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Dynamic Contracting with Long-Term Consequences: Optimal CEO Compensation and Turnover

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Dynamic Contracting with Long-Term Consequences: Optimal CEO Compensation and Turnover*

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Abstract

We examine optimal managerial compensation and turnover policy in a principal-agent model in which the firm output is serially correlated over time. The model captures a learning-by-doing feature: higher effort by the manager increases the quality of the match between the firm and the manager in the future. The optimal incentive scheme entails an inefficiently high turnover rate in the early stages of the employment relationship. The optimal turnover probability depends on the past performance and the likelihood of turnover decreases gradually with superior performance. With good enough past performance, the turnover policy reaches efficiency; the manager is never retained if it is inefficient to do so. The manager's compensation depends on the firm value and the optimal performance-compensation relation increases with past performance.

Keywords: Dynamic moral hazard, managerial turnover, pay for performance

JEL Classification: C73, D82, D86.

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1 Introduction

An extensive literature discusses the impact of the CEO ability on the firm value. Following weak performance, the CEO is replaced in a hope of finding a new manager who is better able to choose the right management strategy to enhance the firm value. However, empirical literature documents that the CEO turnover decisions do not only depend on the firm value, but the probability of the CEO turnover decreases with tenure. In particular, Dikolli, Mayew, and Nanda (2014) report that the turnover probability decreases with superior performance. Furthermore, Cremers and Palia (2011) find that a CEO's pay to performance increases with tenure. Thus, CEO compensation increases with both the firm value and the length of the manager's employment in the firm.

We show that the decrease in turnover probability is a consequence of an asymmetric information problem between the firm owner (principal) and the manager (agent): the manager is compensated for his efforts by promising him a higher job security in the future. Surprisingly, the turnover policy is never too lenient; the manager is never retained if it is inefficient to do so. In contrast, the model predicts an inefficiently high turnover rate for managers with shorter tenure. Moreover, the model predicts an increasing pay-to-performance over time that is consistent with empirical predictions.

To be more concrete, this paper studies the managerial compensation and turnover in a model with a firm and a manager in which the manager's actions have long-term consequences. The value of the firm under the current management depends both on the manager's past efforts and on exogenous circumstances. By investing in the firm, the manager has the opportunity to increase the quality of the match between himself and the firm. The manager's investment is productive and increases the firm value under the current management. This captures the additional value that the right choice of manager adds to the firm profit. The model includes a learning-by-doing component: higher managerial effort today increases the firm profit in all future periods.

Our starting point is a dynamic model with a principal (firm owner) and an agent (manager). Both players are risk-neutral, but the agent is protected by limited liability. Limited liability implies that the contract cannot impose negative payments to the agent, and that eventual losses have to be covered by the principal. The firm produces a stochastic output that is serially correlated over time.[1] The agent can exert effort to increase the quality of the match and thereby the firm profit. Effort is unobserved by the principal and related with opportunity cost for the agent. If the agent shirks, he receives a private benefit. To prevent

[1] We assume that there is a one-to-one mapping between the quality of the match between the players and the output under the current manager.

the agent from enjoying private benefits, the principal has to reward him for good performance and punish him following low outcomes. Since the manager's quality within the current firm is correlated over time, the agent's actions have long-term consequences. Shirking today decreases the firm profit under the current manager for all future periods.

At any point in time, the firm has the opportunity to fire the manager and hire a new one. If the firm replaces the manager, a new one is drawn from a time invariant distribution. Thus, we assume that both the expected firm value and the expected manager's utility following a turnover is constant while the value of the firm under the current manager depends on both the past performance and on the manager's effort. The firm potentially has two motivations to replace the manager. Firstly, the new manager eventually provides a better match to the firm, such that turnover is efficient. Secondly, the turnover threat provides the agent incentives to exert effort. Vice versa, the firm can reward the manager for good performance by promising him a higher level of job security in the future.

Turnover is efficient if the quality of the match falls too low, and the optimal contract never retains the manager unless it is efficient to do so. The optimal dynamic contract between the firm owner and the manager implies that the manager's compensation increases and the probability of turnover decreases with superior performance. In particular, we find that the optimal long-term contract entails an inefficiently high turnover rate that is needed to provide the manager incentives in the beginning of the employment. Following superior performance, the turnover threat is relaxed and the manager is rewarded by promising him a higher job security in future periods. The manager's compensation depends both on the firm value and on the tenure level. Finally, we find strong evidence indicating that tenure decisions correlate with past performance. Tenure does not only reflect manager's ability but also rewards him for his past efforts on the firm development.

We build on the leading continuous-time model by DeMarzo and Sannikov (2006) and extend the model to allow for the output to be serially correlated over time. DeMarzo and Fishman (2007) derive the optimal contract in a discrete-time version of the model and provide theoretical foundations for deriving renegotiation-proof contracts. Biais, Mariotti, Plantin, and Rochet (2007) prove that the discrete time counterpart of the model converges to its continuous-time limit. Allowing for serial correlation is essential to examine efficiency of turnover decisions. With uncorrelated output, turnover is never efficient. In such a framework, an inefficiently high retention rate is never feasible.

The empirical fact that turnover decisions become more lenient over time is often interpreted as a consequence of managerial entrenchment in the literature. In their seminal paper, Shleifer and Vishny (1989) predict that managers attempt

to gain power by choosing specific investments that enhance their value inside the firm when compared to outsiders. As a consequence, the model predicts an inefficiently low turnover rate for managers with longer tenure.

This paper is not the first examining efficiency of turnover decisions using a dynamic principal-agent problem as a starting point. Garrett and Pavan (2012) show than when there is asymmetric information about the manager's quality, the optimal contract entails excessive retention of the manager over time.[2] The result follows since uncertainty about the manager's quality diminishes over time. Interestingly, the result is in sharp contrast with the predictions of our model that suggests that the optimal contract never retains the agent if it is inefficient to do so. Moreover, turnover probability decreases following superior performance, depending on the entire history of past performance and not only on the current quality.

Our model is related to several previous papers examining dynamic CEO compensation. In particular, Sannikov (2014) examines a related model in which the agent's action affects the future output.[3] Our models differ in several important directions. The optimal contract exhibits a similar dynamics with deferred compensation for the agent. However, in Sannikov, the agent's risk-aversion implies that it is optimal to smooth consumption, but determines payment to the agent following the termination of the employment relationship. Because of the agent's risk-neutrality, our model is unable to capture the optimality of smoothing the manager's payoff. However, the payoffs terminate with the termination of the relationship, a fact that is consistent with empirical observations. Finally, the simplicity of our framework allows us to make more detailed predictions of the relationship between the optimal compensation and the turnover policy.

