem.

To show the existence, we consider the $SL(U)$-equivariant morphism

$$\mathbf{P}(S^5U) \setminus \sigma_2(C_5) \longrightarrow^{\pi} \mathbf{P}(S^3U)$$

defined by the covariant g. The fiber of a polynomial

$$z(x, y) = (p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y) \in \mathbf{P}(S^3U)$$

with distinct roots satisfies

$$\pi^{-1}(z) \supseteq \{c_1(p_1x + q_1y)^5 + c_2(p_2x + q_2y)^5 + c_3(p_3x + q_3y)^5 | c_1 \neq 0, c_2 \neq 0, c_3 \neq 0\} \tag{5}$$

by the uniqueness argument and the fact that if some $c_i = 0$ then the corresponding polynomial belongs to $\sigma_2(C_5)$. Hence any polynomial which is a sum of three distinct 5th powers must belong to one of the above fibers, so that its image under π must have three distinct roots. Now an infinitesimal version of the above computation shows that if $a = a^1 + a^{11} + a^3$ where a^{11} is on the tangent line at a^1, then $g(a, x, y)$ has a double root at (x_1, y_1).

In particular if $f \in \mathbf{P}(S^5U)$ cannot be expressed as the sum of three distinct 5th powers then $\pi(f)$ must have a double root. This shows that the equality holds in (5) and it concludes the proof.

Note that the fiber of the general point is the algebraic torus given by the 3-secant $\mathbf{P}^2$ minus three lines. To make everything explicit, denote by T_p^i the ith osculating space at p to C_5, so T_p^1 is the usual tangent line at p. If $z(x, y) = (p_1x + q_1y)^2(p_2x + q_2y) \in \mathbf{P}(S^3U)$ then

$$\pi^{-1}(z) = \langle T^1_{(p_1x+q_1y)^5}, (p_2x + q_2y)^5\rangle \setminus \left(T^1_{(p_1x+q_1y)^5} \cup \langle (p_1x + q_1y)^5, (p_2x + q_2y)^5\rangle\right)$$

while if $z(x, y) = (p_1x + q_1y)^3 \in \mathbf{P}(S^3U)$ then

$$\pi^{-1}(z) = T^2_{(p_1x+q_1y)^5} \setminus T^1_{(p_1x+q_1y)^5}.$$

The last two fibers contain polynomials which can be expressed as sum of more than three powers.

In general we consider the $SL(U)$-equivariant morphism

$$\mathbf{P}(S^{2m+1}U) \setminus \sigma_m(C_{2m+1}) \longrightarrow^{\pi} \mathbf{P}(S^{m+1}U).$$

It follows that the polynomials $f \in \mathbf{P}(S^{2m+1}U)$ which have a unique canonical form as sum of $m + 1$ powers are exactly those lying outside the irreducible hypersurface which is the closure of π^{-1}(discriminant), which has degree $2m(m + 1)$, and it is the Zariski closure of the union of all linear spans $\langle T^1_{p_1}, p_2, \ldots, p_m\rangle$ where p_i are distinct points in C_{2m+1}. If $z \in \mathbf{P}(S^{m+1}U)$ has q distinct roots, then the fiber $\pi^{-1}(z)$ is isomorphic to $\mathbf{P}^m$ minus q hyperplanes.

We emphasize that this argument by Sylvester not only proves the uniqueness of the canonical form of an odd binary form as the sum of powers, but its also gives an algorithm to construct it, up to factor a polynomial equation in one variable.

A proof of Theorem 7.1 using symbolic (umbral) calculus can be found in [24].

7.2. The general case

The cases of small degree and the first exceptions in Theorem 1.1 were known since a long time. The first nontrivial exception of plane quartics was studied by Clebsch [17], who found in 1861 the equation of the degree 6 invariant, which gives the hypersurface $\sigma_5(\mathbf{P}^2, \mathcal{O}(4))$, as we sketched in Section 3. Richmond in [32] listed all the exceptions appearing in Theorem 1.1. For example in the more difficult case, concerning a general cubic in $\mathbf{P}^4$ which is not the sum of seven cubes, the method of Richmond is to construct the rational normal curve through seven points, and then to manipulate the equations of the problem into partial fractions. A sentence from Richmond paper is illuminating: *"It does not appear to be possible to make any general application of the method. I therefore continue to consider special problems"*.

