The small-deformation limit in elasticity and elastoplasticity in the presence of cracks

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Herrn MSc. Pascal Gussmann

Präsidentin der Humboldt-Universität zu Berlin:
Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät:
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Alexander Mielke
2. Prof. Dr. Dorothee Knees
3. Prof. Dr. Friedemann Schuricht

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Abstract

The small-deformation limit in presence of a given crack is considered in three distinct continuum-mechanical models. First, a purely static finite-strain elasticity model is considered in the limit of small loading, where the constraint of global injectivity is shown to converge in the sense of Gamma-convergence to a local constraint of non-interpenetration along the crack. Second, finite-strain deformation plasticity based on the multiplicative decomposition of the strain tensor is shown to Gamma-converge to linearized deformation elastoplasticity with crack conditions. Third, the rate-independent evolution of elastoplasticity is considered with a generalized class of global injectivity constraints for the finite-strain model. On the one hand, neglecting the constraints the evolutionary Gamma-converge to linearized elastoplasticity is proven. On the other hand, a conjecture is made, subject to which the evolutionary Gamma-convergence with constraints still holds.

Zusammenfassung

Der Grenzwert kleiner Deformationen in Anwesenheit eines gegebenen Risses wird in drei verschiedenen kontinuumsmechanischen Modellen betrachtet. Erstens wird für rein statische Elastizität mit finiter Spannung im Grenzwert kleiner Belastung bewiesen, dass die Nebenbedingung globaler Injektivität im Sinne der Gamma-Konvergenz eine lokale Nichtdurchdringungsbedingung auf dem Riss ergibt. Zweitens wird Deformationsplastizität mit finiten Spannungen und multiplikativer Zerlegung des Spannungstensors behandelt und die Gamma-Konvergenz zu linearisierter Deformationsplastizität mit Rissbedingungen gezeigt. Drittens wird die ratenunabhängige Evolution der Elastoplastizität betrachtet mit einer allgemeineren Klasse globaler Injektivitätsbedingungen für den finiten Fall. Hierbei wird einerseits die evolutionäre Gamma-Konvergenz unter Vernachlässigung der Nebenbedinung gezeigt, andererseits eine Vermutung aufgestellt, unter deren Voraussetzung die evolutionäre Gamma-Konvergenz auch mit Rissbedingungen gilt.

Contents

1 Introduction

In [DMNP02] Dal Maso, Negri, and Percivale proved the Γ-convergence of finite-strain elasticity to small-strain linearized elasticity under the assumptions of small loadings. Later, this result was extended to different settings, e.g. to situations with much weaker coercivity conditions by Agostiniani, Dal Maso, and DeSimone [ADMD12], to multi-well energies by Schmidt [Sch08], or to materials with residual stresses by Paroni and Tomassetti [PT09, PT11]. Also evolutionary problems were treated, e.g. in elastoplasticity by Mielke and Stefanelli [MS13] and in crack propagation by Negri and Zanini [NZ14]. This dissertation discusses extensions of the results in [DMNP02] to three different settings where the reference domain Ω has a crack Γ_{Cr} of a certain class including cracks with kinks, see Section 2.2 for details. Namely, Chapter 2 deals with (static,) pure elasticity, Capter 3 with deformation plasticity and Chapter 4 with the full evolution of (rate-independend) elastoplasticity.

The presence of the crack destroys the Lipschitz property of the cracked domain $\Omega_{\mathrm{Cr}} := \Omega \setminus \Gamma_{\mathrm{Cr}}$ and therefore crucial tools, such as the well-known rigidity estimate from [FJM02], have to be adapted to the setting of cracked domains, see Proposition 2.5. More importantly, the setting of domains with cracks requires to introduce an additional constraint of global injectivity of the deformations $v : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$. A crucial step for the small-deformation Γ-limit is to show that this particular *global injectivity condition* leads to a *local non-interpenetration condition* along the crack Γ_{Cr}.

In [CN87] Ciarlet and Nečas proposed the condition $\int_\Omega \det \nabla v(x)\,\mathrm{d}x \leq \mathrm{Vol}\,(v(\Omega))$, where $\mathrm{Vol}(A)$ denotes the d-dimensional volume. This condition has been used in various applications, e.g. by Giacomini and Ponsiglione [GP08] in the SBV-theory for brittle materials or by Mariano and Modica [MM09] in the theory of weak diffeomorphisms to describe deformations in "complex bodies". In [GMS98, Prop. 3.2.1], Giaquinta, Modica, and Souček showed that the above condition is equivalent to the condition

$$\int_\Omega \varphi(v(x))\,|\det \nabla v(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y \quad \text{for all } \varphi \in \mathrm{C_c}(\mathbb{R}^d,\mathbb{R}) \text{ with } \varphi \geq 0, \tag{1.1}$$

which will be simply called *GMS condition.*

This latter condition turns out to be an appropriate formulation for our purpose. In particular, assuming that $v_\varepsilon : \Omega \to \mathbb{R}^d$ satisfy (1.1) we will deduce that a weak limit $u_0 : \Omega \to \mathbb{R}^d$ for $\varepsilon \to 0$ of the rescaled displacements

$$u_\varepsilon :\ x\ \mapsto\ \frac{1}{\varepsilon}(v_\varepsilon(x) - x)$$

satisfies the local jump condition on the crack

$$0 \leq [\![u_0(x)]\!]_{\Gamma_{\mathrm{Cr}}} := \big(u_0^+(x) - u_0^-(x)\big)\cdot \nu(x), \tag{1.2}$$

where ν is the normal vector on Γ_{Cr} pointing up and u_0^+ and u_0^- are the traces of u_0 on Γ_{Cr} from the upper and the lower side, respectively, see Theorems 2.10, 3.5 and 4.10. This condition will be called *local non-interpenetration.*

Our analysis is based on energies of integral type, e.g. $\mathcal{E}(v) = \int_\Omega W(\nabla v(x))\,\mathrm{d}x$ in the case of finite pure elasticity in Chapter 2. Apart from classical assumptions on the elastic energy density W, such as coercivity and local orientation preservation, for the derivation of the linearized theory, we need to impose conditions on the quadratic behavior of W near the identity matrix $F = I$:

$$\exists\, \mathbb{C} \geq 0 \text{ with } \mathbb{C}^\top = \mathbb{C}\ \forall\, \delta > 0\ \exists\, r_\delta > 0\ \forall A \in B_{r_\delta}(0) \subset \mathbb{R}^{d\times d}: \\ \Big| W(I{+}A) - \frac{1}{2}\langle A, \mathbb{C}A\rangle \Big| \ \leq\ \delta\, \langle A, \mathbb{C}A\rangle. \tag{1.3}$$

To take the small-deformation limit one considers *small deformations* of the form $v_\varepsilon = \mathrm{id}{+}\varepsilon u_\varepsilon$ for small parameters $\varepsilon > 0$, where u_ε remains bounded in a suitable function space. As the quadratic behavior of W around I suggests, the scaling of $W(\nabla v_\varepsilon) = W(I{+}\varepsilon\nabla u)$ by $\frac{1}{\varepsilon^2}$ will be appropriate to obtain linearized elasticity in the bulk, namely the condition of quadratic behavior above implies pointwise continuous convergence of the rescaled energy densities

$$\forall A_\varepsilon \to A_0\colon \quad \overline{W}_\varepsilon(A_\varepsilon) := \frac{1}{\varepsilon^2} W(I{+}\varepsilon A_\varepsilon)\ \to\ \frac{1}{2}\big|A_0\big|_{\mathbb{C}}^2,$$

where the notation of the semi-norm $\big|A\big|_{\mathbb{C}}^2 := \langle A, \mathbb{C}A\rangle$ implicitly assumes the symmetry of $\mathbb{C}$ and the frame indifference of W implies $\big|A\big|_{\mathbb{C}} = \big|A^{\mathrm{sym}}\big|_{\mathbb{C}}$, see the discussion after (2.1).

The results from Chapter 2 are published online first in [GM18]. The small-deformations limit in (static pure) elasticity in the spirit of [DMNP02] is considered, i.e. for the rescaled stored energies now completed by the constraint of global injectivity by the GMS condition

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_{\Omega_{\mathrm{Cr}}} \overline{W}_\varepsilon(\nabla u)\,\mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{\varepsilon^2} W\big(I{+}\varepsilon\nabla u(x)\big)\,\mathrm{d}x & \text{if } v = \mathrm{id}{+}\varepsilon u \text{ satisfies (1.1)},\\ \infty & \text{otherwise,} \end{cases}$$

and for an external loading ℓ, one is interested in solutions to the minimization problem of the total energy

$$u_\varepsilon \in \operatorname*{Argmin}_{u\in\mathcal{U}} \big(\mathcal{G}_\varepsilon(u)\big) := \operatorname*{Argmin}_{u\in\mathcal{U}} \big(\mathcal{F}_\varepsilon(u) - \langle \ell, u\rangle\big)$$

over a suitably defined function space $\mathcal{U}$ and in the hypothetical convergence $u_\varepsilon \to u_0$ to a minimizer u_0 of a limit functional $\mathcal{G}_0$.

The question of convergence of minimizers of a series functionals to minimizers of limiting functionals can be answered by the theory of Γ-convergence. In Theorem 2.1 *Mosco-convergence* of $\mathcal{F}_\varepsilon$ in $\mathcal{U} \subset \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ is proved, where Dirichlet boundary conditions are implemented in the function space, i.e. Γ-convergence in both strong and weak topology:

$$\mathcal{F}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{F} \ \text{ in } \mathcal{U}, \text{ i.e.}$$
$$\forall\, u_\varepsilon \rightharpoonup u \text{ in } \mathcal{U}: \qquad \mathcal{F}_0(u) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon),$$
$$\forall\, \bar{u} \in \mathcal{U}\ \exists\, \bar{u}_\varepsilon \to \bar{u}: \quad \mathcal{F}_0(\bar{u}) \geq \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon).$$

The proved limit functional $\mathcal{F}_0$ is the quadratic limit functional from [DMNP02]

$$\widetilde{\mathcal{F}}_0(u) = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2} |\nabla u^{\mathrm{sym}}|^2_{\mathbb{C}} \,\mathrm{d}x$$

equipped with the constraint of local non-interpenetration:

$$\mathcal{F}_0(u) = \begin{cases} \widetilde{\mathcal{F}}_0(u) & \text{if } u \text{ satisfies (1.2) } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}}, \\ \infty & \text{otherwise.} \end{cases}$$

The following two Chapters 3 and 4 aim to lift the results from Chapter 2 to models involving plasticity. In particular the model of the small-deformation limit in rate-independent elastoplasticity as in [MS13] is considered. Finite elastoplasticity is commonly based on the multiplicative split

$$\nabla v = F_{\mathrm{el}} F_{\mathrm{pl}}$$

of the deformation gradient ∇v into the elastic part $F_{\mathrm{el}} \in \mathrm{GL}_+(d)$ and the irreversible plastic part $F_{\mathrm{pl}} \in \mathrm{SL}(d)$. The stored energy in integral form without constraint reads as

$$\widetilde{\mathcal{E}}(v, F_{\mathrm{pl}}) = \int_{\Omega \setminus \Gamma_{\mathrm{Cr}}} W_{\mathrm{el}}(\nabla v F_{\mathrm{pl}}^{-1}) \,\mathrm{d}x + \int_{\Omega \setminus \Gamma_{\mathrm{Cr}}} W_{\mathrm{h}}(F_{\mathrm{pl}}) \,\mathrm{d}x,$$

where W_{el} is a frame indifferent elastic potential as considered in the pure elastic case in Chapter 2 with $F_{\mathrm{pl}} = I$ and W_{h} describes hardening. Dissipative effects are modeled by a suitably defined dissipation distance $\mathcal{D}\colon \mathrm{SL}(d) \times \mathrm{SL}(d) \to [0, \infty]$, that is given in terms of a positively 1-homogenous dissipation potential R by

$$\mathcal{D}(F_{\mathrm{pl}}, \hat{F}_{\mathrm{pl}}) := \mathcal{D}(I, F_{\mathrm{pl}} \hat{F}_{\mathrm{pl}}^{-1}) := \inf \int_{\Omega \setminus \Gamma_{\mathrm{Cr}}} \int_0^1 R(\dot{P} P^{-1}) \,\mathrm{d}t \,\mathrm{d}x,$$

where the infimum is taken over the set of all smooth trajectories $P\colon [0,1] \to \mathbb{R}^{d \times d}$ connecting F_{pl} and $\hat{F}_{\mathrm{pl}}$.

Given the stored energy $\mathcal{E}$, the dissipation distance $\mathcal{D}$ and an external loading force $\ell\colon [0,T] \times \Omega \setminus \Gamma_{\mathrm{Cr}} \to \mathbb{R}^d$ one can study the full evolution e.g. by the concept of energetic solutions as we will do in Chapter 4. However, Chapter 3 will restrain to a certain subproblem. A common approach to the full evolution with continuous time is to consider a time discretization by a partition $0 = t^0 < \ldots < t^N = T$, solve the (iterative) incremental minimization problems

$$(v^i, F_{\mathrm{pl}}^i) \in \operatorname*{Argmin}_{v, F_{\mathrm{pl}}} \Big(\mathcal{E}(v, F_{\mathrm{pl}}) - \int_{\Omega \setminus \Gamma_{\mathrm{Cr}}} (v - \mathrm{id}) \ell(t^i) \,\mathrm{d}x + \mathcal{D}(F_{\mathrm{pl}}^{i-1}, F_{\mathrm{pl}}) \Big)$$

and investigate the convergence of the right-continuous, piecewise-constant interpolants

$$(\bar{v}^{(N)}, \bar{F}_{\mathrm{pl}}^{(N)}) = (v^i, F_{\mathrm{pl}}^i) \text{ on } [t_\varepsilon^{i-1}, t_\varepsilon^i)$$

as the diameter $\tau^{(N)} := \max\{t^i - t^{i-1}\} \overset{N \to \infty}{\to} 0$ vanishes. These incremental problems for a fixed time step τ and fixed initial plastic load $F_{\mathrm{pl}}^{(\tau)}$, from now on called *one-step minimization problems*, are the interest in Chapter 3:

For given $F_{\mathrm{pl}}^{(\tau)}$ and ℓ_τ find minimizers of the functional:
$\mathcal{G}(v, F_{\mathrm{pl}}) = \mathcal{E}(v, F_{\mathrm{pl}}) - \langle v - \mathrm{id}, \ell_\tau \rangle + \mathcal{D}(F_{\mathrm{pl}}^{(\tau)}, F_{\mathrm{pl}})$.

This deformation plasticity fits the framework proposed by Ortiz and Repetto in [OR99] and Carstensen, Hackl and Mielke in [CHM02] and is used by a broad community ever since (see e.g. [CO05], [CT05], [KZ10], [CDK13] or [AD14]).

In addition to the energy densities W_{el} and W_{h} admitting quadratic expansions by tensors $\mathbb{C}$ and $\mathbb{H}$ respectively as in (1.3), we assume small loadings and small deformations as in Chapter 2 as well as small plastic strains:

$$\ell_\varepsilon = \varepsilon\ell, \quad v_\varepsilon = \mathrm{id} + \varepsilon u, \quad F_{\mathrm{pl},\varepsilon} = I + \varepsilon z.$$

As in the pure elastic case the integral parts of stored energies are rescaled by $\frac{1}{\varepsilon^2}$

$$\widetilde{\mathcal{E}}_\varepsilon(u,z) = \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} \frac{1}{\varepsilon^2} W_{\mathrm{el}}\big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\big)\,\mathrm{d}x + \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} \frac{1}{\varepsilon^2} W_{\mathrm{h}}(I+\varepsilon z)\,\mathrm{d}x$$

and completed by the GMS condition:

$$\mathcal{E}_\varepsilon(u,z) = \begin{cases} \widetilde{\mathcal{E}}_\varepsilon(u,z) & \text{if } v = \mathrm{id}+\varepsilon u \text{ fulfills GMS-condition (1.1)}, \\ \infty & \text{otherwise.} \end{cases}$$

For the dissipation a rescaling by $\frac{1}{\varepsilon}$ will be suited to obtain the desired convergences:

$$\mathcal{D}_\varepsilon(\hat{z}, z) = \frac{1}{\varepsilon}\mathcal{D}(I+\varepsilon\hat{z}, I+\varepsilon z).$$

The different scaling of $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ despite them sharing the same physical dimension can be heuristically explained by noting, that with decreasing ε not only should the quantity of the dissipation be scaled by $\frac{1}{\varepsilon^2}$, but also the yield stress should be of the same order of magnitude as the plastic strain $F_{\mathrm{pl},\varepsilon} = I + \varepsilon z$, for plastic deformations to be still observable. Thus by rate-independence and 1-homogeneity of R another ε in the scaling of the dissipation may be expected.

Choosing some data $(\ell^{(\tau)}, \hat{z}^{(\tau)})$ for the external load and initial plastic strain now gives rise to a series of one-step minimization problems for the total energy

$$\mathcal{G}_\varepsilon^{(\tau)}(u,z) := \mathcal{E}_\varepsilon(u,z) - \langle \ell^{(\tau)}, u\rangle + \mathcal{D}_\varepsilon(\hat{z}^{(\tau)}, z)$$

on a suitably defined state space $\mathcal{Q} = \mathcal{U}\times\mathcal{Z}$, where the notation shall suggest the correspondance to the fixed time step $\tau > 0$. The dissipation $\mathcal{D}_\varepsilon(\hat{z}^{(\tau)}, \cdot)$ in the context of the one-step problem is considered a part of the total energy $\mathcal{G}_\varepsilon^{(\tau)}$, as in the corresponding solution concept, dissipation $\mathcal{D}_\varepsilon(\hat{z}^{(\tau)}, \cdot)$ and stored energy $\mathcal{E}_\varepsilon$ play roles of equal rights. This is in contrast to the concept of energetic solutions in the full evolution in Chapter 4, where stored energy and dissipation play very different roles. Actually for the sake of simplicity Chapter 3 will restrict to the choice $\hat{z}^{(\tau)} = 0$, as the goal is not utmost generality, but rater a proof of concept for the use of the GMS condition in the small-deformation limit in the presence of plastic strain. The question, whether the series of minimizers

$$(u_\varepsilon, z_\varepsilon) \in \operatorname*{Argmin}_{(u,z)\in\mathcal{Q}}\big(\mathcal{G}_\varepsilon^{(\tau)}(u,z)\big)$$

converges to a minimizer of some limiting one-step minimization problem is fully answered by Theorem 3.1, where Mosco-convergence of $\mathcal{G}_\varepsilon^{(\tau)}$ to

$$\mathcal{G}_0^{(\tau)}(u,z) = \mathcal{E}_0(u,z) - \langle \ell^{(\tau)}, u\rangle + \mathcal{D}_0(\hat{z}^{(\tau)}, z)$$

is proved, where

$$\mathcal{D}_0(\hat{z}, z) = \int_{\Omega_{\mathrm{Cr}}} R(z - \hat{z})\,\mathrm{d}x$$

and the linearized stored energy

$$\mathcal{E}_0(u, z) = \begin{cases} \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}|\nabla u - z|_{\mathbb{C}}^2\,\mathrm{d}x + \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}|z|_{\mathbb{H}}^2\,\mathrm{d}x & \text{if } u \text{ fulfills (1.2) } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}}, \\ \infty & \text{otherwise,} \end{cases}$$

is quadratic, has the constraint of local non-interpenetration (1.2) and displays an additive splitting for the linearized elastic tensor $A_{\mathrm{el}} = \nabla u - z$.

