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Closed-Range Composition Operators on $\mathbb{A}^2$ and the Bloch Space

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Abstract. For any analytic self-map φ of $\{z : |z| < 1\}$ we give four separate conditions, each of which is necessary and sufficient for the composition operator C_φ to be closed-range on the Bloch space $\mathcal{B}$. Among these conditions are some that appear in the literature, where we provide new proofs. We further show that if C_φ is closed-range on the Bergman space $\mathbb{A}^2$, then it is closed-range on $\mathcal{B}$, but that the converse of this fails with a vengeance. Our analysis involves an extension of the Julia-Carathéodory Theorem.

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Preliminaries

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1\}$ and let $\mathbb{T}$ denote the unit circle $\{z : |z| = 1\}$. We let A denote two-dimensional Lebesgue measure on $\mathbb{D}$. The Bergman space $\mathbb{A}^2$ is the collection of functions f that are analytic in $\mathbb{D}$ such that

$$||f||^2_{\mathbb{A}^2} := \int_{\mathbb{D}} |f|^2 \, dA < \infty.$$

As a closed subspace of $L^2(A)$, $\mathbb{A}^2$ forms a Hilbert space with respect to the inner product $<f, g> := \int_{\mathbb{D}} f\overline{g} dA$. The Bloch space $\mathcal{B}$ is the collection of functions f that are analytic in $\mathbb{D}$ such that

$$||f||_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Now $||\cdot||_{\mathcal{B}}$ defines a norm on $\mathcal{B}$, and under this norm $\mathcal{B}$ forms a Banach space. Moreover, $||f||_{\mathbb{A}^2} \leq 3||f||_{\mathcal{B}}$ for any function f that is analytic in $\mathbb{D}$, and hence $\mathcal{B} \subseteq \mathbb{A}^2$. A function φ that is analytic in $\mathbb{D}$ and that satisfies $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ is called an *analytic self-map* of $\mathbb{D}$. Analytic automorphisms of $\mathbb{D}$ are Möbius transformations of the form $z \mapsto c\frac{\alpha - z}{1-\bar{\alpha}z}$, where c is some unimodular constant and α is some point in $\mathbb{D}$; we let $\varphi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$. The so-called *pseudohyperbolic metric* on $\mathbb{D}$ is given by $\rho(z,w) = |\varphi_w(z)|$; and is indeed a metric. For any z in $\mathbb{D}$ and any r, $0 < r < 1$, we let $D(z,r)$ denote the pseudohyperbolic disk of radius r about z, namely, $\{w \in \mathbb{D} : \rho(z,w) < r\}$. Now if φ is an analytic self-map of $\mathbb{D}$, then the composition operator C_φ, given by $C_\varphi(f) := f \circ \varphi$, is a bounded operator on both $\mathbb{A}^2$ and $\mathcal{B}$. This result for the Bloch space is a simple consequence of the Schwarz-Pick Lemma (cf., [7, page 2]), and for the Bergman space case one may consult [13, page 17]. Moreover, if φ is not constant, then C_φ is one-to-one on these spaces and hence, by the Open Mapping Theorem, is closed-range if and only if it is bounded below. For any analytic self-map φ of $\mathbb{D}$, define τ_φ on $\mathbb{D}$ by

$$\tau_\varphi(z) := \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}.$$

For $\varepsilon > 0$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}$ and let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. We say that F_ε satisfies the *reverse Carleson condition* if there exist s and c, $0 < s,\, c < 1$, such that

$$A(F_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all z in $\mathbb{D}$; cf., [10] for seminal work regarding this condition. It has been shown that C_φ is closed-range on $\mathcal{B}$ if and only if there exists $\varepsilon > 0$ such that F_ε satisfies the reverse Carleson condition; cf. [9] and [3]. In fact, in [3] it is shown that, what appears to be a weaker condition than the one stated above, is indeed equivalent. To be specific, if there exist $\varepsilon > 0$ and s, $0 < s < 1$, such that $F_\varepsilon \cap D(z,s) \neq \emptyset$ for all z in $\mathbb{D}$, then C_φ is closed-range on $\mathcal{B}$. One of the first results of this paper adds one more equivalent condition to this list, and we give a brief and rather novel proof that each of the three conditions are equivalent to C_φ being closed-range on $\mathcal{B}$; see Theorem 2.2. We then turn to connections between the Bloch and Bergman space settings. In the Bergman space setting there is an analogue of Λ_ε that takes center stage. Indeed, if φ is an analytic self-map of $\mathbb{D}$ and $\varepsilon > 0$, then we let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and let $G_\varepsilon = \varphi(\Omega_\varepsilon)$. In [1] it is shown that C_φ is closed-range on $\mathbb{A}^2$ if and only if there exists $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition; that is, there exist s and c, $0 < s,\, c < 1$, such that

$$A(G_\varepsilon \cap D(z,s)) \geq cA(D(z,s)),$$

for all z in $\mathbb{D}$. Here, we establish an extension of the Julia-Carathéodory Theorem (see Theorem 3.4) and use it to show that if C_φ is closed-range on $\mathbb{A}^2$, then there exist ε and s, $0 < \varepsilon,\, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$; see Theorem 3.5. From this we easily have the implication that if C_φ is closed-range on $\mathbb{A}^2$, then it is also closed-range on $\mathcal{B}$; see Corollary 3.6. We also (by examples) show that the converse of Corollary 3.6 fails, without remedy. Indeed, we construct a *thin* Blaschke product that fixes zero and that has no angular derivative anywhere on $\mathbb{T}$, whence C_B is norm preserving on $\mathcal{B}$ and yet is compact on $\mathbb{A}^2$; see Example 3.8. And we also construct a univalent analytic self-map h of $\mathbb{D}$ that has no unimodular nontangential boundary values on $\mathbb{T}$, and thus has no angular derivative anywhere on $\mathbb{T}$ (whence, C_h is compact on $\mathbb{A}^2$), such that C_h is closed-range on $\mathcal{B}$; see Example 3.10. We close the paper with a result that follows easily from work done in [1] and a remark concerning Fredholm operators; see Sect. 4.

