STABILITY OF CRYSTALLINE EVOLUTIONS

MATTEO NOVAGA
Dipartimento di Matematica, Università di Pisa, via Buonarroti 2
56127 Pisa, Italy
novaga@mail.dm.unipi.it

EMANUELE PAOLINI
Dipartimento di Matematica, Università di Firenze viale Morgagni 67a
50134 Firenze, Italy
paolini@math.unifi.it

In this paper we analyze the stability properties of the Wulff-shape in the crystalline flow. It is well known that the Wulff-shape evolves self-similarly, and eventually shrinks to a point. We consider the flow restricted to the set of convex polyhedra, we show that the crystalline evolutions may be viewed, after a proper rescaling, as an integral curve in the space of polyhedra with fixed volume, and we compute the Jacobian matrix of this field. If the eigenvalues of such a matrix have real part different from zero, we can determine if the Wulff-shape is stable or unstable, i.e., if all the evolutions starting close enough to the Wulff-shape converge or not, after rescaling, to the Wulff-shape itself.

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1. Introduction

In the last few years, a considerable effort has been done in the analysis of geometric evolution problems, like for example the mean curvature flow. This research is motivated by various applications coming from the theory of phase transitions and crystal growth,[12,11] and from the denoising problem in image reconstruction.[2,7] In some applications, particular directions are preferred by the evolving set, thus leading to anisotropic evolutions. In this paper we are concerned with a typical example of such evolution, the so called *crystalline mean curvature flow*.[11,5,3]

This motion corresponds to a weighted L^2–gradient flow for a (crystalline) surface energy of the type

$$P_\phi(E) := \int_{\partial E} \phi^o(\nu) d\mathcal{H}^{n-1}, \tag{1.1}$$

where the function $\phi^o : \mathbb{R}^n \to \mathbb{R}$ is positive definite, positively one-homogeneous and piecewise linear. Functions ϕ^o with these properties are called *crystalline anisotropies*. The fact that the function ϕ^o is not strictly convex makes more difficult the study of the functional and of the related flow, since usual elliptic operators are replaced by degenerate monotone operators.

A dual function $\phi : \mathbb{R}^n \to \mathbb{R}$ defined by $\phi(\xi) = \sup\{\langle \xi, \nu \rangle : \phi^o(\nu) \leq 1\}$, is naturally associated to the function ϕ^o. The function ϕ is a crystalline anisotropy too and its unit ball $W_\phi := \{\xi : \phi(\xi) \leq 1\}$ is a convex polyhedron which is usually called the Wulff-shape associated to the energy (1.1).

It turns out that the Wulff-shape is always a homothetic solution of the anisotropic mean curvature flow (even if ϕ^o is not piecewise linear). In the case of Euclidean curvature flow for embedded curves, i.e. when $\phi^o(x) = |x|$ and the space dimension is $n = 2$, Gage and Hamilton proved[4] that the evolution exists up to an extinction time and that, if we suitably rescale the evolution (e.g. in such a way that the volume is kept fixed), the curves converge to a circle. Shortly afterwards, Huisken proved[6] that any convex set in $\mathbb{R}^n$ moving under the Euclidean mean curvature flow, shrinks to a point in finite time, and after rescaling converges to a ball.

Surprisingly, there are some obstructions to the extension of these results to the crystalline setting. Despite the difficulties coming from the weak convexity of the function ϕ^o, in two dimensions the analysis is much easier because the evolution of a regular curve reduces to a simple ODE. For such evolutions many properties are known, among them: the validity of a comparison principle and the existence of the evolution up to a maximal time when the curve shrinks to a point.[5] From the work of Stancu,[9] we also know that a stability result holds for two–dimensional crystalline evolutions; more precisely, under the assumption that the anisotropy is symmetric, i.e. $\phi^o(-x) = \phi^o(x)$, and that the Wulff-shape is not a quadrilateral, any embedded regular curve converges, after rescaling, to the Wulff-shape. On the contrary, if the Wulff-shape is a square, any rectangle shrinks self-similarly under the related crystalline flow, hence it is a stationary point for the rescaled flow.

However, as shown in the examples discussed in[8,10] this result is not true in higher dimensions. For example when the Wulff-shape is a cube any rectangular prism (also very close to W_ϕ) has a rescaled evolution which becomes very different from W_ϕ, and in most of the cases converges to a straight line or to a plane. The cube is not the only unstable Wulff-shape, but we show that there exist many other examples.

In the present paper we study the validity of Huisken's convergence theorem for crystalline anisotropies in $\mathbb{R}^3$. In particular we give an explicit (but rather complicated) formula by means of which we can determine if a given Wulff-shape is stable or unstable.

The plan of the paper is the following. As a first step we restrict ourselves to a finite dimensional case by considering only polyhedral sets obtained from the Wulff-shape by not too large translations of the facets. If the Wulff-shape has N facets, every such polyhedron can be identified by a point $x \in \mathbb{R}^N$. We also assume that the Wulff-shape is *simple*, which means that the number of facets meeting at a vertex is exactly three. Non-simple polyhedra can suddenly change the number of edges at the beginning of the evolution and hence it is more complicated to deal with them. However, in Section 5 we study a non-simple polyhedron by approximating it with simple polyhedra.

We let $\Omega \subset \mathbb{R}^N$ be the open set of the points corresponding to all the polyhedra considered and $\bar{x} \in \Omega$ be the point corresponding to the Wulff-shape. We notice that the class of polyhedral sets is not stable under crystalline mean curvature flow, since "facet-breaking" phenomena may occur,[3,13] hence the evolution we study coincides with the crystalline evolution provided such singularities do not appear, however this doesn't happen if the evolving set is close enough to the Wulff-shape. In this setting the crystalline evolution of a polyhedron becomes the solution $x(t)$ to the system of ODE's

$$\dot{x} = \kappa(x)$$

where $\kappa\colon \Omega \to \mathbb{R}^N$ is the vector field representing the crystalline mean curvature. We express this by saying that $x(t)$ is an integral curve of the vector field κ.