Note that the different splittings mark a regime change. In the finite case the deformations v are continuous maps on the manifold Ω and a combination of two deformations v_1 and v_2 should be done by composition $v_2 \circ v_1$. Since by chain rule the gradient of a composition of maps is the product of the gradients, a multiplicative splitting $\nabla v = F_{\mathrm{el}} F_{\mathrm{pl}}$ corresponds to a compositional splitting $v = v_{\mathrm{el}} \circ v_{\mathrm{pl}}$ of the maps, although the notation of the multiplicative splitting does not assume the tensors F_{el} and F_{pl} to be gradients. In the linearized setting however, the displacements u should be read as tangent fields on the manifold Ω, thus one should distinguish the space $\mathbb{R}^d \supset \Omega$ in which Ω is embedded from its tangent spaces $\mathbb{R}^d = T_x\Omega$. Tangent fields combine by addition, thus the multiplicative split from the finite case translates to the additive split in the linear case. The same difference may be observed in the transformation behavior of the GMS condition and the local non-interpenetration. For a bijective transformation $T \colon \Omega \to \hat{\Omega}$ between manifolds as in Section 2.2 continuous maps $v \colon \Omega \to \mathbb{R}^d$ on the manifold Ω are transformed to continuous maps on the manifold $\hat{\Omega}$ by

$$(T^{-1})^* \colon v \mapsto \hat{v} = v \circ T^{-1},$$

and for v satifying the GMS condition by smoothness and bijectivity of T the GMS condition for $\hat{v}$ follows with integral transformation. However, in differential geometry a vector field u is transformed by the gradient, this in the context of continuum mechanics is called the *Piola transform* (see (2.10)), which preserves the non-interpenetration (1.2) (see (2.11)).

Chapter 4 considers the full rate-independent elastoplasticity by the concept of energetic solutions. Starting from [MT04], this concept has been used in many different rate-independent contexts and recently an exhaustive presentation of the theory of rate-independent systems is available in [MR15]. A strategy for the small-deformation limit is presented using the abstract theory of evolutionary Γ-convergence for rate-independent systems. This theory was developed in [MRS08] and states, that for the convergence of solution trajectories in addition to separate $\liminf$ estimates on the energy and dissipation also a $\limsup$ estimate on the transition cost $\mathcal{T}_\varepsilon$ is needed (see (4.18)). The latter is obtained by the construction of a *mutual recovery sequence* which is much more involved than the construction of the (common) recovery sequence in Chapter 3. This is due to the fact, that in the deformation plasticity one needs a sequence $(\bar{u}_\varepsilon, \bar{z}_\varepsilon)$ for any state $(\bar{u}_0, \bar{z}_0)$, that is simultaneously a (usual) recovery sequence for both the stored energy and the dissipation, while a mutual recovery sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)$ aims to recover competitors $(\hat{u}_0, \hat{z}_0)$, which are to be compared to a given sequence of states $(u_\varepsilon, z_\varepsilon)$ by the transition cost $\mathcal{T}_\varepsilon$.

For an external loading $\ell\colon [0,T] \to \mathcal{U}'$, which is now given on a whole time interval, the same state space $\mathcal{Q}$ and dissipation distance $\mathcal{D}_\varepsilon$ as in Chapter 3 are considered. The rescaled stored energies however are slightly modified. The part of the stored energies without constraint $\widetilde{\mathcal{E}}_\varepsilon$ is identical to the case of deformation plasticity, but we propose a relaxed slightly weaker constraint, since the full GMS-condition seems to be still too difficult for the mutual recovery sequence (see Remark 4.15 on Cnjecture 4.14). Namely, for $\delta > 0$ we introduce the weaker δ-GMS condition, which allows for interpenetration in a δ-neighborhood $U_\delta(\Gamma_{\mathrm{Cr}})$ of Γ_{Cr}:

$$\int\limits_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} \varphi(v(x))\,\big|\det\nabla v(x)\big|\,\mathrm{d}x \le \int\limits_{\mathbb{R}^d}\varphi(y)\,\mathrm{d}y \qquad \text{for all } \varphi\in \mathrm{C}_0(\mathbb{R}^d,\mathbb{R}) \text{ with } \varphi\ge 0. \quad (1.4)$$

Choosing an exponent $\alpha \in (0,\infty]$ the δ-GMS condition with $\delta(\varepsilon) = \varepsilon^\alpha$ is imposed in the finite case on the rescaled total energies, the notation again suggesting the dependence on the choice of α:

$$\widetilde{\mathcal{G}}_\varepsilon(t,u,z) = \widetilde{\mathcal{E}}_\varepsilon(u,z) - \langle \ell(t), u\rangle,$$
$$\mathcal{G}_\varepsilon^{(\alpha)}(t,u,z) = \begin{cases} \widetilde{\mathcal{G}}_\varepsilon(t,u,z) & \text{if id}+\varepsilon u \text{ fulfills } \varepsilon^\alpha\text{-GMS-condition (1.4)},\\ \infty & \text{otherwise.}\end{cases}$$

For $\alpha = \infty$ we use the convention $\varepsilon^\alpha = \varepsilon^\infty := 0$, such that the ε^α-GMS condition (1.4) becomes the original GMS condition (1.1).

In the case without self contact on a reference configuration with Lipschitz boundary, the evolutionary Γ-convergence of finite elastoplasticity to linearized elastoplasticity in the small-defortmation limit was shown in [MS13]. The results of Chapter 4 include the extension of that to the case of the non Lipschitz domain Ω_{Cr} but without the constraints, i.e. the evolutionary Γ-convergence $(\mathcal{Q},\widetilde{\mathcal{G}}_\varepsilon,\mathcal{D}_\varepsilon) \xrightarrow{\Gamma} (\mathcal{Q},\widetilde{\mathcal{G}}_0,\mathcal{D}_0)$ is shown in Theorem 4.19. Furthermore for the energetic systems with constraints $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha)},\mathcal{D}_\varepsilon)$ on one hand the lim inf estimate needed in the proof of the evolutionary Γ-convergence is shown to hold for $\alpha > 1$, on the other hand the lim sup estimate obtained from the mutual recovery sequence is proven for $\alpha < \beta < 1$, where $\beta := \frac{2p-2d}{2p-2d+pd}$ is the exponent, that emerged proving a priori estimates in Propositions 2.9 and 3.3. The Conjecture 4.14 is posed and discussed in Remark 4.15, assuming which we are able to prove the mutual recovery sequence with constraint for the conjectured $\alpha_{\mathrm{Con}} > 1$. This in turn enables us to prove the evolutionary Γ-convergence with constraint $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})},\mathcal{D}_\varepsilon) \xrightarrow{\Gamma} (\mathcal{Q},\mathcal{G}_0,\mathcal{D}_0)$ in Theorem 4.20.

2 Linearized elasticity as Gamma-limit of finite elasticity

2.1 Introduction

The starting point of this Chapter is pure elasticity with stored elastic energies of integral type, i.e. $\mathcal{E}(y) = \int_\Omega W(\nabla y(x))\,\mathrm{d}x$, where we want to combine the small-deformations limit in the spirit of [DMNP02] with a cracked domain $\Omega_{\mathrm{Cr}} := \Omega \setminus \Gamma_{\mathrm{Cr}}$, which will be specified in Section 2.2. In finite-strain elasticity, the classical assumptions for W are coercivity, i.e. p-growth from below as in (2.1c), frame indifference (2.1b), and the determinant constraint giving local orientation preservation, see (2.1a). For the derivation of the linearized theory, we need to impose conditions on the quadratic behavior of W near the identity matrix $F = I$. With $\mathrm{GL}_+(d) := \{ A \in \mathbb{R}^{d\times d} \mid \det A > 0 \}$ and $\mathrm{SO}(d) := \{ R \in \mathbb{R}^{d\times d} \mid R^\top R = I,\ \det(R) = 1 \}$ the following conditions on the stored-energy density $W : \mathbb{R}^{d\times d} \to [0,\infty]$ are posed:

$$\forall\, F \in \mathbb{R}^{d\times d} \setminus \mathrm{GL}_+(d) : \qquad W(F) = \infty; \tag{2.1a}$$

$$\forall\, F \in \mathbb{R}^{d\times d},\ R \in \mathrm{SO}(d) : \quad W(RF) = W(F); \tag{2.1b}$$

$$\left.\begin{aligned} &\exists\, p > d,\ c_W, C_W > 0\ \forall\, F \in \mathbb{R}^{d\times d} : \\ &\qquad\qquad W(F) \geq c_W \max\left\{ \mathrm{dist}(F, \mathrm{SO}(d))^2, |F|^p - C_W \right\}; \end{aligned}\right\} \tag{2.1c}$$

$$\left.\begin{aligned} &\exists\, \mathbb{C} \geq 0 \text{ with } \mathbb{C}^\top = \mathbb{C}\ \forall\, \delta > 0\ \exists\, r_\delta > 0\ \forall A \in B_{r_\delta}(0) \subset \mathbb{R}^{d\times d} : \\ &\qquad\qquad \Big| W(I{+}A) - \frac{1}{2}\langle A, \mathbb{C}A\rangle \Big| \ \leq\ \delta\, \langle A, \mathbb{C}A\rangle. \end{aligned}\right\} \tag{2.1d}$$

In particular, condition (2.1d) states that $A \mapsto \frac{1}{2}\langle A, \mathbb{C}A\rangle$ is the second order Taylor expansion of W around I. It implies $W(I) = 0$, $\partial_F W(I) = 0$ and $\partial_F^2 W(I) = \mathbb{C}$, where the second part yields that the material is stress free and, if W would be C^2 in a neighborhood of I, from the third part the assumed symmetry of $\mathbb{C}$ could be deduced. Moreover the semi norm given by $|A|_{\mathbb{C}}^2 := \frac{1}{2}\langle A, \mathbb{C}A\rangle$ is equivalent to the semi norm $A \mapsto |A^{\mathrm{sym}}|$ as on the one hand the frame indifference (2.1b) implies $\mathbb{C}A = \mathbb{C}A^{\mathrm{sym}}$ for every $A \in \mathbb{R}^{d\times d}$ and on the other hand the first part of assumption (2.1c) being $W(F) \geq c_W \,\mathrm{dist}^2(F, \mathrm{SO}(d))$ and assumption (2.1d) imply

$$c_W |A^{\mathrm{sym}}| \leq |A|_{\mathbb{C}}^2, \tag{2.2}$$

which will give uniform convexity of the linearized energy by Korn inequality (see [MS13] for the details).

To take the small-deformation limit one considers *small deformations* of the form $v_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ for small parameters $\varepsilon > 0$, where u_ε remains bounded in a suitable function space. As the above discussed quadratic behavior of W around I suggests, the scaling of $W(\nabla v_\varepsilon) = W(I+\varepsilon\nabla u)$ by $\frac{1}{\varepsilon^2}$ will be appropriate to obtain linearized elasticity in the bulk, namely in Lemma 2.13 the pointwise continuous convergence is shown:

$$\forall A_\varepsilon \to A_0: \quad \overline{W}_\varepsilon(A_\varepsilon) := \frac{1}{\varepsilon^2} W(I+\varepsilon A_\varepsilon) \;\to\; \frac{1}{2}|A_0|^2_{\mathbb{C}}. \tag{2.3}$$

The correspondingly rescaled elastic energies (cf. [DMNP02]) without GMS condition read

$$\widetilde{\mathcal{F}}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon^2} W(x, I+\varepsilon\nabla u(x))\,\mathrm{d}x$$

while we are interested in the elastic energy with the GMS condition (1.1), namely

$$\mathcal{F}_\varepsilon : \mathcal{U} \to \mathbb{R}\cup\{\infty\}, \quad u \mapsto \begin{cases} \widetilde{\mathcal{F}}_\varepsilon(u) & \text{if id}+\varepsilon u \text{ satisfies (1.1)}, \\ \infty & \text{otherwise}, \end{cases} \tag{2.4}$$

where Γ_{Dir} and $\mathcal{U}$ are specified in (2.13) such that $u \in \mathcal{U}$ implies $(u-g)|_{\Gamma_{\mathrm{Dir}}} = 0$. The functional $\widetilde{\mathcal{F}}_\varepsilon$ is the one considered in [DMNP02], and it is shown to Γ-converge to

$$\widetilde{\mathcal{F}}_0(u) = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}\langle e(u), \mathbb{C}e(u)\rangle\,\mathrm{d}x, \quad \text{where } e(u) := (\nabla u)^{\mathrm{sym}} := \frac{1}{2}(\nabla u + (\nabla u)^\top).$$

The main result of this chapter is the Mosco convergence (i.e. Γ-convergence with respect to both weak and strong H^1-topology) of $\mathcal{F}_\varepsilon$ to the functional $\mathcal{F}_0$, which is obtained from $\widetilde{\mathcal{F}}_0$ by adding the local non-interpenetration condition (1.2), namely

$$\mathcal{F}_0 : \mathcal{U} \to \mathbb{R}\cup\{\infty\}, \quad u \mapsto \begin{cases} \widetilde{\mathcal{F}}_0(u) & \text{if } u \text{ satisfies (1.2) } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}}, \\ \infty & \text{otherwise}. \end{cases} \tag{2.5}$$

The equi-coercivity of the functionals $\mathcal{F}_\varepsilon$ is directly implied by the equi-coercivity of $\widetilde{\mathcal{F}}_\varepsilon$, once the rigidity result of [FJM02] has been generalized to our class of crack domains $\Omega_{\mathrm{Cr}} := \Omega\backslash\Gamma_{\mathrm{Cr}}$ as specified in Section 2.2. Thus, the coercivity (2.1c) and the energy bound $\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \le C < \infty$ imply $\|u_\varepsilon\|_{\mathrm{H}^1} \le C$ and $\|\varepsilon u_\varepsilon\|_{\mathrm{L}^p} \le C$, which gives $\|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} \le C\varepsilon^\beta$ for some $\beta > 0$, see Proposition 2.9. Our main Theorem 2.1 states the following Γ-convergence:

$$\begin{aligned} &\mathcal{F}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{F} \;\text{ in } \mathcal{U}, \text{ i.e.} \\ &\qquad \forall\, u_\varepsilon \rightharpoonup u \text{ in } \mathcal{U}: \qquad \mathcal{F}_0(u) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon), \\ &\qquad \forall\, \widetilde{u} \in \mathcal{U}\ \exists\, \widetilde{u}_\varepsilon \to \widetilde{u}: \quad \mathcal{F}_0(\widetilde{u}) \ge \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\widetilde{u}_\varepsilon). \end{aligned} \tag{2.6}$$

Section 2.4 provides the liminf estimate (in the weak topology of $\mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$), where because of the result in [DMNP02] it remains to establish the local non-interpenetration condition (1.2) as a limit of the global condition (1.1), which is not too difficult, see Theorem 2.10. The construction of recovery sequences for the limsup estimate (now in the strong topology of H^1) is more delicate, as in general (even for very smooth) displacements $u \in \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ satisfying the local non-interpenetration condition (1.2) the associated close-to-identity deformation $v_\varepsilon = \mathrm{id} + \varepsilon u$ does not satisfy the GMS condition (1.1) for global injectivity, see Example 2.16. On the one hand, our construction of recovery sequences invokes an approximation of functions in $\mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ satisfying (1.2) by functions in $W^{1,\infty}(\Omega_{\rm Cr};\mathbb{R}^d)$ still satisfying (1.2), which is reminiscent to the density results in Proposition 2.19 for convex constraints derived in [HR15, HRR16]. On the other hand, we have to use an artificial forcing apart of the two crack sides to be able to guarantee (1.1), see Proposition 2.17.

2.2 Transformation and main result

Throughout this dissertation considers a reference configuration with a Lipschitz domain Ω and a given crack $\Gamma_{\rm Cr}$ on which the displacements $u \in H^1(\Omega_{\rm Cr},\mathbb{R}^d)$ may have jumps. We expect that our theory works for general domains Ω and cracks $\Gamma_{\rm Cr}$ that are piecewise $C^{1,{\rm Lip}}$, if all the edges and corners are non-degenerate. However, to avoid an overload of technicalities we concentrate on the essential difficulties that arise by (i) smooth pieces of the crack, (ii) by the edge of the crack, (iii) by kinks inside a crack, and (iv) through the intersection of the crack with the boundary $\partial\Omega$.

Thus, we define a model domain $\hat\Omega$ with a model crack $\hat\Gamma_{\rm Cr}$ that displays all these difficulties and then consider all domains Ω with cracks $\Gamma_{\rm Cr}$ that are obtained by a bi-Lipschitz mapping $T:\Omega\to\hat\Omega$ such that $\hat\Gamma_{\rm Cr}=T(\Gamma_{\rm Cr})$.

Conditions on the model pair $(\hat\Omega,\hat\Gamma_{\rm Cr})$**.** The conditions essentially say that $\hat\Omega_{\rm Cr}=\hat\Omega\backslash\hat\Gamma_{\rm Cr}$ can be written as the union of two Lipschitz domains A_+ and A_- that have a nontrivial intersection $A_+\cap A_-$, which is a Lipschitz set again, and that define $\hat\Gamma_{\rm Cr}$ as the intersection of the boundaries ∂A_+ and ∂A_-, where we understand Lipschitz boundary as locally being the preimage of a plane under a bi-Lipschitz chart. Using the upward normal vector $\widehat\nu\in\mathbb{S}^{d-1}$ of the crack $\hat\Gamma_{\rm Cr}$, the outward normal vector $\widehat n\in\mathbb{S}^{d-1}$ on $\partial\Omega$ and the standard normal base $(e_j)_{1\le j\le d}$ of $\mathbb{R}^d$, the precise assumptions are the following.

$$\hat\Omega\subset\mathbb{R}^d \text{ is a bounded Lipschitz domain;} \tag{2.7a}$$

$$\left.\begin{aligned}
&\hat\Gamma_{\rm Cr}:=\Big(\big([0,1]\times\{0\}\times\mathbb{R}^{d-2}\big)\cup\big(\{0\}\times[0,\infty]\times\mathbb{R}^{d-2}\big)\Big),\\
&\hat\Gamma_{\rm edge}:=\{(1,0)\}\times\mathbb{R}^{d-2},\\
&\hat\Gamma_{\rm kink}:=\{(0,0)\}\times\mathbb{R}^{d-2},
\end{aligned}\right\} \tag{2.7b}$$

$$\left.\begin{aligned}
&\text{the sets } \widehat A_+ := \{\,\widehat x\in\hat\Omega \mid (\widehat x_1>0,\widehat x_2>0)\ \text{ or }\ \widehat x_1>1\,\}\\
&\text{and } \widehat A_- := \{\,\widehat x\in\hat\Omega \mid \widehat x_1<0\ \text{ or }\ \widehat x_1>1\ \text{ or }\ \widehat x_2<0\,\}\text{ as well as}\\
&\widehat A_+\cap\widehat A_- \text{ and } \widehat A_-\setminus\widehat A_+ \text{ have Lipschitz boundary}
\end{aligned}\right\} \tag{2.7c}$$

$$\left.\begin{aligned}
&\textit{Transversality of } \hat\Gamma_{\rm Cr}\text{: } \partial\widehat\Omega \text{ and } \hat\Gamma_{\rm Cr} \text{ intersect transversally, i.e.}\\
&\exists\,\delta>0\quad \forall\,\widehat x_0\in\partial\widehat\Omega\cap\hat\Gamma_{\rm Cr}\setminus(\hat\Gamma_{\rm edge}\cup\hat\Gamma_{\rm kink})\quad \exists\,\varrho>0:\\
&\qquad \big(\widehat n(\widehat x)\cdot\widehat\nu(\widehat x_0)\big)^2\le 1-\delta \text{ for } \mathcal{H}^{d-1}\text{-a.e. } \widehat x\in\partial\widehat\Omega\cap B_\varrho(\widehat x_0).
\end{aligned}\right\} \tag{2.7d}$$