Regarding the Bloch Space

Recall that, for any analytic self-map φ of $\mathbb{D}$ and any $\varepsilon > 0$,

$$\tau_\varphi(z) := \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}, \quad \text{and} \quad \Lambda_\varepsilon := \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}.$$

Lemma 2.1. *For any $\varepsilon > 0$ there exist r and s, $0 < r, s < 1$, such that if $z \in \Lambda_\varepsilon$, then*

i) $D(z,r) \subseteq \Lambda_{\frac{\varepsilon}{2}}$,
ii) *φ is univalent in $D(z,r)$ and*
iii) $D(\varphi(z), s) \subseteq \varphi(D(z,r))$.

Proof. (i) By [8], τ_φ is Lipschitz with respect to the pseudohyperbolic metric. Indeed, there is a positive constant c, independent of φ and of z and w in $\mathbb{D}$, such that

$$|\tau_\varphi(z) - \tau_\varphi(w)| \le c\rho(z,w).$$

Let $r = \frac{\varepsilon}{2c}$ and suppose that $|\tau_\varphi(w)| \le \frac{\varepsilon}{2}$. Then, for z in Λ_ε,

$$\frac{\varepsilon}{2} \;<\; ||\tau_\varphi(z)| - |\tau_\varphi(w)|| \;\le\; |\tau_\varphi(z) - \tau_\varphi(w)| \;\le\; c\rho(z,w).$$

Therefore, if $z \in \Lambda_\varepsilon$ and $\rho(z,w) < \frac{\varepsilon}{2c}$, then $w \in \Lambda_{\frac{\varepsilon}{2}}$.

(ii) Suppose that $a \in \Lambda_\varepsilon$ and $\alpha := \varphi(a)$. Notice that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is an analytic self-map of the unit disk that maps 0 to 0 and that

$$|(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)| = |\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}(0)| = |\tau_{\varphi \circ \varphi_a}(0)| = |\tau_\varphi(a)| > \varepsilon.$$

We argue that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is univalent in $\{z : |z| < r\}$; where, as in (i), $r := \frac{\varepsilon}{2c}$. Multiplying $\varphi_\alpha \circ \varphi \circ \varphi_a$ by an appropriate unimodular constant we may assume that $(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)$ is a positive real number (greater than ε). And using the facts that $\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}$ is Lipschitz with respect to the pseudohyperbolic metric, with the same Lipschitz constant c, and that $\varphi_\alpha \circ \varphi \circ \varphi_a$ maps 0 to 0, we find that

$$\operatorname{Re}((\varphi_\alpha \circ \varphi \circ \varphi_a)'(z)) > \frac{\varepsilon}{2}, \tag{2.1.1}$$

whenever $|z| < r$. Now let z and w be distinct points both of which have modulus less than r, and define γ on $[0,1]$ by $\gamma(t) = (1-t)z + tw$. Then, by (2.1.1),

$$0 \neq (w-z) \cdot \int_0^1 (\varphi_\alpha \circ \varphi \circ \varphi_a)'(\gamma(t))dt = (\varphi_\alpha \circ \varphi \circ \varphi_a)(w) - (\varphi_\alpha \circ \varphi \circ \varphi_a)(z),$$

and hence $\varphi_\alpha \circ \varphi \circ \varphi_a$ is univalent in $\{z : |z| < r\}$. It now follows that φ is univalent in $D(a, r)$.

(iii) Given the terminology of part (ii), $h(z) := \frac{1}{r\varepsilon}(\varphi_\alpha \circ \varphi \circ \varphi_a)(rz)$ is analytic and univalent in $\mathbb{D}$, $h(0) = 0$ and $|h'(0)| > 1$. Therefore, by the Koebe One-Quarter Theorem (cf., [13, page 154]),

$$\left\{z : |z| < \frac{1}{4}\right\} \subseteq h(\mathbb{D}).$$

From this it follows that

$$\left\{z : |z| < \frac{r\varepsilon}{4}\right\} \subseteq (\varphi_\alpha \circ \varphi \circ \varphi_a)(\{z : |z| < r\}).$$

With $s := \frac{r\varepsilon}{4}$ we then find that $D(\varphi(a), s) \subseteq \varphi(D(a, r))$. □

As before, let φ be an analytic self-map of $\mathbb{D}$, let $\tau_\varphi(z) = \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}$ and for $\varepsilon > 0$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}$ and let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. We now give two conditions, each of which is equivalent to C_φ being closed-range on $\mathcal{B}$; cf., [9] and [3], or Theorem 2.2 below.

(∗) There exist $\varepsilon > 0$ and constants c and s, $0 < c, s < 1$, such that $A(F_\varepsilon \cap D(z, s)) \geq cA(D(z, s))$ for all z in $\mathbb{D}$.

(#) There exist $\varepsilon > 0$ and s, $0 < s < 1$, such that $F_\varepsilon \cap D(z, s) \neq \emptyset$ for all z in $\mathbb{D}$.

Theorem 2.2. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

i) *C_φ is closed-range on $\mathcal{B}$.*
ii) *Condition (∗) holds.*
iii) *Condition (#) holds.*
iv) *There are constants r, s and c, $0 < r, s, c < 1$, such that, for any w in $\mathbb{D}$, there exists z_w in $\mathbb{D}$ with the property: φ is univalent on $D(z_w, s)$, $\varphi(D(z_w, s)) \subseteq D(w, r)$ and $A(\varphi(D(z_w, s))) \geq c(1 - |w|^2)^2$.*