The second step is to renormalize such evolutions up to rescaling and translation. To achieve this we consider a $N-4$-dimensional manifold $\mathcal{M} \subset \Omega$ which represents all the polyhedra which have the same volume and the same barycenter of the Wulff-shape (so that $\bar{x} \in \mathcal{M}$). We then consider a projection $\pi\colon \Omega \to \mathcal{M}$ which maps a polyhedron x to the only polyhedron $\pi(x) \in \mathcal{M}$ which is homothetic to x. We then show that the rescaled evolution $y(t) = \pi(x(t))$ is itself, up to reparameterization, the integral curve of a $N-4$-dimensional vector field $\eta \in T\mathcal{M}$ (Theorem 3.2). In Theorem 3.1 we point out that every other choice for rescaling (e.g. by keeping constant the perimeter instead of the volume) is, in some sense, equivalent.

The last step is to study the stability of the rescaled evolution $\dot{y} = \eta(y)$. Clearly $\eta(\bar{x}) = 0$ since the Wulff-shape is self-similar under the crystalline evolution. Hence the stability of the Wulff-shape can be determined by inspecting the eigenvalues of the Jacobian matrix $d\eta$ (Theorem 4.1). In fact if the real part of all the eigenvalues is negative, then the Wulff-shape is stable. On the other hand, if at least one eigenvalue has positive real part, then the Wulff-shape is unstable. Notice that even if the Wulff-shape is stable, there might be some rescaled evolutions which do not converge to the Wulff-shape. Indeed, there are evidences of anisotropies in which the Wulff-shape is stable but there exist different stationary solutions of the rescaled flow.[8] In Section 4 we explain how to compute the Jacobian matrix $d\eta$ in terms of the geometry of the Wulff-shape.

We conclude the paper discussing some numerical computation in which the eigenvalues of the Jacobian matrix are computed for some polyhedron. The diagrams show that if the number of faces is high enough and the shape is roughly spherical then it is likely that the Wulff-shape is stable. There are also some evidences that the stability as well as instability properties are preserved by small C^0-perturbation of the W_ϕ, in particular we expect the existence of regular and strictly convex anisotropies which do not satisfy the convergence theorem.

2. Notation

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a positively one-homogeneous, convex and coercive function, i.e. suppose that

(1) $\phi(\lambda v) = \lambda\phi(v)$ for all $v \in \mathbb{R}^3$ and $\lambda \geq 0$;
(2) $\phi(v + w) \leq \phi(v) + \phi(w)$;
(3) $\phi(v) = 0$ if and only if $v = 0$.

Let $W_\phi := \{v \in \mathbb{R}^3 : \phi(v) \leq 1\}$ be the unit ball with respect to ϕ. W_ϕ turns out to be a convex closed neighborhood of 0 and is usually called *Wulff-shape* relative to ϕ.

Let $F_\phi := \{\xi \in \mathbb{R}^3 : \langle \xi, v\rangle \leq 1 \quad \forall v \in W_\phi\}$. The bracket product $\langle \cdot, \cdot\rangle$ is the usual inner product of $\mathbb{R}^3$: $\langle(\xi^1, \xi^2, \xi^3), (v_1, v_2, v_3)\rangle := \xi^1 v_1 + \xi^2 v_2 + \xi^3 v_3$. F_ϕ is usually called *Frank Diagram* and is itself a unit ball with respect to a dual norm ϕ^o which is defined by

$$\phi^o(\xi) := \max_{\phi(v)\leq 1} \langle \xi, v\rangle.$$

It can be easily proved that ϕ^o is again positively one-homogeneous, convex and coercive, moreover $\phi^{oo} = \phi$, $F_\phi = W_{\phi^o}$ and $W_\phi = F_{\phi^o}$.

Given a set $E \subset \mathbb{R}^3$ with Lipschitz boundary, we can define its perimeter with respect to the anisotropy ϕ:

$$P_\phi(E) := \int_{\partial E} \phi^o(\nu_E(x))\, \mathrm{d}\mathcal{H}^2(x).$$

Here $\nu_E(x)$ is the external unit normal to E and $\mathcal{H}^2$ is the Hausdorff area measure.

When $\phi(x) = |x|$ (the usual Euclidean norm of $\mathbb{R}^3$) we have $W_\phi = F_\phi = \overline{B_1}$ (the closed unit ball) and P_ϕ is the usual Euclidean perimeter $P_\phi(E) = \mathcal{H}^2(\partial E)$. In this case the *mean curvature* of E can be viewed as minus the gradient of P_ϕ with respect to the L^2-norm of the variations of E.[3]

2.1. *The crystalline setting*

From now on we restrict to the case when W_ϕ (and hence F_ϕ) is a polyhedron (crystalline case), i.e. when the function ϕ is piecewise linear. We call a polyhedron *simple* if all its vertices are the intersection of exactly three facets; we also say that a polyhedron is *simplicial* if every facet has three edges. In order to simplify the computations, we shall assume that W_ϕ is simple, which in turn implies that F_ϕ is simplicial.

Notice that there is a one-to-one correspondence between the vertices of F_ϕ and the facets of W_ϕ, and that the vector representing a vertex of F_ϕ is orthogonal to the corresponding facet of W_ϕ. Moreover, if ξ is a vertex of F_ϕ, then the facet of W_ϕ corresponding to ξ is contained in the plane $\{v : \langle \xi, v\rangle = 1\}$ which has distance $1/|\xi|$ from the origin.

We shall only consider sets E which are simple convex polyhedra with the same number of facets of W_ϕ and such that every facet of E is parallel to a corresponding facet of W_ϕ. In fact E will be regarded as a variation of W_ϕ, obtained by moving the facets parallel to themselves. In this setting, the space of sets we consider can be identified with an open subset of $\mathbb{R}^N$, where N is the number of facets of W_ϕ.

Let $\xi_1, \ldots, \xi_N$ be the vertices of F_ϕ. Given $x \in \mathbb{R}^N$ we define

$$E(x) := \{v \in \mathbb{R}^3 : \langle \xi_k, v\rangle \le x_k, \quad \forall k = 1, \ldots, N\} = \{v \in \mathbb{R}^3 : \Xi \cdot v \le x\},$$

where $\Xi := \text{rows}\{\xi_1, \ldots, \xi_N\}$ is the matrix generated by the vectors $\xi_1, \ldots, \xi_N$. Notice that $W_\phi = E(\bar{x})$ where $\bar{x} = (1, 1, \ldots, 1) \in \mathbb{R}^N$.