This paper is not the first one to examine moral hazard with long-term consequences of the agent's action. Our model can be seen as a generalization of He (2009), who derives the optimal contract in a principal-agent model in which the agent takes a hidden action to affect the firm scale. The agent's outside option depends on the firm scale, which allows for an elegant characterization of the optimal contract using the firm scale as a state variable. We extend the model to allow for the outside option to be independent of the firm value. The optimal contract implies a richer dynamics, but is described by a partial differential equation, which, unfortunately, comes with some loss of tractability. An alternative approach to model correlation of output was proposed by Kwon (2011) who derives the optimal contract in discrete time by taking joint limits of pairs of sets.[4]

[2]Empirical evidence provides stronger support on the turnover decisions reflecting differences in managerial abilities than managerial entrenchment, see for example Rose and Shepard (1997).

[3]The model builds on Sannikov (2008) in which the agent's action only has an instantaneous effect on the output. Sannikov (2012) provides an excellent survey of the earlier literature.

[4]See also Kwon (2016).

While the discrete-time approach allows for a more intuitive characterization of the recursive contract, the tractability of the continuous-time framework allows us to consider a model with more than two states of the world that is needed to make predictions of efficiency of the retention decision.

Our solution concepts borrow from DeMarzo and Sannikov (2011) who examine a related model in which the players learn about the unknown mean of the cash-flow process. Our framework differs from theirs in several important dimensions. First, in our model the firm value is known at equilibrium, but stochastically changes over time. Thus our model has no learning at equilibrium, but the players are contracting in a changing environment. Besides, in DeMarzo and Sannikov the agent has persistent private information while the firm value is known in our framework.

In general, the optimal contracting problem can be arbitrarily complex, because the optimal decision might depend on the entire history of past returns. To reduce the complexity, we adopt a classical approach that uses the agent's continuation value as a state variable to summarize the dependence of the optimal contract on the entire history. The reduction allows us to characterize the optimal contract in a complex economic environment, and is justified by Spear and Srivastava (1987) who show that the agent's continuation value is a sufficient statistic for the history in the optimal contract. Besides, because of the serial correlation, we need a second state variable, the current output, to summarize the firm value.

To model serial correlation of output, we adopt an approach that defines the cash-flow as the level of a diffusion process, rather than as the increment. In the principal-agent framework, the approach was first introduced by Williams (2011) who examines a model in which the output is persistent, and the agent is risk-averse. Strulovici (2011) extends the model to allow for the players to renegotiate the contract. Again, our model differs from these important papers by the information structure: the agent has no persistent information in our framework. Besides, Strulovici (2011); Williams (2011) allow for a more general utility function, but do not consider turnover.

2 The Model

We examine a game in which a principal (she) hires an agent (he) who is necessary to operate a company. Time is continuous and the time horizon is infinite. The principal has access to unlimited funds, but the agent is protected by limited liability. In our framework this implies that the agent cannot make negative payments. Both players discount the future by a common rate $r > 0$.

At any time t during the employment of the current agent, the firm produces an output x_t. The output is stochastic and changes following a standard Brownian

motion. Formally, the output at period t is

$$x_t = x_0 + \int_0^t a_s ds + \int_0^t dZ_s, \tag{1}$$

where Z is a standard Brownian motion and $a_t \leq \mu$ denotes the agent's effort. At time 0, the output process starts from the initial value x_0 that is common knowledge. (1) implies that $E[x_t|x_0] = x_0$, i.e., the output follows a Markov process.[5]

The output process $\{x_t : t \geq 0\}$ is observed by both players, but the effort $\{a_t : t \geq 0\}$ is the agent's private information. (1) implies that actions have long-term consequences. If the agent shirks today, the firm value is lower in all future periods. Thus, the model entails a learning-by-doing component: a higher effort today increases the expected match value between the principal and the agent in the future. This increases the expected output in the future and therefore the firm value.

Effort is costly for the agent and shirking generates an instantaneous private benefit of

$$\lambda(\mu - a_t)dt.$$

We assume that $\lambda \in (0,1)$ such that private benefits are inefficient. Hence shirking is related with a social loss of $1-\lambda$ for each unit of labor.

We abstract away from private savings. Since both players are risk neutral and discount the future by the same rate, the assumption is without loss of generality. Any contract with private savings can be replicated by a contract that includes no private savings that yields the same payoff to the players.[6]

The principal has the possibility to terminate the agent's contract and hire a new agent. If the agent is fired, the principal hires a new agent under which the firm value is stochastic and depends on the quality of the match with the new agent. We assume that the expected firm value under the new agent is independently and identically distributed over time. Termination of the contract is irreversible such that the old agent returns to the pool of new agents and the match quality becomes unobserved. Without loss of generality, we normalize the expected payoff under the new agent to 0, net of the hiring cost of the new agent and possible firing cost. If the agent is fired, he receives an expected payoff of 0, which includes the expected cost of finding a new employment and the expected profit under the new employment contract.

In the beginning of each employment relationship, the players agree on a contract. The contract specifies a nonnegative payment process $\{dW_t \geq 0 : t \geq 0\}$ from

[5]The assumption is in sharp contrast with DeMarzo and Sannikov (2006) in which the output is $dx_t = \mu dt + \sigma dZ_t$. The expected flow of output is $E[dx_t] = \mu dt$ which is independent of the past performance.

[6]See DeMarzo and Fishman (2007); DeMarzo and Sannikov (2006) for a formal argument. The extension to our framework is straightforward.

the principal to the agent, and a stopping time τ at which the agent's contract is terminated. Both the payment process and the stopping time are adapted to the filtration generated by the public history of the past outcomes, $\mathscr{F}_t = \sigma\{x_s : 0 \leq s \leq t\}$. We assume that both players can fully commit to the contract.

For any incentive compatible contract, the principal's total expected profit is the discounted stream of output x_t minus the payments dW_t to the agent

$$v_0 = E\left[\int_0^\tau e^{-rt}(x_t dt - dW_t)\right], \tag{2}$$

where the output x_t depends on the agent's action a_t as described in (1).

Let U_0 denote the agent's promised utility from the contract at time 0. The promise-keeping constraint at time 0 guarantees that the agent receives his expected payoff consisting of the discounted stream of payments

$$U_0 = E\left[\int_0^\tau e^{-rt} dW_t\right] \tag{3}$$

under the contract that implements full effort, $a_t = \mu$ for all $t \leq \tau$.

The agent's incentive compatibility constraint guarantees that the agent receives at least the same utility for exerting full effort from t onwards than for any arbitrary effort strategy with $\{a_t \leq \mu : t \leq \tau\}$ with an additional stream of private benefits $\lambda(\mu - a_t)$. Formally,

$$U_0 \geq E^a\left[\int_0^\tau e^{-rt}(dW_t + \lambda a_t dt)\right] \tag{4}$$

almost surely, for all t, and for all for feasible strategies $a_t \leq \mu$.

The optimal contract determines an intertemporal payment rule and a turnover policy that satisfy the agent's promise-keeping condition, the incentive compatibility constraints, and the limited liability constraints, and maximize the principal's expected payoff. Formally, it chooses a nonnegative payment process W and a stopping time τ to maximize (2) subject to the constraints (3) and (4), and the nonnegativity constraints on the payment process.

