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Nonoscillatory Solutions for Nonlinear Discrete Systems

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Abstract. We investigate some asymptotic properties of the nonlinear forced difference system

$$\begin{aligned} &\Delta(r_k \Phi_\alpha(\Delta x_k)) - \sigma\varphi_k f(y_{k+1}) = \sigma\hat{\varphi}_k,\\ &\Delta(q_k \Phi_\beta(\Delta y_k)) - \psi_k g(x_{k+1}) = \hat{\psi}_k. \end{aligned}$$

In particular we give necessary and sufficient conditions for existence of the so-called regularly decaying solutions and thereby we complete the results presented in [10].

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1 Introduction

In [10] the authors investigated certain discrete asymptotic boundary value problems on the discrete interval $[m, \infty) := \{m, m+1, ..\}, m \in \mathbb{Z}$, associated to the

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nonlinear forced difference system

$$\begin{aligned} \Delta(r_k\Phi_\alpha(\Delta x_k)) - \sigma\varphi_k f(y_{k+1}) &= \sigma\hat{\varphi}_k, \\ \Delta(q_k\Phi_\beta(\Delta y_k)) - \psi_k g(x_{k+1}) &= \hat{\psi}_k. \end{aligned} \tag{1}$$

In particular, necessary and sufficient conditions for the existence of the so-called *strongly decaying solutions* of (1) were presented. The principal aim of this contribution is to complete those results examining also the existence of the so-called *regularly decaying solutions*. For the definitions of these concepts see Definition 1. In system (1) we assume that $\{\varphi_k\}, \{\psi_k\}, \{r_k\}, \{q_k\}$ are real positive sequences defined for any $k \geq m$, the forcing terms $\{\widehat{\varphi}_k\}, \{\widehat{\psi}_k\}$ are real nonnegative sequences defined for any $k \geq m$, $\Phi_p(u) = |u|^{p-1}\operatorname{sgn} u$ with $p > 1$ is the one-dimensional p-laplacian operator, $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are monotone continuous functions and $\sigma \in \{-1, 1\}$.

There are several motivations for the investigation of (1):

- System (1) arises in the discretization process of differential systems with p-laplacian operator. For results concerning the role of system (1) in various applications, we refer the reader to [6], [9], [12] and to the references contained therein.
- The study of system (1) is motivated also by certain results obtained for scalar second order difference equations of the form

$$\Delta(r_k\Phi_\alpha(\Delta x_k)) = \sigma\varphi_k f(x_{k+1}),$$

which attracted a considerable attention in recent years, see, e.g., [2], [3], [4], [7], [8], [11], [15]. Other interesting contributions can be found also in the monograph [1].

- System (1) can be rewritten in the form of a fourth-order nonlinear equation of the type

$$\Delta^2(r_k\Delta^2 u_k) + p_k f(u_{k+2}) = 0.$$

For results from this point of view see for instance [13], [14].

Here we are interested in the existence of positive nonoscillatory solutions, which are asymptotically decreasing towards zero. A *nonoscillatory solution* of (1) is a vector sequence $(x, y) = (\{x_k\}, \{y_k\})$ satisfying (1) for $k \geq m$, and such that both components $\{x_k\}, \{y_k\}$ are eventually of fixed sign.

Definition 1. A solution (x, y) of (1) is said to be

– *decaying*, if x, y are eventually positive decreasing and

$$\lim_{k\to\infty} x_k = 0 = \lim_{k\to\infty} y_k;$$

– *regularly decaying*, if it is decaying and

$$\lim_{k\to\infty} r_k\Phi_\alpha(\Delta x_k) = -A_x, \quad \lim_{k\to\infty} q_k\Phi_\beta(\Delta y_k) = -B_y,$$

where A_x, B_y are positive constants;

– *strongly decaying*, if it is decaying and $\lim_{k\to\infty} q_k\Phi_\beta(\Delta y_k) = 0$,

$$\lim_{k\to\infty} r_k\Phi_\alpha(\Delta x_k) = \begin{cases} -C_x & \text{for } \sigma = -1, \\ 0 & \text{for } \sigma = 1, \end{cases}$$

where C_x is a positive constant.