So $E(x)$ is a polyhedron with no more than N facets and it has exactly N facets if x is close enough to $\bar{x}$. Given k, j, i different indices in $\{1, 2, \ldots, N\}$ we define the facets, edges and vertices of $E(x)$ as

$$E_k(x) := \{v \in E(x) : \langle \xi_k, v\rangle = x_k\},$$

$$E_{kj}(x) := E_k(x) \cap E_j(x), \qquad E_{kji}(x) := E_k(x) \cap E_j(x) \cap E_i(x),$$

together with their measures

$$m(x) := \mathcal{H}^3(E(x)), \qquad m_k(x) := \mathcal{H}^2(E_k(x)),$$

$$m_{kj}(x) := \mathcal{H}^1(E_{kj}(x)), \qquad m_{kji}(x) := \mathcal{H}^0(E_{kji}(x)).$$

Moreover, we define the *barycenters* of the elements of $E(x)$ as

$$b(x) := \frac{1}{m(x)} \int_{E(x)} z \, \mathrm{d}\mathcal{H}^3(z), \qquad b_k(x) := \frac{1}{m_k(x)} \int_{E_k(x)} z \, \mathrm{d}\mathcal{H}^2(z),$$

$$b_{kj}(x) := \frac{1}{m_{kj}(x)} \int_{E_{kj}(x)} z \, \mathrm{d}\mathcal{H}^1(z).$$

Finally, we consider the sets of relevant indices of facets, edges and vertices:

$$I_1 := \{1, 2, \ldots, N\}, \qquad I_2 := \{(k, j) \in I_1^2 : 0 < m_{kj}(\bar{x}) < \infty\},$$

$$I_3 := \{(k, j, i) \in I_1^3 : m_{kji}(\bar{x}) = 1\}.$$

Notice that $m_{kj} = \infty$ only if $k = j$ and $m_{kji} \in \{0, 1, \infty\}$. Clearly $\#I_1$ is the number of facets of W_ϕ, $\#I_2$ is twice the number of edges and, since W_ϕ is simple, $\#I_3$ is six times the number of vertices.

We now restrict our study to the following subset of $\mathbb{R}^N$:

$$\Omega := \{x \in \mathbb{R}^n : m_{kji}(x) = m_{kji}(\bar{x}), \quad \forall (k, j, i) \in I_3\}.$$

This definition implies that for every $x \in \Omega$ the polyhedron $E(x)$ has the same number of facets, edges and vertices of $W_\phi = E(\bar{x})$. Being W_ϕ simple the set Ω is open and clearly $\bar{x} \in \Omega$.

In this finite dimensional setting the anisotropic perimeter P_ϕ is given by

$$P(x) := P_\phi(E(x)) = \sum_{k \in I_1} \phi^o(\xi_k/|\xi_k|) m_k(x) = \sum_k \frac{m_k(x)}{|\xi_k|}$$

so that $P : \Omega \to \mathbb{R}^+$. It turns out that Ω should not be considered flat. Indeed on Ω we put the Riemannian metric which corresponds to the L^2-Φ-norm of variations of the sets $E(x)$. So given $x \in \Omega$ and $v \in T_x\Omega = \mathbb{R}^N$, we can consider the polyhedron $E(x+v)$ as a variation of $E(x)$. The L^2-Φ-norm of the variation is then

$$\|v\|_x := \left(\sum_k \frac{m_k(x)}{|\xi_k|} v_k^2\right)^{\frac{1}{2}}$$

which is induced by the following scalar product

$$(v,w)_x := \sum_{k=1}^N \frac{m_k(x)}{|\xi_k|} v_k w_k, \qquad v, w \in T_x\Omega.$$

This scalar product turns Ω into a Riemannian manifold of dimension N.

We can now define the *crystalline curvature* vector field $\kappa : \Omega \to \mathbb{R}^N$ simply by

$$\kappa(x) := -\nabla_x P(x)$$

where $(\nabla_x P(x), v)_x = \sum_j (\nabla P)_j v_j m_j / |\xi_j| = \frac{\partial P}{\partial v}(x)$. So we obtain

$$\kappa_j(x) = -\frac{|\xi_j|}{m_j(x)} \frac{\partial P}{\partial x_j}(x).$$

3. Evolution and limit shape

The mean curvature evolution flow is then given (at least for a short time, due to possible facet-breaking phenomena) by the solution of the following system of ODE's:

$$\begin{cases} \dot{x}(t) = \kappa(x(t)), \\ x(0) = x_0. \end{cases} \tag{3.1}$$

Notice that, since the function $\kappa = -\nabla_x P$ belongs to $C^\infty(\Omega)$, for any initial datum $x_0 \in \Omega$ problem (3.1) has a unique solution $x(t) \in \Omega$ defined on a maximal time interval $[0, T[$.

First of all, notice that given $x \in \Omega$ and $t \in \mathbb{R}$ it holds $P(tx) = t^2 P(x)$ which implies

$$2P(x) = \frac{d}{dt} P(tx)_{|t=1} = (\nabla P(x), x)_x = -(\kappa(x), x)_x.$$

In particular $\kappa(x) \neq 0$ for all $x \in \Omega$ (recall that $x \in \Omega$ implies $P(x) > 0$). Moreover, if $x(t)$ is a solution to (3.1) we get

$$(\dot{x}(t), x(t))_{x(t)} = -2P(x(t)) < 0,$$

hence there exists a time $T > 0$ such that

$$\lim_{t \to T^-} x(t) \in \partial\Omega.$$

Also notice that

$$\frac{d}{dt}P(x(t)) = \big(\nabla P(x(t)), \kappa(x(t))\big)_{x(t)} = -(\kappa(x(t)), \kappa(x(t)))_{x(t)} < 0;$$

hence $P(x(t))$ is a strictly decreasing function.