3 First-Best Solution

We first revisit the solution to the problem when there is no asymmetric information. Since both players are risk-neutral and discount the future by the same rate, the intertemporal allocation of payments is irrelevant for efficiency. The goal is to determine the optimal match value at which the old agent's contract is terminated and a new agent is hired.

The first-best value s solves a standard real option problem. The objective is to choose an optimal stopping time τ, measurable with respect to the output process x_t at time t, to maximize the expected discounted flow of output

$$s_0 = E\left[\int_0^\tau e^{-rt} x_t dt\right]. \tag{5}$$

The optimal turnover decision is a threshold policy: the old agent is retained so long as the match value stays above a certain threshold, $x_t \geq x^*$, and as soon as it reaches x^*, it is efficient to turn over the agent.

The efficient turnover policy is the unique solution of the Hamilton-Jacobi-Bellman equation

$$rs(x) = x + \mu s_x(x) + \frac{1}{2} s_{xx}(x) \tag{6}$$

with the boundary conditions $s(x^*) = 0$ and $s_x(x^*) = 0$. The value function $s(x)$ and the first-best optimal turnover value x^* can be solved explicitly. The results are summarized in the following proposition

Proposition 1. *The agent is retained as long as the match value stays above the threshold $x^* = 1/\alpha - \mu/r$ with $\alpha = -\mu - \sqrt{\mu^2 + 2r}$, and a new agent is hired as soon as the match value reaches x^*. The firm value at any point $x \in [x^*, \infty)$ is*

$$s(x) = \frac{x + \mu/r}{r} - \frac{x^* + \mu/r}{r} e^{\alpha(x - x^*)}. \tag{7}$$

Proof of Proposition 1. We verify that the turnover policy described in Proposition 1 maximizes the firm value. Consider the process

$$S_t = \int_0^t e^{-rs} x_s ds + e^{-rt} s(x_t). \tag{8}$$

We show that S_t is a supermartingale for an arbitrary turnover policy, and a martingale if the optimal turnover policy is chosen.

Using Itô's lemma with (1) on (8), we find that

$$e^{rt} dS_t = x_t dt - rs(x_t)dt + \mu s_x(x) + \frac{1}{2} s_{xx}(x_t)dt + s_x(x_t)dZ_t.$$

By (6), S_t is a martingale when $x_t \geq x^*$ and a supermartingale if $x_t < x^*$. The firm value at time 0 satisfies

$$E\left[\int_0^\tau e^{-rt} x_t dt\right] = E[S_\tau] \leq S_0 = s(x_0)$$

with equality if and only if the optimal stopping time is chosen; i.e. τ is reached as x_t reaches x^*. □

4 Incentive Compatibility

In this section, we derive necessary and sufficient conditions that guarantee that the agent exerts full effort at the optimal contract. Since effort is unobserved, the principal does not know if fluctuations of output are consequences of the agent's actions or of exogenous events. This gives rise to a moral hazard problem in our framework. For the agent to be willing to exert effort, he needs to be compensated for his lost private benefit.

To provide the agent incentives to exert effort, the principal has to let his continuation value vary with the fluctuations of the output. Following the standards in the literature, we start by specifying how the agent's continuation value U_t depends on his actions. In continuous time, the agent's continuation value admits a convenient representation as a stochastic process.

At $t \leq \tau$, the agent's continuation value is

$$U_t = E_t \left[\int_t^\tau e^{-r(s-t)} (dW_s + \lambda(\mu - a_s)ds) \right] \tag{9}$$

for any incentive compatible contract that implements the effort $\{a_s \leq \mu : t \leq s \leq \tau\}$ by the agent. (9) is sometimes called the agent's promise-keeping constraint. It is a book-keeping constraint that guarantees that the agent receives his promised value at the optimal contract.

We first determine how the agent's continuation value evolves for an arbitrary effort strategy $\{\tilde{a}_s : 0 \leq s \leq \tau\}$ when the principal wants to implement the particular effort $\{a_s : 0 \leq s \leq \tau\}$. Then we derive conditions for the agent to choose full effort, $\tilde{a}_t = \mu$ for all $t \leq \tau$. Later we will show that it is indeed optimal for the principal to implement full effort.

The following lemma describes how the agent's promised value U_t evolves in response to his action

Lemma 1. *Fix a contract $\{a, W, \tau\}$ with $U_t < \infty$ for all t. The process U_t is the agent's continuation value from the contract if and only if the following conditions are satisfied. (i) At $t \leq \tau$, U_t admits the representation*

$$dU_t = rU_t dt - dW_t + \beta_t(dx_t - a_t). \tag{10}$$

β is a progressively measurable process in L^* that describes the sensitivity of the agent's continuation value to his effort.[7] *(ii) U_t satisfies the transversality condition $\lim_{s\to\infty} E_t[1_{s\leq\tau} e^{-rs} U_{t+s}] = 0$ almost everywhere.*

Proof. See, for example, Sannikov (2008). □

[7] A process β is in L^* if $E\left[\int_0^t 1_{s\leq\tau}\beta_s^2 ds\right] < \infty$.

To derive conditions, that guarantee that the agent exerts full effort, we compare his payoff from the full effort strategy $a_t = \mu$ to the payoff from an arbitrary effort strategy with $\tilde{a}_t \leq \mu$. If the agent shirks, he earns an additional flow of private benefits

$$\lambda(\mu - \tilde{a}_t)dt. \tag{11}$$

However, he loses a flow compensation of

$$\beta_t(\mu - \tilde{a}_t)dt \tag{12}$$

that he would have received from the principal if he had exerted higher effort. By comparing (11) and (12), we find that, for the agent to be willing to exert effort, his continuation value has to increase by $\beta_t \geq \lambda$ for each unit of effort required.

The result is summarized in the following proposition

Proposition 2. *A necessary and sufficient condition for $a_t = \mu$ to be incentive compatible is that $\beta_t \geq \lambda$ for all $t \leq \tau$.*

Proof. Using the standard argument we can write the agent's incentive compatibility constraint as

$$U_0 \geq U_0 + E^{\tilde{a}}\left[\int_0^\tau e^{-rt}(dW_t + \lambda(\mu - \tilde{a}_t) - dW_t - \beta_t(\mu - \tilde{a}_t)dt)\right],$$

or,

$$-E^{\tilde{a}}\left[\int_0^\tau e^{-rt}(\beta_t - \lambda)(\mu - \tilde{a}_t)dt\right] \leq 0.$$

Since $\tilde{a}_t \leq \mu$, $\tilde{a}_t = \mu$ is incentive compatible if and only if $\beta_t \geq \lambda$. □

5 Optimal Contract

We next turn to the principal's problem of determining the optimal contract. This section presents a heuristic discussion; the Appendix verifies the optimal contract. Since shirking is always inefficient, the optimal contract implements full effort by the agent. The incentive constraints bind and the payments are delayed until the first-best optimal turnover policy can be reached. Then the agent is rewarded with a compensation that depends on the firm value.