To the best of our knowledge, the first paper which faces the problem (with $n \geq 2$) in general was published by Campbell in 1892 [9] on the "Messenger of Mathematics", a journal which stopped being published in 1928 and was absorbed by the Oxford Quarterly Journal. Campbell is better known for the Campbell–Hausdorff formula for multiplication of exponents in Lie algebras. He proved an equivalent form of the second Terracini Lemma 2.3 for linear systems of plane curves by looking at the Jacobian of the system. Campbell deduced that if a union X of k double points does not impose independent conditions on plane curves of degree d, then every curve C of degree d through X has to be a double curve, and d is even. The correct conclusion is that C contains a double component, but it is easy to complete this argument, as we saw in Theorem 2.4 and we repeat in a while. The idea of Campbell was to add t points in order that $3k + t = \binom{d+2}{2}$ and he found also the other equation $k + t = \binom{(d/2)+2}{2} - 1$. This system has only the two solutions

$$\begin{cases} d = 2 \\ k = 2 \\ t = 0 \end{cases} \quad \text{and} \quad \begin{cases} d = 4 \\ k = 5 \\ t = 0 \end{cases}$$

which give the two exceptions of Theorem 1.1 for $n = 2$.

Campbell then considered the case $n = 3$ and he claimed that if a union X of double points does not impose independent conditions on surfaces of degree d, then every surface C of degree d through X has to be a double surface, and d is even. Although the conclusion is correct, the argument given by Campbell seems to be wrong, otherwise it should work also when $n = 4$, but in this case the only cubic singular at seven points is actually reduced. This fourth exceptional case in the list of Theorem 1.1 was probably not known to Campbell. It is worth to remark that Campbell proved in the same paper that the only Veronese surfaces which are weakly defective (in the modern notation, according to [14]) are given by the linear systems $|\mathcal{O}(d)|$ with $d = 2, 4$ or 6. His argument is a slight modification of the previous one, and it seems essentially correct.

Campbell concluded by applying his theorem to the canonical forms of general hypersurfaces as sums of powers, and he got that the expected number of summands is attained, with the only exceptions of Theorem 1.1 (here $n \leq 3$). He did not apply Lasker Proposition 2.1. His more indirect approach, which uses the Jacobian, seems essentially equivalent to Proposition 2.1.

Campbell paper was not quoted by Richmond, we do not know if this is a signal of the rivalry between Oxford and Cambridge.

In Italy the problem was faced in the same years by the school of Corrado Segre. Palatini, a student of Segre, attacked the general problem, and was probably not aware of Campbell's results. The paper [29] is contemporary to [32], and treats the same problem of the defectivity of the system of cubics in $\mathbf{P}^4$. Palatini's argument that shows the defectivity is geometrical, and resembles the one we have sketched in Section 3. A proof of Theorem 1.1 in the case $n = 2$ is given in [30]. We sketch the argument of Palatini in the case $d = 7$, which is direct, in opposition with the ones of Campbell and Terracini which rely on infinitesimal computations. Palatini's aim is to prove that the 12-secant spaces to the 7-Veronese embedding of $\mathbf{P}^2$ fill the ambient space $\mathbf{P}^{35}$. Denote by D_p a plane curve of degree p. Palatini first proved the following preliminary lemma.

Lemma 7.2 (*Palatini*). (i) *Assume $p_1, \ldots, p_{12}$ are general points in $\mathbf{P}^2$ and $p_{13}, \ldots, p_{24}$ are chosen such that $h^0(7H - \sum_{i=1}^{24} p_i) = 36 - 24 + 1 = 13$ (one more than the expected value). Then $p_1, \ldots, p_{24}$ are the complete intersection of a D_4 with a D_6.*