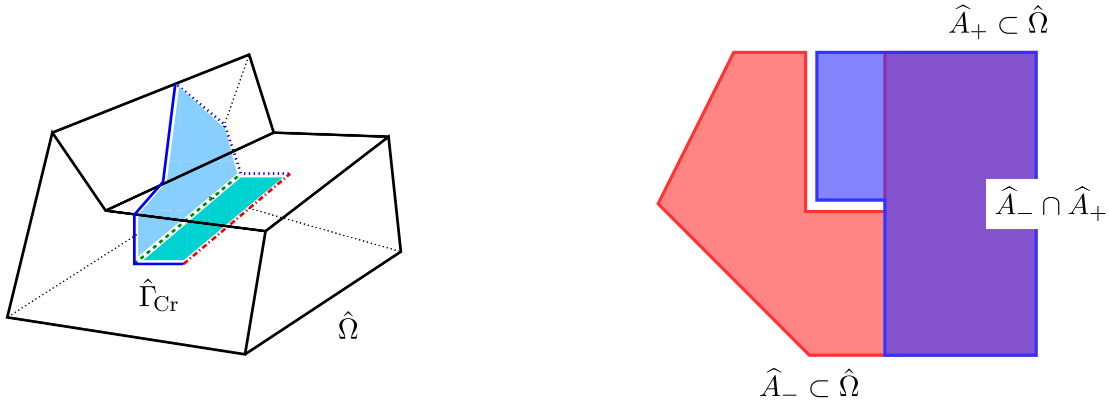

Figure 2.1: Left: Crack $\hat{\Gamma}_{\mathrm{Cr}}$ (areas shaded in light blue) inside the domain $\hat{\Omega}$, the crack edge $\hat{\Gamma}_{\mathrm{edge}}$ is red, the crack kink $\hat{\Gamma}_{\mathrm{kink}}$ is green lying between the two shaded areas, and $\partial\Omega \cap \hat{\Gamma}_{\mathrm{Cr}}$ is blue. Right: Decomposition of a planar $\hat{\Omega}$ into overlapping Lipschitz domains $\widehat{A}_+$ and $\widehat{A}_-$ according to (2.7c).

$$\left.\begin{array}{l} \textit{Transversality of } \hat{\Gamma}_{\mathrm{edge}} \text{ and } \hat{\Gamma}_{\mathrm{kink}}\text{:} \\ \hat{\Gamma}_{\mathrm{edge}} \text{ and } \hat{\Gamma}_{\mathrm{kink}} \text{ intersect with } \partial\widehat{\Omega} \text{ transversally, i.e.} \\ \exists\, \delta > 0 \quad \forall \widehat{x}_0 \in (\hat{\Gamma}_{\mathrm{edge}} \cup \hat{\Gamma}_{\mathrm{kink}}) \cap \partial\widehat{\Omega} \quad \exists \varrho > 0: \\ \qquad (\widehat{n}(\widehat{x})\cdot e_1)^2 + (\widehat{n}(\widehat{x})\cdot e_2)^2 \leq 1-\delta \text{ for } \mathcal{H}^{d-2}\text{-a.e. } \widehat{x} \in \partial\widehat{\Omega} \cap B_\varrho(\widehat{x}_0). \end{array}\right\} \tag{2.7e}$$

The conditions on $(\hat{\Omega}, \hat{\Gamma}_{\mathrm{Cr}})$ are illustrated in Figure 2.1. The model crack $\hat{\Gamma}_{\mathrm{Cr}}$ defined in (2.7b) contains two special subsets, namely (i) the crack edge $\hat{\Gamma}_{\mathrm{edge}}$ and (ii) the crack kink $\hat{\Gamma}_{\mathrm{kink}}$. For all other points we have the well-defined crack normal $\nu(\widehat{x}) = (1, 0, .., 0)^\top \in \mathbb{R}^d$ or $(0, 1, 0, .., 0)^\top$, respectively. Conditions (2.7d) and (2.7e) ask that the crack $\hat{\Gamma}_{\mathrm{Cr}}$ and its edge $\hat{\Gamma}_{\mathrm{edge}}$ and kink $\hat{\Gamma}_{\mathrm{kink}}$ to not meet the boundary $\partial\hat{\Omega}$ tangentially.

The decomposition $\hat{\Omega}_{\mathrm{Cr}} \subset \widehat{A}_+ \cup \widehat{A}_-$ in (2.7c) will be used for three purposes, namely (i) for the derivation of a rigidity result for the cracked domain, (ii) to construct enough good test functions for deriving the jump condition in Theorem 2.10, and (iii) for distinction of different cases in Proposition 2.17.

The domains Ω and the cracks Γ_{Cr} for which we will formulate our theory are now obtained by a bi-Lipschitz mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ that is additionally $\mathrm{C}^{1,\mathrm{Lip}} = \mathrm{W}^{2,\infty}$. Thus, the conditions on the pair $(\Omega, \Gamma_{\mathrm{Cr}})$ or the cracked domain $\Omega_{\mathrm{Cr}} := \Omega \setminus \Gamma_{\mathrm{Cr}}$ are the following:

Assumptions on $(\Omega, \Gamma_{\mathrm{Cr}})$**:**

$$\begin{array}{l} (\hat{\Omega}, \hat{\Gamma}_{\mathrm{Cr}}) \text{ satisfy (2.7) and there exists a bi-Lipschitz map } T: \mathbb{R}^d \to \mathbb{R}^d \\ \text{such that } \ \hat{\Omega} = T(\Omega), \quad \hat{\Gamma}_{\mathrm{Cr}} = T(\Gamma_{\mathrm{Cr}}), \quad \text{and} \quad T \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d; \mathbb{R}^d). \end{array} \tag{2.8}$$

Note that the true crack Γ_{Cr} will be piecewise $\mathrm{C}^{1,\mathrm{Lip}}$, since we allowed for a kink in $\hat{\Gamma}_{\mathrm{Cr}}$. In [LT11] the quasistatic evolution of fracture in linearized elasticity is developed, where cracks may occur along arbitrary paths that have $\mathrm{C}^{1,\mathrm{Lip}}$ regularity, which is the same regularity needed piecewise for our analysis.

As a first consequence of this assumption we see that $\Omega_{\rm Cr}$ can also be decomposed similarly to $\hat{\Omega}_{\rm Cr}$ in (2.7c). Defining $A_\pm := T^{-1}(\widehat{A}_\pm)$ with $\widehat{A}_\pm$ from (2.7c) we have that

$$\left.\begin{aligned} &A_+, A_- \subset \Omega \text{ are Lipschitz domains with } A_+ \cup A_- = \Omega_{\rm Cr} \\ &\text{such that } A_+ \cap A_- \text{ and } A_- \setminus A_+ \text{ are also Lipschitz domains.} \end{aligned}\right\} \tag{2.9}$$

This overlapping covering of $\Omega_{\rm Cr}$ in assumption (2.9) is used for three different purposes. First, it allows us to extend the rigidity result from Lipschitz domains to our crack domains $\Omega_{\rm Cr}$, see Corollary 2.6. Second, it allows us to derive the jump condition (1.2) in Theorem 2.10 by applying the divergence theorem on a disjoint cover given by A_+ and $A_- \setminus A_+$. Finally, and third, we use it in Proposition 2.17 for the construction of injective close-to-identity deformations.

The assumption that $T : \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz mapping means that it is bijective and that both T and T^{-1} are Lipschitz continuous. The additional condition $T \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d;\mathbb{R}^d)$ then implies $T^{-1} \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d;\mathbb{R}^d)$. A diffeomorphism $v : \Omega \to \mathbb{R}^d$ can be transformed to a mapping on $\hat{\Omega}$ via the transform

$$\widehat{v}(\widehat{x}) = T\big(v(T^{-1}(\widehat{x}))\big) \quad \text{or} \quad v(x) = T^{-1}\big(\widehat{v}(T(x))\big).$$

In particular, for $\widehat{v}_{\varepsilon,\widehat{u}} := \mathrm{id} + \varepsilon\widehat{u} : \hat{\Omega} \to \mathbb{R}^d$ we find the expansion

$$v_\varepsilon(x) = T^{-1}\big(\widehat{v}_{\varepsilon,\widehat{u}}(T(x))\big) = x + \varepsilon \nabla T(x)^{-1}\widehat{u}(T(x)) + O(\varepsilon^2),$$

The mapping from $\widehat{u}$ to the corresponding term in v_ε is called the *Piola transform P_T for vector fields*, cf. also [KMZ08, KS12]. Under the assumption (2.8) the mapping

$$P_T : \left\{ \begin{array}{ccc} \mathrm{H}^1(\hat{\Omega}) & \to & \mathrm{H}^1(\Omega) \\ \widehat{u} & \mapsto & u : x \mapsto \nabla T(x)^{-1}\widehat{u}(T(x)) \end{array} \right. \tag{2.10}$$

is a bijective bounded linear mapping as well as its inverse $P_{T^{-1}} : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\hat{\Omega})$.

The Piola transform is especially useful for us, as it also transforms the local non-interpenetration condition in the correct way, see e.g. [KMZ08, KS12]. If $\widehat{\nu}(\widehat{x})$ is the normal vector at $\widehat{x} \in \hat{\Gamma}_{\rm Cr}$, then it is related to the normal vector $\nu(x)$ at $x = T^{-1}(\widehat{x}) \in \Gamma$ via

$$\nu(x) = \frac{1}{|\nabla T(x)^\top \widehat{\nu}\big((T(x))\big|} \nabla T(x)^\top \widehat{\nu}(T(x)) \;\text{ or }\; \widehat{\nu}(T(x)) = \frac{1}{|\nabla T(x)^{-\top}\nu(x)|}\nabla T(x)^{-\top}\nu(x).$$

Thus, for the jump over the crack we obtain the relation

$$\begin{aligned} [\![u]\!]_{\Gamma_{\rm Cr}}(x) &= \big(u^+(x) - u^-(x)\big)\cdot\nu(x) \\ &= \big(\nabla T(x)^{-1}\widehat{u}^+(T(x)) - \nabla T(x)^{-1}\widehat{u}^-(T(x))\big)\cdot\nu(x) \\ &= \big(\widehat{u}^+(T(x)) - \widehat{u}^-(T(x))\big)\cdot\nabla T(x)^{-\top}\nu(x) \\ &= |\nabla T(x)^{-\top}\nu(x)|\, [\![\widehat{u}]\!]_{\widehat{\nu}}(T(x)). \end{aligned} \tag{2.11}$$

Thus, the jumps translate correctly if we take into account the prefactor that associates with the stretching of surface elements.

For future use of the above assumptions on $(\Omega, \Gamma_{\rm Cr})$ we derive the following well-known consequences, which will be employed below in our theory of Γ-convergence:

$$\left.\begin{array}{l}\Omega \text{ Lipschitz domain, and for all } x_0 \in \partial\Omega \text{ there exists an open} \\ \text{neighborhood } U \subset \mathbb{R}^d \text{ of } x_0 \text{ and a bi-Lipschitz } \Psi_{x_0}\colon U \to V \subset \mathbb{R}^d \\ \text{such that } U \cap \Omega \subset \Psi_{x_0}^{-1}(\{v \in V \mid v \cdot e_d > 0\}) \text{ and} \\ U \cap \partial\Omega \subset \Psi_{x_0}^{-1}(\{v \in V \mid v \cdot e_d = 0\});\end{array}\right\} \tag{2.12a}$$

$$\left.\begin{array}{l}\textbf{transversality of } \Gamma_{\rm Cr} \textbf{ and } \partial\Omega\textbf{:} \text{ for all } x_0 \in \Gamma_{\rm Cr} \cap \partial\Omega \text{ there exist} \\ \widehat{\eta}_{x_0} \in \mathbb{S}^{d-1},\ \kappa > 0, \text{ and } U \text{ and } \Psi_{x_0} \text{ as in (2.12a), such that} \\ \quad \begin{array}{lll} \text{(i)} & \nabla\Psi_{x_0}(x)^\top e_d \cdot \nabla T(x)^{-1}\, \widehat{\eta}_{x_0} \geq \kappa & \mathcal{L}^d\text{-a.e. in } U \cap \Omega, \\ \text{(ii)} & \widehat{\eta}_{x_0} \cdot \nabla T(x)^{-\top}\nu(x) = 0 & \mathcal{H}^{d-1}\text{-a.e. in } U \cap \Gamma_{\rm Cr}, \\ \text{(iii)} & \widehat{\eta}_{x_0} \in \{(0,0)\} \times \mathbb{R}^{d-2} & \text{if } x_0 \in \partial\Omega \cap \Gamma_{\rm edge}, \end{array} \\ \text{where } \Gamma_{\rm edge} := T^{-1}(\hat{\Gamma}_{\rm edge}) \text{ with } \hat{\Gamma}_{\rm edge} := \{(1,0)\} \times \mathbb{R}^{d-2}.\end{array}\right\} \tag{2.12b}$$

Note that condition (ii) in (2.12b) simply means $\widehat{\eta}_{x_0} \cdot \widehat{\nu}(T(x)) = 0$, where $\widehat{\nu}$ takes one of the values e_1, $e_2 \in \mathbb{R}^d$, or even both values if $T(x_0) \in \hat{\Gamma}_{\rm kink}$. Hence, this condition follows directly from (2.7d), but we will use the form as given in (2.12b) for a full neighborhood. Similarly, condition (iii) in (2.12b) is a direct consequence of (2.7e).

Note that the angle of $\frac{\pi}{2}$ at the kink of $\hat{\Gamma}_{\rm Cr}$ is not essential and will be varied by the mapping $\nabla T^{-1}(\hat{x})$ for $\hat{x} \in \hat{\Gamma}_{\rm Cr} \cap \hat{\Omega}_{\rm Cr}$. Furthermore the choice of $\hat{\Gamma}_{\rm Cr} = T(\Gamma_{\rm Cr}) \subset \hat{\Omega}$ in (2.8) is just an example as easy as possible while still showing the crucial difficulties. We expect that the theory works for any Lipschitz surface that is piecewise $C^{1,\rm Lip}$. The proofs and constructions are made with the intention to be adaptable to other special situations.

The transversality condition (2.12b) requires the crack $\Gamma_{\rm Cr}$ and the boundary $\partial\Omega$ to intersect transversally. Technically it enables us to use the following implicit function theorem for Lipschitz maps to conclude $\partial\hat{\Omega}$ being a graph in the direction η, which is parallel to $\hat{\Gamma}_{\rm Cr}$ in a whole open neighborhood of $T(x_0)$. You can interpret this graphically when having in mind the fact, that normal vectors transform by the cofactor of the gradient. Then equation (i) of (2.12b) can be read as the vector field $\eta_{x_0} = \nabla T(x)^{-\top}\widehat{\eta}_{x_0}$, which is constant on the flat configuration $\hat{\Omega} \setminus \hat{\Gamma}_{\rm Cr}$ having an angle bounded away from $\frac{\pi}{2}$ to the normal on the boundary, which is given by $\nabla\Psi_{x_0}(x)e_d = \nabla\Psi_{x_0}(x)(0,...,0,1)^\top$. The last two requirements specify that for $x_0 \in \Gamma_{\rm Cr}$ or $x_0 \in \Gamma_{\rm edge}$ the vector $\widehat{\eta}_{x_0}$ is tangential to $\hat{\Gamma}_{\rm Cr}$ or $\hat{\Gamma}_{\rm edge}$ respectively.

To collect all the assumptions we now specify the boundary conditions in terms of the part $\Gamma_{\rm Dir} \subset \partial\Omega$, where the Dirichlet boundary conditions $(u - g_{\rm Dir})|_{\Gamma_{\rm Dir}} = 0$ are imposed.

$$\begin{aligned} &\overline{\Gamma_{\rm Dir}} \cap \overline{\Gamma_{\rm cr}} = \emptyset, \quad \mathcal{H}^{d-1}(\Gamma_{\rm Dir}) > 0, \quad g_{\rm Dir} \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^d) \\ &\mathcal{U} := \mathrm{clos}_{\mathrm{H}^1(\Omega_{\rm Cr})}\Big(\Big\{ u \in \mathrm{W}^{1,\infty}(\Omega_{\rm Cr}; \mathbb{R}^d) \mid (u - g_{\rm Dir})|_{\Gamma_{\rm Dir}} = 0 \Big\} \Big). \end{aligned} \tag{2.13}$$

Note that we chose Dirichlet boundary conditions just for simplicity and other boundary conditions may be considered as well.

Theorem 2.1 (Mosco convergence $\mathcal{F}_\varepsilon \overset{\rm M}{\to} \mathcal{F}_0$)**.** *Let assumptions* (2.1)*,* (2.8)*, and* (2.13) *be satisfied and* $\mathcal{F}_\varepsilon$ *and* $\mathcal{F}_0$ *defined as in* (2.4) *and* (2.5)*. Then* $\mathcal{F}_\varepsilon$ *Mosco-converges to* $\mathcal{F}_0$ *in the* $\mathrm{H}^1(\Omega_{\rm Cr}; \mathbb{R}^d)$ *topology.*

The proof of this result is the content of the following sections. In particular, the liminf estimate is established in Proposition 2.14, and the limsup estimate in Theorem 2.20.

The following result is a weak version of the implicit function theorem (see [Cla90]) that will be needed to represent the boundary $\partial\Omega$ near a point $x_0 \in \partial\Omega \cap \Gamma_{\mathrm{Cr}}$, see Corollary 2.3.

Theorem 2.2 (Special version of Implicit Function Theorem)**.** *Let $U_m \subset \mathbb{R}^m$, $U_n \subset \mathbb{R}^n$ be open sets, $a \in U_m$, $b \in U_n$ and $F : U_m \times U_n \to \mathbb{R}^n$ be a Lipschitz map with $F(a, b) = 0$. Suppose there exists a constant $K > 0$ such that for all $x \in U_m$ and $y_1, y_2 \in U_n$ it holds*

$$|F(x, y_1) - F(x, y_2)| \geq K|y_1 - y_2|. \tag{2.14}$$

Then there exists an open neighborhood V_m of a, $V_m \subset U_m$ and a Lipschitz map $\varphi : V_m \to \mathbb{R}^n$ such that $\varphi(a) = b$ and

$$F^{-1}(0) = \{(x, \varphi(x)) | x \in V_m\}.$$

Proof. We will sketch the proof briefly.

By (2.14), which is a Lipschitz analog of the invertibility of $\nabla_y F$ in the differentiable version of the inverse function theorem, the map $f : U_m \times U_n \supset \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}, (x, y) \mapsto (x, \delta F(x, y))$ is bi-Lipschitz for $0 < \delta < \|\nabla F\|_{L^\infty}^{-1}$. In particular f is continuous, injective and maps an open subset of $\mathbb{R}^{m+n}$ to $\mathbb{R}^{m+n}$, thus by Brouwer's invariance of domain theorem f is an open map, i.e. $f(U_m \times U_n)$ is open in $\mathbb{R}^{m+n}$ and f^{-1} is continuous. Consider the embedding $e_m : \mathbb{R}^m \to \mathbb{R}^{m+n}, x \mapsto (x, 0)$ and the projection $p_n : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n, (x, y) \mapsto y$. Both e_m and p_n are Lipschitz continuous , thus $\varphi := p_m \circ f^{-1} \circ e_m$ defines a Lipschitz map on $V_m := e_m^{-1}(f(U_m \times U_n))$, which is open by continuity of e_m and f^{-1}. Because of the assumption $F(a, b) = 0$ we have $a \in V_m$ and $\varphi(a) = b$. Regarding the claimed equality $F^{-1}(0) = \{(x, \varphi(x)) | x \in V_m\}$ we get on the one hand the inclusion "$\supset$" from $F(x, \varphi(x)) = 0$, which follows by construction of φ. On the other hand for every $(x, y) \in U_m \times U_n$ with $F(x, y) = 0$ we have $f(x, y) = (x, 0)$ such that x lies in the domain V_m of φ by construction of V_m, which gives the other inclusion "$\subset$". □

We are now able to write the boundary $\partial\hat{\Omega}$ near $\widehat{x}_0 \in \partial\hat{\Omega} \cap \hat{\Gamma}_{\mathrm{Cr}}$ as a Lipschitz graph over the plane $\widehat{P}_{\widehat{x}_0}$ through $\widehat{x}_0 = T(x_0)$ that is normal to $\widehat{\eta}_{\widehat{x}_0}$. This construction will be needed in the proof of Proposition 2.19.

Corollary 2.3. *Let $\widehat{x}_0 = T(x_0) \in \hat{\Gamma}_{\mathrm{Cr}} \cap \partial\hat{\Omega}$ and U and $\widehat{\eta}_{x_0}$ as in the transversality condition (2.12b). Set $\widehat{P}_{x_0} := \{\, \widehat{x} \in \mathbb{R}^d \mid (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0} = 0 \,\}$. Then, there is an open neighborhood $\widehat{V}$ of $T(x_0)$ and a Lipschitz continuous function $\varphi_{x_0} : \widehat{V} \cap \widehat{P}_{x_0} \to \mathbb{R}$ such that the function*

$$\widehat{g} : \widehat{V} \to \mathbb{R};\ \widehat{g}(\widehat{x}) := \varphi_{x_0}\big(\widehat{x} - \big[(\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}\big]\, \widehat{\eta}_{x_0}\big) - (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}$$

characterizes $\partial\hat{\Omega}$ locally via $\widehat{g}(\widehat{x}) > 0$ for $\widehat{x} \in \hat{\Omega}$, $\widehat{g}(\widehat{x}) = 0$ for $\widehat{x} \in \partial\hat{\Omega}$, and $\widehat{g}(\widehat{x}) < 0$ for $\widehat{x} \in \mathbb{R}^d \setminus \mathrm{clos}(\hat{\Omega})$.