5.1 Contract Dynamics

Using the results obtained in Section 4, we can write the optimal contracting problem as one of maximizing the expected flow of outputs x_t minus the flow of payments W_t to the agent

$$v(U,x) = \max E\left[\int_0^\tau e^{-rt}(x_t dt - dW_t)\right]$$

subject to the state variables x_t and U_t evolving according to (1) and (10), the transversality condition

$$\lim_{s\to\infty} E_t[1_{s\le\tau} e^{-rs} U_{t+s}] = 0$$

almost everywhere, and the feasibility, incentive compatibility and limited liability constraints

$$a_t \le \mu, \;\; \beta_t \ge \lambda \;\text{ and }\; dW_t \ge 0 \;\text{ for all }\; t \le \tau.$$

5.1.1 Payment Policy

The optimal contract relies on an inefficient turnover threat to provide the agent incentives. In particular, if the agent's continuation value falls too low, his contract is terminated and a new agent is hired. Delaying payments to the agent increases his continuation value for the future, which reduces the risk of inefficient turnover. Delaying payments is optimal until the efficient turnover policy can be reached.

Since the agent is risk-neutral, the marginal cost of providing him immediate income is -1. The marginal cost of increasing the agent's continuation value is lower whenever the inefficient termination threat is reduced. Indeed, we find that if[8]

$$v_U(U,x) \ge -1, \tag{13}$$

it is optimal to set $dW_t = 0$. The payments to the agent are delayed, and the principal rewards him by promising a higher continuation value for the future.[9]

Delaying payments to the agent is optimal until his continuation value grows high enough such that the first-best optimal turnover policy can be reached. Let $U^{FB}(x_t)$ denote the smallest continuation value of the agent that reaches the first-best optimal turnover policy when the current match value is x_t. If $U_t < U^{FB}(x_t)$, the principal can gain by increasing the agent's continuation value. The payments to the agent are delayed until U_t reaches $U^{FB}(x_t)$, and the first-best solution is attainable. Thereafter $U_t = U^{FB}(x_t)$, and the agent is rewarded with immediate income. The optimal contract avoids inefficient turnover and attains the first-best solution. Lemma 2 below verifies that the first-best boundary is finite.

5.1.2 Full Effort Implemented

We next discuss the optimal contract in the region (U,x) in which the first-best optimal turnover policy cannot be attained. Then $dW_t = 0$ and the principal's

[8] See the discussion below Lemma 3 in the Appendix for a more precise explanation.

[9] Notice that since $v_U(U^{FB},x) = -1$ and $v(U,x)$ is concave, the optimal contract that maximizes the principal's profit *does not* reach the first-best solution at time 0.

optimal choice of $\{a,\beta\}$ at any point can be derived from her Hamilton-Jacobi-Bellman equation

$$rv(U,x) = \max_{a\leq\mu,\beta\geq\lambda} \Big\{ x + rUv_U(U,x) + av_x(U,x) + \frac{\beta^2}{2}v_{UU}(U,x) + \beta v_{Ux}(U,x) + \frac{1}{2}v_{xx}(U,x) \Big\}. \quad (14)$$

The boundary condition is $v(0,x) = 0$. Because of limited liability, U cannot become negative. If the agent's continuation value decreases to 0, the only way to provide him incentives to exert effort is to terminate his contract. Besides, since the agent is risk-neutral, it is (weakly) suboptimal to deliver him income after τ. Therefore, it is without loss of generality to concentrate on contracts that terminate the agent's contract if his continuation value hits 0.

We show that the optimal contract implements full effort by the agent. Implementing full effort is optimal because cash-flows are valuable. Additional effort increases expected cash-flow, which is anticipated by the principal. For any incentive compatible contract, the agent's total value follows a martingale and any deviation from the expected value becomes as a surprise. As a consequence, all additional rents can be extracted from the agent. Indeed, (14) implies that implementing the full effort $a = \mu$ is optimal since

$$v_x(U,x) \geq 0, \quad (15)$$

as confirmed by Lemma 3 in the Appendix.

5.1.3 Incentive Constraints Bind

To show that the incentive constraints bind at the optimum, we adopt the following approach, borrowed from DeMarzo and Sannikov (2011). We compare the principal's profit from the contract with $\beta = \lambda$ with her profit from any other contract with $\beta \geq \lambda$. We show that the contract with $\beta = \lambda$ attains the highest feasible profit for the principal.

For $\beta = \lambda$, the principal's Hamilton-Jacobi-Bellman equation (14) can be written as

$$rv(U,x) = x + rUv_U(U,x) + \mu v_x(U,x) + \frac{\lambda^2}{2}v_{UU}(U,x) + \lambda v_{Ux}(U,x) + \frac{1}{2}v_{xx}(U,x). \quad (16)$$

By comparing (14) and (16), we find that $\beta = \lambda$ is optimal if and only if

$$\frac{1}{2}(\beta-\lambda)^2 v_{UU}(U,x) + (\beta-\lambda)(\lambda v_{UU}(U,x) + v_{Ux}(U,x)) \leq 0. \quad (17)$$

We analyze the two terms of (17) separately.

The first term of (17) is familiar from related models with independently distributed output, see in particular, DeMarzo and Sannikov (2006). As we can see from (10), β determines the volatility of the agent's continuation value U_t. If U_t is more volatile, the risk that it hits 0 is higher, an event that corresponds to the termination of the agent's contract. Excess volatility of the agent's continuation value increases the risk of inefficient turnover, which is costly for the principal. Indeed, Lemma 6 in the Appendix shows that

$$(\beta - \lambda)^2 v_{UU}(U,x) \leq 0, \tag{18}$$

which is maximized at $\beta = \lambda$. The principal optimally exposes the firm to the minimal inefficient contract termination risk that is necessary to sustain incentives.

The second term in (17) is new, and arises because the agent's actions have long-term consequences. Recall that $v_U(U,x)$ describes the marginal cost of providing the agent incentives. The second term in (17) describes how the marginal cost changes when the state variables U and x increase, but the instantaneous turnover risk is held constant.[10] Marginal cost of providing incentives (weakly) decreases in this direction since the boundary moves away from the first-best. Indeed, Lemma 7 in the Appendix together with $\beta \geq \lambda$ implies that

$$(\beta - \lambda)(\lambda v_{UU}(U,x) + v_{Ux}(U,x)) \leq 0. \tag{19}$$

(17) together with (18) and (19) imply that $\beta = \lambda$ at the optimum. Principal always terminates the manager's contract at the lowest match value that is compatible with incentives.

5.1.4 Characterization of the Optimal Contract

The optimal contract relies on an inefficient turnover threat to provide the agent incentives. It delays payments in the beginning, and rewards him for high higher outputs by increasing his continuation value for the future. With good past performance, the continuation value eventually becomes high enough such that the first-best solution can be reached. In particular, the turnover policy becomes efficient. The agent starts to receive payments contingent on the firm value.

