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p-Groups with All the Elements of Order p in the Center*

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Abstract. Resting on a suitable base of the quotients of the λ_i-series for the free groups on r generators, we get, for p odd, a class of TH-p-groups (the groups in the title) ${}^nG_r$ with arbitrary large derived length. We prove that every TH-p-group G with r generators and exponent p^n is a quotient of ${}^nG_r$ and a product of m cyclic groups, where $p^m = |\Omega_1(G)|$. At last we describe the TH-p-groups of exponent p^2.

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1 Introduction

Let G be a p-group and p an odd prime. We denote by $\Omega_i(G)$ the subgroup of G generated by the elements of order dividing p^i, and we call G a *TH-p-group* if all its elements of order p are central, i.e., $\Omega_1(G) \leq Z(G)$. This name was introduced by the authors in [2] in acknowledgment to Thompson who first obtained some classical results for the number of generators of these groups (see [4, III, 12.2]). In [2], it was shown that several properties of the regular p-groups hold also for the class of TH-p-groups. There we characterized the TH-p-groups G with $|\Omega_1(G)| = p^2$ and exhibited some other examples of TH-p-groups obtaining only metabelian groups.

In this paper, following a suggestion of C.M. Scoppola, we construct a class of TH-p-groups ${}^nG_r$ (see Definition 3.1) with arbitrary large derived length (see Theorems 3.2 and 3.3).

This construction rests on the properties of the central series $\lambda_i(F_r)$ of the free group F_r on r generators and on the behaviour of a particular base

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of the elementary abelian quotients $\lambda_i(F_r)/\lambda_{i+1}(F_r)$ (see Theorem 2.5(c)). The explicit determination of a base for $\lambda_i(F_r)/\lambda_{i+1}(F_r)$ has interest in itself and we devote Section 2 to this, resting on methods and results in [5] and [1].

Similar goals for the central series $\kappa_n(F_r)$, the Jennings–Lazard–Zassenhaus series of G, were reached by C.M. Scoppola in [6] (compare Lemma 1.11 and Proposition 2.5 in [6] with (a) and (c) of Theorem 2.5).

In Section 4, we observe that each TH-p-group with r generators and exponent p^n is a quotient of the group ${}^nG_r$ (Theorem 4.1). Moreover, we obtain some new general result about the structure of TH-p-groups which turn out to be a suitable product of cyclic groups (Theorem 4.3). Finally, in Section 5, we describe the TH-p-groups of exponent p^2.

The notation is standard. We indicate by $\gamma_i(G)$ the i-th term of the lower central series of a group G. Throughout this paper, p will be always an odd prime.

2 The λ_i-Series of the Free Groups

We recall the construction of the central series $\lambda_i(G)$ of a group G and the properties of this series which we intend to use, collecting them from [5] and [1].

Definition 2.1. For any group G, put

$$\lambda_i(G) := \gamma_1(G)^{p^{i-1}}\gamma_2(G)^{p^{i-2}}\cdots\gamma_i(G) \quad (i \geq 1).$$

Thus, $\lambda_i(G)$ is a characteristic subgroup of G and

$$G = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \geq \cdots .$$

Theorem 2.2. [5, 1] *For any $i \in \mathbb{N}$, the following properties hold*:

(a) $[\lambda_i(G), \lambda_j(G)] \leq \lambda_{i+j}(G)$;
(b) $\lambda_i(G) = [\lambda_{i-1}(G), G]\,\lambda_{i-1}(G)^p$;
(c) $[\lambda_i(G), G] = \gamma_2(G)^{p^{i-1}}\cdots\gamma_{i+1}(G)$;
(d) *if $G/\gamma_j(G)$ is torsion free, then* $\lambda_i(G)\cap\gamma_j(G) = \gamma_j(G)^{p^{i-j}}\cdots\gamma_i(G)$;
(e) $\lambda_i(G)^{p^j} \leq \lambda_{i+j}(G)$;
(f) *the λ_i-series is central and $\lambda_i(G)/\lambda_{i+1}(G)$ is an elementary abelian p-group.*

Definition 2.3. Let F_r be the free group on r free generators $x_1, x_2, \ldots, x_r$, and $\mathcal{A} = (a_1, a_2, \ldots, a_n)$ be an ordered subset of F_r. We denote by $\mathcal{A}^p$ the ordered subset $\mathcal{A}^p = (a_1^p, a_2^p, \ldots, a_n^p)$. Moreover, if $H \trianglelefteq F_r$, we denote with $\mathcal{A}$ mod H the ordered subset $(a_1H, a_2H, \ldots, a_nH)$ of F_r/H.