Similarly, the boundary $\partial\Omega$ near a point $x_0 \in \Gamma_{\mathrm{Cr}} \cap \partial\Omega$ can be characterized by the function $g = \widehat{g} \circ T^{-1}$, where $\widehat{g}$ is obtained as above for $\widehat{x}_0 = T(x_0)$.

Proof. Take Ψ_{x_0} as in the transversality condition (2.12b) and introduce local coordinates $z \in \widehat{P}_{x_0}$ and $y \in \mathbb{R}$ providing a unique representation of $\widehat{x} \in \mathbb{R}^d$ via $\widehat{x} = z + y\widehat{\eta}_{x_0}$. The map

$$F : U \cap \widehat{P}_{x_0} \times \mathbb{R} \to \mathbb{R};\ F(z,y) := e_d \cdot \Psi_{x_0}(T^{-1}(z+y\widehat{\eta}_{x_0})),$$

is Lipschitz and satisfies $F^{-1}(0) \subset \partial\Omega$. Moreover, applying the chain rule, we obtain the transversality condition $\frac{\partial}{\partial y}F(z,y) \geq \kappa$.

As $\widehat{P}_{x_0}$ can be identified with $\mathbb{R}^{d-1}$, the special version of the Implicit Function Theorem 2.2 is applicable and we obtain the Lipschitz function φ_{x_0} such that $F(z,y) = 0$ can locally be expressed as $y = \varphi_{x_0}(z)$.

The remaining assertions follow by simple computations. □

2.3 Coercivity via rigidity

The equi-coercivity of the $\mathcal{F}_\varepsilon$ is directly implied by the equi-coercivity of the $\widetilde{\mathcal{F}_\varepsilon}$, since $\mathcal{F}_\varepsilon \geq \widetilde{\mathcal{F}_\varepsilon}$ holds. For extending the proof of the equi-coercivity of $\widetilde{\mathcal{F}_\varepsilon}$ from [DMNP02] we have to generalize the rigidity estimate from [FJM02] from Lipschitz domains to domains with cracks. For this we will use the overlapping decomposition $\Omega_{\mathrm{Cr}} = A_+ \cup A_-$ from (2.9).

Definition 2.4 (Rigidity domains)**.** *A domain $\widetilde{\Omega} \subset \mathbb{R}^d$ is called a* rigidity domain*, if*

$$\exists\, C > 0\ \forall v \in \mathrm{H}^1(\widetilde{\Omega}, \mathbb{R}^d):\quad \inf_{R \in \mathrm{SO}(d)} \|\nabla v - R\|^2_{L^2(\widetilde{\Omega})} \leq C \big\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\big\|^2_{L^2(\widetilde{\Omega})}. \quad (2.15)$$

The smallest such constant we call rigidity constant $\mathcal{R}(\widetilde{\Omega})$.

In [FJM02] it is proved, that every bounded Lipschitz domain is a rigidity domain. Furthermore a doubling argument can be found therein similar to the one used in the following proof.

Proposition 2.5. *Let $A, B \subset \mathbb{R}^d$ be bounded rigidity domains such that $A \cap B$ is a rigidity domain with positive volume. Then $A \cup B$ is a rigidity domain, and we have*

$$\mathcal{R}(A\cup B) \leq (2+4\mu_A)\mathcal{R}(A) + (2+4\mu_B)\mathcal{R}(b) + 4(\mu_A+\mu_B)\mathcal{R}(A\cap B),$$

where $\mu_A = \mathrm{Vol}(A)/\mathrm{Vol}(A\cap B) \geq 1$ and $\mu_B = \mathrm{Vol}(B)/\mathrm{Vol}(A\cap B) \geq 1$.

Proof. We fix $v \in \mathrm{H}^1(A\cup B, \mathbb{R}^d)$ and denote by $R_A, R_B, R_{A\cap B} \in \mathrm{SO}(d)$ the minimizers $R \in \mathrm{SO}(d)$ in (2.15) on the corresponding domains. Hence on $A\cup B$ we obtain the estimate

$$\int_{A\cup B} |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x \leq I_A + I_B, \quad \text{where } I_D := \int_D |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x.$$

Writing shortly $\delta(F) := \mathrm{dist}(F, \mathrm{SO}(d))^2$ we can estimate:

$$\begin{aligned} I_A &\leq 2\int_A |\nabla v(x) - R_A|^2 \,\mathrm{d}x + 2\int_A |R_A - R_{A\cap B}|^2 \,\mathrm{d}x \\ &\leq 2\mathcal{R}(A)\int_A \delta(\nabla v(x))\,\mathrm{d}x + 2\mu_A \int_{A\cap B} |R_A - R_{A\cap B}|^2 \,\mathrm{d}x, \end{aligned}$$

where we used that R_A is the minimizer for the set A, that $|R_A - R_{A\cap B}|$ is constant and the definition of μ_A. For the second term of I_A we have

$$\int_{A\cap B} |R_A - R_{A\cap B}|^2 \,\mathrm{d}x \le 2\int_{A\cap B} |R_A - \nabla v(x)|^2 \,\mathrm{d}x + 2\int_{A\cap B} |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x$$
$$\le 2\mathcal{R}(A)\int_A \delta(\nabla v(x))\,\mathrm{d}x + 2\mathcal{R}(A\cap B)\int_{A\cap B} \delta(\nabla v(x))\,\mathrm{d}x.$$

Together we find $I_A \le \big((2+4\mu_A)\mathcal{R}(A) + 4\mu_A\mathcal{R}(A\cap B)\big)\int_{A\cup B} \delta(\nabla v(x))\,\mathrm{d}x$.

Interchanging A and B we find the analogous estimate for I_B, and the result follows. □

In this form, the rigidity estimate applies to our situation by our assumption (2.7c) on the decomposition of Ω in two overlapping Lipschitz domains. We simply apply the above proposition to $\Omega_{\mathrm{Cr}} = A\cup B$ with $A = A_+$ and $B = A_-$, see (2.9).

Corollary 2.6 (Ω_{Cr} is a rigidity domain)**.** *Let* $(\Omega, \Gamma_{\mathrm{Cr}})$ *satisfy* (2.8)*. Then,* $\Omega_{\mathrm{Cr}} = \Omega \setminus \Gamma_{\mathrm{Cr}}$ *is a rigidity domain, i.e. there is a constant* $C > 0$ *such that*

$$\forall\, v \in \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\ \exists\, R \in SO(d): \quad \|\nabla v - R_v\|_{L^2(\Omega_{\mathrm{Cr}})} \le C\big\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\big\|_{L^2(\Omega_{\mathrm{Cr}})}. \tag{2.16}$$

Before proving coercivity, let us note the following quantitative statement on the rotations showing up when applying the rigidity estimate to small deformations $v_\varepsilon = \mathrm{id} + \varepsilon u$. In [DMNP02] as well as for us, it is a main step in the proof of the equi-coercivity. Moreover, we will need it for proving Theorem 2.10 on the local non–interpenetration in the next chapter. The main point is to show that for mappings $v_\varepsilon = \mathrm{id} + \varepsilon u$ the corresponding rotation matrices $R_{\mathrm{id}+\varepsilon u}$ that are minimizers in the rigidity estimate are also close to the identity matrix $I \in \mathbb{R}^{d\times d}$. For this we use the boundary conditions $u|_{\Gamma_{\mathrm{Dir}}} = g$.

Lemma 2.7. *Let* $\Omega, \Gamma_{\mathrm{Cr}}$*, and* W *satisfy the assumption* (2.8) *and* (2.1) *and fix* $g_{\mathrm{Dir}} \in \mathrm{W}^{1,\infty}(\Omega)$*. Then, there exist constants* $C_{\mathcal{F}}, C_{\mathcal{R}} > 0$ *such that for all* $\varepsilon \in\,]0,1[$ *and all* $u \in \mathcal{U}$ *the following holds:*

$$\int_{\Omega_{\mathrm{Cr}}} |I + \varepsilon\nabla u(x) - R_{\mathrm{id}+\varepsilon u}|^2 \,\mathrm{d}x \le C_F \varepsilon^2 \widetilde{\mathcal{F}}_\varepsilon(u), \tag{2.17a}$$

$$|I - R_{\mathrm{id}+\varepsilon u}|^2 \le C_R \varepsilon^2 \Big(\widetilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\mathrm{Dir}}} |g_{\mathrm{Dir}}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{2.17b}$$

where R_v *denotes the minimizer* $R \in \mathrm{SO}(d)$ *in* (2.16) *for fixed* $v \in \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$*.*

Proof. Combining the coercivity of W in (2.1c) with the rigidity constant from Corollary 2.6 we obtain (2.17a) with $C_F = \mathcal{R}(\Omega_{\mathrm{Cr}})/c_W$.

To derive the second estimate we set $R_\varepsilon := R_{\mathrm{id}+\varepsilon u}$ and $\zeta_\varepsilon := \fint_{\Omega_{\mathrm{Cr}}} (x + \varepsilon u(x) - R_\varepsilon x)\,\mathrm{d}x$. By continuity of the traces and Poincaré's inequality we find

$$\int_{\Gamma_{\mathrm{Dir}}} |(x+\varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon|^2 \,\mathrm{d}\mathcal{H}^{d-1} \le C_2 \big\|(x+\varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon\big\|_{H^1(\Omega_{\mathrm{Cr}},\mathbb{R}^d)}$$
$$\le C_3 \int_{\Omega_{\mathrm{Cr}}} |(I + \varepsilon\nabla u(x)) - R_\varepsilon|^2 \,\mathrm{d}x \le C_4 \varepsilon^2 \widetilde{\mathcal{F}}_\varepsilon(u)$$

with $C_4 = C_F C_3$. Exploiting $u|_{\Gamma_{\rm Dir}} = g_{\rm Dir}$ and the prefactor ε we obtain

$$\int_{\Gamma_{\rm Dir}} |(I{-}R_\varepsilon)x - \zeta_\varepsilon|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq C_5\varepsilon^2\big(\,\widetilde{\mathcal{F}_\varepsilon}(u)\, + \int_{\Gamma_{\rm Dir}} |g_{\rm Dir}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\big).$$

Note that $R_\varepsilon{-}I$ is an element of the closed cone K generated by $\mathrm{SO}(d) - I$, on which Lemma 3.3 from [DMNP02] applies (see the derivation of (3.14) therein). Thus

$$|I{-}R_\varepsilon|^2 \leq C_6 \min_{\zeta\in\mathbb{R}^d} \int_{\Gamma_{\rm Dir}} |(I{-}R_\varepsilon)x - \zeta|^2 \,\mathrm{d}\mathcal{H}^{d-1},$$

and the estimate (2.17b) follows with $C_R = C_6 C_5$. □

Now we can proof the equi-coercivity of $\widetilde{\mathcal{F}_\varepsilon}$ on $\mathcal{U}$.

Proposition 2.8 (First a priori bound)**.** *Assume that* $\Omega, \Gamma_{\rm Cr}$*, and* W *satisfy* (2.8) *and* (2.1)*. Then, there exists* $c_\mathcal{F}, C_\mathcal{F} > 0$ *such that*

$$\forall\, \varepsilon \in\,]0,1[\ \forall\, u \in \mathcal{U}: \quad \widetilde{\mathcal{F}_\varepsilon}(u) \geq c_\mathcal{F}\|u\|^2_{\mathrm{H}^1} - C_\mathcal{F}.$$

Proof. By the first part of assumption (2.1c) on W and Corollary 2.6 we have

$$\begin{aligned}\|(I{+}\varepsilon\nabla u) - R_\varepsilon\|^2_{L^2} &\leq C_1 \int_{\Omega_{\rm Cr}} \mathrm{dist}^2(I{+}\varepsilon\nabla u(x), \mathrm{SO}(d))\,\mathrm{d}x \\ &\leq C_2 \int_{\Omega_{\rm Cr}} W(I{+}\varepsilon\nabla u(x))\,\mathrm{d}x \leq C_2\varepsilon^2\mathcal{F}_\varepsilon(u).\end{aligned}$$

Using both estimates from Lemma 2.7 we proceed to obtain

$$\varepsilon^2\|\nabla u\|^2_{L^2} \leq 2\big(\|I{-}R_\varepsilon\|^2_{L^2} + \|I{+}\varepsilon\nabla u - R_\varepsilon\|^2_{L^2}\big) \leq \varepsilon^2 C_3\Big(\mathcal{F}_\varepsilon(u) + \int_{\Gamma_{\rm Dir}} |g|^2 \,\mathrm{d}\mathcal{H}^{d-1}\Big)$$

with $C_3 = 2C_F + 2C_R$. Dividing by ε^2 and exploiting the boundary conditions in $\mathcal{U}$ as well as Poincaré's inequality we arrive at the desired result. □

The above result shows that sequences $(u_\varepsilon)_\varepsilon$ with bounded energy $\widetilde{\mathcal{F}_\varepsilon}(u_\varepsilon) \leq C < \infty$ are bounded in $\mathrm{H}^1(\Omega_{\rm Cr};\mathbb{R}^d)$. The next results provides a weaker, but still useful a priori bound, which implies that $\varepsilon u_\varepsilon$ converges to 0 in $\mathrm{L}^\infty(\Omega;\mathbb{R}^d)$ for energy bounded sequences.

Proposition 2.9 (Second a priori bound)**.** *Let* W *satisfy assumption* (2.1)*. Consider a sequence* $(u_\varepsilon)_{\varepsilon>0}$ *with* $\sup_{\varepsilon>0}\widetilde{\mathcal{F}_\varepsilon}(u_\varepsilon) \leq C_* < \infty$*. Then, there exists a constant* $C > 0$ *such that*

$$\|\varepsilon u_\varepsilon\|_{\mathrm{W}^{1,p}} \leq C \quad \textit{and} \quad \|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} \leq C\varepsilon^\beta \tag{2.18}$$

with $\beta = 1$ *for* $d = 1$*,* $\beta \in\,]0,1[$ *arbitrary for* $d = 2$*, and* $\beta = \frac{2(p-d)}{2p-2d+pd} \in\,]0,1[$ *for* $d \geq 3$*.*

Proof. The first estimate in (2.18) follows directly from the coercivity (2.1c) for W:

$$\varepsilon^2 C_* \geq \varepsilon^2\widetilde{\mathcal{F}_\varepsilon}(u_\varepsilon) \geq \int_{\Omega_{\rm Cr}} c_W\big(|I{+}\varepsilon\nabla u_\varepsilon(x)|^p {-} C_W\Big)\,\mathrm{d}x \geq \frac{c_W}{2}\,\|\varepsilon\nabla u_\varepsilon\|^p_{\mathrm{L}^p} - \overline{C}.$$

Using Poincaré's inequality for $u_\varepsilon \in \mathcal{U}$ we obtain a uniform bound in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$.

For the second estimate in (2.18) we use the Gagliardo-Nirenberg interpolation estimate for $f = \varepsilon u_\varepsilon$, where we crucially exploit $p > d$ as provided in (2.1c):

$$\|f\|_{\mathrm{L}^\infty} \leq C\|f\|^\theta_{\mathrm{W}^{1,p}}\|f\|^{1-\theta}_{\mathrm{H}^1}.$$

For $d = 1$ we can take $\theta = 0$ because $\mathrm{H}^1 \subset \mathrm{L}^\infty$, and for $d = 2$ any $\theta \in\,]0,1]$ is sufficient. For $d \geq 3$ we can choose $\theta = \frac{pd}{2p-2d+pd} \in\,]0,1[$, and the result follows by using Proposition 2.8, which gives $\|f\|^{1-\theta}_{\mathrm{H}^1} \leq \varepsilon^{1-\theta}C$. □

2.4 The liminf estimate

In contrast to the equi-coercivity the Γ-$\liminf$ estimate for $\mathcal{F}_\varepsilon$ does not follow directly from the Γ-$\liminf$ estimate for $\widetilde{\mathcal{F}_\varepsilon}$, since we have to consider the case $\mathcal{F}_0(u) = \infty$ carefully, i.e. we have to show that the global injectivity condition (1.1) generates the local non-interpenetration condition (1.2) in the limit $\varepsilon \to 0$. This is the content of the following result.

Theorem 2.10 (Local non-interpenetration)**.** *Consider $u_\varepsilon, u_0 \in \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ such that $u_\varepsilon \stackrel{H^1}{\rightharpoonup} u_0$ and $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$; then $[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ holds $\mathcal{H}^{d-1}$-a.e. on Γ_{Cr}.*

To prove this theorem we will first prove the following linearization result concerning the determinant of $I{+}\varepsilon\nabla u$:

Lemma 2.11. *There exists $C_{\det} > 0$ depending on Ω, Γ_{Dir}, Γ_{Cr} and the exponent $p > d$ and constants from assumption* (2.1c) *such that*

$$\begin{aligned}&\forall\, \varepsilon \in \,]0,1[\ \forall\, u \in \mathcal{U}:\\ &\qquad \int_{\Omega_{\mathrm{Cr}}} \big|\det\big(I{+}\varepsilon\nabla u(x)\big) - 1{-}\varepsilon \operatorname{div} u(x)\big|\,\mathrm{d}x \leq \varepsilon^2\, C_{\det}\big(\widetilde{\mathcal{F}_\varepsilon}(u) + C_{\det}\big).\end{aligned} \tag{2.19}$$

Proof. For matrices $A \in \mathbb{R}^{d\times d}$ we have

$$\big|\det(I{+}A) - (1{+}\operatorname{tr} A)\big| \leq C_d\big(|A|^2 + |A|^d\big), \tag{2.20}$$

where we will insert $A = \varepsilon\nabla u(x)$. To control the term $|A|^d$ we will use $W(I{+}A)$ and $|I{+}A|^p \geq \frac{1}{2}|A|^p - C_1$, which yields

$$W(I{+}A) \geq c_W\big(|I{+}A|^p - C_W\big) \geq \frac{c_W}{2}\big(|A|^p - C_2\big). \tag{2.21}$$

Using $W(F) \geq 0$ we even have $W(I{+}A) \geq \frac{c_W}{2}\big[|A|^p - C_2\big]_+$, where $[a]_+ := \max\{a, 0\}$. Because of $p > d \geq 2$ there exists $C_* > 0$ such that

$$t^d \leq C_*\big(t^2 + (t^p{-}C_2)_+\big) \qquad \text{for all } t \geq 0.$$

Combining this $t = |A|$ with (2.20) and (2.21) and setting $C_3 = C_d(C_*{+}1)(C_W{+}1)$, where C_W is taken from (2.1c), we arrive at:

$$\begin{aligned}\big|\det(I{+}A) - (1{+}\operatorname{tr} A)\big| &\leq C_d(C_* + 1)\big(|A|^2 + [|A|^p - C_2]_+\big)\\ &\leq C_3\big(|A|^2 + W(I + A)\big) \qquad \text{for all } A \in \mathbb{R}^{d\times d}.\end{aligned} \tag{2.22}$$

Thus, inserting $A = \varepsilon\nabla u(x)$ and integrating over Ω_{Cr} results in

$$\int_{\Omega_{\mathrm{Cr}}} \Big|\det\big(I{+}\varepsilon\nabla u(x)\big) - \big(1{+}\varepsilon \operatorname{div} u(x)\big)\Big|\,\mathrm{d}x \leq \varepsilon^2 C_3\Big(\|\nabla u\|^2_{\mathrm{L}^2} + \widetilde{\mathcal{F}_\varepsilon}(u)\Big).$$

Together with Proposition 2.8 we see that the assertion holds with $C_{\det}$ chosen as the maximum of $C_3(c_{\mathcal{F}} + 1)/c_{\mathcal{F}}$ and $C_{\mathcal{F}}/(c_{\mathcal{F}} + 1)$. □

With this lemma at hand, we are now able to complete the proof of the main theorem of this section. The idea is to consider the GMS condition (1.1) for global injectivity for $y_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ with non-negative test functions $\varphi \in \mathrm{C}^\infty(\Omega)$. Dividing by ε and passing to the limit with the help of the above lemma one can derive the relation $\int_{\Omega_{\mathrm{Cr}}} \operatorname{div}(\varphi u)\,\mathrm{d}x \geq 0$, which provides the local non-interpenetration condition (1.2).