The final step is to verify that the first-best boundary is finite when the output follows the Markov process (1).[11] In particular, we consider a payment schedule

[10] To see this, consider the case in which the agent is always rewarded with immediate payments. In that case one can show that the distance between the current match value and the one at which turnover occurs, stays constant, cf. Lemma 7 in the Appendix.

[11] If the output process is uncorrelated over time but has an unbounded support, an unbounded continuation value is needed to implement the first-best solution. Cf. DeMarzo and Sannikov (2006); Biais et al. (2007).

that compensates the agent for high performance by making him an immediate payment. That is,

$$dW_t = rU_t dt. \tag{20}$$

We now show that the agent's continuation value, that is needed to implement the first-best solution, is bounded above by[12]

$$\bar{U}_t = \lambda(x_t - x^*). \tag{21}$$

In particular, (21) is nonnegative for any $x_t \geq x^*$ such that the manager's limited liability constraint is not violated. The next lemma verifies that the resulting contract is incentive compatible for all $x_t \geq x^*$.

Lemma 2. *At the optimal contract, $U^{FB}(x_t) < \infty$ for all $t \geq 0$ and all $x \geq x^*$.*

Proof. As noted in the text, $\bar{U}_t(x_t) \geq 0$ for all $x \geq x^*$ such that x_t hits x^* before U_t hits 0. We show that if $U_t = \bar{U}_t$, the contract is incentive compatible. Consider the process

$$\bar{A}_t = \int_0^t e^{-rt}(dW_t + \lambda(\mu - a_t)) + e^{-rt}\bar{U}_t.$$

We need to show that $\bar{A}_t$ is a supermartingale for all $a_t \leq \mu$.

Using Ito's lemma and (10) with $\beta_t = \lambda$, we find that

$$e^{rt}\bar{A}_t = dW_t + \lambda(\mu - a_t)dt - r\bar{U}_t dt + d\bar{U}_t = \lambda(\mu - a_t)dt + \lambda(dx_t - \mu dt) = \lambda dZ_t.$$

This verifies that, $\overline{A}_t$ is a martingale for all strategies. Then indeed,

$$\bar{A}_t = E_t[\bar{A}_\tau]$$

which implies that

$$\bar{U}_t = E_t\left[\int_t^\tau e^{-r(s-t)}(dW_t + \lambda(\mu - a_t)dt)\right].$$

This implies that if the agent's continuation value is (21), the agent is indifferent between his strategies. In particular, the contract, that implements full effort by the agent, is incentive compatible. □

The optimal contract that implements full effort is summarized in the following theorem

[12]The method borrows from Williams (2011), see also Sannikov (2014) who use a similar argument to verify sufficient conditions for incentive compatibility. Notice that here (21) is a (strict) upper bound for $U^{FB}(x_t)$. In particular, we show that the agent is indifferent between his strategies.

Theorem 1. *Starting from the initial match quality $x_0 > x^*$, and the agent's initial value $U_0 \in (0, U^{FB}(x_0))$, the optimal contract attains the profit $v(U_0, x_0)$ for the principal. The players' values evolve stochastically in response to the fluctuations of the output, and admit the following dynamics:*

1. *When $U_t \in (0, U^{FB}(x))$, it evolves according to*

$$dU_t = rU_t dt + \lambda dZ_t. \tag{22}$$

The payments to the agent are delayed, i.e. $dW_t = 0$. The principal's expected profit at any point is $v(U,x)$, which is the unique solution of the following partial differential equation

$$rv(U,x) = x + rUv_U(U,x) + \mu v_x(U,x) + \frac{\lambda^2}{2} v_{UU}(U,x) + \lambda v_{Ux}(U,x) + \frac{1}{2} v_{xx}(U,x) \tag{23}$$

with the boundary conditions $v_U(U^{FB}(x), x) = -1$, $v^{FB}(x) + U^{FB}(x) = s(x)$ and $v(0,x) = 0$.

2. *When U_t reaches $U^{FB}(x_t)$, the first-best solution is implemented. The agent receives a payment $dW_t = U_t - U^{FB}(x_t)$. At any point, the firm value is $v(x,U) + U = s(x) = (x + \mu/r)/r - (x^* + \mu/r)/re^{\alpha(x-x^*)}$ with $x^* = 1/\alpha - \mu/r$ and $\alpha = -\mu - \sqrt{\mu^2 + 2r}$.*

3. *When U_t reaches 0, the current agent's contract is terminated. The players receive their expected values $v(0,x) = 0$, and $U = 0$ from the new employment contract.*

5.2 Tenure and Job Security

In this section, we discuss formally how the manager's tenure increases job security at the optimal contract. In particular, we argue that the turnover policy becomes more lenient over time. Let $U_0^1 = U_0^*(x_0)$ denote the value at which the principal optimally hires the manager 1 at time 0. Consider a later time $t > 0$ such that the manager 1 is still employed and $x_t = x_0$. Then by (22), we have

$$U_t^1 - U_0^1 = \int_0^t rU_s^1 ds + \lambda \int_0^t dZ_s = \int_0^t rU_s^1 ds \geq 0$$

since $x_t = x_0$. Thus, $U_t^1 \geq U_0^*(x_t)$. Now consider the principal's problem of hiring a different manager of the same match value x_t at time t. Then clearly, $U_t^2(x_t) = U_0^*(x_0) \leq U_t^1$.

Let τ^1 denote the turnover policy determined in the manager 1's contract and τ^2 the turnover policy determined by the manager 2's contract and define $\tau = \tau^1 \wedge \tau^2$.[13] We argue that $\tau^1 \geq \tau^2$. To see this, use (22) to write

$$U_\tau^1 - U_\tau^2 = \int_0^\tau r(U_s^1 - U_s^2)ds + \lambda \int_0^\tau (dZ_s - dZ_s) \geq 0;$$

for any path of the Brownian motion $\{Z_s : t \leq s \leq \tau\}$. This implies that U^2 hits 0 no later than U^1 and, therefore, $\tau^1 \geq \tau^2$. Turnover policy becomes more lenient over time such that the manager's job security increases with tenure. Notice that increasing job security is part of the optimal contract and not a consequence of the agent's actions. This verifies that job security may increase over time even if the manager does not engage in inefficient entrenchment activities.

6 Conclusions

This paper studies the optimal managerial compensation and turnover policy in a framework with learning-by-doing. We find that the optimal contract is of the following form. In the beginning of the relationship, the payments to the manager are delayed, and the contract relies on an inefficient turnover threat to sustain incentive compatibility. Payments to the manager are delayed until his continuation value becomes sufficiently high. Then the first-best solution is implemented, and the manager starts to receive payments.