For brevity, we will often write λ_i and γ_i instead of $\lambda_i(F_r)$ and $\gamma_i(F_r)$, respectively.

Lemma 2.4. *Let* $a_1, a_2, \ldots, a_n \in \lambda_{i-1}$ $(i \geq 2)$. *Then*

$$(a_1 a_2 \cdots a_n)^p \equiv ({a_1}^p {a_2}^p \cdots {a_n}^p) \bmod \lambda_{i+1}.$$

Proof. Let $a_1,\ a_2 \in \lambda_{i-1}$. By the Hall–Petrescu formula, there exist elements $c_k \in \gamma_k(\lambda_{i-1})$ $(k = 2, \ldots, p)$ such that

$${a_1}^p {a_2}^p = (a_1 a_2)^p c_2^{\binom{p}{2}} \cdots {c_{p-1}}^{\binom{p}{p-1}} c_p.$$

Since $i, k \geq 2$, we get $\lambda_{k(i-1)} \leq \lambda_i$, and by (a) and (e) of Theorem 2.2, we obtain $\gamma_k(\lambda_i) \leq \lambda_{ki}$ and ${\lambda_i}^p \leq \lambda_{i+1}$; so whenever $k \leq p-1$, we get

$${c_k}^{\binom{p}{k}} \in \gamma_k(\lambda_{i-1})^p \leq {\lambda_{k(i-1)}}^p \leq {\lambda_i}^p \leq \lambda_{i+1}.$$

For $k = p$, since $i \geq 2$ and $p \geq 3$ imply $p(i-1) \geq i+1$, we get

$$c_p \in \gamma_p(\lambda_{i-1}) \leq \lambda_{p(i-1)} \leq \lambda_{i+1}.$$

Hence,

$${a_1}^p {a_2}^p \equiv (a_1 a_2)^p \bmod \lambda_{i+1}.$$

Now by induction on n, the lemma follows at once. □

Theorem 2.5.

(a) λ_{i-1}/λ_i *may be embedded into* λ_i/λ_{i+1}. *Moreover, there exists a base* $\mathcal{A}_{i-1} \bmod \lambda_i$ *of* λ_{i-1}/λ_i *such that* ${\mathcal{A}_{i-1}}^p \bmod \lambda_{i+1}$ *is independent in* λ_i/λ_{i+1}.

(b) *For each* i, *let* $\mathcal{A}_i \bmod \gamma_{i+1}$ *be a base of* γ_i/γ_{i+1}. *Then*

$$(\mathcal{A}_1^{p^{i-1}} \cup \mathcal{A}_2^{p^{i-2}} \cup \cdots \cup \mathcal{A}_i) \bmod \lambda_{i+1}$$

is a base for λ_i/λ_{i+1}.

(c) *Let* $\mathcal{C}_i$ *denote the set of the basic commutators of weight* i *in a fixed sequence, then*

$$\mathcal{B}_i := (\mathcal{C}_1^{p^{i-1}} \cup {\mathcal{C}_2}^{p^{i-2}} \cup \cdots \cup \mathcal{C}_i) \bmod \lambda_{i+1}$$

is a base of λ_i/λ_{i+1}.

(d) *The map* $\varphi_i : \lambda_{i-1} \to \lambda_i/\lambda_{i+1}$ *defined by* $x\varphi_i = x^p \lambda_{i+1}$ *is a homomorphism and* $\ker \varphi_i = \lambda_i$.

Proof. (a) By a well-known result of Blackburn [5, VIII, 1.9b)], since p is odd, there is an isomorphism

$$\alpha_i : \frac{\gamma_1}{{\gamma_1}^p \gamma_2} \times \cdots \times \frac{\gamma_i}{{\gamma_i}^p \gamma_{i+1}} \longrightarrow \lambda_i/\lambda_{i+1}$$

given by $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_i)\alpha_i = {a_1}^{p^{i-1}} {a_2}^{p^{i-2}} \cdots a_i \lambda_{i+1}$. If μ is the natural immersion of $\frac{\gamma_1}{{\gamma_1}^p\gamma_2} \times \cdots \times \frac{\gamma_{i-1}}{{\gamma_{i-1}}^p\gamma_i}$ into $\frac{\gamma_1}{{\gamma_1}^p\gamma_2} \times \cdots \times \frac{\gamma_i}{{\gamma_i}^p\gamma_{i+1}}$, then the map

$$\pi_i = {\alpha_{i-1}}^{-1}\mu\alpha_i : \lambda_{i-1}/\lambda_i \longrightarrow \lambda_i/\lambda_{i+1}$$

is a monomorphism.