Proof of Theorem 2.10: As $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$ there is a subsequence $(\varepsilon_j, u_{\varepsilon_j})$ such that $\mathrm{id} + \varepsilon_j u_j$ fulfills the GMS-condition (1.1) and $\det(I{+}\varepsilon_j \nabla u_{\varepsilon_j}) > 0$ a.e. on Ω. Hence, by rearranging (1.1), for every $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$ with $\varphi \geq 0$ we have:

$$\begin{aligned}
0 &\geq \frac{1}{\varepsilon_j}\int_{\Omega_{\mathrm{Cr}}} \Big(\varphi(x{+}\varepsilon_j u_j(x)) \det\big(I{+}\varepsilon_j \nabla u_{\varepsilon_j}(x)\big) - \varphi(x)\Big)\,\mathrm{d}x \\
&= \frac{1}{\varepsilon_j}\int_{\Omega_{\mathrm{Cr}}} \varphi(x{+}\varepsilon_j u_j(x))\Big(\det\big(I{+}\varepsilon_j \nabla u_{\varepsilon_j}(x)\big) - \big(1{+}\varepsilon_j \operatorname{div} u_{\varepsilon_j}(x)\big)\Big)\,\mathrm{d}x \\
&\qquad + \int_{\Omega_{\mathrm{Cr}}} \varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) \operatorname{div} u_{\varepsilon_j}(x)\,\mathrm{d}x + \int_{\Omega_{\mathrm{Cr}}} \frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)\Big)\,\mathrm{d}x \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

By Lemma 2.11 and Hölder's inequality the first summand I_1 on the right-hand side is bounded by $\varepsilon_j \|\varphi\|_{\mathrm{L}^\infty} C_{\det}(\alpha{+}C_{\det})$ and thus converges to 0 for $j \to \infty$.

The second summand $I_2 = \int_{\Omega_{\mathrm{Cr}}} \varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) \operatorname{div} u_{\varepsilon_j}(x)\,\mathrm{d}x$ is treated as a pairing of one strongly and one weakly converging sequence in L^2. On one hand, by weak convergence $u_{\varepsilon_j} \rightharpoonup u_0$ we have $\operatorname{div} u_{\varepsilon_j} \rightharpoonup \operatorname{div} u_0$ weakly in $\mathrm{L}^2(\Omega_{\mathrm{Cr}}, \mathbb{R})$. On the other hand by Lipschitz continuity of φ we have

$$\|\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)\|_{\mathrm{L}^2}^2 = \int_{\Omega_{\mathrm{Cr}}} |\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)|^2\,\mathrm{d}x \leq \varepsilon_j^2 \|\nabla\varphi\|_{\mathrm{L}^\infty}^2 \|u_{\varepsilon_j}\|_{\mathrm{L}^2}^2,$$

where on the right-hand side $\|u_{\varepsilon_j}\|_{\mathrm{L}^2}$ is bounded by $u_{\varepsilon_j} \overset{\mathrm{H}^1}{\rightharpoonup} u_0$, such that $\varphi(\mathrm{id}{+}\varepsilon_j u_{\varepsilon_j}) \to \varphi$ strongly in $\mathrm{L}^2(\Omega_{\mathrm{Cr}}, \mathbb{R})$ and thus $I_2 \to \int_{\Omega_{\mathrm{Cr}}} \varphi(x) \operatorname{div} u_0(x)\,\mathrm{d}x$ follows.

Finally, for the integrand of third term $I_3 = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)\Big)\,\mathrm{d}x$ we have pointwise convergence

$$\begin{aligned}
&\frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)\Big) \\
&\qquad = \underbrace{\frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x{+}\varepsilon_j u_0(x))\Big)}_{|\cdot| \leq \|\nabla\varphi\|_{\mathrm{L}^\infty} |u_{\varepsilon_j}(x) - u_0(x)| \to 0} + \frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_0(x) - \varphi(x)\Big) \\
&\qquad \to \nabla\varphi(x) u_0(x)
\end{aligned}$$

as well as the dominating bound

$$\frac{1}{\varepsilon_j}\Big(\varphi(x{+}\varepsilon_j u_{\varepsilon_j}(x)) - \varphi(x)\Big) \leq \|\nabla\varphi\|_{\mathrm{L}^\infty} |u_{\varepsilon_j}(x)| =: g_j(x).$$

By the convergence $u_{\varepsilon_j} \rightharpoonup u_0$ weakly in H^1 we have $u_{\varepsilon_j} \to u_0$ strongly in L^2, such that for the dominating bound g_j we obtain $g_j \to \|\nabla\varphi\|_{\mathrm{L}^\infty} |u_0(x)|$ strongly in L^1. Hence, by the generalized Lebesgue dominated convergence theorem we have the convergence $I_3 \to \int_{\Omega_{\mathrm{Cr}}} \nabla\varphi(x)\cdot u_0(x)\,\mathrm{d}x$.

Altogether the limit $\varepsilon \to 0$ provides three limit values on the right-hand side, namely

$$\begin{aligned} 0 \geq\ 0\ +\ &\int_{\Omega_{\mathrm{Cr}}} \varphi(x)\,\mathrm{div}\,u_0(x)\,\mathrm{d}x + \int_{\Omega_{\mathrm{Cr}}} \nabla\varphi(x)\cdot u_0(x)\,\mathrm{d}x \\ = \int_{\Omega_{\mathrm{Cr}}} &\mathrm{div}(\varphi u_0)(x)\,\mathrm{d}x = -\int_{\Gamma_{\mathrm{Cr}}} \varphi(x)\,[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}(x)\,\mathrm{da}(x). \end{aligned}$$

For the last identity we now restricted to $\varphi \in \mathrm{C_c}(\Omega)$ such that no boundary terms on $\partial\Omega$ are present. Moreover, we have to recall that u lies in $\mathcal{U} \subset \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ such that the upper and lower traces at the crack Γ_{Cr} may be different. Applying the divergence theorem on the Lipschitz sets A_+ and $A_- \setminus A_+$ (see (2.9)) separately, all terms cancel except for the jump along Γ_{Cr}. As $\varphi \geq 0$ was arbitrary, we conclude $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ_{Cr}. □

The Theorem 2.10 takes care of the constraints in the lim inf estimate. The integral quantities we will treat with the following lower-semicontinuity tool, which we cite from [MS13, Lem. 4.2] without proof.

Lemma 2.12 (Lower-semicontinuity tool)**.** *Let $f_0, f_\varepsilon\colon \mathbb{R}^n \to [0,\infty]$ be lower semicontinuous, i.e.*

$$\forall v_0 \in \mathbb{R}^n\colon \quad f_0(v_0) \leq \left\{ \liminf_{\varepsilon\to 0} f_\varepsilon(v_\varepsilon) \mid v_\varepsilon \to v_0 \right\},$$

f_0 convex and $w_\varepsilon \rightharpoonup w_0$ weakly in $\mathrm{L}^1(\Omega,\mathbb{R}^n)$.

Then it holds:

$$\int_\Omega f_0(w_0)\,\mathrm{d}x \leq \liminf_{\varepsilon\to 0} \int_\Omega f_\varepsilon(w_\varepsilon)\,\mathrm{d}x.$$

This tool enables us to conclude lim inf estimates on integrals from pointwise lim inf estimates on the integrands. The following lemma further shows, that a quadratic expansion as in (2.1d) even gives pointwise continuous convergence, which will be also useful when showing upper lim sup estimates.

Lemma 2.13. *Let $f\colon \mathbb{R}^n \to [0,\infty]$ admit a quadratic expansion:*

$$\begin{aligned} \exists\, \mathbb{F} \geq 0 \textit{ with } \mathbb{F}^\top = \mathbb{F}\ \forall\,\delta > 0\ \exists\, r_\delta > 0\ \forall A \in B_{r_\delta}(0) \subset \mathbb{R}^n : \\ \left| f(I{+}A) - \frac{1}{2}\langle A, \mathbb{F}A\rangle \right| \ \leq\ \delta\,\langle A, \mathbb{F}A\rangle. \end{aligned}$$

Then the rescaled functions $\overline{f}_\varepsilon(A) := \frac{1}{\varepsilon^2} f(I{+}\varepsilon A)$ continuously converge to $\overline{f}_0(A) := \frac{1}{2}\langle A, \mathbb{F}A\rangle$:

$$\forall A_\varepsilon \to A_0\colon \quad \overline{f}_\varepsilon(A_\varepsilon) \ \to\ \frac{1}{2}\langle A_0, \mathbb{F}A_0\rangle = \frac{1}{2}|A_0|^2_{\mathbb{F}},$$

Proof. Let $\delta > 0$ be arbitrarily fixed. By $A_\varepsilon \to A_0$ in $\mathbb{R}^n$ on one hand we have

$$\frac{1}{2}\langle A_\varepsilon, \mathbb{F}A_\varepsilon\rangle \to \frac{1}{2}\langle A_0, \mathbb{F}A_0\rangle,$$

on the other hand we have $\varepsilon A_\varepsilon \in B_{r_\delta}(0)$ for sufficiently small ε, thus the quadratic expansion gives

$$\Big|\frac{1}{\varepsilon^2}f(I+\varepsilon A_\varepsilon) - \frac{1}{2}\langle A_\varepsilon, \mathbb{F}A_\varepsilon\rangle\Big| \le \frac{\delta}{2}\langle A_\varepsilon, \mathbb{F}A_\varepsilon\rangle \le \delta\frac{|\mathbb{F}|}{2}|A_\varepsilon|^2 \le \delta|\mathbb{F}|\,|A_0|^2 \quad \text{for } \varepsilon \text{ small enough.}$$

Taking the limit $\varepsilon \to 0$ of the latter and inserting the former we arrive at:

$$\Big|\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}f(I+\varepsilon A_\varepsilon) - \frac{1}{2}\langle A_0, \mathbb{F}A_0\rangle\Big| \le \delta|\mathbb{F}|\,|A_0|^2.$$

Since $\delta > 0$ was arbitrary, the left-hand side hast to be zero and the assertion follows. □

We are now ready for deriving the lim inf part for the Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{F}$.

Proposition 2.14 (Liminf estimate)**.** *For every sequence $\varepsilon_j \to 0$ and $u_j, u \in \mathcal{U}$ with $u_j \rightharpoonup u$ in $\mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ we have*

$$\mathcal{F}_0(u) \le \liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j).$$

Proof. We can assume that $\alpha := \liminf_{j\to\infty}\mathcal{F}_{\varepsilon_j}(u_j) < \infty$, since otherwise the inequality holds trivially. Thus, there is a subsequence (ε_j, u_j) such that $\mathrm{id}+\varepsilon_j u_j$ is globally injective and that $\mathcal{F}_{\varepsilon_j}(u_j) = \widetilde{\mathcal{F}}_{\varepsilon_j}(u_j) \to \alpha$. By Theorem 2.10 we conclude $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0$. Consequently the liminf estimate above reduces to the liminf estimate for $\widetilde{\mathcal{F}}_\varepsilon$:

$$\mathcal{F}_0(u) = \widetilde{\mathcal{F}}_0(u) \le \alpha = \lim_{j\to\infty}\widetilde{\mathcal{F}}_{\varepsilon_j}(u_j) = \lim_{j\to\infty}\mathcal{F}_{\varepsilon_j}(u_j).$$

Because the integrand of $\widetilde{\mathcal{F}}_0$, namely $A \mapsto \frac{1}{2}|A|^2_{\mathbb{C}}$ is convex, by Lemma 2.12 it suffices to show the pointwise lim inf estimate of the respective densities. From assumption (2.1d) Lemma 2.13 even gives equality of the pointwise limit. Thus the assertion is proved. □

2.5 The limsup estimate

Showing the lim sup estimate in (2.6) amounts in the construction of a recovery sequence $u_\varepsilon \to u$ converging strongly in $\mathcal{U} \subset \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$. In the case without constraints (1.1) or (1.2) the limsup estimate for the Γ-convergence $\widetilde{\mathcal{F}}_\varepsilon \xrightarrow{\Gamma} \widetilde{\mathcal{F}}_0$ is much simpler since for $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ we can take the constant recovery sequence $u_j = u$. Then, the extension to general $u \in \mathcal{U}$ follows by density and the strong continuity of $\widetilde{\mathcal{F}}_0$, see [DMNP02, Prop. 4.1].

Due to the constraints (1.1) and (1.2) in the functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$, respectively, we have to do some extra work. First, setting

$$\mathcal{J} := \{\, u \in \mathcal{U} \mid [\![u]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0 \,\}$$

we have to show that $\mathrm{W}^{1,\infty} \cap \mathcal{J}$ is dense in $\mathcal{J}$ with respect to the H^1 norm. Second we have to overcome the problem, that not every $u \in \mathrm{W}^{1,\infty} \cap \mathcal{J}$ is close-to-identity injective in the following sense.

Definition 2.15. *We say that a displacement $u \in \mathrm{H}^1(\Omega_{\mathrm{Cr}},\mathbb{R}^d)$ is* close-to-identity injective*, if the following holds:*

$$\exists \varepsilon_u > 0\, \forall \varepsilon < \varepsilon_u : \quad \mathrm{id} + \varepsilon u \textit{ satisfies the GMS condition.}$$

The set of close-to-identity injective displacements we annotate as:

$$\mathcal{I} := \{\, u \in \mathcal{U} \mid u \text{ is close-to-identity injective} \,\}.$$

Note that Theorem 2.10 gives the inclusion $\mathcal{I} \subset \mathcal{J}$. The following example of a displacement u with positive jump condition $[\![u]\!]_{\widetilde{\nu}} > 0$ that is not close-to-identity injective shows, that equality of $\mathcal{I}$ and $\mathcal{J}$ cannot be expected.

Example 2.16 (Non-injectivity). *Consider the domain* $\widetilde{\Omega} = \,]-1,1[^2 \subset \mathbb{R}^2$, *the crack* $\widetilde{\Gamma}_{\mathrm{cr}} = \{0\} \times [0,\infty[$, *the cracked domain* $\widetilde{\Omega}_{\mathrm{cr}} := \widetilde{\Omega} \setminus \widetilde{\Gamma}_{\mathrm{cr}}$ *and the displacement*

$$u : \widetilde{\Omega}_{\mathrm{cr}} \to \mathbb{R}^2;\ u(x_1,x_2) = \begin{cases} (0,0)^\top & \text{for } x_2 < 0, \\ (x_2+(x_2)^2, x_2)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 > 0, \\ (x_2, 0)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 < 0. \end{cases}$$

Then, $u \in \mathrm{W}^{1,\infty}(\widetilde{\Omega}_{\mathrm{cr}};\mathbb{R}^2)$, *and along the crack we have* $\widetilde{\nu}(0,x_2) = e_1 = (1,0)^\top$ *and the jump* $[\![u]\!]_{\widetilde{\nu}}(0,x_2) = (x_2)^2 > 0$, *except on the crack tip* $\widetilde{\Gamma_{\mathrm{edge}}} = (0,0)^\top$.

However, $v_\varepsilon := \mathrm{id}+\varepsilon u$ *is not injective for any* $\varepsilon > 0$ *near the crack tip. To see this, we set* $x_\varepsilon^+ = \big((\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2}\big)^\top$ *and* $x_\varepsilon^- = \big(-(\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2}\big)^\top$ *which lie in the first and second quadrant, respectively. We have* $v_\varepsilon(x_\varepsilon^+) = \big(\frac{\varepsilon^2}{2} + 3(\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2}\big)^\top = v_\varepsilon(x_\varepsilon^-)$,*which violates injectivity. Even more, we see that the second quadrant is mapped to the set* $\{\, y \in \mathbb{R}^2 | y_2 \geq 0,\ y_1 < \varepsilon y_2 \,\}$ *while the first quadrant is mapped to* $\{\, y \in \mathbb{R}^2 \mid y_2 \geq 0,\ y_1 > h_\varepsilon(y_2) \,\}$ *with* $h_\varepsilon(z) = \varepsilon z(1+\varepsilon+z)/(1+\varepsilon)^2$. *Thus, each point in the area*

$$\{\, (y_1,y_2) \mid 0 < y_2 < \varepsilon(1+\varepsilon),\ \varepsilon y_2 > y_1 > h_\varepsilon(y_2) \,\}$$

has two preimages.

The main problem in handling domains with cracks is that of the missing Lipschitz property. For Lipschitz domains Ω we have $\mathrm{C}^{\mathrm{Lip}}(\Omega) = \mathrm{W}^{1,\infty}(\Omega)$ with an estimate

$$\mathrm{Lip}_\Omega(u) \leq C_\Omega \|\nabla u\|_{\mathrm{L}^\infty(\Omega)}. \tag{2.23}$$

For convex domains one has $C_\Omega = 1$ but for general domains the constant depends on the relation between Euclidean distance and the inner distance

$$d_\Omega : \Omega \times \Omega \to \mathbb{R};\ d_\Omega(x,\widetilde{x}) = \inf\{\, \mathrm{Length}(\gamma) \mid \gamma \text{ connects } x \text{ with } \widetilde{x} \text{ inside } \Omega \,\}.$$

Then, the chain rule guarantees $|u(x) - u(\widetilde{x})| \leq \|\nabla u\|_\infty d_\Omega(x,\widetilde{x})$. Thus, we can choose $C_\Omega = \sup\{\, d_\Omega(x,\widetilde{x})/|x-\widetilde{x}| \mid x,\widetilde{x} \in \Omega,\ x \neq \widetilde{x} \,\}$ in (2.23).

In a domain Ω_{Cr} with a crack, we obviously have $C_{\Omega_{\mathrm{Cr}}} = \infty$, since points x^+ and x^- on two opposite sides may have arbitrary small Euclidean distance $|x^+ - x^-|$ but large inner distance $d_{\Omega_{\mathrm{Cr}}}(x^+,x^-)$. This explains the difficulty in proving global injectivity, since for a close-to-identity mapping $v_\varepsilon = \mathrm{id}+\varepsilon u$ we have

$$|v_\varepsilon(x^+) - v_\varepsilon(x^-)| \geq |x^+ - x^-| - \varepsilon|u(x^+) - u(x^-)| \geq |x^+ - x^-| - \varepsilon\|\nabla u\|_{\mathrm{L}^\infty(\Omega_{\mathrm{Cr}})} d_{\Omega_{\mathrm{Cr}}}(x^+,x^-).$$

Thus, for Lipschitz domains Ω with $C_\Omega < \infty$ the global injectivity follows easily if $\varepsilon\|\nabla u\|_{\mathrm{L}^\infty(\Omega)} C_\Omega \leq 1/2$, but for cracked domains Ω_{Cr} we have to be much more careful.

Indeed, we have to require that our functions $u \in \mathcal{J} \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}; \mathbb{R}^d)$ also have a crack opening that is bounded from below linearly by the distance of the points on the crack from the edge Γ_{edge}. In the next result, we will show that we can achieve this by a suitable forcing apart.