An important result about this evolution law is that for $x_0 = \bar{x}$ the solution to (3.1) is the following:

$$x(t) = \sqrt{1-2t}\bar{x}.$$

We notice that the evolution is *self similar*, defined for $t < T = 1/2$ and that for $t \to T^-$ we get $x(t) \to 0 \in \partial\Omega$. So the set $E(x(t))$ tends to $\{0\}$ (in the sense of Hausdorff convergence of sets) as $t \to T^-$. But we are more interested in the *shape* of $E(x(t))$ which, in this case, is always W_ϕ.

As *limit shape* we mean the limits of a sequence of compact sets up to translation an rescaling. More precisely: given a sequence E_k of compacts, non-empty subsets of $\mathbb{R}^3$, we say that a compact set $E \neq \emptyset$ is a *limit shape* of E_k if there exist a sequence $\lambda_k > 0$ and a sequence $z_k \in \mathbb{R}^3$ such that

$$\lambda_k(E_k - z_k) \to E$$

with respect to the Hausdorff distance of compact sets.

Obviously if E is a limit shape than also $\lambda E + z$ is a limit shape for all $\lambda > 0$ and $z \in \mathbb{R}^3$. Moreover every singleton $\{z\}$ is always a limit shape. The following theorem shows that apart from these pathologies the limit shape is unique.

Theorem 3.1. *Let E_k, E, F be compact subsets of $\mathbb{R}^3$, let $\lambda_k, \mu_k \in \mathbb{R}$, with $\lambda_k, \mu_k > 0$, and $x_k, y_k \in \mathbb{R}^3$. Suppose that*

$$\lambda_k(E_k - x_k) \to E, \qquad \mu_k(E_k - y_k) \to F$$

and suppose moreover that $0 < \operatorname{diam} E, \operatorname{diam} F < +\infty$. Then there exist $\lambda > 0$ and $x \in \mathbb{R}^3$ such that

$$E + x = \lambda F.$$

Proof. Let $\lambda = \operatorname{diam} E / \operatorname{diam} F$. Since

$$\lambda_k \operatorname{diam} E_k = \operatorname{diam}(\lambda_k(E_k - x_k)) \to \operatorname{diam} E,$$

$$\mu_k \operatorname{diam} E_k = \operatorname{diam}(\mu_k(E_k - y_k)) \to \operatorname{diam} F$$

we get $\lambda_k/\mu_k \to \lambda$. So

$$\lambda_k(E_k - x_k) + \lambda_k(x_k - y_k) = \lambda_k(E_k - y_k) = \lambda\mu_k(E_k - y_k) \to \lambda F$$

and since $\lambda_k(E_k - x_k) \to E$ we get that $\lambda_k(x_k - y_k)$ converges to some $x \in \mathbb{R}^3$ with $E + x = \lambda F$. $\square$

3.1. *Renormalized evolution*

We now want to define translations and rescaling on $\Omega \subset \mathbb{R}^N$. Notice that given $x \in \Omega$ and $\lambda > 0$ one has $\lambda x \in \Omega$ and

$$E(\lambda x) = \lambda E(x).$$

Given $z \in \mathbb{R}^3$, we define the translation operator $\tau_z : \Omega \to \Omega$ so that

$$E(\tau_z(x)) = E(x) + z.$$

This implies that τ_z can be defined in coordinates as

$$(\tau_z(x))_k := x_k + \langle \xi_k, z \rangle$$

that is $\tau_z(x) = x + \Xi z$.

Let now define the $(N-4)$-dimensional manifold $\mathcal{M} \subset \Omega$ which corresponds to properly rescaled polyhedra

$$\mathcal{M} := \{x \in \Omega : m(x) = m(\bar{x}),\ b(x) = b(\bar{x})\}$$

and let $\pi : \Omega \to \mathcal{M}$ be renormalization defined as

$$\pi(x) := \left(\frac{m(x)}{m(\bar{x})}\right)^{-\frac{1}{3}} (\tau_{b(\bar{x})-b(x)}(x))$$

so that $m(\pi(x)) = m(\bar{x})$ and $b(\pi(x)) = b(\bar{x})$. Notice that $\pi(\Omega) = \mathcal{M}$ and that $\pi_{|\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is the identity map on $\mathcal{M}$.

Our aim is to study the stability of the self-similar evolutions, where self-similar means here stationary for the flow up to homothety and translation. Given a solution to $\dot{x}(t) = \kappa(x(t))$ we consider the rescaled evolution $\pi(x(t))$ and, in the following theorem, we show that this evolution can be regarded as an integral curve in $\mathcal{M}$ with respect to the vector field $\eta \in T\mathcal{M}$ given by $\eta(y) := d\pi_y \cdot \kappa(y)$, for all $y \in \mathcal{M}$. So we can study the stability of $\pi(x(t))$ by inspecting the eigenvalues of the Jacobian $(N-4) \times (N-4)$-dimensional matrix $d\eta$.

Theorem 3.2. *Given $x_0 \in \Omega$ let $x(t)$ ($t \in [0,T)$) and $y(s)$ ($s \in [0,S)$) be respectively the maximal solutions to the following Cauchy problems:*

$$\begin{cases} \dot{x}(s) = \kappa(x(s)) \\ x(0) = x_0 \end{cases} \qquad \begin{cases} \dot{y}(s) = \eta(y(s)) \\ y(0) = \pi(x_0) \end{cases}$$

Then there exists a reparameterization $t \mapsto s(t)$ such that $\pi(x(t)) = y(s(t))$.