Let $\bar{a}_j \in \frac{\gamma_j}{{\gamma_j}^p\gamma_{j+1}}$ and $x = (1, \dots, 1, \bar{a}_j, 1, \dots, 1)\alpha_{i-1} \in \lambda_{i-1}/\lambda_i$. Then $x = {a_j}^{p^{i-j-1}}\lambda_i$ and we have

$$x\pi_i = {a_j}^{p^{i-j}}\lambda_{i+1} \in \lambda_i/\lambda_{i+1}.$$

Now if $\mathcal{C}_j \bmod {\gamma_j}^p\gamma_{j+1}$ is a base of $\gamma_j/{\gamma_j}^p\gamma_{j+1}$, then

$$\bigcup_{j=1}^{i-1}((\mathcal{C}_j \bmod {\gamma_j}^p\gamma_{j+1})\alpha_{i-1}) = \bigcup_{j=1}^{i-1}({\mathcal{C}_j}^{p^{i-1-j}} \bmod \lambda_i)$$

is a base, say $\mathcal{A}_{i-1} \bmod \lambda_i$ of λ_{i-1}/λ_i. Hence, we obtain that

$$(\mathcal{A}_{i-1} \bmod \lambda_i)\pi_i = \bigcup_{j=1}^{i-1}({\mathcal{C}_j}^{p^{i-j}} \bmod \lambda_{i+1}) = {\mathcal{A}_{i-1}}^p \bmod \lambda_{i+1}$$

is a base of $\operatorname{Im}\pi_i$.

(b) First of all, we observe that if $\mathcal{A}_i \bmod \gamma_{i+1}$ is a base of γ_i/γ_{i+1}, since γ_i/γ_{i+1} is a torsion-free abelian group, $\mathcal{A}_i \bmod {\gamma_i}^p\gamma_{i+1}$ is a base of the elementary abelian group $\gamma_i/{\gamma_i}^p\gamma_{i+1}$.

Next the base $(\mathcal{A}_1 \bmod {\gamma_1}^p\gamma_2) \cup \cdots \cup (\mathcal{A}_i \bmod {\gamma_i}^p\gamma_{i+1})$ of

$$\frac{\gamma_1}{{\gamma_1}^p\gamma_2} \times \cdots \times \frac{\gamma_i}{{\gamma_i}^p\gamma_{i+1}}$$

is taken onto the base $(\mathcal{A}_1^{p^{i-1}} \cup {\mathcal{A}_2}^{p^{i-2}} \cup \cdots \cup \mathcal{A}_i) \bmod \lambda_{i+1}$ of λ_i/λ_{i+1} by the isomorphism α_i which we mentioned in the proof of (a).

(c) By the well-known Hall's basis theorem, $\mathcal{C}_i \bmod \gamma_{i+1}$ is a base of γ_i/γ_{i+1} (see [3, Chapter 11]). Thus, (c) follows from (b).

(d) By Lemma 2.4, the map $\varphi_i : \lambda_{i-1} \to \frac{\lambda_i}{\lambda_{i+1}}$ given by $x\varphi_i = x^p\lambda_{i+1}$ is a homomorphism.

We prove $\ker\varphi_i = \lambda_i$. Clearly, $\lambda_i \leq \ker\varphi_i$, and we only need to show that for $x \in \lambda_{i-1}$, the condition $x^p \in \lambda_{i+1}$ implies $x \in \lambda_i$. By (a), we can choose a base $\mathcal{A}_{i-1} \bmod \lambda_i$ for λ_{i-1}/λ_i with $\mathcal{A}_{i-1} = (a_1, a_2, \dots, a_t)$ such that $\mathcal{A}_{i-1}^p \bmod \lambda_{i+1}$ is independent in λ_i/λ_{i+1}. Let $x \equiv a_1^{\alpha_1}a_2^{\alpha_2}\cdots a_t^{\alpha_t} \bmod \lambda_i$ for suitable $0 \leq \alpha_i < p$. By Lemma 2.4, we get

$$x^p \equiv a_1^{\alpha_1 p}a_2^{\alpha_2 p}\cdots a_t^{\alpha_t p} \bmod \lambda_{i+1},$$

hence, $x^p \in \lambda_{i+1}$ implies

$$a_1^{\alpha_1 p}a_2^{\alpha_2 p}\cdots a_t^{\alpha_t p} \equiv 1 \bmod \lambda_{i+1}.$$

Since $\mathcal{A}_{i-1}^p \bmod \lambda_{i+1}$ is independent, we obtain $\alpha_1 = \alpha_2 = \cdots = \alpha_t = 0$. □

Note that the monomorphism of elementary abelian groups

$$\bar{\varphi}_i : \lambda_{i-1}/\lambda_i \longrightarrow \lambda_i/\lambda_{i+1}$$

induced by φ_i coincides with the π_i introduced in the proof of (a) since they agree on a base of λ_{i-1}/λ_i.