Proposition 2.17. *For every $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \cap \mathcal{J}$ there exists a sequence $u_k \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \cap \mathcal{U}$ with $u_k \overset{H^1}{\to} u$, such that each u_k is close-to-identity injective, ie.:*

$$\forall\, k \in \mathbb{N}\ \exists\, \varepsilon_k > 0\ \forall\, \varepsilon \in\,]0, \varepsilon_k[: \quad \mathrm{id}{+}\varepsilon u_k \textit{ satisfies } (1.1). \tag{2.24}$$

Proof. Motivated by the above example we will use the displacement $\widehat{\varphi}_{\delta,\eta} : \hat{\Omega}_{\mathrm{Cr}} \to \mathbb{R}^d$, which forces to two sides of the crack $\hat{\Gamma}_{\mathrm{Cr}}$ apart. For two small parameters $\delta, \eta > 0$ we set $\widehat{\varphi}_{\delta,\eta}(\widehat{x}) = \delta\lambda_\eta(\widehat{x})\widehat{n} \in \mathrm{H}^1(\hat{\Omega}_{\mathrm{Cr}}, \mathbb{R}^d)$ with $\widehat{n} = (1, 1, 0, ..., 0)^\top \in \mathbb{R}^d$. The scalar function $\lambda_\eta \in \mathrm{W}^{1,\infty}(\hat{\Omega}_{\mathrm{Cr}})$ is given by

$$\lambda_\eta(x_1, x_2, \ldots, x_d) = \begin{cases} 0 & \text{if } x_1 > 1, \\ \min\left\{1, \frac{1}{\eta}(1{-}x_1)\right\} & \text{for } x_1 \in\,]0,1] \text{ and } x_2 > 0, \\ -\min\left\{1, \frac{1}{\eta}(1{-}x_1)\right\} & \text{for } x_1 \in\,]0,1] \text{ and } x_2 < 0, \\ -1 & \text{for } x_1 \le 0. \end{cases}$$

Hence the jump of λ_η grows linearly with slope $1/\eta$ with the distance from $\hat{\Gamma}_{\mathrm{edge}}$ and then saturates at the values ± 1.

We now choose an exponent $\alpha \in\,]1, 2[$ and a positive sequence $\delta_k \to 0$ and set $\eta_k = \delta_k^\alpha$. With this we define $\widehat{\varphi}_k := \widehat{\varphi}_{\delta_k,\eta_k}$ on $\hat{\Omega}_{\mathrm{Cr}}$. Using the pullback of $\widehat{\varphi}_k$ to the reference configuration Ω via the Piola transform

$$\varphi_k(x) = \nabla T(x)^{-1}\widehat{\varphi}_k(T(x)), \tag{2.25}$$

see (2.8). Moreover, using (2.13) we can choose a cut-off function $\gamma \in \mathrm{W}^{1,\infty}(\Omega; [0, 1])$ that is 1 on a neighborhood of Γ_{Cr} and vanishes on Γ_{Dir}. With this we define the required sequence

$$u_k \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d); \quad x \mapsto u_k(x) = u(x) + \gamma(x)\varphi_k(x).$$

Note that the boundary value on Γ_{Dir} is not changed, i.e. $u_k \in \mathcal{U}$.

To show the convergence $u_k = u + \gamma\varphi_k \overset{\mathrm{H}^1}{\to} u$ we need the smallness of $\gamma\varphi_k$. Using

$$\|\gamma\varphi_k\|_{\mathrm{H}^1(\Omega_{\mathrm{Cr}})} \le \|\gamma\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\nabla T^{-1}\|_{\mathrm{W}^{1,\infty}(\hat{\Omega})}\|\widehat{\varphi}_k\|_{\mathrm{H}^1(\hat{\Omega}_{\mathrm{Cr}})}.$$

will give the first condition for α

$$\|\widehat{\varphi}_k\|^2_{\mathrm{L}^2(\hat{\Omega})} \le \mathrm{Vol}(\hat{\Omega})\, |\widehat{n}|^2\, \delta_k^2$$

$$\|\nabla\widehat{\varphi}_k\|^2_{\mathrm{L}^2(\hat{\Omega}_{\mathrm{Cr}})} \le \int_{\hat{\Omega}\cap\{1-\eta_k \le x_1 \le 1\}} \Big(\delta_k/\eta_k\Big)^2 \mathrm{d}x \le \mathrm{diam}(\hat{\Omega})^{d-1}\, \delta_k^{2-\alpha},$$

where we used $\eta_k = \delta_k^\alpha$. Because of $\alpha < 2$ we have $\|u_k{-}u\|_{\mathrm{H}^1} \to 0$ as desired.

Let us now come to the global invertibility. We establish the existence of $\varepsilon_k > 0$ by a contradiction argument. For this, we can keep k fixed for most parts of the proof (namely up to and including (2.32)) and assume there is a sequence $\varepsilon_j \to 0$ such that id+$\varepsilon_j u_k$ is not globally invertible for all $j \in \mathbb{N}$. Thus, there exist $x_j, y_j \in \Omega_{\mathrm{Cr}}$ with $(\mathrm{id}{+}\varepsilon_j u_k)(x_j) = (\mathrm{id}{+}\varepsilon_j u_k)(y_j)$, i. e.

$$0 \neq x_j - y_j = \varepsilon_j\big(u_k(y_j) - u_k(x_j)\big). \tag{2.26}$$

By boundedness of Ω there is a (not relabeled) subsequence, such that x_j and y_j both converge. Since (2.26) gives $|x_j{-}y_j| \leq \varepsilon_j \|u_k\|_{\mathrm{L}^\infty(\Omega_{\mathrm{Cr}})} \leq \varepsilon_j(\|u\|_{L^\infty(\Omega_{\mathrm{Cr}})}{+}3\delta_k)$, these two limits are the same, from now denoted by z_∞. We next establish the following claim:

Claim: The point z_∞ lies in the crack edge $\Gamma_{\mathrm{edge}} = T^{-1}\big(\{(1,0)\} \times \mathbb{R}^{d-2}\big)$, and the convergence gives a very specific picture, i.e. $T(x_j){\cdot}e_2 > 0$, $T(y_j){\cdot}e_2 < 0$, $T(x_j){\cdot}e_1 < 1$, $T(y_j){\cdot}e_1 < 1$, and

$$\frac{|x_j - y_j|}{(1 - T(x_j){\cdot}e_1) + (1 - T(y_j){\cdot}e_1)} \;\to\; 0 \;\; \text{as } j \to \infty. \tag{2.27}$$

That means that x_j and y_j converge to z_∞ by approaching the crack asymptotically from left above and from left below, respectively.

A major part of the proof of the claim is due to Lipschitz continuity. If both, x_j and y_j, are in A_+ or both are in A_-, then with $L_k := \mathrm{Lip}_{A_\pm}(u_k)$ we would obtain

$$|x_j{-}y_j| = \varepsilon_j|u_k(y_j){-}u_k(x_j)| \leq \varepsilon_j L_k d_U(x_j, y_j) \leq \varepsilon_j L_k C_U |x_j{-}y_j|.$$

For $\varepsilon_j L_k C_U < 1$ this implies $x_j = y_j$, which contradicts (2.26). Thus, we have $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ or vice versa. Using $x_j, y_j \to z_\infty$ we conclude $z_\infty \in \Gamma_{\mathrm{Cr}}$.

For the subsequent arguments we choose the notation such that always $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$. If $\widehat{z}_\infty := T(z_\infty) \in \hat{\Gamma}_{\mathrm{Cr}} \setminus \hat{\Gamma}_{\mathrm{kink}}$ we have a normal vector to $\hat{\Gamma}_{\mathrm{Cr}}$ given by

$$\widehat{\nu} = \begin{cases} e_1 := (1, 0, \dots, 0) & \text{for } e_1 \cdot z_\infty = 0, \\ e_2 := (0, 1, 0, \dots, 0) & \text{for } e_2 \cdot z_\infty = 0. \end{cases}$$

By the above choice $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ we obtain

$$\big(T(x_j) - T(y_j)\big) \cdot \widehat{\nu} > 0 \tag{2.28}$$

for sufficiently big $j \in \mathbb{N}$. Thus, exploiting the smoothness of T across the crack and the relation (2.26) again we obtain

$$\begin{aligned} 0 < \frac{1}{\varepsilon_j}\big(T(x_j){-}T(y_j)\big) \cdot \widehat{\nu} \;&=\; \int_0^1 \nabla T\big(x_j{+}t(y_j{-}x_j)\big)\,\mathrm{d}t\, \frac{1}{\varepsilon_j}(x_j{-}y_j) \cdot \widehat{\nu} \\ &\overset{(2.26)}{=} \int_0^1 \nabla T\big(x_j{+}t(y_j{-}x_j)\big)\,\mathrm{d}t\, \big(u_k(y_j){-}u_k(x_j)\big) \cdot \widehat{\nu}. \end{aligned}$$

Passing to the limit $j \to \infty$ we find the jump condition

$$0 \leq \nabla T(z_\infty)\big(u_k^-(z_\infty){-}u_k^+(z_\infty)\big) \cdot \widehat{\nu} = \big(u_k^-(z_\infty){-}u_k^+(z_\infty)\big) \cdot \nabla T(z_\infty)^\top \widehat{\nu}.$$

However, because of the non-interpenetration condition $[\![u_k]\!]_{\Gamma_{\mathrm{Cr}}} = [\![u]\!]_{\Gamma_{\mathrm{Cr}}} + [\![\varphi_k]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$, where $[\![\varphi_k]\!]_{\Gamma_{\mathrm{Cr}}} > 0$ except on the crack edge, we have

$$\begin{aligned} &\big(u_k^+(z_\infty) - u_k^-(z_\infty)\big) \cdot \nabla T(z_\infty)^{\mathrm{T}} \widehat{\nu} \geq 0, \\ &\text{where equality holds if and only if } z_\infty \in \Gamma_{\mathrm{edge}}. \end{aligned} \tag{2.29}$$

Thus, we conclude that z_∞ cannot lie in $\Gamma_{\mathrm{Cr}} \setminus (\Gamma_{\mathrm{kink}} \cup \Gamma_{\mathrm{edge}})$.

It remains to exclude $z_\infty \in \Gamma_{\mathrm{kink}}$. If this would be the case, then both (2.28) and (2.29) still hold for some $\widehat{\nu}$ but for different reasons. One the one hand, using $x_j \in A_+$ and $y_j \in A_-$ for all j, there is a subsequence such that condition (2.28) holds for either $\widehat{\nu} = e_1$ or $\widehat{\nu} = e_2$. On the other hand, (2.29) holds for both $\widehat{\nu} = e_1$ and $\widehat{\nu} = e_2$ by continuity of u_k. Thus, we similarly conclude $z_\infty \notin \Gamma_{\mathrm{kink}}$, and $z_\infty \in \Gamma_{\mathrm{edge}}$, which is the first part of the above claim.

From here on let $\widehat{U} := B_\varrho(T(z_\infty)) \subset \hat{\Omega}$ with $\varrho < 1$, such that $\widehat{U}$ does cannot touch Γ_{kink}. Then, $T(x_j)$, $T(y_j) \in \widehat{U}$ for j big enough, and $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ gives

$$T(x_j) \cdot e_1 < 1, \quad T(x_j) \cdot e_2 > 0, \quad T(y_j) \cdot e_1 < 1, \quad T(y_j) \cdot e_2 < 0,$$

which is the second part of the above claim.

To see the last part of the claim note that we have either (2.27) as claimed or there is a subsequence (not relabeled) such that

$$(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1) \le C\, |x_j - y_j| \tag{2.30}$$

with some positive constant C independent of j. We assume now (2.30) in order to generate a contradiction. Indeed, the smallness of the quantities on the left-hand side allow us to exploit the Lipschitz continuity of u_k on $T^{-1}(\{\, \widehat{x} \in \widehat{U} \mid \widehat{x}_1 \ge 1 \,\})$, which is the domain to the right of the crack edge containing the intersection $A_+ \cap A_-$. Introducing the projections

$$x_j' := T^{-1}\Big(T(x_j) + (1 - T(x_j) \cdot e_1)\, e_1\Big) \quad \text{and} \quad y_j' := T^{-1}\Big(T(y_j) + (1 - T(y_j) \cdot e_1)\, e_1\Big),$$

we can compare them with x_j and y_j, respectively, as well as x_j' and y_j' to each other:

$$\begin{aligned}
\frac{1}{\varepsilon_j}|x_j - y_j| &= |u_k(x_j) - u_k(y_j)| \\
&\le |u_k(x_j) - u_k(x_j')| + |u_k(x_j') - u_k(y_j')| + |u_k(y_j') - u_k(y_j)| \\
&\le L_k\Big(|x_j - x_j'| + |x_j' - y_j'| + |y_j' - y_j|\Big) \\
&\le L_k\Big(2|x_j - x_j'| + |x_j - y_j| + 2|y_j' - y_j|\Big) \\
&\le L_k\Big(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty}\big(|T(x_j) - T(x_j')| + |T(y_j') - T(y_j)|\big)\Big) \\
&\le L\Big(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty}\big((1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1)\big)\Big) \\
&\overset{(2.30)}{\le} L\,|x_j - y_j|\,\Big(1 + 2\|\nabla T^{-1}\|_{L^\infty} C\Big)\,.
\end{aligned}$$

After dividing by $|x_j - y_j| \ne 0$, we see that this contradicts $\varepsilon_j \to 0$, such that (2.30) must be false, and hence (2.27) and the whole above claim is established.

We still have to produce a contradiction to show that (2.26) is false. But now we can use the relations in the above claim, in particular the convergence (2.27). To this end, we will use the assumption $\alpha > 1$ in the definition $\eta_k = \delta_k^\alpha$.

In the following calculation we use the abbreviation $A_j := \int_0^1 \nabla T(x_j{+}t(y_j{-}x_j))\,\mathrm{d}t \in \mathbb{R}^{d\times d}$ and insert relation (2.26) (recall $u_k = u+\gamma\varphi_k$ with $\gamma \equiv 1$ in a neighborhood of Γ_{Cr}):

$$\begin{aligned}
0 &\le \frac{1}{\varepsilon_j}(T(x_j){-}T(y_j))\cdot e_2 = \frac{1}{\varepsilon_j}A_j(x_j{-}y_j)\cdot e_2 \overset{(2.26)}{=} A_j\big(u_k(y_j){-}u_k(x_j)\big)\cdot e_2 \qquad (2.31)\\
&= A_j\Big(\big(u(y_j){-}u(y_j')\big) + \big(u(y_j'){-}u(x_j')\big) + \big(u(x_j'){-}u(x_j)\big) + \big(\varphi_k(y_j){-}\varphi_k(x_j)\big)\Big)\cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty}\|\nabla u\|_{\mathrm{L}^\infty}\big(|y_j{-}y_j'| + |y_j'{-}x_j'| + |x_j'{-}x_j|\big) + A_j\big(\varphi_k(y_j){-}\varphi_k(x_j)\big)\cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty}\|\nabla u\|_{\mathrm{L}^\infty}\Big(|x_j{-}y_j| + 2\big(|x_j{-}x_j'| + |y_j{-}y_j'|\big)\Big) + A_j\big(\varphi_k(y_j){-}\varphi_k(x_j)\big)\cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty}\|\nabla u\|_{\mathrm{L}^\infty}\Big(|x_j{-}y_j| + 2\|\nabla T^{-1}\|_{\mathrm{L}^\infty}\big((1{-}T(x_j){\cdot}e_1){+}(1{-}T(y_j){\cdot}e_1)\big)\Big)\\
&\qquad + A_j\big(\varphi_k(y_j){-}\varphi_k(x_j)\big)\cdot e_2
\end{aligned}$$

Dividing by $(1{-}T(x_j)\cdot e_1) + (1{-}T(y_j)\cdot e_1)$ and taking the limit $j\to\infty$, the assumed convergence (2.27) implies that the first summand of the right-hand side converges to the constant $C_u := 2\|\nabla T\|_{\mathrm{L}^\infty}\|\nabla u\|_{\mathrm{L}^\infty}\|\nabla T^{-1}\|_{\mathrm{L}^\infty}$, which is independent of k. The idea is now to show that for our choice of $\alpha > 1$ the second summand makes the right-hand side negative for sufficiently small δ_k, which then produces a contradiction.

For this, we exploit the definition of φ_k via the function λ_{η_k} and the choices $x_j \in A_+\setminus A_-$ and $y_j \in A_-\setminus A_+$. Since x_j and y_j are near Γ_{edge} we obtain

$$\lambda_\eta(T(x_j)) = \frac{1}{\eta}(1{-}T(x_j){\cdot}e_1) \quad \text{and} \quad \lambda_\eta(T(y_j)) = -\frac{1}{\eta}(1{-}T(y_j){\cdot}e_1).$$

Inserting this with $\eta = \delta_k^\alpha$ we find

$$\begin{aligned}
&A_j\big(\varphi_k(y_j){-}\varphi_k(x_j)\big)\cdot e_2 = \delta_k A_j\Big(\nabla T(y_j)^{-1}\lambda_{\delta_k^\alpha}(T(y_j))\widehat{n}{-}\nabla T(x_j)^{-1}\lambda_{\delta_k^\alpha}(T(x_j))\widehat{n}\Big)\cdot e_2\\
&= \delta_k A_j\Big({-}\frac{1}{\delta_k^\alpha}\nabla T(y_j)^{-1}(1{-}T(y_j){\cdot}e_1)\widehat{n} - \frac{1}{\delta_k^\alpha}\nabla T(x_j)^{-1}(1{-}T(x_j){\cdot}e_1)\widehat{n}\Big)\cdot e_2\\
&= -\delta_k^{1-\alpha}\Big((1{-}T(y_j){\cdot}e_1)\, e_2{\cdot}A_j\nabla T(y_j)^{-1}\widehat{n} + (1{-}T(x_j){\cdot}e_1)\, e_2{\cdot}A_j\nabla T(x_j)^{-1}\widehat{n}\Big).
\end{aligned}$$

The matrices $A_j\nabla T(y_j)^{-1}$ and $A_j\nabla T(x_j)^{-1}$ converge to $I \in \mathbb{R}^{d\times d}$ by dominated convergence and continuity of ∇T, thus we have $e_2{\cdot}A_j\nabla T(x_j)^{-1}\widehat{n} \to e_2{\cdot}\widehat{n} = 1$ and similarly for y_j. Because both $(1{-}T(x_j){\cdot}e_1)$ and $(1{-}T(y_j){\cdot}e_1)$ are positive, this implies the convergence

$$\delta_k^{\alpha-1}\frac{A_j\big(\varphi_k(y_j){-}\varphi_k(x_j)\big){\cdot}e_2}{(1{-}T(x_j){\cdot}e_1) + (1{-}T(y_j){\cdot}e_1)} \;\to\; -1 \quad \text{for } j\to\infty. \qquad (2.32)$$

Inserting this into (2.31) divided by $(1{-}T(x_j){\cdot}e_1) + (1{-}T(y_j){\cdot}e_1) > 0$ we obtain $0 \le 2C_u{-}\frac{1}{2}\delta_k^{1-\alpha}$ for each fixed k in the limit $j\to\infty$. Thus, making δ_k smaller if necessary, we arrive at a contradiction, because $\delta_k \to 0$ and $\alpha > 1$.

This shows that (2.26) cannot hold for $\varepsilon_j \to 0$. Thus, the existence of $\varepsilon_k > 0$ is established, and Proposition 2.17 is proved. □

The Proposition 2.17 shows that the set of near-identity injective displacements with bounded gradient $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\cap\mathcal{I}$ is dense in $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\cap\mathcal{J}$. To further extend the achieved knowledge from $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\cap\mathcal{J}$ to the general case $u\in\mathcal{J}$, we have to show that all functions $u\in\mathcal{J}$ can be approximated by $u_k\in\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\cap\mathcal{J}$, i.e. we have to approximate under the convex constraint of local non-interpenetration. Similar approximation results for more classical state constraints are contained in [HR15, HRR16].

To handle our conditions of non-negativity of jumps over the crack we can use a reflection and decomposition into odd and even parts. To simplify the reading of the following proof, we illustrate this idea by a simple scalar two-dimensional problem.

Example 2.18 (Straight crack in $\mathbb{R}^2$). *We consider* $\Omega=\mathbb{R}^2$, $\Gamma_{\mathrm{Cr}}=\mathbb{R}\times\{0\}$, *and a function* $u\in\mathrm{H}^1(\Omega\setminus\Gamma_{\mathrm{Cr}})$ *with* $[\![u]\!]_{\Gamma_{\mathrm{Cr}}}\geq 0$. *To find a smooth approximation we define*

$$u^{\mathrm{even}}(x)=\frac{1}{2}(u(x_1,x_2)+u(x_1,-x_2))\quad\text{and}\quad u^{\mathrm{odd}}(x)=\frac{1}{2}(u(x_1,x_2)-u(x_1,-x_2)),$$

such that $u=u^{\mathrm{even}}+u^{\mathrm{odd}}$, $[\![u^{\mathrm{even}}]\!]_{\Gamma_{\mathrm{Cr}}}=0$, *and* $[\![u^{\mathrm{odd}}]\!]_{\Gamma_{\mathrm{Cr}}}=2u^{\mathrm{odd}}(\cdot,0^+)=[\![u]\!]_{\Gamma_{\mathrm{Cr}}}$.