Our results deliver interesting insights to the optimal turnover policy that are in contrast with previous theoretical findings. In particular, the manager is never retained if it is inefficient to do so. The optimal contract implies an inefficiently high turnover rate in the beginning of the employment relationship and the likelihood of performance-related dismissal decreases with superior performance. Also, the relative pay for performance increases with past performance. The resulting dynamics is well in line with empirical evidence.

[13] We let $\tau^1 \wedge \tau^2$ denote the minimum of the stopping times τ^1 and τ^2.

7 Appendix

The appendix verifies that the contract conjectured in Section 5 is optimal for the principal. We show that it is optimal for the principal to request full effort by the agent and solve the principal's optimization problem subject to the incentive constraints. The method, that we use to show that the incentive constraints bind, borrow from DeMarzo and Sannikov (2011). We show that the principal's profit is maximized if she sets $\beta_t = \lambda$ for all $t \leq \tau$.

The proof proceeds as follows

- Show that marginal cost of implementing effort is nonnegative.
- Show that the principal's value function is concave in the agent's value and nonincreasing in the agent's share of the implemented effort.
- Verify that the contract in Theorem 1 attains the highest feasible profit in the principal's problem.

We start by introducing a series of Lemmas that examine some key properties of the principal's value function $v(U_0, x_0)$. We then use the properties to verify that the contract described in Theorem 1 is optimal for the principal.

The next lemma show that the marginal value of implementing effort is nonnegative

Lemma 3. *$v(U,x)$ is nondecreasing in x.*

Proof. Consider the processes $(U_s^i, x_s^i)_{s\geq 0}$, $i = 1,2$, that follow (1) and (22) starting from the values that satisfy

$$x_0^1 - x_0^2 = \delta > 0 \text{ and } U_0^1 = U_0^2 \geq 0. \tag{24}$$

Let $\tau^1 \equiv \tau(U^1)$ and $\tau^2 \equiv \tau(U^2)$. (22) together with (24) implies that $U_t^1 = U_t^2$ for all t, and therefore, $\tau \equiv \tau^1 = \tau^2$. Then

$$v(U,x^1) - v(U,x^2) = E\left[\int_0^\tau e^{-rt}(x_t^1 - x_t^2)dt\right] = E\left[\int_0^\tau e^{-rt}\delta dt\right] \geq 0$$

almost surely. The result follows since δ was chosen arbitrarily. □

The next Lemmas show that the principal's value is concave in the agent's value and that principal's value is nonincreasing in the agent's marginal share of implemented effort. The methods that we use to derive the properties borrow from DeMarzo and Sannikov (2011). In particular, we show that the principal's value function satisfies the following conditions

$$v_{UU}(U,x) \leq 0, \tag{25}$$

and

$$\lambda v_{UU}(U,x) + v_{Ux}(U,x) \leq 0 \tag{26}$$

for all (U,x). Moreover, since $v_{UU}(U_t,x_t) \leq 0$ and $v_U(U_t,x_t) = -1$ if $U_t \geq U^{FB}(x_t)$, $v_U(U_t,x_t) \geq -1$ for all U_t.[14]

The intuition behind the results is straightforward. As discussed in the text, (25) implies that the principal wants to minimize the inefficient turnover threat. If the agent's continuation value hits 0, the limited liability constraint becomes binding and the contract is terminated. The higher the volatility of the agent's continuation value, the more it fluctuates, and higher is the risk that 0 is hit. The principal's value function is concave in the agent's continuation value.

Recall that $v_U(U,x)$ describes the principal's marginal cost of providing the agent incentives. (26) describes how the marginal cost changes if we increase U and x such that $dU/dx = \lambda$. The marginal cost is is nonincreasing since the possibility of delaying payments results in reduction of the termination risk.

The next lemma provides useful information about the evolution of $v_U(U,x)$

Lemma 4. *Given that the state of the world (U_t,x_t) evolves according to (1) and (22), $v_U(U_t,x_t)$ is a martingale.*

Proof. Differentiating (23) with respect to U, we find that

$$rv_U(U,x) = rv_U(U,x) + rUv_{UU}(U,x) + \mu v_{xU}(U,x) + \frac{\lambda^2}{2}v_{UUU}(U,x) + \lambda v_{UUx}(U,x) + \frac{1}{2}v_{Uxx}(U,x)$$

or,

$$0 = rUv_{UU}(U,x) + \mu v_{xU}(U,x) + \frac{\lambda^2}{2}v_{UUU}(U,x) + \lambda v_{UUx}(U,x) + \frac{1}{2}v_{Uxx}(U,x). \tag{27}$$

Moreover, applying Itô's lemma with (1) and (22) on $v_U(U_t,x_t)$, we can see that the right hand side of (27) corresponds to its drift when $U_t \leq U^{FB}(x_t)$. Therefore, $v_U(U_t,x_t)$ is a martingale on that range.

Furthermore, if $U_t \geq U^{FB}(x_t)$, $v_U(U_t,x_t) = -1$ such that $v_U(U_t,x_t)$ is again a martingale. The result follows by combining. □

The next lemma shows the marginal gain from continuation is higher for higher levels of x_t

Lemma 5. *$v_U(0,x)$ is increasing in x.*

[14]The result can be seen graphically: Since $v_U(U_t,x_t)$ is (weakly) decreasing, and we know that it is -1 from $U_t \geq U^{FB}(x_t)$ onwards, we must have $v_U(U_t,x_t) \geq -1$. Otherwise $v_U(U_t,x_t)$ would not be decreasing.

Proof. Consider two processes $(U_t^i, x_t^i)_{s \geq t}$, starting from the values $U_0^1 = U_0^2 = \varepsilon$, and $x_0^1 = x_0^2 + \delta$, for some $\varepsilon > 0$ and $\delta > 0$. Let $\tau^1 \equiv \tau(U^1)$ and $\tau^2 \equiv \tau(U^2)$.

Suppose first that $U_\tau^i < U^{FB}(x_\tau^i)$. Since $U_0^1 = U_0^2$, we have by (22) for any corresponding path of the Brownian motion that

$$U_t^1 - U_t^2 = \underbrace{U_0^1 - U_0^2}_{=0} + \int_0^t r\underbrace{(U_s^1 - U_s^2)}_{=0} ds + \int_0^t \lambda \underbrace{(dx_s^1 - dx_s^2)}_{=0} = 0.$$

Now since $U_t^1 = U_t^2$ for all t, $\tau(U^1) = \tau(U^2) \equiv \tau$. Substituting in the principal's value function, we find that

$$\begin{aligned} v(\varepsilon, x_0^1) - v(\varepsilon, x_0^2) &= E\left[\int_0^\tau e^{-rt}(x_t^1 - x_t^2)dt\right] - U_0^1 + U_0^2 \\ &= E\left[\int_0^\tau e^{-rt}(x_0^1 - x_0^2)dt\right] - \varepsilon + \varepsilon = E\left[\int_0^\tau e^{-rt}\delta dt\right] > 0. \end{aligned}$$