Proof. (i) $\Longrightarrow$ (iii). Since any Frostman shift of φ (i.e., $\varphi_\alpha \circ \varphi$, where $\alpha \in \mathbb{D}$) gives rise to a closed-range composition operator on $\mathcal{B}$ if and only if φ does, we may assume that $\varphi(0) = 0$. Now suppose that (iii) does not hold. Then we can find sequences $\{r_n\}_{n=1}^\infty$, where $0 < r_n < 1$ and $\lim_{r\to\infty} r_n = 1$, and $\{w_n\}_{n=1}^\infty$ in $\mathbb{D}$, where $\lim_{n\to\infty} |w_n| = 1$, such that

$$\sup\{|\tau_\varphi(z)| : z \in \varphi^{-1}(D(w_n, r_n))\} \longrightarrow 0,$$

as $n \to \infty$. Let $\Delta_n = \varphi^{-1}(D(w_n, r_n))$ and let $D_n = \mathbb{D}\backslash\Delta_n$; for $n = 1, 2, 3, \ldots$. Now

$$\begin{aligned} ||\varphi_{w_n} \circ \varphi||_{\mathcal{B}/C} &:= \sup\{(1 - |z|^2)|(\varphi_{w_n} \circ \varphi)'(z)| : z \in \mathbb{D}\} \\ &= \sup\{[1 - \rho^2(w_n, \varphi(z))]|\tau_\varphi(z)| : z \in \mathbb{D}\} \\ &\leq \sup\{[1 - \rho^2(w_n, \varphi(z))]|\tau_\varphi(z)| : z \in \Delta_n\} \\ &\quad + \sup\{[1 - \rho^2(w_n, \varphi(z))]|\tau_\varphi(z)| : z \in D_n\} \longrightarrow 0, \end{aligned}$$

as $n \to \infty$. Yet $||\varphi_{w_n}||_{\mathcal{B}/C} = 1$, for all n. By Theorem 0 of [9] it now follows that C_φ is not closed-range on $\mathcal{B}$.

(iii) $\implies$ (ii). We assume (iii), that (#) holds. Then, by Lemma 2.1, $(*)$ holds for $\frac{\varepsilon}{2}$.

(ii) $\implies$ (i). This follows immediately from Proposition 1 and Theorem 1 of [9].

At this point we have established the equivalence of (i), (ii) and (iii).

(iii) $\implies$ (iv). This follows immediately from Lemma 2.1.

(iv) $\implies$ (iii). Assuming (iv),

$$\int_{D(z_w,s)} |\varphi'(z)|^2 dA(z) \geq c(1-|w|^2)^2,$$

and hence

$$\int_{D(z_w,s)} \frac{|\varphi'(z)|^2}{(1-|w|^2)^2}\, dA(z) \geq c.$$

Thus we can find a positive constant ε, dependent only on r and s, such that

$$\int_{D(z_w,s)} |\tau_\varphi(z)|^2 dA(z) \geq \varepsilon^2 A(D(z_w,s)).$$

Therefore, $|\tau_\varphi(z)| \geq \varepsilon$ for some z in $D(z_w,s)$, and hence $F_\varepsilon \cap D(w,r) \neq \emptyset$ for each w in $\mathbb{D}$; which gives us (iii). The proof is now complete. $\square$

A special case of our next result is given by Theorem 2 of [9]; namely, the case that φ is a univalent, analytic self-map of $\mathbb{D}$. As is indicated in the proof of Theorem 2.2, if $f \in \mathcal{B}$, then $||f||_{\mathcal{B}/C} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$.

Corollary 2.3. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is closed-range on $\mathcal{B}$ if and only if there exists $\delta > 0$ such that, for all α in $\mathbb{D}$, $||\varphi_\alpha \circ \varphi||_{\mathcal{B}/C} \geq \delta$.*

Proof. We may assume that $\varphi(0) = 0$ here since any Frostman shift of φ gives rise to a closed-range composition operator on $\mathcal{B}$ if and only if φ does, and since the collection of analytic automorphisms of $\mathbb{D}$ forms a group under the operation of composition. Moreover, notice that $||\varphi_\alpha||_{\mathcal{B}/C} = 1$ for all α in $\mathbb{D}$. So, if C_φ is closed-range on $\mathcal{B}$, then, by Theorem 0 of [9], there exists $\delta > 0$ such that $||\varphi_\alpha \circ \varphi||_{\mathcal{B}/C} \geq \delta$ for all α in $\mathbb{D}$. Conversely, suppose that there exists $\delta > 0$ such that $||\varphi_\alpha \circ \varphi||_{\mathcal{B}/C} \geq \delta$ for all α in $\mathbb{D}$. Then, by Proposition 2 of [9], (iii) of Theorem 2.2 holds and hence C_φ is closed-range on $\mathcal{B}$. $\square$

The Context of $\mathbb{A}^2$ Versus that of $\mathcal{B}$

Let φ be an analytic self-map of $\mathbb{D}$ and, for $\varepsilon > 0$, let $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$, let $G_\varepsilon = \varphi(\Omega_\varepsilon)$ and let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. By the Julia-Carathéodory Theorem (cf., [13, page 57]), φ has an angular derivative at each point ξ in K, which we denote by $\varphi'(\xi)$. Indeed, $\varphi'(\xi) = \zeta\overline{\xi}d$, where $\zeta := \varphi(\xi) := \angle\lim_{z\to\xi}\varphi(z)$ and d is given by

$$d := \liminf_{z\to\xi} \frac{1-|\varphi(z)|}{1-|z|} \ \left(= \liminf_{z\to\xi} \frac{1-|\varphi(z)|^2}{1-|z|^2}\right).$$

The Julia-Carathéodory Theorem tells us that $d > 0$. And since $\xi \in K$, $d \leq \frac{1}{\varepsilon}$.