3 The TH-p-Groups ${}^nG_r$

Definition 3.1. For all positive integers n and $r \geq 2$, we set

$${}^nG_r = \frac{F_r}{\lambda_{n+1}(F_r)}.$$

Theorem 3.2. *The groups ${}^nG_r$ have the following properties*:

(a) *${}^nG_r$ is a finite p-group of order $p^{|\mathcal{B}_1|+\cdots+|\mathcal{B}_n|}$.*
(b) *The nilpotency class of ${}^nG_r$ is n.*
(c) *The exponent of ${}^nG_r$ is p^n.*
(d) *$\lambda_i({}^nG_r) = \lambda_i/\lambda_{n+1}$ for all $i \in \mathbb{N}$.*
(e) *The derived length $\mathrm{dl}\,({}^nG_r)$ of ${}^nG_r$ is $\log_2 \frac{n+1}{3} + 2$ when $\log_2 \frac{n+1}{3}$ is an integer, otherwise we have*

$$\lceil \log_2 \frac{n+1}{3} \rceil + 1 \leq \mathrm{dl}\,({}^nG_r) \leq \lceil \log_2 \frac{n+1}{3} \rceil + 2,$$

where $\lceil a \rceil$ denotes the upper integral part of the real number a.

Proof. (a) The order of ${}^nG_r$ follows immediately from Theorem 2.5(c).

(b) Since $\gamma_{n+1}({}^nG_r) = \frac{\gamma_{n+1}\lambda_{n+1}}{\lambda_{n+1}} = 1$, the nilpotency class of ${}^nG_r$ is at most n. On the other side, $\mathcal{C}_n \bmod \lambda_{n+1}$ generates $\gamma_n({}^nG_r) = \frac{\gamma_n\lambda_{n+1}}{\lambda_{n+1}}$, and by Theorem 2.5(c), $\mathcal{C}_n \bmod \lambda_{n+1}$ is included in a base of λ_n/λ_{n+1}, so $\gamma_n({}^nG_r)$ does not reduce to the identity.

(c) Since ${F_r}^{p^n} = {\lambda_1}^{p^n} \leq \lambda_{n+1}$ by Theorem 2.2(e), we get $\exp {}^nG_r \leq p^n$. But ${\mathcal{C}_1}^{p^{n-1}} \bmod \lambda_{n+1}$ is included in a base of λ_n/λ_{n+1}, hence $\exp {}^nG_r = p^n$.

(d) We prove that $\lambda_i({}^nG_r) = \lambda_i/\lambda_{n+1}$ by induction on i. The inductive hypothesis gives

$$\lambda_{i-1}({}^nG_r) = \lambda_{i-1}/\lambda_{n+1},$$

hence by Theorem 2.2(b), we get

$$\lambda_i(F_r/\lambda_n) = [\lambda_{i-1}(F_r/\lambda_n),\, F_r/\lambda_{n+1}]\, \lambda_{i-1}(F_r/\lambda_{n+1})^p$$
$$= \Big[\frac{\lambda_{i-1}}{\lambda_{n+1}}, \frac{F_r}{\lambda_{n+1}}\Big] \Big(\frac{\lambda_{i-1}}{\lambda_{n+1}}\Big)^p = \frac{[\lambda_{i-1}, F_r]}{\lambda_{n+1}}\, \frac{{\lambda_{i-1}}^p \lambda_{n+1}}{\lambda_{n+1}} = \frac{\lambda_i}{\lambda_{n+1}}.$$

This also holds for $i = 1$ by the definition of ${}^nG_r$.

(e) Since ${}^nG_r{}^{(k)} \leq \gamma_{2^k}({}^nG_r) \leq \lambda_{2^k}({}^nG_r) = \lambda_{2^k}/\lambda_{n+1}$ by (d), we have ${}^nG_r{}^{(k)} = 1$ whenever $k \geq \log_2(n+1)$, hence $\mathrm{dl}\,({}^nG_r) \leq \lceil \log_2(n+1) \rceil$.