Lemma 3.1. *Let $x \in \Omega$. Then*

$$d\pi_x \cdot \kappa(x) = \left(\frac{m(x)}{m(\bar{x})}\right)^{-\frac{2}{3}} d\pi_{\pi(x)} \cdot \kappa(\pi(x)).$$

Proof. Let $x_0 \in \Omega$ be a given point. Given $x \in \Omega$ define $\alpha(x) := (m(\bar{x})/m(x))^{\frac{1}{3}} \in \mathbb{R}$ and $\beta(x) = b(\bar{x}) - b(x) \in \mathbb{R}^3$. Recall that $\pi(x) = \alpha(x)\tau_{\beta(x)}(x)$. Consider the application $T : \Omega \to \Omega$ defined by

$$T(x) := \alpha(x_0)\tau_{\beta(x_0)}(x) = \alpha(x_0)(x + \Xi\beta(x_0)).$$

Clearly $T(x_0) = \pi(x_0)$. Moreover $\pi(T(x)) = \pi(x)$ for all $x \in \Omega$. Differentiating the last equation in x_0 we get

$$d\pi_{\pi(x_0)} \cdot dT_{x_0} = d\pi_{x_0}.$$

Notice now that $dT_x = \alpha(x_0)Id$ and recall that $\kappa(x) = \alpha(x)\kappa(\pi(x))$. So we obtain

$$d\pi_{x_0} \cdot \kappa(x_0) = \alpha^2(x_0) d\pi_{\pi(x_0)} \cdot \kappa(\pi(x_0))$$

which is the claim of the lemma. □

Proof. [Theorem 3.2] Let $\alpha(x)$ be defined as in Lemma 3.1 and let

$$s(t) = \int_0^t \alpha^2(x(\tau))\,\mathrm{d}\tau.$$

Clearly $s(t)$ is strictly increasing and hence invertible. Let $t(s)$ be its inverse and define

$$z(s) = \pi(x(t(s))).$$

We claim that $z(s) = y(s)$ which concludes the proof. First notice that $z(0) = \pi(x(0)) = \pi(x_0) = y(0)$. Being $\dot{t}(s) = (\alpha(x(t(s))))^{-2}$ and applying the previous lemma we get

$$\dot{z}(s) = \dot{t}(s) d\pi_{x(t(s))} \cdot \kappa(x(t(s))) = d\pi_{z(s)} \cdot \kappa(z(s)) = \eta(z(s)).$$

So z and y are solutions to the same Cauchy problem therefore by uniqueness we get $z = y$. □

4. The Stability Condition for W_ϕ

Let us assume that $\bar{y} \in \mathcal{M}$ corresponds to a self-similar evolving polyhedron. By the previous results we may state this simply as $\eta(\bar{y}) = 0$ which means that $\bar{y}$ is a fixed point of the dynamical system $\dot{x} = \eta(x)$. We know that $\bar{x}$ (which represents the Wulff-shape) has always this property, but in some cases there may exist self-similar solutions different from W_ϕ.

We say that $\bar{y}$ correspond to a *stable* self-similar evolution if there exists $\rho > 0$ such that for any $x_0 \in B_\rho(\bar{y})$ we have $\lim_{t\to T^-} \pi(x(t)) = \bar{y}$. Conversely, we say that $\bar{y}$ correspond to an *unstable* evolution if there exists $\rho > 0$ such that for any $\rho' < \rho$ there exists $x' \in B_{\rho'}(\bar{y})$ and $\tau = \tau(x')$ such that $x'(\tau) \notin B_\rho(\bar{y})$.

The following result follows by standard analysis on nonlinear ODE's on manifolds.[1]

Theorem 4.1. *If $\eta(\bar{y}) = 0$ and all the eigenvalues of the linear mapping $d\eta_{\bar{y}}$ have strictly negative real part then $\bar{y}$ corresponds to a stable evolution. On the other hand, if $\eta(\bar{y}) = 0$ and the linear mapping $d\eta_{\bar{y}}$ has an eigenvalue with strictly positive real part, then $\bar{y}$ corresponds to an unstable evolution.*

We want to write an explicit formula for $d\eta_{\bar y}$, i.e. an expression which involves only the geometric quantities of the polyhedron $E(\bar y)$: the measures and the barycenters of edges and facets.

In order to make explicit computations of the eigenvalues of $d\eta_{\bar y}$, we want to represent this linear mapping with an $(N-4)\times(N-4)$ matrix. To achieve this we consider an extension $\tilde\eta\colon\Omega\to\mathbb{R}^N$ of $\eta\in T\mathcal{M}$, defined on the whole Ω by the formula

$$\tilde\eta(x) := d\pi_x\cdot\kappa(x).$$

Then we consider any orthonormal basis $(v_1,\dots,v_{N-4})$ of $T_{\bar y}\mathcal{M} = d\pi_{\bar y}(\mathbb{R}^N)$ and let $Q : T_{\bar y}\mathcal{M}\to\mathbb{R}^N$ be the $(N-4)\times N$ matrix with columns $v_1,\dots,v_{N-4}$. The matrix which represents $d\eta_{\bar y}$ in these coordinates is

$$d\eta_{\bar y} = Q^t d\tilde\eta_{\bar y} Q \tag{4.1}$$

and we want to compute this matrix in terms of the geometric description of the Wulff-shape. In order to accomplish this we have to compute the matrices $d\pi_{\bar y}$ and $d\tilde\eta_{\bar y}$.

Given $x\in\Omega$, we have

$$\frac{\partial m(x)}{\partial x_j} = \frac{m_j(x)}{|\xi_j|}.$$

For $(k,j)\in I_2$ we have

$$\frac{\partial m_k(x)}{\partial x_j} = \frac{m_{kj}(x)}{|\xi_j|\sin\alpha_{kj}} \quad\text{with}\quad \alpha_{kj} := \arccos\frac{\xi_k\cdot\xi_j}{|\xi_k|\,|\xi_j|},$$

and

$$\frac{\partial m_k(x)}{\partial x_k} = -\frac{1}{|\xi_k|}\sum_{(j,k)\in I_2}\cot\alpha_{kj}\, m_{kj}(x).$$

From the equation $\frac{\partial}{\partial x_k}\int_{E(x)} z\,\mathrm{d}z = \frac{1}{|\xi_k|}\int_{E_k(x)} z\,\mathrm{d}z$ we get

$$\frac{\partial b(x)}{\partial x_k} = \frac{m_k(x)}{|\xi_k| m(x)}[b_k(x)-b(x)].$$