Note that the constant C_x cannot be zero when $\sigma = -1$, since in this case the quasidifference $r_k\Phi_\alpha(\Delta x_k)$ is eventually negative decreasing.

As already mentioned at the beginning of this section, the main purpose of this paper is to complete the results presented in [10], in which the existence of strongly decaying solutions is considered. To this end we will give here necessary and sufficient conditions for existence of regularly decaying solutions (see Section 2), in view of their crucial role in a variety of physical applications, as already claimed. We will also restate the main results proved in [10] in order to have a comparison with our new results (see Section 3). Note that both the regular case (i.e. with the nonlinearities f, g bounded in a right neighborhood of zero) and the singular case (i.e. with f or g unbounded in a right neighborhood of zero) will be considered. Some remarks and comments are given throughout this contribution.

Finally, nontrivial prototypes of nonlinearities f, g are the one-dimensional Laplacians Φ_γ, Φ_δ, respectively, with $\gamma, \delta \neq 1$ (i.e., with possible singularities at zero). Then (1) leads to the Emden-Fowler type system

$$\begin{aligned} \Delta(r_k\Phi_\alpha(\Delta x_k)) &= \sigma\varphi_k\Phi_\gamma(y_{k+1}) \\ \Delta(q_k\Phi_\beta(\Delta y_k)) &= \psi_k\Phi_\delta(x_{k+1}) \end{aligned} \tag{2}$$

as a special case.

Some of our main results remain valid for more general systems of the form

$$\begin{aligned} \Delta(r_k\alpha(\Delta x_k)) &= \sigma F(k, y_{k+1}, x_{k+1}) \\ \Delta(q_k\beta(\Delta y_k)) &= G(k, x_{k+1}, y_{k+1}), \end{aligned} \tag{3}$$

where the functions α and β are monotone continuous increasing with $\alpha(0) = 0 = \beta(0)$, and F, G are positive continuous on $\{m, m+1, \dots\} \times (0, \varepsilon] \times (0, \varepsilon]$ with some $\varepsilon > 0$ and bounded with respect to the third variable.

2 Regularly Decaying Solutions

Denote with $\alpha*$ the conjugate number of α i.e. $1/\alpha + 1/\alpha* = 1$, and analogously for β. The following necessary and sufficient condition holds:

Theorem 2. *System* (1) *has at least one regularly decaying solution if and only if*

$$\sum_{k=m}^{\infty} \Phi_{\alpha*}\left(\frac{1}{r_k}\right) < \infty, \quad \sum_{k=m}^{\infty} \Phi_{\beta*}\left(\frac{1}{q_k}\right) < \infty, \tag{4}$$

$$\sum_{k=m}^{\infty} \hat{\varphi}_k < \infty, \quad \sum_{k=m}^{\infty} \hat{\psi}_k < \infty, \tag{5}$$

and there exist $A > 0$, $B > 0$ such that

$$\sum_{k=m}^{\infty} \varphi_k\, f\left(\sum_{j=k+1}^{\infty} \Phi_{\beta*}\left(\frac{A}{q_j}\right)\right) < \infty, \tag{6}$$

$$\sum_{k=m}^{\infty} \psi_k\, g\left(\sum_{j=k+1}^{\infty} \Phi_{\alpha*}\left(\frac{B}{r_j}\right)\right) < \infty. \tag{7}$$

In addition, if $\sigma = 1$, then this solution is decreasing for any $k \geq m$.