(ii) *Conversely, if $Z = D_4 \cap D_6$ then $h^0(I_Z(7)) = 13$.*

Proof. By assumption a septic D_7 which contains 23 of the given points contains also the last one. Let D_3 be the cubic through $p_1, \dots, p_9$. Let D_4 be a quartic through $p_{11}, \dots, p_{24}$; by assumption it contains also p_{10}. Considering the cubic through $p_1, \dots, p_8, p_{10}$, it follows that D_4 contains also p_9, and continuing in this way, all the points are contained in D_4. The general sextic D_6 through $p_1, \dots, p_{24}$ does not contain D_4 as a component. Indeed let D_1 be the line through p_1 and p_2. Let D_6 be a sextic through $p_4, \dots, p_{24}$, by assumption it contains also p_3. Starting from other lines, such a D_6 contains all the 24 points. Then $H^0(6H - \sum_{i=1}^{24} p_i) = H^0(6H - \sum_{i=4}^{24} p_i)$ which has dimension $\geq 28 - 21 = 7 > 6 = h^0(2H)$. This proves (i). Part (ii) is today obvious from the Koszul complex. □

By duality, a 12-secant space π corresponds to the linear system of D_7 through 12 points $p_1, \dots, p_{12}$. Consider all the other 12-secant spaces which meet our π. These correspond to collections of 12 points $p_{13}, \dots, p_{24}$ such that $h^0(7H - \sum_{i=1}^{24} p_i) = 13$. By Lemma 7.2 these collections of 12 points are parametrized by the pairs (D_4, E) where D_4 is a quartic through $p_1, \dots, p_{12}$ and $p_1 + \cdots + p_{12} + E$ is a divisor cut on D_4 by a sextic. There are ∞^2 quartic curves and by Riemann–Roch formula E has 9 parameters, so that there are ∞^{11} 12-secant spaces which meet our π. This means that for a general point of π there are only finitely many 12-secant spaces, hence the 12-secant variety has the expected dimension as we wanted. Closing the paper [30], Palatini wrote: *"si può già prevedere che l'impossibilità di rappresentare una forma s-aria generica con la somma di potenze di forme lineari contenenti un numero di costanti non inferiore a quello contenuto nella forma considerata, si avrà soltanto in casi particolari".*[1] Then he listed the particular cases known to him, and they are exactly the exceptions of Theorem 1.1. So this sentence can be considered as the first conjecture of the statement of Theorem 1.1.

At the end of [30] it is proved that the expression of the general element of $\sigma_7(V^{5,2})$ has a sum of seven 5th powers is unique. This fact was proved also by Richmond [32], and also Hilbert knew and claimed it in a letter to Hermite in 1888 [21]. For recent results about the uniqueness of canonical forms see [27].

The work of Terracini is a turning point in this story. In his celebrated paper [36] Terracini introduced new techniques to attack the problem, and in particular he proved (what today are called) the first and the second Terracini lemmas, as we have stated in Section 2. These results are not difficult to prove, but they represent a new viewpoint on the subject. Terracini got them in an elegant way, as a natural state of things. In [36] Terracini was actually interested in a different direction. Before his work there were two different characterizations of the Veronese surface. Del Pezzo proved in 1887 that the Veronese surface in $\mathbf{P}^5$ is the unique surface such that any two of its tangent planes meet each other. Severi proved in 1901 that the Veronese surface in $\mathbf{P}^5$ is the unique surface such that its secant variety does not fill the ambient space. Is this only a coincidence? Terracini's approach allows to unify these two results, indeed thanks to the first Terracini lemma the results of Del Pezzo and Severi turn out to be equivalent. This was probably not a surprise because the Severi proof was deeply inspired by the Del Pezzo proof. But this opens another story that we do not pursue here.

In 1915 Terracini, with the paper [37] realized that his two lemmas allow to attack the problem raised by Palatini. Terracini obtained in few lines at page 93 Theorem 1.1 in the case $n = 2$. His argument is the following. The general ternary form of degree d is sum of the expected number $k = \lceil \frac{(d+2)(d+1)}{6} \rceil$ of dth powers of linear form if and only if there is no plane curve having double points at general $p_1, \dots, p_k$. On the other hand if there is such a curve, by Lemma 2.3 it has to contain as a component a double curve of degree $2l$ through $p_1, \dots, p_k$. Hence we have the inequality

$$k \leq \frac{l(l+3)}{2}$$

so that we get the inequality

$$\left\lceil \frac{(d+2)(d+1)}{6} \right\rceil \leq \frac{d}{4} \left(\frac{d}{2} + 3 \right)$$

which gives $d = 2$ or $d = 4$ as we wanted. This is the third published proof of Theorem 1.1 in the case $n = 2$, and the reader will notice that it is a refinement of Campbell proof.