Proposition 3.1. *Given the terminology of the above discussion, φ is continuous on $\overline{\Omega}_\varepsilon$ and φ' is continuous on K.*

Proof. The continuity of φ on $\overline{\Omega}_\varepsilon$ was established in [1]; see Remark 2.6 in this reference. Now let $\{\xi_n\}_{n=1}^\infty$ be a sequence in K that converges to ξ_0 in K, and let $d_n = |\varphi'(\xi_n)|$, for $n = 0, 1, 2, \ldots$. Since φ is continuous on K, the continuity of φ' on K will follow if we show that $d_n \longrightarrow d_0$, as $n \to \infty$. Now by the discussion just prior to this proposition, $\{d_n\}_{n=1}^\infty$ is bounded. And so, passing to a subsequence if necessary, we may assume that $d_n \longrightarrow d$, as $n \to \infty$. Thus our goal here is to show that $d = d_0$. To this end, by the Julia-Carathéodory Theorem we can find a sequence $\{r_n\}_{n=1}^\infty$ in $(0, 1)$, such that $\lim_{n\to\infty} r_n = 1$ and $|d_n - \frac{1-|\varphi(r_n\xi_n)|}{1-r_n}| < \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Hence, $\{r_n\xi_n\}_{n=1}^\infty$ is a sequence in $\mathbb{D}$ that converges to ξ_0 and $\{\frac{1-|\varphi(r_n\xi_n)|}{1-r_n}\}_{n=1}^\infty$ converges to d. Julia's Theorem (cf., [13, page 63]) now tells us that $d = d_0$. □

We now set the stage for two subsequent results.

Discussion 3.2. For any point ξ in $\mathbb{T}$ and any θ, $0 < \theta < \pi$, we let $S(\xi, \theta)$ denote the interior of closed convex hull of $\{\xi\} \cup \{z : |z| \leq \sin(\frac{\theta}{2})\}$. We call $S(\xi, \theta)$ the *Stolz region* based at ξ with vertex angle θ. For our purposes here it is sufficient that we keep the vertex angles of our Stolz regions fixed at $\frac{\pi}{2}$, though our arguments carry through for any fixed θ in the aforementioned range. Let φ be an analytic self-map of $\mathbb{D}$ and, for $\varepsilon > 0$, let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. Define W_ε by

$$W_\varepsilon = \bigcup_{\xi \in K} S(\xi, \tfrac{\pi}{2}).$$

Suppose that $\{z_n\}_{n=1}^\infty$ is a sequence in W_ε that converges to a point ξ_0 in K. So, we can find a sequence $\{\xi_n\}_{n=1}^\infty$ in K such that $z_n \in S(\xi_n, \frac{\pi}{2})$ (for $n = 1, 2, 3, \ldots$) and $\lim_{n\to\infty} \xi_n = \xi_0$. Now

- $\zeta_n := \varphi(\xi_n) := \angle\lim_{z\to\xi_n} \varphi(z)$, and
- $\angle\lim_{z\to\xi_n} \varphi'(z) =: \varphi'(\xi_n) = \zeta_n\overline{\xi}_n d_n$ – *the angular derivative of φ at ξ_n*, where $d_n := |\varphi'(\xi_n)|$.

By Proposition 3.1, $\varphi'(\xi_n) \longrightarrow \varphi'(\xi_0) = \zeta_0\overline{\xi}_0 d_0$, as $n \to \infty$, where $\zeta_0 := \varphi(\xi_0) := \angle\lim_{z\to\xi_0} \varphi(z)$ and $d_0 := |\varphi'(\xi_0)|$. Since $0 < d_0 < \infty$, we can find $M > 1$ such that $\frac{1}{M} \leq d_n \leq M$ for all n.

Lemma 3.3. *Assuming the terminology of Discussion 3.2, for any $\varepsilon > 0$, there exist s, $0 < s < 1$, and N (in $\mathbb{N}$) such that*

$$\left| d_n - \tfrac{1-|\varphi(z)|}{1-|z|} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Proof. If not, then we can find $d \neq d_0$, a subsequence $\{\xi_{n_k}\}_{k=1}^\infty$ of $\{\xi_n\}_{n=1}^\infty$ and a sequence $\{z'_k\}_{k=1}^\infty$ such that

- $z'_k \in S(\xi_{n_k}, \frac{\pi}{2})$ for all k,
- $|z'_k - \xi_{n_k}| \longrightarrow 0$ and hence $|z'_k - \xi_0| \longrightarrow 0$ (as $k \to \infty$), and
- $(1 - |\varphi(z'_k)|)/(1 - |z'_k|) \longrightarrow d$, as $k \to \infty$.

By Julia's Theorem this would then tell us that

$$d = |\varphi'(\xi_0)| = d_0;$$

a contradiction. □

Theorem 3.4. *Assuming the terminology of Discussion 3.2, φ' is continuous on $\overline{W}_\varepsilon$.*

Proof. Our proof here is based on Lemma 3.3 and some observations concerning the proof of the Julia-Carathéodory Theorem in [13]. By Proposition 3.1, all we need to show is that, given the hypothesis of Discussion 3.2, $\varphi'(z_n) \longrightarrow \varphi'(\xi_0)$, as $n \to \infty$.

Claim A. *For any $\varepsilon > 0$ there exist s, $0 < s < 1$, and N (in $\mathbb{N}$) such that*

$$\left| \zeta_n \bar{\xi}_n d_n - \frac{\zeta_n - \varphi(z)}{\xi_n - z} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

To justify this claim we first observe that, by Lemma 3.3, for any $\eta > 0$, there exist σ, $0 < \sigma < 1$, and ν (in $\mathbb{N}$) such that

$$\left| d_n - \frac{1 - |\varphi(r\xi_n)|}{1 - r} \right| < \eta \quad \text{and} \quad \left| d_n - \frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} \right| < \eta \qquad (3.4.1)$$

provided $\sigma \leq r < 1$ and $n \geq \nu$. Mimicking the proof of JC (1) $\Longrightarrow$ JC (2) (in Sect. 4.5 of [13]), for $n \geq \nu$ we carry the discussion to the right half-plane $\{w : \mathrm{Re}(w) > 0\}$. Let φ_n and ψ_n be the Möbius transformations given by $\varphi_n(z) := \frac{\xi_n + z}{\xi_n - z}$ and $\psi_n(z) := \frac{\zeta_n + z}{\zeta_n - z}$. Define Φ_n and γ_n on $\{w : \mathrm{Re}(w) > 0\}$ by $\Phi_n(w) := (\psi_n \circ \varphi \circ \varphi_n^{-1})(w)$ and $\gamma_n(w) := \Phi_n(w) - c_n w$, where $c_n := \frac{1}{d_n}$. Now by (3.4.1), if $n \geq \nu$ and $\sigma \leq r < 1$, then

$$d_n - \eta < \frac{1 - |\varphi(r\xi_n)|}{1 - r}, \frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} < d_n + \eta$$

and hence, by Julia's Theorem and with $w_{n,r} := \varphi_n(r\xi_n)$ $(= \frac{1+r}{1-r})$,

$$\frac{1}{d_n} \leq \frac{\mathrm{Re}(\Phi_n(w_{n,r}))}{\mathrm{Re}(w_{n,r})} = \left(\frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} \right) \frac{(1 - r)^2}{|\zeta_n - \varphi(r\xi_n)|^2} < \frac{d_n + \eta}{(d_n - \eta)^2}.$$