Next, suppose that $U_t^2 = U^{FB}(x_t^2)$. Then it follows by (22) that $U_t^1 \geq U_t^2$. Then we find that $\tau^1 \geq \tau^2$, and

$$\begin{aligned} v(\varepsilon, x_0^1) - v(\varepsilon, x_0^2) &= E\left[\int_0^{\tau^1} e^{-rt} x_t^1 dt\right] - E\left[\int_0^{\tau^2} e^{-rt} x_t^2 dt\right] - \varepsilon + \varepsilon \\ &= E\left[\int_0^{\tau^2} e^{-rt}\delta dt\right] + E\left[\int_{\tau^2}^{\tau^1} e^{-rt} x_t^1 dt\right] > 0. \end{aligned}$$

The second term in the last line is nonnegative since

$$E\left[\int_{\tau^2}^{\tau^1} e^{-rt} x_t^1 dt\right] = v(U_{\tau^2}^1, x_{\tau^2}^1) + U_{\tau^2}^1 \geq v(U_{\tau^2}^1, x_{\tau^2}^1) \geq 0.$$

The first inequality follows since $U_{\tau^2}^1 \geq 0$ by limited liability. The second inequality follows since $v(U_{\tau^2}^1, x_{\tau^2}^1) \geq 0$; otherwise, the principal would be better of terminating the contract.

Then

$$v(U^1, x^1) - v(U^2, x^2) = E\left[\int_0^{\tau^1} e^{-rt} x_t^1 dt\right] - E\left[\int_0^{\tau^2} e^{-rt} x_t^2 dt\right] - (U_0^1 - U_0^2) = \delta > 0.$$

By combining we find that

$$\begin{aligned} & v\left(\varepsilon, x_0^1\right) > v\left(\varepsilon, x_0^2\right) \\ \iff & v\left(\varepsilon, x_0^1\right) > v\left(\varepsilon, x_0^2\right) \\ \iff & v\left(\varepsilon, x_0^1\right) - v\left(0, x_0^1\right) > v\left(\varepsilon, x_0^2\right) - v\left(0, x_0^2\right). \end{aligned}$$

The result follows since δ and ε were chosen arbitrarily. □

The next lemma implies that $v_{UU}(U,x) \leq 0$.

Lemma 6. *$v_U(U,x)$ is nonincreasing in U.*

Proof. We show that for any x_0, and for any two values $U_0^1 > U_0^2$, $v_U(U_t^1, x_t) \leq v_U(U_t^2, x_t)$. Consider two processes $(U_s^i, x_s^i)_{s\geq 0}$, $i = 1,2$. Suppose that $x_0^i = x_0$, but that $U_0^1 = U_0^2 + \delta$. Let $\tau^1 \equiv \tau(U^1)$ and $\tau^2 \equiv \tau(U^2)$ denote the first time at which each process reaches 0.

We need to consider three cases, depending on if U_t^i has reached $U^{FB}(x_t^i)$ before τ. First, suppose that $U^{FB}(x_t^1) = U^{FB}(x_t^2)$ for all $t \leq \tau$. Then

$$x_{\tau^1}^1 = x_{\tau^2}^2 = x^*.$$

Second, suppose that $U^{FB}(x_t) = U_t^1 > U_t^2$ for all $t \leq \tau$. Then

$$x^* = x_{\tau^1}^1 < x_{\tau^2}^2.$$

Third, suppose that $U_t^i < U^{FB}(x_t)$ for all $t \leq \tau$. Since $U^{FB}(x_t) > U_t^1 > U_t^2$, it holds by (22) for any such path of the Brownian motion that

$$U_t^1 - U_t^2 = \underbrace{U_0^1 - U_0^2}_{=\delta} + \int_0^t r\underbrace{(U_s^1 - U_s^2)}_{\geq 0} ds + \int_0^t \lambda\underbrace{(dx_s - dx_s)}_{=0} \geq \delta.$$

That is, $U_t^1 > U_t^2$ for all $t \leq (\tau^1 \wedge \tau^2)$. Therefore, for any path of the Brownian motion, the process U^1 always reaches 0 later than the process U^2. This implies that $\tau^1 > \tau^2$.

Moreover, (22) implies that

$$U_{\tau^1}^1 = U_0^1 + \int_0^{\tau^1} rU_s^1 ds + \int_0^{\tau^1} \lambda dx_s^1,$$
$$U_{\tau^2}^2 = U_0^2 + \int_0^{\tau^2} rU_s^2 ds + \int_0^{\tau^2} \lambda dx_s^2.$$

By combining, reorganizing, and using the assumption that $x_0^1 = x_0^2$ we find that

$$\lambda(x_{\tau^2}^2 - x_{\tau^1}^1) = U_0^1 - U_0^2 + \int_0^{\tau^2} r(U_s^1 - U_s^2) ds + \int_{\tau^2}^{\tau^1} rU_s^1 ds \geq \delta,$$

from which it follows that $x_{\tau^1}^1 < x_{\tau^2}^2$.

Combining the results we find that $x_{\tau^1}^1 \leq x_{\tau^2}^2$. Using Lemmas 4 and 5, we obtain

$$v_U(U_0^1, x_0) = E[v_U(0, x_{\tau^1}^1)] \leq E[v_U(0, x_{\tau^2}^2)] = v_U(U_0^2, x_0).$$

with equality only if $U_0^i = U^{FB}(x_0^i)$. □

The next lemma implies that $\lambda v_{Ux}(U,x) + v_{UU}(U,x) \leq 0$

Lemma 7. *$v_U(U,x)$ is nonincreasing in the direction in which U and x increase according to $dU/dx = \lambda$.*

Proof. Consider the processes $(U_s^i, x_s^i)_{s\geq 0}$, $i = 1,2$, that follow (1) and (22) starting from the values that satisfy

$$x_0^1 - x_0^2 = \delta > 0 \text{ and } U_0^1 - U_0^2 = \lambda\delta.$$

Again, let $\tau^1 \equiv \tau(U^1)$ and $\tau^2 \equiv \tau(U^2)$.

If $U_t^1 = U^{FB}(x_t^1) \geq U_t^2$, we can repeat the steps in the proof of Lemma 6 to show that

$$x^* = x_{\tau^1}^1 \leq x_{\tau^2}^2.$$

Suppose that $U_t^i < U^{FB}(x_t^i)$. Reasoning along the same lines than earlier, we can use (22) to write

$$U_t^1 - U_t^2 = \underbrace{U_0^1 - U_0^2}_{=\lambda\delta} + \int_0^t r\underbrace{(U_s^1 - U_s^2)}_{\geq 0}ds + \int_0^t \lambda\underbrace{(dx_s^1 - dx_s^2)}_{=0} \geq \lambda\delta$$

Hence, for any path of the Brownian motion and $t > 0$,

$$U_t^1 - U_t^2 \geq \lambda\delta.$$

Moreover, $\tau^1 > \tau^2$.