[1] One can expect that the impossibility of representing a general form in s variables as a sum of powers of linear forms containing a number of constants not smaller than the number of constants contained in the given forms, holds only in a few particular cases.

Terracini observed also in [37] that the exceptional case of cubics in $\mathbf{P}^4$ is solved by the consideration that given seven points in $\mathbf{P}^4$, the rational quartic through them is the singular locus of its secant variety, which is the cubic hypersurface defined by the invariant J in the theory of binary quartics.

In [38] Terracini got a proof of Theorem 1.1 for $n = 3$. In the introduction he finally quoted the paper of Campbell, so it is almost certain that he was not aware of it when he wrote the article [37]. Terracini gave to Campbell the credit to have stated correctly Theorem 1.1 in the cases $n = 2$ and $n = 3$. We quote from [38]: *"Questa proposizione fu dimostrata per la prima volta in modo completo dal Palatini* [30], *vedi un'altra dimostrazione nella mia nota* [37]; *ma già l'aveva enunciata parecchi anni prima Campbell* [9] *deducendola con considerazioni poco rigorose, considerazioni che divengono anche meno soddisfacenti quando il Campbell passa ad estendere la sua ricerca alle forme quaternarie".*[2]

This claim about the lack of rigor is interesting, because after a few years the Italian school of algebraic geometry received the same kind of criticism, especially from the Bourbaki circle. The concept of the measure of rigor, invoked by Terracini, is also interesting. Indeed we can agree even today that Campbell argument was essentially correct in the case $n = 2$, but it was wrong in the case $n = 3$.

Terracini's paper [38] represents a change in the writing style. All the lemmas and the theorems are ordered and numbered, differently from all the papers quoted above. His proof is by induction on the degree, and he uses what we called in Section 4 the Castelnuovo sequence, by specializing as many points as possible on a plane. We saw in Section 4 that there is an arithmetic problem which makes the argument hard when the number of double points is near to a critical bound. Terracini's argument plays with linear systems with vanishing jacobian. His approach was reviewed and clarified by Roé, Zappalà and Baggio in [33], during the 2001 Pragmatic School directed by Ciliberto and Miranda. It seems to us that they also filled a small gap at the end of Terracini's proof, obtaining a rigorous proof of Theorem 1.1 in the case $n = 3$. It seems also that this approach does not generalize to higher values of n.

In 1931 it appeared in the paper [8] of Bronowski, at that time in Cambridge. He took the statement of Theorem 1.1 from [30] and he claimed to give a complete proof of it. The argument of Bronowski is based on the possibility to check if a linear system has vanishing jacobian by a numerical criterion. This criterion already fails in the exceptional case of cubics in $\mathbf{P}^4$, and Bronowski tried to justify this fact arguing that the cases $n = 2$ and $n = 3$ are special ones. However it is hard to justify his approach of considering the base curve of the system. In his nice MacTutor biography on the web, accounting a very active life, it is written: *"In 1933 he (Bronowski) published a solution of the classical functional Waring problem, to determine the minimal n such that a general degree d polynomial f can be expressed as a sum of d-th powers of n linear forms, but his argument was incomplete".* We agree with this opinion.

In 1985 Hirschowitz [22] gave a proof of Theorem 1.1 in the cases $n = 2$ and $n = 3$, which makes a step beyond the classical proofs, apparently not known to him at that time. He used the powerful language of zero-dimensional schemes in the degeneration argument, this is the last crucial key to solve the general problem. In 1988 Alexander used the new tools introduced by Hirschowitz and in [2] he proved Theorem 1.1 for $d \geq 5$ with a very complicated but successful inductive procedure. He needed only a limited number of cases for $d \leq 4$ in the starting point of the induction. In the following years Alexander and Hirschowitz got Theorem 1.1 for $d = 4$ [3] and finally in [5] they settled the case $d = 3$, so obtaining the first complete proof of Theorem 1.1. This proof, which in its first version covered more than 150 pages, can be celebrated as a success of modern cohomological theories facing with a long standing classical problem. In 1993 Ehrenborg and Rota [19], not aware of the work by Alexander and Hirschowitz, posed the problem of Theorem 1.1 as an outstanding one.