Proof. The "if part". Choose an integer $T \geq m$ such that

$$\sum_{k=T}^{\infty} \varphi_k\, f\left(\sum_{j=k+1}^{\infty} \Phi_{\beta*}\left(\frac{1}{q_j}\right)\right) < 1/2, \quad \sum_{k=T}^{\infty} \psi_k\, g\left(\sum_{j=k+1}^{\infty} \Phi_{\alpha*}\left(\frac{1}{r_j}\right)\right) < 1/2.$$

Denote with ℓ_T^∞ the Banach space of all bounded sequences defined for $k \geq T$, endowed with the topology of the supremum norm, and consider the set $\Omega \subset \ell_T^\infty \times \ell_T^\infty$ given by

$$\Omega = \Bigg\{(u,v) = (\{u_k\},\{v_k\}) \in \ell_T^\infty \times \ell_T^\infty \,:\, \sum_{j=k+1}^{\infty} \Phi_{\alpha*}\left(\frac{M_1}{r_j}\right) \leq$$
$$\leq u_k \leq \sum_{j=k+1}^{\infty} \Phi_{\alpha*}\left(\frac{M_2}{r_j}\right), \sum_{j=k+1}^{\infty} \Phi_{\beta*}\left(\frac{N_1}{q_j}\right) \leq v_k \leq \sum_{j=k+1}^{\infty} \Phi_{\beta*}\left(\frac{N_2}{q_j}\right)\Bigg\},$$

where M_1, N_1, M_2, N_2 are suitable positive constants which will be determined later. Consider the operator $\mathcal{T} : \Omega \to \ell_T^\infty \times \ell_T^\infty$ defined by

$$\mathcal{T}(u,v) = (\mathcal{T}_1(v), \mathcal{T}_2(u)) = (\{(\mathcal{T}_1(v))_k\}, \{(\mathcal{T}_2(u))_k\}),$$

where

$$(\mathcal{T}_1(v))_k = \sum_{j=k}^{\infty} \Phi_{\alpha*}\left(\frac{M_{\tau(\sigma)}}{r_j} + \frac{\sigma}{r_j}\sum_{i=j}^{\infty}\left[\varphi_i f(v_{i+1}) + \hat{\varphi}_i\right]\right),$$

$$(\mathcal{T}_2(u))_k = \sum_{j=k}^{\infty} \Phi_{\beta*}\left(\frac{N_1}{q_j} + \frac{1}{q_j}\sum_{i=j}^{\infty}\left[\psi_i g(u_{i+1}) + \hat{\psi}_i\right]\right)$$

with $\tau(\sigma) = 1$ for $\sigma = 1$ and $\tau(\sigma) = 2$ for $\sigma = -1$. In order to show that $\mathcal{T}$ has a fixed point in Ω, for the Schauder-Tychonoff fixed point theorem it is sufficient to verify that: (i) Ω is a nonempty, closed and convex subset of $\ell_T^\infty \times \ell_T^\infty$, (ii) $\mathcal{T}(\Omega) \subseteq \Omega$, (iii) $\mathcal{T}(\Omega)$ is relatively compact, (iv) $\mathcal{T}$ is continuous in Ω.

The validity of (i) is obvious, and furthermore it is not difficult to show that $\mathcal{T}$ maps Ω into itself, i.e. (ii) holds. To show it, if f, g are nondecreasing it is sufficient to choose $M_1 = N_1 = 1/2$ and $M_2 = N_2 = 1$. If f, g are nonincreasing we choose $M_1 = N_1 = 1, M_2 = N_2 = 3/2$, and finally, when f is nondecreasing and g is nonincreasing, or vice versa, it is sufficient to choose $M_1 = N_2 = 1, M_2 = 3/2, N_1 = 1/2$, or $N_1 = M_2 = 1, M_1 = 1/2, N_2 = 3/2$, respectively.

(iii) To show that $\mathcal{T}(\Omega)$ is relatively compact it is sufficient to prove that $\mathcal{T}(\Omega)$ is uniformly Cauchy in the topology of $\ell_T^\infty \times \ell_T^\infty$ by [4, Theorem 3.3], i.e. for any $\varepsilon > 0$ there exists $N \geq T$ such that for any $k, l \geq N$ it holds

$$|(\mathcal{T}_1(v))_k - (\mathcal{T}_1(v))_l| < \varepsilon \quad \text{and} \quad |(\mathcal{T}_2(u))_k - (\mathcal{T}_2(u))_l| < \varepsilon$$

for $(u, v) \in \Omega$. The details are left to the reader.

(iv) The continuity of $\mathcal{T}$ in Ω can be proved using a similar argument to that in the proof of [10, Theorem 1], namely using the discrete analogue of the Lebesgue dominated convergence theorem, since the series occuring in the definition of the operator $\mathcal{T}$ are totally convergent.