Therefore, if n is sufficiently large (allowing η to be sufficiently small), one can force

$$\frac{\mathrm{Re}(\gamma_n(w_{n,\sigma}))}{\mathrm{Re}(w_{n,\sigma})}$$

to be less than any prescribed positive real number; and $w_{n,\sigma} = \frac{1+\sigma}{1-\sigma}$, which clearly does not vary with n. We let $w_{n,\sigma}$ play the role of w_0 in the proof of JC (1) $\Longrightarrow$ JC (2) (in Sect. 4.5 of [13]). And since the image under φ of any compact subset of $\mathbb{D}$ is a compact subset of $\mathbb{D}$, $\{|\gamma_n(w_{n,\sigma})|\}_{n=1}^{\infty}$ is bounded. Thus, following through with the argument in [13], we find that, for any

$\tau > 0$, there is a positive real number R such that if $w \in \varphi_n(S(\xi_n, \frac{\pi}{2}))$ and $|w| > R$, then

$$\left|\frac{\gamma_n(w)}{w}\right| < \tau, \tag{3.4.2}$$

provided n is sufficiently large. Now, via the correspondence $w = \varphi_n(z)$, routine calculations give that

$$\frac{w+1}{\Phi_n(w)+1} = \xi_n \bar{\zeta}_n \left(\frac{\zeta_n - \varphi(z)}{\xi_n - z}\right),$$

and hence,

$$\left|\frac{\gamma_n(w)+1}{w+1}\right| = \left|\frac{\xi_n - z}{\zeta_n - \varphi(z)} - \frac{\xi_n \bar{\zeta}_n c_n w}{w+1}\right|.$$

We now find that Claim (A) follows from (3.4.2).

Claim B. *For any $\varepsilon > 0$ there exist s, $0 < s < 1$, and N (in $\mathbb{N}$) such that*

$$|\zeta_n \bar{\xi}_n d_n - \varphi'(z)| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Now Claim (B) follows directly from Claim (A) and the proof of JC (2) $\Longrightarrow$ JC (3) (in Sect. 4.6 of [13]). And by Claim (B) and the fact that $\varphi'(\xi_n) \longrightarrow \varphi'(\xi_0)$, as $n \to \infty$, we find that

$$\varphi'(z_n) \longrightarrow \varphi'(\xi_0),$$

as $n \to \infty$; which completes our proof. □

Theorem 3.5. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^2$, then there exist ε and s, $0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$.*

Proof. Suppose that C_φ is closed-range on $\mathbb{A}^2$. Then there exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies the reverse Carleson condition; cf., [1]. In particular, $\mathbb{T} \subseteq \overline{G}_\varepsilon$. So, for each point v_0 in $\mathbb{T}$, we can find a sequence $\{w_n\}_{n=1}^\infty$ in Ω_ε such that $\{\varphi(w_n)\}_{n=1}^\infty$ converges to v_0. Passing to a subsequence if necessary, we may assume that $\{w_n\}_{n=1}^\infty$ converges to some point ω_0 in $K := \mathbb{T} \cap \overline{\Omega}_\varepsilon$. Therefore, by Julia's Theorem, $v_0 = \varphi(\omega_0) := \angle \lim_{w \to \omega_0} \varphi(w)$. Thus, $\varphi(K) = \mathbb{T}$. We proceed indirectly and suppose that the conclusion of this theorem fails. Then we can find a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D} \backslash \{0\}$, such that $\{|z_n|\}_{n=1}^\infty$ converges to 1 and

$$\sup\{|\tau_\varphi(w)| : \varphi(w) = z_n\} \longrightarrow 0, \tag{3.5.1}$$

as $n \to \infty$. Since $\varphi(K) = \mathbb{T}$, there exists $\{\xi_n\}_{n=1}^\infty$ in K such that $\varphi(\xi_n) = \zeta_n := \frac{z_n}{|z_n|}$, for $n = 1, 2, 3, \ldots$. Passing to a subsequence if need be, we may assume that $\{\xi_n\}_{n=1}^\infty$ converges to some point ξ_0 in K. Since, by Proposition 3.1, φ is continuous on K, indeed, continuous on $\overline{\Omega}_\varepsilon$, we find that $\{\zeta_n\}_{n=1}^\infty$ converges to $\zeta_0 := \varphi(\xi_0)$. Now, by Theorem 3.4 and its proof, there exist δ and s, $0 < \delta$, $s < 1$, and N in $\mathbb{N}$ such that

$$|\tau_\varphi(z)| \geq \delta,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$. Moreover, by Claim (A) in the proof of Theorem 3.4 (that speaks to the conformality of φ at ξ_n), we can find σ, $0 < \sigma < 1$, and ν in $\mathbb{N}$ such that

$$\{r\zeta_n : \sigma \leq r < 1\} \subseteq \varphi(\{z \in S(\xi_n, \frac{\pi}{2}) : |z| > s\}),$$

whenever $n \geq \nu$. Since $z_n \in \{r\zeta_n : \sigma \leq r < 1\}$, if n is sufficiently large, we find that (3.5.1) above cannot occur; and our proof is complete. □

Our next result is an immediate consequence of Theorem 3.5 and Theorem 2.2; and so we state it without proof.