In 1997 Alexander and Hirschowitz themselves got a strong simplification of their proof in [6], working for $d \geq 5$. By reading [6] it is very clear the role of the *dime* and the *degue*, see the Remark at the end of Section 6. Later Chandler (see [12]) simplified further the proof by Alexander and Hirschowitz in the case $d \geq 4$, with the help of the Curvilinear Lemma 6.1. In [13] she got a simpler proof also in the case $d = 3$.

Recently a different combinatorial approach to the problem succeeded in the case $n = 2$. The idea is to degenerate the Veronese surface to a union of d^2 planes, as we learned from two different talks in 2006 by R. Miranda and S. Sullivant. If in the union of planes we can locate k points on k different planes in such a way that the corresponding planes are transverse, then by semicontinuity the dimension of the k-secant variety is the expected one. A proof of

[2] This proposition was completely proved for the first time by Palatini [30], see another proof in my note [37]; however Campbell [9] already stated it several years before, deducing it in a not very rigorous way, and his argument becomes even less satisfactory when Campbell tries to extend his research to quaternary forms.

Theorem 1.1 in the case $n = 2$ along these lines was published by Draisma [18]. The proof reduces to a clever tiling of a triangular region. This proof was extended to $n = 3$ in Brannetti's thesis [7]. At present it is not clear if this approach, which is related to tropical geometry, can be extended to $n \geq 4$.

We believe that the work on this beautiful subject will continue in the future. Besides the higher multiplicity case mentioned in the introduction, we stress that the equations of the higher secant varieties $\sigma_k(V^{d,n})$ are still not known in general for $n \geq 2$, and their knowledge could be useful in the applications.

Acknowledgement

Both authors are partially supported by Italian MUR and are members of GNSAGA.