Thus the Schauder fixed point theorem can be applied and the operator $\mathcal{T}$ has a fixed point $(x, y) \in \Omega$. It is easy to see that (x, y) is a regularly decaying solution of (1) for $k \geq T$.

To show that this solution can be extended to the left in a decreasing manner for $\sigma = 1$ we use the fact that system (1) is actually a recurrence relation. We proceed in the same way as it is done in the proof of [10, Theorem2].

The "only if part". Let (x, y) be a regularly decaying solution of (1). Then there exist positive constants M_1, M_2, N_1, N_2 and $T \geq m$ such that

$$\begin{aligned} \textstyle\sum_{j=k}^\infty \Phi_{\alpha*}\left(\frac{M_1}{r_j}\right) \leq x_k \leq \sum_{j=k}^\infty \Phi_{\alpha*}\left(\frac{M_2}{r_j}\right), \\ \textstyle\sum_{j=k}^\infty \Phi_{\beta*}\left(\frac{N_1}{q_j}\right) \leq y_k \leq \sum_{j=k}^\infty \Phi_{\beta*}\left(\frac{N_2}{q_j}\right) \end{aligned} \tag{8}$$

for $k \geq T$. Assume f, g nondecreasing; the remaining cases can be treated similarly. Summing twice both equations in (1) from k to ∞ we get

$$\begin{aligned} x_k = \sigma \textstyle\sum_{j=k}^\infty \Phi_{\alpha*}\left(\frac{1}{r_j}\sum_{i=j}^\infty(\varphi_i f(y_{i+1}) + \hat{\varphi}_i)\right), \\ y_k = \textstyle\sum_{j=k}^\infty \Phi_{\beta*}\left(\frac{1}{q_j}\sum_{i=j}^\infty(\psi_i g(x_{i+1}) + \hat{\psi}_i)\right). \end{aligned} \tag{9}$$

Now (8) and (9) imply the conditions (4), (5), (6) and (7), that leads to the assertion. □

Remark 3. Note that the necessary and sufficient conditions in Theorem 2 are the same for both cases $\sigma = \pm 1$ in spite of the fact that different sign condition causes a different dynamical behavior as regards other types of solutions.

Remark 4. Using similar arguments to those in the above proof, we are able to prove the "if part" for more general system (3), where the first condition in (4) is replaced by $\sum_{k=m}^{\infty} \alpha^{-1}(1/r_j) < \infty$, α^{-1} being the inverse of the function α. Instead of (6) we suppose the existence of certain "upper" functions $\bar{\varphi}, \bar{F}$ such that $F(k,u,v) \le \bar{\varphi}_k \bar{F}(u)$ on $\{m, m+1, \dots\} \times (0,\varepsilon] \times (0,\varepsilon]$, where $\bar{\varphi}, \bar{F}$ satisfy

$$\sum_{k=m}^{\infty} \bar{\varphi}_k \, \bar{F}\bigg(\sum_{j=k+1}^{\infty} \beta^{-1}\left(\frac{1}{q_j}\right)\bigg) < \infty,$$

β^{-1} being the inverse of β. The latter condition in (4) and (7) would be rearranged in a similar way. Obviously, a necessary condition in the above sense cannot be stated in this case.

Remark 5. Closer examination of the proof enables us to obtain an asymptotic estimate for regularly decaying solutions. Indeed, it is not difficult to see that x_k is asymptotic to $\sum_{j=k}^{\infty} \Phi_{\alpha *}(1/r_j)$, while y_k is asymptotic to $\sum_{j=k}^{\infty} \Phi_{\beta *}(1/q_j)$.

Remark 6. Finally note that, besides the regular case, the statement of Theorem 2 includes the singular one as well, i.e. when f, g are unbounded in a right neigborhood of zero. It should be emphasized that we do not require any additional conditions to treat this case.

3 Strongly Decaying Solutions

For sake of completeness in this section we recall some results that were proved in [10] in order to have "complementary" statements. We start with a necessary and sufficient criterion guaranteeing the existence of strongly decaying solutions of system (1) with $\sigma = -1$.