Corollary 3.6. *Let φ be an analytic self-map of $\mathbb{D}$. If C_φ is closed-range on $\mathbb{A}^2$, then it is also closed-range on $\mathcal{B}$.*

A slight modification of the proof of Theorem 3.5 gives us the following rather surprising result. It also can be viewed as a byproduct of the nice behavior of φ on $\overline{W}_\varepsilon$, as indicated by Theorem 3.4.

Theorem 3.7. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

i) *C_φ is closed-range on $\mathbb{A}^2$.*

ii) *There exist ε, s and c, $0 < \varepsilon, s, c < 1$, such that*

$$A(G_\varepsilon \cap D(z, s)) \geq cA(D(z, s)),$$

for all z in $\mathbb{D}$.

iii) *There exist ε and s, $0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq G_\varepsilon$.*

Proof. The equivalence between (i) and (ii) was established in [1]. And clearly (iii) implies (ii). So we need only establish that (i) implies (iii). To this end, assume that C_φ is closed-range on $\mathbb{A}^2$ and mimic the proof of Theorem 3.5, replacing $|\tau_\varphi(z)|$ by $\frac{1-|z|^2}{1-|\varphi(z)|^2}$, throughout. The argument carries over with this modification to gives us (iii). □

By Theorem 2.5 of [1], the only univalent analytic self-maps of $\mathbb{D}$ that give rise to closed-range composition operators on $\mathbb{A}^2$ are the analytic automorphisms of $\mathbb{D}$. This is in contrast with the Bloch space setting. Indeed, if ψ is any conformal mapping from $\mathbb{D}$ one-to-one and onto $\mathbb{D} \setminus [0, 1)$, then C_ψ is closed-range on $\mathcal{B}$; cf., Example 2 of [9]. So, the converse of Corollary 3.6 fails. Our next two examples show that the converse fails with a vengeance. Our first is an example of a *thin* Blaschke product B that fixes zero and has no angular derivative at any point of the unit circle $\mathbb{T}$; and by thin we mean that $(1 - |a_n|^2)|B'(a_n)| \longrightarrow 1$, as $n \to \infty$, where $\{a_n\}_{n=1}^{\infty}$ are the zeros of B. Therefore, C_B is norm preserving on $\mathcal{B}$ (cf., [5], or [11]) and yet is compact on $\mathbb{A}^2$ (cf., [13, pages 52 and 195]). And since C_B is compact and not of finite rank on $\mathbb{A}^2$, it is not closed-range on $\mathbb{A}^2$. This first example is a factor of the one produced by J. Shapiro on page 185 of [13].

Example 3.8. Let B^* be the Blaschke product constructed by J. Shapiro on page 185 of [13] and let $\{a_n\}_{n=1}^{\infty}$ be the zeros of B^*. Associated with each a_n is an arc I_n of length $\frac{1}{n}$ of the form $I_n = \{e^{i\theta} : \theta_n \leq \theta \leq \theta_{n+1}\}$. The zeros a_n

are given by: $a_n := r_n e^{i\omega_n}$, where $r_n := 1 - \frac{1}{n^2}$ and $\omega_n := \frac{1}{2}(\theta_n + \theta_{n+1})$. A theorem of O. Frostman (cf., [13, page 183]) is then used to show that B^* has no angular derivative anywhere on $\mathbb{T}$. For each positive integer ν, we define the ν^{th} "layer" of zeros of B^* as $[a_\nu] := \{a_\nu, a_{\nu+1}, \ldots, a_{N_\nu}\}$, where N_ν is the unique positive integer that satisfies:

$$\mathbb{T} \subseteq \bigcup_{n=\nu}^{N_\nu} I_n, \text{ yet } \mathbb{T} \not\subseteq \bigcup_{n=\nu}^{N_\nu - 1} I_n.$$

Since $\sum_{n=\nu}^{N_\nu - 1} \frac{1}{n} < 2\pi$, it follows that $N_\nu < 540\nu$. For any positive integer ν, let B_ν be the Blaschke product with (simple) zeros $[a_\nu]$. For any a_k in $[a_\nu]$, let $B_\nu^{\hat{k}}$ denote B_ν with the Blaschke factor involving a_k deleted. And choose a_{k^*} in $[a_\nu] \setminus \{a_k\}$ such that $\rho(a_k, a_{k^*}) \le \rho(a_k, a_l)$, whenever $a_l \in [a_\nu] \setminus \{a_k\}$. Then, for such l,

$$\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \ge - \frac{(1 - r_k^2)(1 - r_{k^*}^2)}{1 - 2 r_k r_{k^*} \cos(\theta_k - \theta_{k^*}) + r_k^2 r_{k^*}^2}.$$

Now, $|\theta_k - \theta_{k^*}| \ge \frac{1}{4k}$ and so, for ν sufficiently large,

$$1 - 2 r_k r_{k^*} \cos(\theta_k - \theta_{k^*}) + r_k^2 r_{k^*}^2 \ge \frac{1}{20k^2}.$$

Hence,

$$\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \ge - \frac{80}{(k^*)^2} \ge - \frac{80}{\nu^2},$$

independent of k and l in our range here. Therefore,

$$\begin{aligned} 0 &> \sum_{k \ne l = \nu}^{N_\nu} \left(\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \right) \\ &\ge (540\nu)(-\frac{80}{\nu^2}) = -\frac{43,200}{\nu} \longrightarrow 0, \end{aligned}$$

as $\nu \to \infty$; uniformly in k, $\nu \le k \le N_\nu$. From this it follows that

$$|B_\nu^{\hat{k}}(a_k)| \longrightarrow 1, \tag{3.7.1}$$

as $\nu \to \infty$; uniformly in k, $\nu \le k \le N_\nu$. Now since B^* is a Blaschke product,

$$|B_\nu| \longrightarrow 1 \tag{3.7.2}$$

uniformly on compact subsets of $\mathbb{D}$, as $\nu \to \infty$. And since, for any fixed ν, B_ν is a finite Blaschke product,