References

[1] H. Abo, G. Ottaviani, C. Peterson, Induction for secant varieties of Segre varieties, math.AG/0607191.
[2] J. Alexander, Singularités imposables en position générale à une hypersurface projective, Compositio Math. 68 (3) (1988) 305–354.
[3] J. Alexander, A. Hirschowitz, Un lemme d'Horace différentiel: application aux singularités hyperquartiques de $\mathbf{P}^5$, J. Algebraic Geom. 1 (3) (1992) 411–426.
[4] J. Alexander, A. Hirschowitz, La méthode d'Horace éclatée: application à l'interpolation en degrée quatre, Invent. Math. 107 (1992) 585–602.
[5] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Alg. Geom. 4 (2) (1995) 201–222.
[6] J. Alexander, A. Hirschowitz, Generic Hypersurface Singularities, Proc. Indian Acad. Sci. Math. Sci. 107 (2) (1997) 139–154.
[7] S. Brannetti, Degenerazioni di varietà toriche e interpolazione polinomiale, Tesi di Laurea Specialistica, Roma Tor Vergata, 2007.
[8] J. Bronowski, The sum of powers as canonical expression, Proc. Camb. Phil. Soc. 29 (1933) 69–82.
[9] J.E. Campbell, Note on the maximum number of arbitrary points which can be double points on a curve, or surface, of any degree, Messenger Math. XXI (1891–1892) 158–164.
[10] M.V. Catalisano, A.V. Geramita, A. Gimigliano, Ranks of tensors, secant varieties of Segre varieties and fat points, Linear Algebra Appl. 355 (2002) 263–285. Erratum, Linear Algebra Appl. 367 (2003), 347-348.
[11] M.V. Catalisano, A.V. Geramita, A. Gimigliano, Secant varieties of Grassmann varieties, Proc. Amer. Math. Soc. 133 (3) (2005) 633–642. math.AG/0208166.
[12] K. Chandler, A brief proof of a maximal rank theorem for generic double points in projective space, Trans. Amer. Math. Soc. 353 (5) (2001) 1907–1920.
[13] K. Chandler, Linear systems of cubics singular at general points of projective space, Compos. Math. 134 (2002) 269–282.
[14] L. Chiantini, C. Ciliberto, Weakly defective varieties, Trans. AMS 354 (2002) 151–178.
[15] C. Ciliberto, Geometrical aspects of polynomial interpolation in more variables and of Waring's problem, in: ECM, vol. I (Barcelona, 2000), in: Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 289–316.
[16] C. Ciliberto, A. Hirschowitz, Hypercubiques de P^4 avec sept points singuliers génériques, C. R. Acad. Sci. Paris Sér. I Math. 313 (3) (1991) 135–137.
[17] A. Clebsch, Über Curven fierter Ordnung, J. Reine Angew. Math. 59 (1861) 125–145.
[18] J. Draisma, A tropical approach to secant dimensions, J. Pure Appl. Algebra 212 (2008) 349–363.
[19] R. Ehrenborg, G.C. Rota, Apolarity and canonical forms for homogeneous polynomials, European J. Combin. 14 (3) (1993) 157–181.
[20] S. Gundelfinger, Zur Theorie der binären Formen, J. Reine Angew. Math. 100 (1886) 413–424.
[21] D. Hilbert, Lettre addressée à M. Hermite, in: Gesam. Abh., vol. II, pp. 148–153.
[22] A. Hirschowitz, La méthode d'Horace pour l'interpolation à plusieurs variables, Manuscripta Math. 50 (1985) 337–388.
[23] A. Iarrobino, V Kanev, Power sums, Gorenstein algebras, and determinantal loci, in: Lecture Notes in Mathematics, vol. 1721, Springer,1999.
[24] J. Kung, G. Rota, The invariant theory of binary forms, Bull. Amer. Math. Soc. (N.S.) 10 (1) (1984) 27–85.
[25] E. Lasker, Zur Theorie der kanonischen Formen, Math. Ann. 58 (1904) 434–440.
[26] B. McGillivray, A probabilistic algorithm for the secant defect of Grassmann varieties, Linear Algebra Appl. 418 (2006) 708–718.
[27] M. Mella, Singularities of linear systems and the Waring problem, Trans. Amer. Math. Soc. 358 (2006) 5523–5538.
[28] G. Ottaviani, An invariant regarding Waring's problem for cubic polynomials (in preparation).
[29] F. Palatini, Sulla rappresentazione delle forme ed in particolare della cubica quinaria con la somme di potenze di forme lineari, Atti R. Accad. Sc. Torino 38 (1902–1903) 43–50.
[30] F. Palatini, Sulla rappresentazione delle forme ternarie mediante la somma di potenze di forme lineari, Rend. Accad. Lincei V t. 12 (1903) 378–384.
[31] B. Reichstein, On Waring's problem for cubic forms, Linear Algebra Appl. 160 (1992) 1–61.
[32] H.W. Richmond, On canonical forms, Q. J. Math. 33 (1902) 331–340.
[33] J. Roé, G. Zappalà, S. Baggio, Linear systems of surfaces with double points: Terracini revisited, Le Matematiche 56 (2001) 269–280.
[34] F. Russo, Tangents and Secants to Algebraic Varieties, Publicações Matemáticas do IMPA.24 Colóquio Brasileiro de Matemática (IMPA), Rio de Janeiro, 2003.
[35] J.J. Sylvester, On a remarkable discovery in the theory of canonical forms and of hyperdeterminants, Philos. Mag. II (1851) 391–410. (Collected Math. Papers, vol. I).

[36] A. Terracini, Sulle V_k per cui la varietà degli $S_h h+1$-secanti ha dimensione minore dell'ordinario, Rend. Circ. Mat. Palermo 31 (1911) 392–396. Selecta vol. I.

[37] A. Terracini, Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari, Ann. Mat. 24 (1915) Selecta vol. I.

[38] A. Terracini, Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari, Atti Accad. Sci. Torino 51 (1915–1916) Selecta vol. I.

[39] F.L. Zak, Tangents and Secants of Algebraic Varieties, in: Translations of Mathematical Monographs, vol. 127, AMS, Providence, RI, 1993.