Theorem 7 ([10], Theorem 1). *System* (1) *with* $\sigma = -1$ *has strongly decaying solutions if and only if condition* (5) *and*

$$\sum_{k=m}^{\infty} \Phi_{\alpha *}\left(\frac{1}{r_k}\right) < \infty$$

hold, and there exists a constant $A > 0$ *such that*

$$\sum_{k=m}^{\infty} \varphi_k f\bigg(\sum_{j=k+1}^{\infty} \omega_j(A)\bigg) < \infty, \tag{10}$$

where

$$\omega_k(A) = \Phi_{\beta *}\bigg\{\frac{1}{q_k}\sum_{j=k}^{\infty}\bigg[\psi_j \, g\bigg(\sum_{i=j+1}^{\infty} \Phi_{\alpha *}\left(\frac{A}{r_j}\right)\bigg) + \hat{\psi}_j\bigg]\bigg\} < \infty.$$

Remark 8. In the contrast to the existence of regularly decaying solutions, the first condition in (4) is not necessary in Theorem 7. On the other hand, if the first condition in (4) holds, then condition (10) can be relaxed to simpler (but only sufficient) conditions (6), (7). Thus we have analogous sufficient conditions guaranteeing the existence of both regular and strongly decaying solutions of (1) with $\sigma = -1$. See Remark 13 for additional information concerning the case $\sigma = 1$.

Remark 9. The proof of Theorem 7 gives an asymptotic estimate for strongly decaying solutions. However, in the contrast to the case in Section 2, we have an asymptotic estimate only for the first component. In fact, x_k is asymptotic to $\sum_{j=k}^{\infty} \Phi_{\alpha*}(1/r_j)$.

In the case of system (1) with $\sigma = 1$, the following existence result holds.

Theorem 10 ([10], Theorem 2). *Let f, g be nondecreasing. Suppose that*

$$\text{at least one of the forcing terms } \{\hat{\varphi}_k\}, \{\hat{\psi}_k\} \text{ is eventually nontrivial.} \tag{11}$$

If

$$\sum_{k=m}^{\infty} \Phi_{\alpha*}\left(\frac{1}{r_k}\sum_{j=k}^{\infty}(\varphi_j + \hat{\varphi}_j)\right) < \infty, \ \sum_{k=m}^{\infty} \Phi_{\beta*}\left(\frac{1}{q_k}\sum_{j=k}^{\infty}(\psi_j + \hat{\psi}_j)\right) < \infty, \tag{12}$$

then system (1) *with $\sigma = 1$ has at least one strongly decaying solution that is decreasing for any $k \geq m$.*

Remark 11. Observe that condition (4) is not necessary. On the other hand, Theorem 10 requires the convergence of the series $\sum_{j=m}^{\infty} \varphi_j$ and $\sum_{j=m}^{\infty} \psi_j$.

When at least one of the series $\sum_{j=m}^{\infty} \varphi_j$ and $\sum_{j=m}^{\infty} \psi_j$ diverges, we can state the following criterion.

Theorem 12 ([10], Theorem 3). *Let f, g be nondecreasing. If* (11)*,* (4)*,* (5)*,* (6)*,* (7) *hold, then system* (1) *with $\sigma = 1$ has at least one strongly decaying solution that is decreasing for any $k \geq m$.*

Remark 13. If f, g are nondecreasing and (11) holds, then the conditions of Theorem 2 are sufficient for the existence of both regularly and strongly decaying solution of (1) with $\sigma = 1$ and with $\sigma = -1$ as well.

Remark 14. In the contrast to Theorems 2 and 7, the presence of forcing terms plays an important role in Theorems 10 and 12 for proving the existence of strongly decaying solutions. On the other hand, as it is shown in [10], there exist discrete nonforced systems having strongly decaying solutions. The problem of finding sufficient conditions for nonforced discrete systems to have singular solutions remains open; in the continuous case this problem can be solved using the concept of *singular solution* (see [12]) that however has no discrete counterpart.

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