Moreover, from $\frac{\partial}{\partial x_j}\int_{E_k(x)} z\,\mathrm{d}z = \frac{1}{|\xi_j|\sin\alpha_{kj}}\int_{E_{kj}(x)} z\,\mathrm{d}z$ we obtain

$$\frac{\partial b_k(x)}{\partial x_j} = \frac{1}{|\xi_j|\sin\alpha_{kj}}\frac{m_{kj}(x)}{m_k(x)}[b_{kj}(x)-b_k(x)],$$

while from $\frac{\partial}{\partial x_k}\int_{E_k(x)} z\,\mathrm{d}z = \frac{v_k}{|\xi_k|^2}m_k(x) - \frac{1}{|\xi_k|}\sum_{(j,k)\in I_2}\cot\alpha_{kj}\int_{E_{kj}(x)} z\,\mathrm{d}z$ we get

$$\frac{\partial b_k(x)}{\partial x_k} = \frac{\xi_k}{|\xi_k|^2} - \frac{1}{|\xi_k|}\sum_{(j,k)\in I_2}\cot\alpha_{kj}[b_{kj}(x)-b_k(x)]\frac{m_{kj}(x)}{m_k(x)}.$$

Finally, for $(k,j,i)\in I_3$ we have

$$\frac{\partial m_{kj}(x)}{\partial x_i} = \frac{|\xi_k\times\xi_j|}{|(\xi_k\times\xi_j)\cdot\xi_i|},$$

while, for $(k,j) \in I_2$,

$$\frac{\partial m_{kj}(x)}{\partial x_k} = \sum_{(i,j,k)\in I_3} \frac{(\xi_j \times \xi_i)\cdot(\xi_k \times \xi_j)}{|\xi_k \times \xi_j|\,|(\xi_i \times \xi_j)\cdot \xi_k|}.$$

Notice that, since $E(x)$ is simple, the last sum is composed by only two terms.

Now we deal with the derivatives of π. Recall that $\pi_k(x) = [m(x)/m(\bar{x})]^{-\frac{1}{3}}$ $\{x_k + \xi_k \cdot [b(\bar{x}) - b(x)]\}$, hence

$$\begin{aligned}\frac{\partial \pi_k(x)}{\partial x_j} = \left[\frac{m(x)}{m(\bar{x})}\right]^{-\frac{4}{3}} & \frac{m_j(x)}{m(\bar{x})|\xi_j|} \\ & \cdot \left\{-\frac{x_k}{3} + \delta_{kj}|\xi_j|\frac{m(x)}{m_j(x)} + \xi_k \cdot \left[\frac{4}{3}b(x) - \frac{1}{3}b(\bar{x}) - b_j(x)\right]\right\},\end{aligned} \tag{4.2}$$

where $\delta_{kj} = 1$ if $k = j$ and 0 otherwise.

We can now compute the components of $d\tilde{\eta}_{\bar{y}} = d(d\pi \cdot \kappa)_{\bar{y}}$, which are

$$\frac{\partial \tilde{\eta}_j(\bar{y})}{\partial x_i} = \sum_{k=1}^{N} \frac{\partial^2 \pi_j(\bar{y})}{\partial x_i \partial x_k}\kappa_k(\bar{y}) + \sum_{k=1}^{N} \frac{\partial \pi_j(\bar{y})}{\partial x_k}\frac{\partial \kappa_k(\bar{y})}{\partial x_i}, \tag{4.3}$$

where

$$\kappa_k(x) = -\frac{|\xi_j|}{m_k(x)}\frac{\partial P(x)}{\partial x_k} = -\sum_{(j,k)\in I_2} \frac{m_{kj}(x)}{m_k(x)}\frac{|\xi_k| - |\xi_j|\cos\alpha_{kj}}{|\xi_k|\,|\xi_j|\sin\alpha_{kj}}. \tag{4.4}$$

So, gathering equations (4.2), (4.3) and (4.4) we are able to compute the matrix (4.1) and hence the eigenvalues and eigenvectors of the linear mapping $d\eta_{\bar{y}}$ in any point $\bar{y} \in \mathcal{M}$.

5. Numerical Computations

In this section we compute the eigenvalues of the matrix $d\eta_{\bar{x}}$ for some class of polyhedra, and we discuss the results. Our computation shows that, at least in these classes, such matrix has always real eigenvalues; we don't know if this is a general property of the matrix $d\eta_{\bar{x}}$.

In Figure 1 we plot the eigenvalues corresponding to the prisms which have a regular n-agonal basis, with $n \in \{3,\ldots,30\}$. The ratio between the radius of the basis and the height of the prism does not affect the eigenvalues.

Both the upper and lower diagram show the eigenvalues, but the lower one has been magnified in the y-axis. We recall that for a polyhedron with N facets (in this case $N = n + 2$) the corresponding matrix $d\eta_{\bar{x}}$ has dimension $(N-4) \times (N-4)$ and hence has $N - 4$ eigenvalues. We notice in particular that every prism has the eigenvalue 1, which means that every prism is unstable. Examining the corresponding eigenvector one can realize that this instability corresponds to a "straightening" of the prism in the vertical direction. We also notice that: the cube ($n = 4$) has 1 as a double eigenvalue, the pentagonal prism has three positive eigenvalues (hence three different instabilities), and the hexagonal prism has 0 as