Next,

$$\begin{aligned} & U_{\tau^2}^1 - \underbrace{U_{\tau^2}^2}_{=0} \geq \lambda\delta \\ \iff & U_{\tau^2}^1 - \underbrace{U_{\tau^1}^1}_{=0} \geq \lambda\delta, \end{aligned}$$

or, using (22) again

$$-(U_{\tau^1}^1 - U_{\tau^2}^1) = -\int_{\tau^2}^{\tau^1} (rU_t^1 dt + \lambda dx_t^1) \geq \lambda\delta.$$

By rewriting we find that

$$\lambda(x_{\tau^2}^1 - x_{\tau^1}^1) \geq \lambda\delta + \int_{\tau^2}^{\tau^1} rU_t^1 dt \geq \lambda\delta$$

from which it follows that $x_{\tau^2}^1 \geq x_{\tau^1}^1 + \delta$.

Moreover, it follows from the assumption that $x_0^1 - x_0^2 = \delta$ and (1) that $x_{\tau^2}^1 = x_{\tau^2}^2 + \delta$. By combining the results, we can conclude that $x_{\tau^2}^2 > x_{\tau^1}^1$.

Finally, combine the results to find that $x_{\tau^1}^1 \leq x_{\tau^2}^2$. Using Lemmas 4 and 5,

$$v_U(U_0^1, x_0^1) = E[v_U(0, x_{\tau^1}^1)] \leq E[v_U(0, x_{\tau^2}^2)] = v_U(U_0^2, x_0^2).$$

with equality only if $U_0^i = U^{FB}(x_0^i)$. □

We use the properties of the value function $v(U_0, x_0)$ to verify the contract in Theorem 1 The result is summarized in the following proposition

Proposition 3. *The optimal contract attains the profit $v(U_0, x_0)$ for the principal, where $v(U_0, x_0)$ is as defined in Theorem 1. Any alternative contract attains at most the profit $v(U_0, x_0)$.*

Proof. We verify that

- The payments to the agent are delayed until the contract achieves the first-best solution.
- The contract with $\beta_t = \lambda$ attains the profit $v(U_0, x_0)$ for the principal.
- An arbitrary contract with $\beta_t \geq \lambda$ attains at most the profit $v(U_0, x_0)$ for the principal.

Consider the process

$$P_t = e^{-rt} v(U_t, x_t) + \int_0^t e^{-rs}(x_s ds - dW_s). \tag{28}$$

The state variables x_t and U_t evolve according to (1) and (10). To prove the result of Proposition 3, we show that for an arbitrary $a_t \leq \mu$, $\beta_t \geq \lambda$ and $dW_t \geq 0$, P_t is a supermartingale. It is a martingale if and only if $a_t = \mu$, $\beta_t = \lambda$ for all $t \leq \tau$, and $dW_t = 0$ whenever $v_U(U_t, x_t) > -1$.

Using Itô's lemma on (28), taking the expectations, and multiplying by e^{rt}, we can write

$$\begin{aligned} e^{rt} E[dP_t] = x_t dt - dW_t - r v(U_t, x_t) dt + (r U_t dt - dW_t - \lambda(\mu - a_t) dt) v_U(U_t, x_t) + a_t v_x(U, x) dt \\ + \frac{\beta_t^2}{2} v_{UU}(U_t, x_t) dt + \beta_t v_{Ux}(U_t, x_t) dt + \frac{1}{2} v_{xx}(U_t, x_t) dt. \end{aligned}$$

Adding (23) we find that

$$\begin{aligned} e^{rt} E[dP_t] = -r(1 + v_U(U_t, x_t)) dW_t - (\mu - a_t)(\lambda v_U(U, x) + v_x(U, x)) dt \\ + \frac{1}{2}\left(\beta_t^2 - \lambda^2\right) v_{UU}(U_t, x_t) dt + \beta_t \left(\beta_t - \lambda\right) v_{Ux}(U_t, x_t) dt. \end{aligned}$$

or,

$$e^{rt}E[dP_t] = -r(1+v_U(U_t,x_t))dW_t - (\mu - a_t)(\lambda v_U(U,x) + v_x(U,x))dt$$
$$+ \frac{1}{2}\left(\beta_t - \lambda\right)^2 v_{UU}(U_t,x_t)dt + \left(\beta_t - \lambda\right)(\lambda v_{UU}(U_t,x_t)dt + v_{Ux}(U_t,x_t))\,dt. \quad (29)$$

First, Lemma 3 together with the feasibility constraint $a_t \leq \mu$ implies that

$$-(\mu - a_t)(\lambda v_U(U,x) + v_x(U,x)) \leq 0,$$

with equality only if $a_t = \mu$. Next, since $w_t \geq 0$, $v_U(U,x) \geq -1$ implies that

$$-r(1+v_U(U_t,x_t))w_t \leq 0,$$

with equality only if $w_t = 0$ whenever $v_U(U_t,x_t) > -1$.

Moreover, since

$$v_{UU}(U_t,x_t) \leq 0$$

by Lemma 6, the second term of (29) is nonpositive, and it is 0 if $\beta_t = \lambda$. Finally, since $\beta_t \geq \lambda$, and

$$\lambda v_{UU}(U_t,x_t) + v_{Ux}(U_t,x_t) \leq 0$$

by Lemma 7, the last term of (29) is always nonpositive, and it is 0 if $\beta_t = \lambda$. Therefore, P_t is a supermartingale for an arbitrary contract that satisfy the feasibility, limited liability and incentive compatibility constraints, and a martingale if the contract in Theorem 1 is chosen.

Moreover, notice that for $U_t \geq U^{FB}(x_t)$, $v_U(U_t,x_t) = -1$, and $v_{UU}(U_t,x_t) = v_{Ux}(U_t,x_t) = 0$. Therefore, (23) can be rewritten as

$$r\underbrace{(v(U_t,x_t) + U_t)}_{=s(x_t)} = x + \frac{1}{2}\underbrace{v_{xx}(U_t,x_t)}_{=s_{xx}(x_t)}.$$

From the analysis of Section 3 we know that the principal optimally lets U_t hit 0 at the same time that x_t hits x^*. That is, whenever U_t reaches $U^{FB}(x_t)$, it is optimal to implement the first-best solution as described in Proposition 1.

The next step is to evaluate the principal's profit for an arbitrary contract. That is,

$$E\left[\int_0^\tau e^{-rs}(x_s ds - dW_s)ds\right] = E[P_\tau] \leq P_0 = v(U_0,x_0),$$

with equality if and only if the optimal contract is chosen. This implies that the principal achieves the profit P_0 if he chooses the optimal contract, and at most the profit P_0 if he chooses any other contract that satisfies the constraints of the program. Therefore, the contract in Theorem 1 attains the highest feasible profit for the principal. □

Theorem 1 now follows by combining the results.

Proof of Theorem 1. Follows from Proposition 3. □

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