We can easily approximate u^{even} *by* $v_k\in\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^2)$, *since it lies in* $\mathrm{H}^1(\mathbb{R}^2)$. *For* u^{odd} *we don't want to smoothen the jump along* Γ. *Hence, we define a "positive extension via reflection" as follows:*

$$\widetilde{u}(x_1,x_2)=\begin{cases} u^{\mathrm{odd}}(x_1,x_2) & \text{for } x_2>0,\\ \max\{0,u^{\mathrm{odd}}(x_1,-x_2)\} & \text{for } x_2<0.\end{cases}$$

Because of $\widetilde{u}(\cdot,0^+)=u^{\mathrm{odd}}(\cdot,0^+)=\frac{1}{2}[\![u]\!]_{\Gamma_{\mathrm{Cr}}}\geq 0$ *we conclude that* $[\![\widetilde{u}]\!]_{\Gamma_{\mathrm{Cr}}}=0$, *which implies* $\widetilde{u}\in\mathrm{H}^1(\mathbb{R}^2)$. *Defining convolution kernels* $\psi_k\in\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R}^2)$ *with* $\psi_k\geq 0$, $\int_{\mathbb{R}^2}\psi_k\,\mathrm{d}y=1$, *and* supp$(\psi_k)\subset B_{1/k}((0,-1/k))\subset\mathbb{R}\times\,]-\infty,0[$ *we can define* $\widetilde{v}_k=\psi_k*\widetilde{u}\in\mathrm{C}^\infty(\mathbb{R}^2)$ *and check that* $\widetilde{v}_k\to\widetilde{u}$ *in* $\mathrm{H}^1(\mathbb{R}^2)$ *and that* $\widetilde{v}_k(x_1,0)\geq 0$, *because* $\widetilde{u}(x_1,x_2)\geq 0$ *for* $x_2\leq 0$ *and the kernel* ψ_k *also has its support in* $\mathbb{R}\times\,]-\infty,0[$. *Thus, setting*

$$u_k(x_1,x_2)=v_k(x_1,x_2)+\mathrm{sign}(x_2)\,\widetilde{v}_k(x_1,|x_2|)$$

we obtain $u_k\in\mathrm{C}^\infty(\Omega\setminus\Gamma_{\mathrm{Cr}})$ *with* $u_k\to u$ *in* $\mathrm{H}^1(\Omega\setminus\Gamma_{\mathrm{Cr}})$ *and* $[\![u_k]\!]_{\Gamma_{\mathrm{Cr}}}\geq 0$.

The analogous construction for our general $\Gamma_{\mathrm{Cr}}\subset\Omega$ works similarly by mapping the displacements $u:\Omega_{\mathrm{Cr}}\to\mathbb{R}^d$ via the Piola transform onto displacements $\widehat{u}:\hat{\Omega}_{\mathrm{Cr}}\to\mathbb{R}^d$, where the positivity of the jumps is preserved, see (2.11). To simplify the proof we introduce some notation for mollifiers and shifts. We choose a fixed convolution kernel $\psi\in\mathrm{C}_{\mathrm{c}}(\mathbb{R}^d)$ with supp$\psi\subset B_1(0)$, $\psi\geq 0$, $\int_{\mathbb{R}^d}\psi\,\mathrm{d}x=1$, and $\psi(x)=\psi(\widetilde{x})$ if $|x|=|\widetilde{x}|$. With this we define the mollifier M_k^a with shift vector $a\in\mathbb{R}^d$ via

$$(M_k^a u)(x)=\int_{|z|\leq 1}\psi(z)u(x-\frac{1}{k}(z-a))\,\mathrm{d}z=\int_{|y-x|<1/k}k^d\psi(k(x-y))u(y+\frac{1}{k}a)\,\mathrm{d}y.$$

The shift vector a will be chosen differently above and below a crack to avoid intersecting the crack, see e.g. (2.33).

Of course, we can take full advantage that the crack $\hat{\Gamma}_{\mathrm{Cr}}$ is piecewise flat. The only point that is more delicate arises for points in the intersection of Γ_{Cr} and $\partial\Omega$.

Proposition 2.19. *Let* $u\in\mathcal{U}$ *with* $[\![u]\!]_{\Gamma_{\mathrm{Cr}}}\geq 0$, *then there is a sequence* $u_k\in\mathcal{U}\cap\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}},\mathbb{R}^d)$ *with* $[\![u_k]\!]_{\Gamma_{\mathrm{Cr}}}\geq 0$ *such that* $u_k\to u$ *in* $\mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$.

Proof. First, we show that it suffices to consider the case $(\hat{\Omega}, \hat{\Gamma}_{\text{Cr}})$ instead of the more general $(\Omega, \Gamma_{\text{Cr}})$. For this we can use the Piola transforms $P_T : \mathrm{H}^1(\hat{\Omega}) \to \mathrm{H}^1(\Omega)$ from (2.10). With the inverse mapping $(P_T)^{-1} = P_{T^{-1}}$. Since, T and T^{-1} lie in $\mathrm{W}^{2,\infty}$ we see that P_T is also a linear bounded map from $\mathrm{W}^{1,\infty}(\Omega_{\text{Cr}}, \mathbb{R}^d)$ into $\mathrm{W}^{1,\infty}(\hat{\Omega}_{\text{Cr}}, \mathbb{R}^d)$ with linear bounded inverse $P_{T^{-1}}$. Thus, for the given $u \in \mathcal{U}$ with $[\![u]\!]_{\Gamma_{\text{Cr}}} \geq 0$ we may consider $\widehat{u} := P_{T^{-1}}u \in \mathrm{H}^1(\hat{\Omega})$ with $[\![\widehat{u}]\!]_{\widehat{\nu}} \geq 0$. If we find approximations $\widehat{u}_k$, then $u_k = P_T\widehat{u}_k$ provides the desired sequence.

Second, we observe that it suffices to show the assertion locally in a neighborhood U of each point $x^* \in \operatorname{clos}(\hat{\Omega})$ because by compactness we have a finite cover of such neighborhoods and recombination by partition of unity gives the result. In all cases we consider $U = B_\delta(x^*) \cap \hat{\Omega}$ and may consider $\widehat{u}$ with $\operatorname{supp}(\widehat{u}) \subset B_\varepsilon(x^*)$ for some $\varepsilon \in]0, \delta[$. Thus, convolutions $M_k^a\widehat{u}$ will be well defined for k sufficiently large, as longs as $\operatorname{supp}(\widehat{u}) + B_{1/k}(a)$ stays inside of $B_\delta(x^*) \cap \hat{\Omega}_{\text{Cr}}$.

We now discuss the occurring different cases.

Bulk points in Ω_{Cr}: For $x^* \in \hat{\Omega}_{\text{Cr}}$ and a ball $B_\delta(x^*) \Subset \hat{\Omega}_{\text{Cr}}$ the convolution $\widehat{u}_k = M_k^0\widehat{u}$ is smooth and converges in $\mathrm{H}^1(B_\delta(x^*), \mathbb{R}^d)$ to $\widehat{u}$.

Free boundary: For the case $x^* \in \partial\hat{\Omega} \setminus (\hat{\Gamma}_{\text{Cr}} \cup \hat{\Gamma}_{\text{Dir}})$ we extend $\widehat{u}$ to the outside of $\hat{\Omega}$ first. For this we take a ball $B_\delta(x^*)$ with $B_{2\delta}(x^*) \cap \hat{\Gamma}_{\text{Cr}} = \emptyset$ and by the Lipschitz property of $\partial\hat{\Omega}$ there is a bi-Lipschitz chart $\Psi : B_\delta(x^*) \to \mathbb{R}^d$ with $\hat{\Omega} \cap B_\delta(x^*) \subset \Psi^{-1}(\{y_d > 0\})$, $\partial\hat{\Omega} \cap B_\delta(x^*) \subset \Psi^{-1}(\{y_d = 0\})$, and $B_\delta(x^*) \setminus \operatorname{clos}(\hat{\Omega}) \subset \Psi^{-1}(\{y_d < 0\})$. An $\mathrm{H}^1(B_\delta(x^*), \mathbb{R}^d)$-extension $\widetilde{u}$ of $\widehat{u}$ is now given by $\widetilde{u}(x) = u(\Psi^{-1}(R(\Psi(x))))$, where $R(y) = (y_1, \dots, y_{d-1}, |y_d|)$. The desired approximations are then given by $\widehat{u}_k = M_k^0\widetilde{u}|_{B_\delta(x^*) \cap \hat{\Omega}}$.

Dirichlet part of the boundary: For $x^* \in \Gamma_{\text{Dir}}$ there exists $B_\delta(x^*)$ disjoint from the crack Γ_{Cr}, and by definition of $\mathcal{U}$ there is a $\mathrm{W}^{1,\infty}$-sequence coinciding with g on $\hat{\Gamma}_{\text{Dir}}$.

Flat parts of the crack: For $x^* \in \hat{\Gamma}_{\text{Cr}} \setminus (\hat{\Gamma}_{\text{edge}} \cup \hat{\Gamma}_{\text{kink}} \cup \partial\hat{\Omega})$ we proceed similarly as in Example 2.18. Since x^* is neither a point in $\partial\hat{\Omega}$ nor in the crack kink $\hat{\Gamma}_{\text{kink}}$ or the crack edge $\hat{\Gamma}_{\text{edge}}$, we can assume, without loss of generality, that $x^* \in \{0\} \times]0, \infty[\times \mathbb{R}^{d-2}$ with $\widehat{\nu} = e_1$, the case $x^* \in]0, 1[\times \{0\} \times \mathbb{R}^{d-2}$ with $\widehat{\nu} = e_2$ is analogous. We take $B_\delta(x^*)$ that touches neither of the critical parts.

For a fixed $n \in \{2, \dots, d\}$ we approximate the component $v = \widehat{u}^{[n]}$ of $\widehat{u} = (\widehat{u}^{[1]}, \dots, \widehat{u}^{[d]})$ simply via

$$v_k(x) = M_k^{\operatorname{sign}(x_1)e_1}v \quad \text{for } x \in B_\delta(x^*) \setminus \hat{\Gamma}_{\text{Cr}}, \tag{2.33}$$

where we can use that the parts left and right of the crack at $x_1 = 0$ are independent (no jump conditions. The shift vectors $\pm e_1$ take care that mollifications never touch the crack.

For $n = 1$ we need to be more careful since $v = \widehat{u}^{[1]}$ has to have a positive jump over the crack, namely $v(0^+, \cdot) - v(0^-, \cdot) = [\![\widehat{u}]\!]_{\widehat{\nu}} \geq 0$. We define the odd and even parts via

$$v^{(i)}(x_1, \dots, x_d) = \frac{1}{2}\Big(v(x_1, \dots, x_d) + (-1)^i\, v(-x_1, x_2, \dots, x_d)\Big).$$

The even function $v^{(0)}$ lies in $\mathrm{H}^1(B_\delta(x^*))$, because it has no jump, thus we can approximate $v^{(0)}$ by the even functions $v_k^{(0)} = M_k^0 v^{(0)}$.

The odd function $v^{(1)}$ has a positive jump which needs to be preserved. Hence we restrict it to the semi-ball with $x_1 > 0$ and use the nonnegative extension of Example 2.18, namely $x \mapsto \max\{0, v^{(1)}(-x_1, x_2, \ldots)\}$ for $x_1 < 0$. This leads to $\widetilde{v} \in \mathrm{H}^1(B_\delta(x^*))$, which is nonnegative for $x_1 < 0$. Thus, the mollifications $\widetilde{v}_k = M_k^{-e_1}\widetilde{k}$ converge to $\widetilde{v}$, and the shift vector $-e_1$ guarantees that $\widetilde{v}_k$ is nonnegative for $x_1 \leq 0$, which implies that the trace of $\widetilde{v}_k$ on $B_\delta(x^*) \cap \{x_1 = 0\}$ is nonnegative.

The desired approximations for $v = u^{[1]}$ are then given by

$$u_k^{[1]}(x) = v_k(x) = M_k^0 v^{(0)}(x) + \mathrm{sign}(x_1) v_k^{(1)}(|x_1|, x_2, \ldots, x_d).$$

Crack edge: For a point $x^* \in \hat{\Gamma}_{\mathrm{edge}} \backslash \partial\hat{\Omega}$ we have $\widehat{\nu} = e_2$. For a $\delta \in \,]0,1[$ with $B_{2\delta}(x^*) \cap \partial\hat{\Omega} = \emptyset$ we proceed similarly. For $n \neq 2$ we consider the component $v = \widehat{u}^{[n]}$, which may have an arbitrary jump along $B_\delta(x^*) \cap \{x_1 < 1 \text{ and } x_2 = 0\}$ but has no jump along $\{x_1 > 1 \text{ and } x_2 = 0\}$.

To handle this case we work with a continuously varying shift vector $a_k(x)$ as follows. Let $h(x) = \max\{0, \min\{x, 1\}\}$ and set

$$a_k : B_\delta(x^*) \setminus \hat{\Gamma}_{\mathrm{Cr}} \to \mathbb{R}^d; \; x \mapsto \mathrm{sign}(x_2) h(\sqrt{k}(1-x_1)) e_2 + \sqrt{k}\, e_1.$$

The main observation is that $x \mapsto \frac{1}{k} a_k(x)$ is a function in $\mathrm{W}^{1,\infty}(B_\delta(x^*) \backslash \hat{\Gamma}_{\mathrm{Cr}}; \mathbb{R}^d)$ with norm bounded by $C/\sqrt{k}$. Moreover, for all $x \in B_\delta(x^*) \setminus \hat{\Gamma}_{\mathrm{Cr}}$ the convolution integration domain $x + B_{1/k}(a_k(x))$ does not intersect $\hat{\Gamma}_{\mathrm{Cr}}$. Thus, $v_k = M_k^{a_k(x)} v$ is well-defined, smooth on $B_\delta(x^*) \backslash \hat{\Gamma}_{\mathrm{Cr}}$, and converges to v.

The case $v = \widehat{u}^{[2]}$ is more difficult, since we need to maintain the non-negativity of the jump. Using the even and odd parts

$$v^{(i)}(x_1, \ldots, x_d) = \frac{1}{2}\Big(u(x_1, \ldots, x_d) + (-1)^i\, v(x_1, -x_2, x_3, \ldots, x_d)\Big),$$

we see that the even part $v^{(0)}$ lies in $\mathrm{H}^1(B_\delta(x^*))$, so we use the mollifications $v_k^{(0)} = M_k^0 v^{(0)}$.

The odd part $v^{(1)}$ is delicate, since we need non-negativity of the jump for $x_1 < 1$ and no jump for $x_1 > 1$. For this we restrict $v^{(1)}$ to the upper semi-ball $B_\delta(x^*) \cap \{x_2 > 0\}$ and extend it to a function $w \in \mathrm{H}^1(B_\delta(x^*))$ which is 0 in $\{x_1 > 1 \text{ and } x_2 < 0\}$. For this, we define a piecewise affine bi-Lipschitz S map between the triangle $\{x \in \mathbb{R}^2 | 0 \leq x_2 \leq 1 - |x_1 - 1|\}$ and the square $[0,1] \times [-1,0]$ via

$$S(x_1, x_2) = \big(\, \min\{0, x_1\} - x_2 \,,\, \min\{0, 1 - x_1\} - x_2 \,\big)$$

This mapping keeps $(1,0)$ fixed, is the identity on the line $L_1 := [0,1] \times \{0\}$, and maps the line $L_2 := [1,2] \times \{0\}$ to the line $L_3 := \{1\} \times [-1,0]$. Thus, setting

$$w(x) = \begin{cases} v^{(1)}(x) & \text{for } x_2 > 0, \\ \max\{0, v^{(1)}(S^{-1}(x_1, x_2), x_3, \ldots)\} & \text{for } x_2 < 0 \text{ and } x_1 < 1, \\ 0 & \text{for } x_2 < 0 \text{ and } x_1 > 1, \end{cases}$$

we find that $w \in \mathrm{H}^1(B_\delta(x^*))$, since the traces on L_1, L_2, and L_3 match by construction. Thus, as w is nonnegative for $x_2 < 0$ and even 0 if additionally $x_1 > 1$, we see that the approximation

$$w_k = M_k^{e_1 - e_2} w \;\; \text{satisfies } w_k \to w \in \mathrm{H}^1(B_\delta(x^*)$$

and is still nonnegative for $x_2 < 0$ and even 0 if additionally $x_1 > 1$.

As above we conclude that $v_k = v_k^{(0)} + \mathrm{sign}(x_2) w_k$ lies in $\mathrm{W}^{1,\infty}(B_\delta(x^*) \backslash \hat{\Gamma}_{\mathrm{Cr}})$ and converges to $v = \widehat{u}^{[2]}$.

<u>Crack kink:</u> Let us come to $x^* \in \hat{\Gamma}_{\mathrm{kink}}$ with $B_{2\delta}(x^*) \cap (\partial\hat{\Omega} \cup \hat{\Gamma}_{\mathrm{edge}}) = \emptyset$. We again decompose the components $v = \widehat{u}^{[n]}$ in odd and even parts, but now we have two hyperplanes, so we need four parts with evenness and oddness in x_1 and x_2, respectively. For $i, j \in \{0,1\}$ we set

$$v^{(i,j)}(x) = \frac{1}{4}\Big(v(x_1, x_2, x_3, \dots, x_d) + (-1)^i\, v(-x_1, x_2, x_3, \dots, x_d) + (-1)^j\, v(x_1, -x_2, x_3, \dots, x_d) + (-1)^{i+j}\, v(-x_1, -x_2, x_3, \dots, x_d) \Big).$$

Thus, each function $v^{(i,j)}$ is completely determined by its value in the positive quadrant $Q_+ := \{x \in \mathbb{R}^d \,|\, x_1, x_2 > 0\}$, namely

$$v^{(i,j)}(x_1, x_2, x_3, \dots) = \mathrm{sign}(x_1^i x_2^j) v^{(i,j)}(N(x)), \quad \text{where } N(x) = (|x_1|, |x_2|, x_3, \dots, x_d).$$

Each component will be approximated by functions $v_k^{(i,j)} \in \mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ such that the desired full approximations v_k of v take the form

$$v_k(x) = \sum_{i,j=0}^{1} \mathrm{sign}(x_1^i x_2^j) v_k^{(i,j)}(N(x)) \tag{2.34}$$

However, to guarantee that v_k lies in $\mathrm{W}^{1,\infty}(B_\delta(x^*) \backslash \hat{\Gamma}_{\mathrm{Cr}})$ we have to show that there are no jumps at (i) $\Sigma_1 := \{x_1 = 0 \text{ and } x_2 < 0\}$ and at (ii) $\Sigma_2 := \{x_1 < 0 \text{ and } x_2 = 0\}$. Moreover, for $n \in \{1,2\}$ we need a non-negativity condition on the jump along $C_n := \{x_n = 0 \text{ and } x_{3-n} > 0\}$:

$$\begin{aligned}
\text{(i)}: \quad & d_k^{(1)} := v_k^{(1,0)} - v_k^{(1,1)} \text{ has trace 0 on } -\Sigma_1 = C_1;\\
\text{(ii)}: \quad & d_k^{(2)} := v_k^{(0,1)} - v_k^{(1,1)} \text{ has trace 0 on } -\Sigma_2 = C_2;\\
\text{if } n = 1: \quad & s_k^{(1)} := v_k^{(1,0)} + v_k^{(1,1)} \text{ has a nonnegative trace on } C_1;\\
\text{if } n = 2: \quad & s_k^{(2)} := v_k^{(0,1)} + v_k^{(1,1)} \text{ has a nonnegative trace on } C_2.
\end{aligned}$$

We only explain the case $n = 1$, since the case $n = 2$ is analogous when interchanging x_1 and x_2. The cases $n \geq 3$ are even simpler, since only (i) and (ii) are needed.