$$|B_\nu(z)| \longrightarrow 1 \tag{3.7.3}$$

uniformly in z, as $|z| \to 1^-$. Using (3.7.1)–(3.7.3), one can find a (rapidly) increasing sequence $\{\nu_j\}_{j=1}^\infty$ of positive integers such that $[a_{\nu_k}] \cap [a_{\nu_l}] = \emptyset$ if $k \ne l$, and such that

$$B := \prod_{j=1}^{\infty} B_{\nu_j},$$

whose (simple) zeros we enumerate as $\{\alpha_n\}_{n=1}^{\infty}$, satisfies

$$|B^{\hat{n}}(\alpha_n)| \longrightarrow 1,$$

as $n \to \infty$; where $B^{\hat{n}}$ denotes B with the Blaschke factor involving α_n deleted. And we may assume that $\nu_1 = 1$. Hence, B is a thin Blaschke product that fixes zero. Since the zeros of B consist of infinitely many disjoint layers of the zeros of B^*, one can argue as in [13, page 185], and find that

$$\sum_{n=1}^{\infty} \frac{1-|\alpha_n|}{|\zeta-\alpha_n|^2} = \infty,$$

for each ζ in $\mathbb{T}$. Thus, by a theorem of O. Frostman (cf., [13, page 183]), we conclude that B has no angular derivative at any point in $\mathbb{T}$.

Remark 3.9. The converse of Theorem 3.5 does not hold. Indeed, by Theorem 2.7 of [4], if B is the Blaschke product that we produced in Example 3.8, then

$$\mathbb{D} \subseteq F_{\frac{1}{2}};$$

and yet C_B is far from closed-range on $\mathbb{A}^2$.

We now produce a univalent analytic self-map h of $\mathbb{D}$ that has no angular derivative at any point of $\mathbb{T}$ (whence, C_h is compact on $\mathbb{A}^2$) such that C_h is closed-range on $\mathcal{B}$. This dramatically improves upon our understanding of what is possible in the univalent case; cf., Example 2 of [9]. And since $h(\mathbb{D})$ contains no annulus with outer boundary equal to $\mathbb{T}$ (and similarly for Example 2 of [9]), there is no analogue of Theorem 3.7 in the context of the Bloch space.

Example 3.10. Here we construct a conformal mapping h from $\mathbb{D}$ one-to-one and onto an *infinite ribbon* G that spirals out to $\mathbb{T}$ such that C_h is closed-range on $\mathcal{B}$. So h will have no unimodular nontangential boundary values on $\mathbb{T}$, and thus no angular derivative anywhere on $\mathbb{T}$. We write h as the composition of three conformal mappings:

- $\zeta = i\left(\frac{1+z}{1-z} + e\right)$, which maps $\mathbb{D}$ univalently onto $G_1 := \{\zeta : Im(\zeta) > e\}$,
- $\xi = \log(\zeta)$, which maps G_1 univalently onto a smoothly bounded subregion G_2 of the swath $\{\xi : \mathrm{Re}(\xi) > 1 \text{ and } 0 < \mathrm{Im}(\xi) < \pi\}$ that asymptotically approximates this swath, and
- $w = \xi^i$, which maps G_2 univalently onto an infinite ribbon G that spirals out to $\mathbb{T}$.

Thus, $h(z) = \left[\log\left(i\left(\frac{1+z}{1-z} + e\right)\right)\right]^i$. Clearly h has no unimodular nontangential boundary values on $\mathbb{T}$ and thus has no angular derivative anywhere on $\mathbb{T}$. As we noted just prior to Example 3.8, this tells us that C_h is compact and hence not closed-range on $\mathbb{A}^2$. One may also refer to Theorem 2.5 of [1] to obtain that C_h is not closed-range on $\mathbb{A}^2$. Now let $\Gamma = h([0,1))$, which is an arc of infinite length that spirals out to $\mathbb{T}$. Our strategy in showing that C_h is closed-range on $\mathcal{B}$ is to first establish that there exists $\varepsilon > 0$ such that $\Gamma \subseteq F_\varepsilon$ and then establish that there exists s, $0 < s < 1$, such that

$\Gamma \cap D(z,s) \neq \emptyset$ for all z in $\mathbb{D}$. Theorem 2.2 then gives us the conclusion. In what follows we use the symbol $\sim$ between real-valued functions f and g defined on $[0,1)$ (viz., $f \sim g$) to indicate that there is a constant $M > 1$ such that $\frac{1}{M} f(x) \leq g(x) \leq M f(x)$ for all x in $[0,1)$. Now, for x in $[0,1)$,

$$\begin{aligned} h(x) &= \left[\log\left(i\left(\frac{1+x}{1-x}+e\right)\right)\right]^i \\ &= \left[\log\left(\frac{1+x}{1-x}+e\right)+\frac{i\pi}{2}\right]^i. \end{aligned}$$

Denoting $\log\left(\frac{1+x}{1-x}+e\right)+\frac{i\pi}{2}$ by ξ_x, we have:

$$h(x) = e^{i\log(\xi_x)} = e^{-\arg(\xi_x)} \cdot e^{i\log|\xi_x|}.$$

Hence,

$$1-|h(x)| \sim \arg(\xi_x) \sim \frac{1}{\log\left(\frac{1+x}{1-x}+e\right)}.$$

Thus, for x in $[0,1)$,

$$\frac{1-x}{1-|h(x)|} \sim (1-x)\log\left(\frac{1+x}{1-x}+e\right).$$

And, for such x, $h'(x) = \frac{e^{i\log(\xi_x)}}{\log\left(\frac{1+x}{1-x}+e\right)+\frac{i\pi}{2}} \cdot \frac{2}{(1-x^2)+e(1-x)^2}$; whence

$$|h'(x)| \sim \frac{1}{(1-x)\log\left(\frac{1+x}{1-x}+e\right)}.$$

Evidently, $|\tau_h(x)| \sim 1$, and so there exists $\varepsilon > 0$ such that $\Gamma \subseteq F_\varepsilon$. Now, as x increases to 1 in $[0,1)$, $h(x)$ traverses Γ through infinitely many counterclockwise rotations about 0 as it works its way toward $\mathbb{T}$. To complete our argument here it is important that we obtain a good estimate on the ratio between $1-|h(x')|$ and $1-|h(x)|$, if $[x,x']$ is a subinterval of $[0,1)$ over which h makes precisely one rotation about 0. Recalling that $h(x) = e^{-\arg(\xi_x)} \cdot e^{i\log|\xi_x|}$, we find that this reduces to an examination of