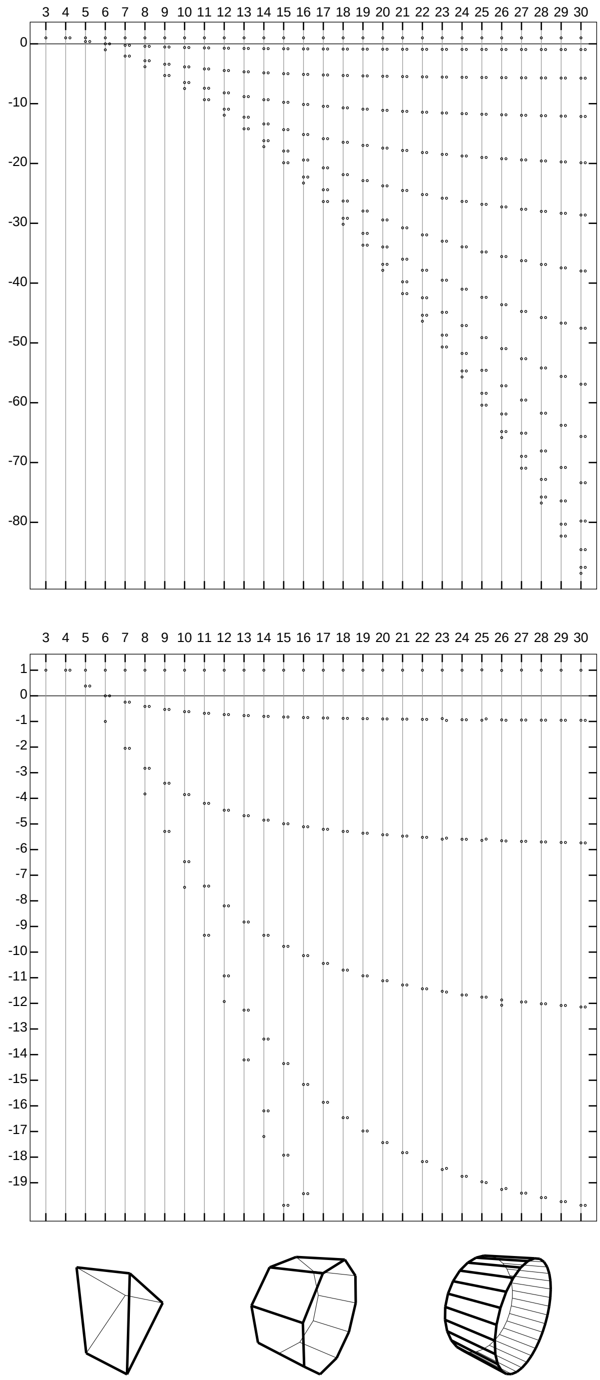

Fig. 1. n-agonal prisms, with $n \in \{3, \ldots, 30\}$.

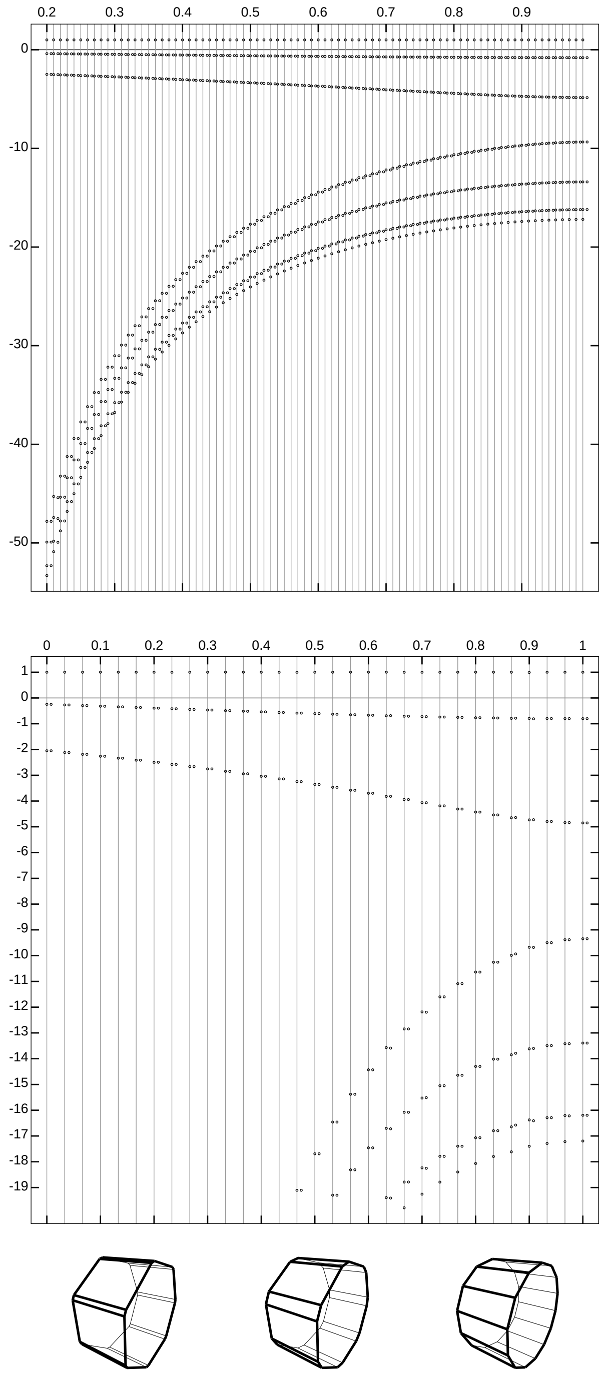

Fig. 2. 14-agonal prisms, with every other side of length $l \in [0, 1]$.