Next let $x_1 < x_2 < \cdots < x_r$ be the basic commutators of weight 1 in F_r. We define inductively the elements $u_i \in F_r$ by setting $u_1 = [x_2, x_1]$ and $u_{i+1} = [[u_i, x_1], u_i]$. It is easily checked by induction on i that the $u_i \in F_r{}^{(i)}$ are basic commutators of weight $w_i = 2^{i-1}3 - 1$. Therefore, $u_i \in \lambda_{w_i}$ and $u_i \notin \lambda_{w_i+1}$ since, by Theorem 2.5(c), the u_i mod λ_{w_i+1} belong to a base of $\lambda_{w_i}/\lambda_{w_i+1}$. This means that $({}^{w_i}G_r)^{(i)} \neq 1$. Thus,

$$i < \mathrm{dl}\,({}^{w_i}G_r) \leq \lceil \log_2(w_i + 1) \rceil = i + 1,$$

hence $\mathrm{dl}\,({}^{w_i}G_r) = i + 1$. If $\log_2 \frac{n+1}{3} = i - 1$ is an integer, we get $n = w_i$, so $\mathrm{dl}\,({}^nG_r) = \log_2 \frac{n+1}{3} + 2$.

If $\log_2 \frac{n+1}{3}$ is not an integer, there exists an integer i such that $w_i < n < w_{i+1}$. It is then clear that

$$i + 1 = \mathrm{dl}\,({}^{w_i}G_r) \leq \mathrm{dl}\,({}^nG_r) \leq \mathrm{dl}\,({}^{w_{i+1}}G_r) = i + 2$$

and $i - 1 < \log_2 \frac{n+1}{3} < i$. The thesis follows. □

Theorem 3.3. *${}^nG_r$ is a TH-p-group on r generators.*

Proof. First of all, we prove that ${}^nG_r$ is a TH-p-group. Since the λ_i-series is central, we only need to show that $x \in F_r$ and $x^p \in \lambda_{n+1}$ imply $x \in \lambda_n$. By contradiction, assume $x \in \lambda_{i-1} - \lambda_i$ with $i \leq n$. Since $x^p \in \lambda_{n+1} \leq \lambda_{i+1}$, by Theorem 2.5(d), we get $x \in \ker \varphi_i = \lambda_i$, against the assumption.

Next we observe that, by Theorem 3.2, $\Phi({}^nG_r) = \lambda_2({}^nG_r)$ coincides with λ_2/λ_{n+1}, and therefore, by Theorem 2.5(c), $|{}^nG_r : \Phi({}^nG_r)| = |\lambda_1 : \lambda_2| = p^r$, which means $d({}^nG_r) = r$. □

Remark 3.4. If G is a TH-p-group of exponent p^n, then $\lambda_{n+1-i}(G) \leq \Omega_i(G)$. In particular $\lambda_{n+1}(G) = 1$.

Indeed, this follows immediately from [5, VIII, 1.6] since the Ω_i-series is central with elementary abelian factors (see [2, 2.3]).

Theorem 3.5. *For $1 \leq i \leq n$ and $r \geq 2$, it holds $\Omega_i({}^nG_r) = \lambda_{n+1-i}({}^nG_r) = \frac{\lambda_{n+1-i}}{\lambda_{n+1}}$. Moreover, $|\Omega_1({}^nG_r)| = p^m$, where m is the number of the basic commutators of weight at most n on r free generators.*

Proof. We proceed by induction on i. It follows from the proof of Theorem 3.3 that for all $x \in F_r$, we have $x^p \in \lambda_{n+1}$ if and only if $x \in \lambda_n$. By Theorem 3.2(d), this yields $\Omega_1({}^nG_r) = \lambda_n/\lambda_{n+1} = \lambda_n({}^nG_r)$. Assume now that $\Omega_i({}^nG_r) = \lambda_{n+1-i}({}^nG_r)$. Then using [2, Lemma 2.2], Theorem 3.2(d) and the case $i = 1$, we have

$$\frac{\Omega_{i+1}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)} = \frac{\Omega_{i+1}({}^nG_r)}{\Omega_i({}^nG_r)} = \Omega_1\left(\frac{{}^nG_r}{\Omega_i({}^nG_r)}\right) = \Omega_1\left(\frac{\lambda_1/\lambda_{n+1}}{\lambda_{n+1-i}/\lambda_{n+1}}\right)$$

$$\cong \Omega_1(\lambda_1/\lambda_{n+1-i}) = \Omega_1({}^{n-i}G_r) = \lambda_{n-i}({}^{n-i}G_r) = \lambda_{n-i}/\lambda_{n-i+1}$$

$$\cong \frac{\lambda_{n-i}/\lambda_{n+1}}{\lambda_{n-i+1}/\lambda_{n+1}} = \frac{\lambda_{n-i}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)}.$$

Thus,

$$\frac{\Omega_{i+1}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)} \cong \frac{\lambda_{n-i}({}^nG_r)}{\lambda_{n+1-i}({}^nG_r)},$$

and since $\lambda_{n-i}({}^nG_r) \leq \Omega_{i+1}({}^nG_r)$ by Remark 3.4, we get that $\Omega_{i+1}({}^nG_r) = \lambda_{n-i}({}^nG_r)$.