$$h^*(y) := e^{-\frac{1}{y}} \cdot e^{i\log(y)},$$

as y in $[1,\infty)$ increases to ∞. Notice that h^* winds through 2π radians on any subinterval of $[1,\infty)$ of the form $[y, e^{2\pi}y]$. And, independent of y, $\frac{1-|h^*(e^{2\pi}y)|}{1-|h^*(y)|}$ is boundedly equivalent to $\frac{1}{e^{2\pi}}$. This then tells us that $\mathbb{D}\setminus\Gamma$ does not contain pseudohyperbolic disks of radius arbitrarily near 1. Hence, there exists s, $0 < s < 1$, such that $\Gamma \cap D(z,s) \neq \emptyset$, for all z in $\mathbb{D}$. Since, as we have shown, $\Gamma \subset F_\varepsilon$, for some $\varepsilon > 0$, we can now refer to Theorem 2.2 and conclude that C_h is closed-range on $\mathcal{B}$.

Closing Remarks

In this final section we give a result in the context of $\mathbb{A}^2$ for singular inner functions and we point out some implications of our work here to the theory of Fredholm operators. In our discussion we let m denote normalized Lebesgue measure on $\mathbb{T}$. Recall that a compact subset E of $\mathbb{T}$ is said to be *porous* if there exists ε, $0 < \varepsilon < 1$, such that whenever I is a arc of $\mathbb{T}$ with $I \cap E \neq \emptyset$, then there is a subarc J of I where $m(J) > \varepsilon m(I)$ and $J \cap E = \emptyset$. In [12] it is shown that E is a porous subset of $\mathbb{T}$ if and only if E has the property: For any singular measure μ supported on E, every nontrivial Frostman shift of the singular inner function S_μ is a Carleson–Newman Blaschke product; that is, a finite product of interpolating Blaschke products. The proof of Corollary 3.11 in [1] also establishes our next result.

Proposition 4.1. *Let E be a porous subset of $\mathbb{T}$. If μ is any singular measure with support in E, then C_{S_μ} is closed-range on $\mathbb{A}^2$.*

Remark 4.2. We close the paper with some thoughts concerning Fredholm operators. We first recall that the little Bloch space $\mathcal{B}_0$ is the collection of functions f in $\mathcal{B}$ for which

$$\lim_{r\to 1} \sup_{r<|z|<1} (1-|z|^2)|f'(z)| = 0.$$

And the Dirichlet space $\mathcal{D}$ is the collection of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in $\mathbb{D}$, such that

$$||f||_{\mathcal{D}}^2 := \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty.$$

An operator between two Banach spaces is called a *Fredholm operator* if its range is closed and both the operator and its adjoint have finite dimensional kernel. If φ is an analytic self-map of $\mathbb{D}$ and C_φ is a Fredholm operator on a Hilbert space of analytic functions that contains $\mathcal{D}$, then φ is a disk automorphism; cf., [6, page 153]. Now $\mathcal{D} \subseteq \mathcal{B}_0$, but we will show that the situation is different for $\mathcal{B}_0$. Indeed, there exists Fredholm composition operators on $\mathcal{B}_0$ whose symbols are not disk automorphisms. The *minimal Besov space* B_1 is the collection of all functions f that are analytic in $\mathbb{D}$ of the form

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \varphi_{w_n}(z), \tag{4.2.1}$$

where $\{w_n\}_{n=1}^{\infty} \subseteq \overline{\mathbb{D}}$, and $\{a_n\}_{n=1}^{\infty} \in l^1$. The norm on B_1 is given by

$$||f||_{B_1} := \inf \left\{ \sum_{n=0}^{\infty} |a_n| : (4.2.1) \text{ holds} \right\}.$$

Now B_1 is a Banach space with respect to this norm and is invariant under disk automorphisms. Under the pairing $(f, g) = \int_{\mathbb{D}} f'(z)\overline{g'(z)}dA(z)$, the dual of $\mathcal{B}_0$ is B_1 and the dual of B_1 is $\mathcal{B}$; cf., [2]. Notice that, for g in $\mathcal{B}_0$ and w in $\mathbb{D}$,

$$(g, \varphi_w) = -\int_{\mathbb{D}} g'(z) \frac{1-|w|^2}{(1-w\overline{z})^2} \, dA(z) = -(1-|w|^2) g'(w),$$

and therefore,

$$(g, C_\varphi^*(\varphi_w)) = <C_\varphi(g), \varphi_w> = -(1-|w|^2)\,(g \circ \varphi)'(w) = -\tau_\varphi(w)(g, \varphi_{\varphi(w)}).$$

If $w \in \mathbb{D}$, then

$$C_\varphi^*(\varphi_w) = -\tau_\varphi(w)\varphi_{\varphi(w)}, \tag{4.2.2}$$

and if $|w| = 1$, then $\varphi_w = w$ and

$$C_\varphi^* \, \varphi_w = 0. \tag{4.2.3}$$

By (4.2.2) and (4.2.3) it is easy to see that the kernel of $C_\varphi^* : B_1 \to B_1$ consists of the constant functions. Also, a non-constant composition operator is always one-to-one, and therefore $C_\varphi : \mathcal{B}_0 \longrightarrow \mathcal{B}_0$ will be a Fredholm operator if it is closed-range. It is shown in [9] that if ψ is a conformal mapping from $\mathbb{D}$ onto $\mathbb{D} \setminus [0, 1)$, then C_ψ is bounded below on $\mathcal{B}$. Any univalent self-map of $\mathbb{D}$ is in $\mathcal{B}_0$, and thus $\psi \in \mathcal{B}_0$ and C_ψ is a Fredholm operator on $\mathcal{B}_0$.

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