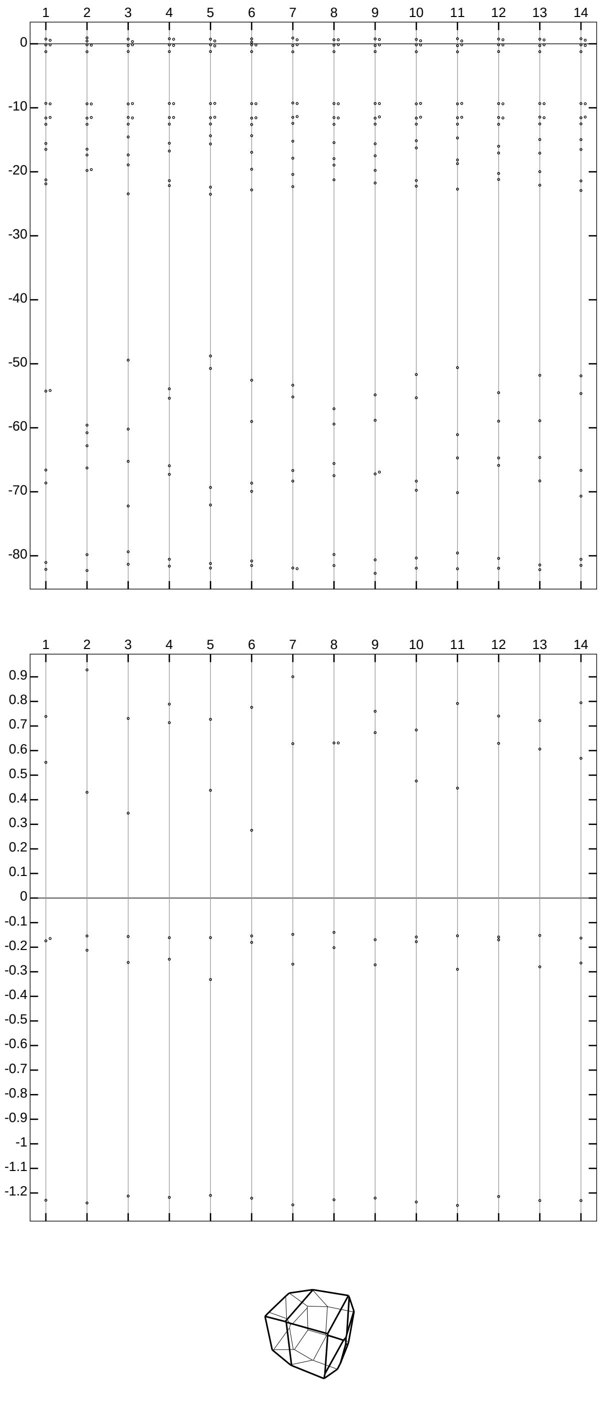

Fig. 3. Fourteen different perturbations of a non simple polyhedron with 24 facets.

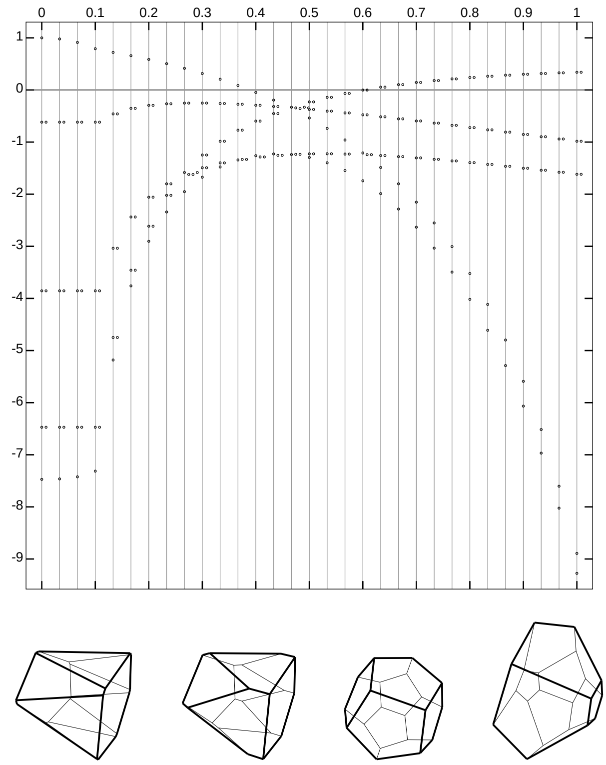

Fig. 4. Dodecahedron with height $h \in [0, 1]$.

a double eigenvalue. For $n \geq 7$ all prisms have the single eigenvalue 1, while all other eigenvalues are negative and possibly double (hence all the other instabilities disappear). It seems also that the set of the eigenvalues converges, as $n \to \infty$, to the expected (infinite) set of eigenvalues for the cylinder.

In Figure 2 we plot the eigenvalues corresponding to a prism which has a polygon with 14 sides as basis. The basis is obtained from a regular heptagon by cutting the vertices so that every new side of the resulting 14-agon is parallel to the sides of the regular one (which we assume has sides of length 1) and has length $l \in [0, 1]$. So for $l = 0$ we have the regular heptagonal prism, while for $l = 1$ we have the regular 14-agonal prism. We observe that the set of eigenvalues converges, as $l \to 0^+$, to the set of eigenvalues of the regular heptagonal prism, in the sense that the first five eigenvalues converge to the five eigenvalues of the heptagonal prism, whereas the others go to $-\infty$.

Considering these pictures, we expect some sort of stability of the eigenvalues under small perturbations (in the Hausdorff distance) of the Wulff-shape, even in the case when the Wulff-shape changes geometry. In particular we would expect that every smooth Wulff-shape which is close enough to an unstable polyhedron is itself unstable for the corresponding anisotropic flow.

In Figure 3 we deal with one of the examples given by Paolini and Pasquarelli[8]. The polyhedron considered is a small perturbation of an hexagonal prism. It has twentyfour facets with unit distance from the origin and whose normal vectors are

$$(\cos(k\alpha)\cos\varepsilon, \sin(k\alpha)\cos\varepsilon, \pm\sin\varepsilon), \quad (\cos(k\alpha)\sin\delta, \sin(k\alpha)\sin\delta, \pm\cos\delta)$$

where $\alpha = \pi/3$, $k \in \{1, \dots, 6\}$, $\varepsilon = 0.1$, $\delta = 0.1$. This polyhedron is not simple, hence our computations do not apply directly. However we can obtain a simple polyhedron by making a small random perturbation of all vertices. In the first diagram of Figure 3 we present the twenty eigenvalues of forteen different random perturbations of the same polyhedron just described. In the second diagram the y-axis has been magnified. It is apparent that there are two positive eigenvalues which correspond to two different directions of instability. With respect to the hexagonal prism (which has eigenvalues: $1, 0, 0, -1$), we find an additional positive eigenvalue, which is due to the fact that we split in two the vertical facets of the prism. The result of the numerical computations is in agreement with the discussion in[8].

In Figure 4 we consider a family of dodecahedra depending on a parameter h. All dodecahedra have two parallel horizontal facets while the angle between the other ten facets and the vertical is given by h radians. For $h = 0$ the two horizontal facets are regular pentagons while the other ten facets are triangles. For small h the two horizontal facets are decagons and the other facets are quadrilaterals. For the other values of h the dodecahedron is composed by pentagons. The value $h \sim 0.47$ gives the regular dodecahedron which has two different negative eigenvalues with multiplicities 5 and 3, hence it is stable. Moreover, close enough to the regular dodecahedron, all the eigenvalues are still negative, hence the corresponding polyhedra are stable. This fact suggests that all polyhedra which are close enough to the sphere should be stable. We also notice that when the dodecahedron changes geometry (for $h \sim 0.1$) the curve of eigenvalues has an angle, while for all the other values this curve is smooth.

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