Finally, by Theorem 3.2(d), we have $|\Omega_1({}^nG_r)| = |\lambda_n({}^nG_r)| = |\lambda_n/\lambda_{n+1}|$, and Theorem 2.5(c) says that there is a base for the elementary abelian group λ_n/λ_{n+1} which consists of as many elements as the number of the basic commutators of weight at most n. □

4 TH-p-Groups

In this section, we want to describe deeply the structure of the TH-p-groups.

Theorem 4.1. *Let G be a TH-p-group with r generators and $\exp G = p^n$. Then G is a quotient ${}^nG_r/N$, where ${}^nG_r{}^{p^{n-1}} \nleq N \leq \Phi({}^nG_r)$.*

Proof. Let F_r be the free group on r free generators and $\psi : F_r \to G$ an epimorphism. We only need to show $\ker\psi \geq \lambda_{n+1}(F_r)$. Since ψ is a homomorphism, we clearly have $\psi(\lambda_i(F_r)) \leq \lambda_i(G)$ for all i, and in particular, by Remark 3.4, we get $\psi(\lambda_{n+1}(F_r)) \leq \lambda_{n+1}(G) = 1$. □

Remark 4.2. Let ${}^nG_r/N$ be a TH-p-group. Then ${}^nG_r/(N \cap \Omega_i({}^nG_r))$ is also a TH-p-group.

In fact, let $\bar{x} = x\lambda_{n+1} \in {}^nG_r$ for some $x \in F_r$, and let $\bar{x}(N \cap \Omega_i({}^nG_r))$ be an element of order p of ${}^nG_r/(N \cap \Omega_i({}^nG_r))$. Then $\bar{x}^p \in N \cap \Omega_i({}^nG_r) = N \cap \lambda_{n+1-i}({}^nG_r)$. In particular, $\bar{x}^p$ is in N, and since ${}^nG_r/N$ is a TH-p-group, this implies $[\bar{x}, {}^nG_r] \leq N$. On the other side, from $\bar{x}^p \in \lambda_{n+1-i}({}^nG_r) = \lambda_{n+1-i}/\lambda_{n+1}$, it follows that $x^p \in \lambda_{n+1-i}$. By the argument used in the proof of Theorem 3.3, we get $x \in \lambda_{n-i}$. Then $[x, F_r] \leq \lambda_{n+1-i}$ and this implies $[\bar{x}, {}^nG_r] \in \lambda_{n+1-i}({}^nG_r) = \Omega_i({}^nG_r)$. Thus, $[\bar{x}, {}^nG_r] \leq N \cap \Omega_i({}^nG_r)$, i.e., $\bar{x}(N \cap \Omega_i({}^nG_r))$ is central in ${}^nG_r/(N \cap \Omega_i({}^nG_r))$.

Theorem 4.3. *Let G be a TH-p-group with $|\Omega_1(G)| = p^m$. Then G is a product of (not in general pairwise permutable) cyclic groups $\langle a_1\rangle, \dots, \langle a_m\rangle$, where $|a_i| \geq |a_{i+1}|$ and $\langle a_i\rangle \cap [\langle a_1\rangle \cdots \langle a_{i-1}\rangle\langle a_{i+1}\rangle \cdots \langle a_m\rangle] = 1$ for $1 \leq i \leq m$.*

Proof. We use induction on the exponent p^n of G. When $n = 1$, G is an elementary abelian p-group and the theorem holds. Next we observe that $G/\Omega_1(G)$ is a TH-p-group of exponent p^{n-1}. Setting

$$p^{m'} = \left|\Omega_1\left(\frac{G}{\Omega_1(G)}\right)\right| = \left|\frac{\Omega_2(G)}{\Omega_1(G)}\right|,$$

by Theorem 2.3 in [2], we have $m' \leq m$. By the inductive hypothesis, we have $\frac{G}{\Omega_1(G)} = \langle \bar{a}_1\rangle \cdots \langle \bar{a}_{m'}\rangle$, where for $1 \leq i \leq m'$, $|\bar{a}_i| \geq |\bar{a}_{i+1}|$ and $\langle \bar{a}_i\rangle \cap [\langle \bar{a}_1\rangle \cdots \langle \bar{a}_{i-1}\rangle\langle \bar{a}_{i+1}\rangle \cdots \langle \bar{a}_{m'}\rangle] = 1$.

Now we set $|\bar{a}_i| = |a_i\Omega_1(G)| = p^{h_i}$ and $H = \langle {a_1}^{p^{h_1}}, \dots, {a_{m'}}^{p^{h_{m'}}}\rangle \leq \Omega_1(G)$.

We claim that $H = \langle {a_1}^{p^{h_1}}\rangle \times \cdots \times \langle {a_{m'}}^{p^{h_{m'}}}\rangle$. Assume the contrary, then for some i and integers $0 \leq x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{m'} < p$, we have

$${a_i}^{p^{h_i}} = {a_1}^{x_1 p^{h_1}} \cdots {a_{i-1}}^{x_{i-1}p^{h_{i-1}}} {a_{i+1}}^{x_{i+1}p^{h_{i+1}}} \cdots {a_{m'}}^{x_{m'}p^{h_{m'}}}.$$

Let

$$b = {a_i}^{-p^{h_i-1}} {a_1}^{x_1 p^{h_1-1}} \cdots {a_{i-1}}^{x_{i-1}p^{h_{i-1}-1}} {a_{i+1}}^{x_{i+1}p^{h_{i+1}-1}} \cdots {a_{m'}}^{x_{m'}p^{h_{m'}-1}},$$

and observe that for all j, ${a_j}^{-p^{h_j-1}} \in \Omega_2(G)$, which is nilpotent of class 2. Then the Hall–Petrescu formula implies $b^p = 1$, so $b \in \Omega_1(G)$, a contradiction. Hence, the thesis follows immediately if $m' = m$. If $m' < m$, since $\Omega_1(G) \leq Z(G)$ is elementary abelian, there exists a subgroup $K = \langle a_{m'+1}\rangle \times \cdots \times \langle a_m\rangle$ of $\Omega_1(G)$ with $\Omega_1(G) = H \times K$. Then $G = \langle a_1\rangle \cdots \langle a_m\rangle$, and the conditions in the theorem are clearly satisfied. □

With the same argument, we can prove the following:

Theorem 4.4. *Let G be a TH-p-group with exponent p^n. If $\mathcal{A} \bmod \Omega_{n-i}(G)$ is a base of $\Omega_{n-i+1}(G)/\Omega_{n-i}(G)$, then $\mathcal{A}^p \bmod \Omega_{n-i-1}(G)$ is independent in $\Omega_{n-i}(G)/\Omega_{n-i-1}(G)$.*

Remark 4.5. Let G be a TH-p-group and $N \lhd G$ with $N \cap G^p = 1$. Then G/N is a TH-p-group. *Actually, in this case, all the elements of order p in G/N are images of elements of order p in G.*

Remark 4.6. Let G/N be a TH-p-group. If there exists an element $x \in G - \Phi(G)$ with $x^p \in N$, then G/N has a cyclic direct factor.

5 TH-p-Groups of Exponent p^2

Disregarding abelian TH-p-groups and TH-p-groups which differ from one another in a cyclic direct factor, we have

Lemma 5.1. *The TH-p-groups of exponent p^2 with r generators are precisely ${}^2G_r$ and the quotients $G = {}^2G_r/N$, where $N \not\geq {}^2G'_r$, $N \leq \Phi({}^2G_r) = \Omega_1({}^2G_r)$, and $N \cap {}^2{G_r}^p = 1$.*

Proof. The third condition guarantees that G is a TH-p-group (see Remark 4.5). Conversely, if G is a non-abelian TH-p-group, then the first condition is fulfilled, the second one follows from Theorem 4.1, and the third one is a consequence of Remark 4.6. □

Now we study in more details the case $r = 3$. We get

$$^2G_3 = \langle x, y, z \mid x^{p^2} = y^{p^2} = z^{p^2} = [y, x]^p = [z, x]^p = [z, y]^p = 1\rangle.$$

Let $H = \langle x^p, y^p, z^p\rangle = {}^2G_3^p$ and $K = \langle [y, x], [z, x], [z, y]\rangle = {}^2G_3'$. By Lemma 5.1, the relevant TH-p-group quotients are obtained by $N \leq \Omega_1(^2G_3) = H \times K$ with $N \cap H = 1$ and $N \cap K \neq K$. From the condition $N \cap H = 1$, it follows that the allowed orders for N are p, p^2, p^3. The possible orders for the commutator subgroup in the correspondent TH-p-groups are p^2 or p^3 in the first case; p, p^2 or p^3 in the others two cases. Thus, we get at least eight non-isomorphic TH-p-groups. But we may observe that there are non-isomorphic TH-p-groups with the same order and the same commutator subgroup order, as we see analyzing the case $|N| = p$.

Remark 5.2. (a) If $N < K$, then $G = {}^2G_3/N$ is isomorphic to the group $G_1 = \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = [w, v] = 1\rangle$.

(b) If $N \cap K = 1$, then $G = {}^2G_3/N$ is isomorphic to one of the following non-isomorphic groups:

$$G_2 = \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = 1,\ [w, v] = u^{-p}\rangle,$$
$$G_3 = \langle u, v, w \mid u^{p^2} = v^{p^2} = w^{p^2} = [v, u]^p = [w, u]^p = 1,\ [w, v] = w^{-p}\rangle.$$

For (a), first of all, we observe that every $k = [y, x]^a[z, x]^b[z, y]^c \in K$ is a commutator. Namely, as $^2G_3$ is nilpotent of class 2, if $c \not\equiv 0 \pmod p$, we have $k = [z^c x^{-a}, yx^{bc^{-1}}]$; while if $c \equiv 0 \pmod p$, we have $k = [y^a z^b, x]$. Let $N = \langle k\rangle$, where $k = [g_2, g_1]$. Since $\Phi(^2G_3) = Z(^2G_3)$, we see that $g_1\Phi(^2G_3)$ and $g_2\Phi(^2G_3)$ are independent, and one among the elements x, y, z, say x, does not belong to the subgroup $\langle g_1, g_2, \Phi(^2G_3)\rangle$. Thus, g_1, g_2, x constitute a base of $^2G_3$, and the map $x \mapsto x$, $y \mapsto g_1$, $z \mapsto g_2$ extends to an automorphism of $^2G_3$ which maps $N_0 = \langle [z, y]\rangle$ onto the subgroup $N = \langle k\rangle$.

In the case (b), the group N is generated by an element of the form $[y, x]^a[z, x]^b[z, y]^c x^{dp}y^{ep}z^{fp}$, where not all of a, b, c and not all of d, e, f are zero. As we showed in (a), the element $[y, x]^a[z, x]^b[z, y]^c$ is a commutator $[g_2, g_1]$, and it is convenient to separate two cases according to $g_3 = x^d y^e z^f \notin \langle g_1, g_2\rangle$ or $g_3 = x^d y^e z^f \in \langle g_1, g_2\rangle$.

Assume $g_3 = x^d y^e z^f \notin \langle g_1, g_2\rangle$. The elements g_1, g_2, g_3 constitute a base for $^2G_3$, and the automorphism determined by $x \mapsto g_3$, $y \mapsto g_2$, $z \mapsto g_1$ takes $N_0 = \langle [z, y]x^p\rangle$ onto N. It follows that $^2G_3/N \cong G_2$.

Assume $g_3 = x^d y^e z^f \in \langle g_1, g_2\rangle$. Then by the explicit form of g_1, g_2 given in (a), we get either $g_3 \in \langle z^{cp}x^{-ap}, y^p x^{bc^{-1}p}\rangle$ or $g_3 \in \langle y^{ap}z^{bp}, x^p\rangle$. Anyway, this implies

$$be = cd + af. \qquad (*)$$

We show that $(*)$ allows to define an automorphism of ${}^2G_3$ which takes the element $[z,y]z^p$ into the element $[y,x]^a[z,x]^b[z,y]^c x^{dp}y^{ep}z^{fp}$. Namely, if $e,f \not\equiv 0 \pmod p$, this automorphism is determined by $x \mapsto x$, $y \mapsto x^{e^{-1}(dcf^{-1}+a)}y^{cf^{-1}}$, $z \mapsto x^d y^e z^f$; while if $f \equiv 0$ and $e \not\equiv 0$, it is determined by $x \mapsto x$, $y \mapsto x^{ae^{-1}}z^{-ce^{-1}}$, $z \mapsto x^a y^e z^f$; and finally, if $f \not\equiv 0$ and $e \equiv 0$, it is determined by $x \mapsto x$, $y \mapsto x^{bf^{-1}}y^{cf^{-1}}$, $z \mapsto x^d y^e z^f$. Observe that, by $(*)$, the only remaining possibility is $c,e,f \equiv 0$, and in this case, the automorphism is given by $x \mapsto x$, $y \mapsto y^{-ad^{-1}}z^{-bd^{-1}}$, $z \mapsto x^d y^e z^f$. It follows that ${}^2G_3/N \cong G_3$.

Finally, it is easily checked directly that there is no isomorphism between G_2 and G_3.

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