The idea is to start from the corresponding $d^{(i)}$ and $s^{(1)}$ for the desired limits $v^{(i,j)}$ and approximate those. The differences $d^{(m)} \in \mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ can be extended by 0 across the plane $C_m = -\Sigma_m \subset \partial Q_+$ such that

$$d_k^{(m)} = M_k^{e_{3-m}-e_m} d^{(m)} \to d^{(m)} \text{ in } \mathrm{H}^1(B_\delta(x^*) \cap Q_+) \quad \text{and } d^{(m)}|_{C_m} = 0.$$

Here the shift vector $-e_m$ guarantees the vanishing trace, while e_{3-m} is used to avoid the other crack part C_{3-m}.

Finally, a positivity preserving extension $\widetilde{s}$ of $s^{(1)}$ across C_1 via $\max\{0, s^{(1)}(-x_1, x_2, \dots)\}$ gives $s_k^{(1)} = M_k^{-e_1+e_2} \widetilde{s}|_{B_\delta(x^*) \cap Q_+}$. Thus, $s_k^{(1)} \to s^{(1)}$ in $\mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ and $s_k^{(1)}|_{C_1} \geq 0$.

With this, $v_k^{(i,j)}$ for $i+j \geq 1$ can be uniquely calculated from $d_k^{(1)}$, $d_k^{(2)}$, and $s_k^{(1)}$, while the even-even function $v^{(0,0)}$ can be approximated arbitrarily. This results in

$$v_k^{(0,0)} = M_k^{e_1+e_2} v^{(0,0)}, \quad v_k^{(1,1)} = \frac{1}{2}(s_k^{(1)} - d_k^{(1)}), \quad v_k^{(1,0)} = d_k^{(1)} + v_k^{(1,1)}, \quad v_k^{(0,1)} = d_k^{(2)} + v_k^{(1,1)}.$$

With this construction, v_k defined in (2.34) gives the desired approximations.

Crack and boundary: For $x^* \in \partial\hat{\Omega} \cap \hat{\Gamma}_{\mathrm{Cr}}$ we again use reflection to extend $\widehat{u}$ from $\hat{\Omega} \cap B_\delta(x^*)$ to the outside but this time specialized by using Corollary 2.3. With U, φ_{x^*}, and η_{x^*} from there, we define the map $R : B_\delta(x^*) \to \hat{\Omega}$ with

$$R(x) = x - 2\max\{0, (x - x^*)\cdot\eta_{x^*} - \varphi_{x^*}(x - \eta_{x^*}\cdot(x - x^*)\eta_{x^*})\}\eta_{x^*},$$

which is Lipschitz continuous and satisfies the property $R^{-1}(U \cap \hat{\Gamma}_{\mathrm{Cr}}) \subset \hat{\Gamma}_{\mathrm{Cr}}$ and if $x^* \in \hat{\Gamma}_{\mathrm{edge}}$ we also have $R^{-1}(U \cap \hat{\Gamma}_{\mathrm{edge}}) \subset \hat{\Gamma}_{\mathrm{edge}}$ .. Thus, we can extend $\widehat{u}$ by $\widehat{u} \circ R \in \mathrm{H}^1(V \setminus \hat{\Gamma}_{\mathrm{Cr}}, \mathbb{R}^d)$ where $V = R^{-1}(\hat{\Omega} \cap U)$ is an open neighborhood of x^*. Now one can proceed as in the case $x^* \in \hat{\Omega} \cap \hat{\Gamma}_{\mathrm{Cr}}$ above.

Thus Proposition 2.19 is established. □

Combining Proposition 2.17 and Proposition 2.19 we see that $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}; \mathbb{R}^d) \cap \mathcal{I}$ is dense in $\mathcal{J}$. We are now ready to proof the desired limsup estimate by constructing a recovery sequence $(u_\varepsilon)_\varepsilon$ that converges strongly in $\mathrm{H}^1(\Omega_{\mathrm{Cr}}; \mathbb{R}^d)$. This result also provides the final part of the proof of the main Theorem 2.1 on the Mosco convergence $\mathcal{F}_\varepsilon \overset{\mathrm{M}}{\to} \mathcal{F}_0$.

Theorem 2.20 (Limsup estimate)**.** *For every $u \in \mathcal{U}$ there exists a sequence (ε_j, u_j) with*

$$\varepsilon_j \to 0, \quad u_j \to u \text{ in } \mathcal{U} \subset \mathrm{H}^1(\Omega_{\mathrm{Cr}}; \mathbb{R}^d), \quad \text{and} \quad \limsup_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) \leq \mathcal{F}_0(u).$$

Proof. For $\mathcal{F}_0(u) = \infty$ there is nothing to show, so we restrict to the case $\mathcal{F}_0(u) < \infty$ which implies $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$.

Case $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$: Applying Proposition 2.17 we obtain a sequence (ε_k, u_k) with $u_k \to u$ in $\mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ such that $v_k = \mathrm{id} + \varepsilon_k u_k$ satisfies the GMS condition (1.1), which implies

$$\mathcal{F}_{\varepsilon_k}(u_k) = \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k) = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x))\,\mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} \overline{W}_\varepsilon(\nabla u_k(x))\,\mathrm{d}x.$$

Since all u_k lie in $\mathrm{W}^{1,\infty}$ we may assume that $\varepsilon_k \|\nabla u_k\|_{\mathrm{L}^\infty} \leq r_{1/2}$ with $r_\delta > 0$ from (2.1d) for $\delta = \frac{1}{2}$. Thus, we have

$$\overline{W}_\varepsilon(\nabla u_k(x)) = \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x)) \leq \left(\frac{1}{2} + \frac{1}{2}\right)|\nabla u_k(x)|_{\mathbb{C}}^2 \leq |\mathbb{C}|\,|\nabla u_k(x)|^2 =: g_k(x).$$

Using $\nabla u_k \to \nabla u$ strongly in $\mathrm{L}^2(\Omega, \mathbb{R}^{d\times d})$ we conclude $g_k \to g$ in $\mathrm{L}^1(\Omega)$, where $g(x) = |\mathbb{C}||\nabla u(x)|^2$. Moreover, we may choose a subsequence such that $\nabla u_k(x) \overset{k\to\infty}{\to} \nabla u(x)$ a.e. in Ω_{Cr}. Using assumption (2.1d) we obtain $\overline{W}_{\varepsilon_k}(\nabla u_k(x)) \to \frac{1}{2}|\nabla u(x)|_{\mathbb{C}}^2$ a.e. in Ω by Lemma 2.13. Now the generalized Lebesgue dominated convergence theorem provides the desired limit, namely

$$\lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k) = \lim_{k\to\infty} \int_{\Omega_{\mathrm{Cr}}} \overline{W}_\varepsilon(\nabla u_k(x))\,\mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}\langle \nabla u(x), \mathbb{C}\nabla u(x)\rangle\,\mathrm{d}x = \mathcal{F}_0(u).$$

General $u \in \mathcal{J}$: For a general $u \in \mathcal{J}$ Proposition 2.19 guarantees the existence of an approximating sequence $u_j \in \mathcal{J} \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$. By the first case there are for each j sequences $(\varepsilon_{j,k}, u_{j,k})_{k\in\mathbb{N}}$ with $u_{j,k} \in \mathcal{J} \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$, $\varepsilon_{j,k} \to 0$, $u_{j,k} \to u_j$, and $\mathcal{F}_{\varepsilon_{j,k}}(u_{j,k}) \to \mathcal{F}_0(u_j)$ as $k \to \infty$.

To construct a diagonal sequence we use the strong continuity of $\mathcal{F}_0$ restricted to the convex set $\mathcal{J}$, namely

$$\exists\, C_F > 0\ \forall\, v \in \mathcal{J} \text{ with } \|v{-}u\|_{\mathrm{H}^1} \leq 1 :\ |\mathcal{F}_0(v) - \mathcal{F}_0(u)| \leq C_F \|v{-}u\|_{\mathrm{H}^1}.$$

With this we can construct a diagonal sequence as follows. For $n \in \mathbb{N}$ we choose $j_n \geq n$ with with $\|u{-}u_{j_n}\|_{\mathrm{H}^1} < 1/n$. Next we choose $k_n \geq n$ with

$$\varepsilon_{j_n,k_n} < 1/n, \quad \|u_{j_n,k_n}{-}u_{j_n}\|_{\mathrm{H}^1} < 1/n, \quad \text{and } |\mathcal{F}_{\varepsilon_{j_n,k_n}}(u_{j_n,k_n}) - \mathcal{F}_0(u_{j_n})| < 1/n.$$

Setting $\widetilde{\varepsilon}_n = \varepsilon_{j_n,k_n}$ and $\widetilde{u}_n = u_{j_n,k_n}$ we obtain $\widetilde{\varepsilon}_n < 1/n$, $\|\widetilde{u}_n - u\|_{\mathrm{H}^1} < 2/n$, and

$$\big|\mathcal{F}_{\widetilde{\varepsilon}_n}(\widetilde{u}_n) - \mathcal{F}_0(u)\big| \leq \big|\mathcal{F}_{\varepsilon_{k_n}}(u_{j_n,k_n}) - \mathcal{F}_0(u_{j_n})\big| + \big|\mathcal{F}_0(u_{j_n}) - \mathcal{F}_0(u)\big| \leq 1/n + C_F/n \to 0.$$

Thus, $(\widetilde{\varepsilon}_n, \widetilde{u}_n)_{n\in\mathbb{N}}$ is a strongly converging recovery sequence for $u \in \mathcal{J}$. □

3 Gamma-convergence for Deformation Plasticity

3.1 Assumptions and main result

In the current chapter we aim to lift the results from Chapter 2 on the small-deformation limit in the case of static pure elasticity on the cracked reference configuration $\Omega_{\mathrm{Cr}} = \Omega \setminus \Gamma_{\mathrm{Cr}}$ to the case of deformation plasticity. We deal with the identical class of reference configurations as in previous Chapter by requiring existence of a transformation $T\colon \mathbb{R}^d \to \mathbb{R}^d$ as in (2.8). The state space

$$\mathcal{Q} := \mathcal{U} \times \mathcal{Z} := \mathcal{U} \times \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d})$$

will contain both the displacement $u \in \mathcal{U}$ and the plastic variable $z \in \mathcal{Z} = \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d})$. Dirichlet boundary conditions are prescribed on $\mathcal{U}$, identically to the previous chapter, in terms of the Dirichlet boundary Γ_{Dir} and Dirichlet data g_{Dir}:

$$\begin{aligned} &\overline{\Gamma_{\mathrm{Dir}}} \cap \overline{\Gamma_{\mathrm{Cr}}} = \emptyset, \quad \mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0, \quad g_{\mathrm{Dir}} \in \mathrm{W}^{1,\infty}(\Omega; \mathbb{R}^d) \\ &\mathcal{U} := \mathrm{clos}_{\mathrm{H}^1(\Omega_{\mathrm{Cr}})}\Big(\{u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}; \mathbb{R}^d) \mid (u-g)|_{\Gamma_{\mathrm{Dir}}} = 0\}\Big). \end{aligned} \tag{3.1}$$

The stored energies $\mathcal{E}_\varepsilon$ include among other things the elastic part $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}$ defined in terms of the elastic energy density W_{el}, on which the assumptions read identical to the assumptions on W in the previous chapter:

$$\forall\, F \in \mathbb{R}^{d\times d} \setminus \mathrm{GL}_+(d)\colon W_{\mathrm{el}}(F) = \infty; \tag{3.2a}$$

$$\forall\, F \in \mathbb{R}^{d\times d},\ R \in \mathrm{SO}(d):\ W_{\mathrm{el}}(RF) = W(F); \tag{3.2b}$$

$$\left.\begin{aligned} &\exists\, p > d, c_{W_{\mathrm{el}}}, C_{W_{\mathrm{el}}} > 0 \forall\, F \in \mathbb{R}^{d\times d} : \\ &\qquad W_{\mathrm{el}}(F) \geq c_{W_{\mathrm{el}}} \max\left\{\mathrm{dist}(F, \mathrm{SO}(d))^2, |F|^p - C_{W_{\mathrm{el}}}\right\}; \end{aligned}\right\} \tag{3.2c}$$

$$\left.\begin{aligned} &\exists\, \mathbb{C} \geq 0 \text{ with } \mathbb{C}^\top = \mathbb{C}\ \forall\, \delta > 0\ \exists\, r_{\mathrm{el}}(\delta) > 0\ \forall A \in B_{r_{\mathrm{el}}(\delta)}(0) \subset \mathbb{R}^{d\times d} : \\ &\qquad \Big|W_{\mathrm{el}}(I+A) - \frac{1}{2}\langle A, \mathbb{C}A\rangle\Big| \ \leq\ \delta\, \langle A, \mathbb{C}A\rangle. \end{aligned}\right\} \tag{3.2d}$$

For a discussion of these assumptions see (2.1). In contrast to pure elasticity in Chapter 2 in plasticity the deformation gradient does not coincide with the elastic tensor, instead the multiplicative split will be considered in $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}$:

$$\nabla v = F_{\mathrm{el}} F_{\mathrm{pl}}. \tag{3.3}$$

Furthermore a hardening energy $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}$ will be defined. For the energy density of the hardening part W_{h} we assume:

$$W_{\mathrm{h}}(P) := \begin{cases} \widetilde{W}_{\mathrm{h}}(P) & \text{if } P \in K, \\ \infty & \text{otherwise;} \end{cases} \tag{3.4a}$$

$$K \text{ is compact in } \mathrm{SL}(d) \text{ and contains a neighborhood of } I; \tag{3.4b}$$

$$\widetilde{W}_{\mathrm{h}} \colon \mathbb{R}^d \to \mathbb{R} \text{ is locally Lipschitz continuous;} \tag{3.4c}$$

$$\left.\begin{array}{l} \exists \mathbb{H}{\geq}0, \mathbb{H}^{\mathrm{T}}{=}\mathbb{H} \quad \forall \delta > 0 \quad \exists r_{\mathrm{h}}(\delta){>}0 \quad \forall A{\in}B_{r_{\mathrm{h}}(\delta)}(0)\colon \\ \qquad |\widetilde{W}_{\mathrm{h}}(I+A) - \frac{1}{2}\langle A, \mathbb{H}A\rangle| \leq \delta|\frac{1}{2}\langle A, \mathbb{H}A\rangle|; \end{array}\right\} \tag{3.4d}$$

$$\exists c_{\mathrm{h}}{>}0 \quad \forall A{\in}\mathbb{R}^{d\times d}\colon W_{\mathrm{h}}(I+A) \geq c_{\mathrm{h}}|A|^2. \tag{3.4e}$$

The conditions (3.4a) and (3.4b) give that W_{h} has the compact effective domain $K = {W_{\mathrm{h}}}^{-1}((-\infty,\infty)) \subset \mathrm{SL}(d)$, where $\mathrm{SL}(d) := \{P \in \mathbb{R}^{d\times d} \mid \det P = 1\}$. This is a rather strong technical assumption, but crucially needed to provide L^∞-bounds essential to control the quotients $F_{\mathrm{el},\varepsilon} = (I+\varepsilon u)(I+\varepsilon z)^{-1}$, as by (3.4b) we can find a constant $c_K > 0$ such that:

$$P \in K \Rightarrow \quad |P| + |P^{-1}| \leq c_K, \tag{3.5}$$

$$P \in \mathrm{SL}(d) \setminus K \Rightarrow \quad |P - I| \geq \frac{1}{c_K}. \tag{3.6}$$

Assumption (3.4d) requires W_{h} to admit a quadratic expansion of the same form as do W_{el} above and W in the previous Chapter. It includes by the implication $\partial_F W_{\mathrm{h}}(I) = 0$ that the material is free of plastic stress and gives the continuous convergence of the rescaled plastic densities $\frac{1}{\varepsilon^2} W_{\mathrm{h}}(I + \varepsilon\cdot)$ by Lemma 2.13. Furthermore, the combination of (3.4d) and (3.4e) gives:

$$\exists c_{\mathbb{H}} > 0 \, \forall A \in \mathbb{R}^{d\times d}\colon \qquad c_{\mathbb{H}}|A|^2 \leq |A|^2_{\mathbb{H}}.$$

The dissipation distance we will define in terms of the dissipation potential $R\colon \mathbb{R}^{d\times d} \to [0,\infty]$, on which the assumptions read:

$$R^{\mathrm{dev}}\colon \mathbb{R}^{d\times d}_{\mathrm{dev}} \to [0,\infty) \text{ convex and positively 1-homogenous;} \tag{3.7a}$$

$$\exists \underline{c}_R, \bar{c}_R{>}0 \, \forall z \in \mathbb{R}^{d\times d}_{\mathrm{dev}}\colon \underline{c}_R|z| \leq R^{\mathrm{dev}}(z) \leq \bar{c}_R|z|; \tag{3.7b}$$

$$R\colon \mathbb{R}^{d\times d} \to [0,\infty]; \qquad R(z) := \begin{cases} R^{\mathrm{dev}}(z) & \text{if } z \in \mathbb{R}^{d\times d}_{\mathrm{dev}}, \\ \infty & \text{otherwise;} \end{cases} \tag{3.7c}$$

where $\mathbb{R}^{d\times d}_{\mathrm{dev}} := \{A \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \mid \operatorname{tr} A = 0\}$. The positive 1-homogeneity of R

$$\forall \lambda > 0\colon \quad R(\lambda z) = \lambda R(z)$$

implements the rate independence of the evolution system considered in Chapter 4, which uses the same dissipation distance and distance function defined in the following:

$$D\colon \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to [0,\infty], \qquad D(P,\hat{P}) := \begin{cases} D(I, \hat{P}P^{-1}) & \text{if } P \text{ is invertible,} \\ \infty & \text{otherwise;} \end{cases}$$

where

$$D(I,\hat{P}) := \inf\Big\{ \int_0^1 R(\dot{P}P^{-1})\,\mathrm{d}t \mid P \in \mathrm{C}^1([0,1],\mathbb{R}^{d\times d}), P(0)=I, P(1)=\hat{P}\Big\}.$$

Note that $D(I,\hat{P}) < \infty$ implies $\det \hat{P} = 1$, because for a trajectory P as above with $R(\dot{P}P^{-1}) < \infty$ a.e. on $[0,1]$ we have $\dot{P}P^{-1} \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ and Jacobi's formula gives $\frac{\mathrm{d}}{\mathrm{d}t}\det P = \det P \operatorname{tr} \dot{P}P^{-1} = 0$, hence $\det \hat{P} = \det P(1) = \det P(0) = 1$. Furthermore, D satisfies the triangle inequality

$$D(P_1,P_3) \le D(P_1,P_2) + D(P_2,P_3), \tag{3.8}$$

which can be proven by taking trajectories $P^{(1,2)}$ and $P^{(2,3)}$ with $P^{(1,2)}(0) = I$, $P^{(1,2)}(1) = P_2P_1^{-1}$, $P^{(2,3)}(0) = I$ and $P^{(2,3)}(1) = P_3P_2^{-1}$ from the definitions of $D(I,P_2P_1^{-1}) = D(P_1,P_2)$ and $D(I,P_3P_2^{-1}) = D(P_2,P_3)$ and inserting the trajectory

$$P^{(1,3)}(t) = \begin{cases} P^{(1,2)}(2t) & \text{for } t \in [0,\frac{1}{2}], \\ P^{(2,3)}(2t-1)P_2P_1^{-1} & \text{for } t \in [\frac{1}{2},1], \end{cases}$$

into the infimum on the right-hand side of the definition of $D(I,P_3P_1^{-1}) = D(P_1,P_3)$. Moreover, for $P,\hat{P} \in \mathrm{SL}(d)$ by (3.7b) we have $D(P,\hat{P}) \le \bar{c}_R|I-\hat{P}P^{-1}| \le c|P-\hat{P}|$, which together with the triangle inequality implies continuity of D. Finally let us note that by inserting $P(t) = \exp(At)$ into the infimum of the definition of $D(I,\exp(A))$ one gets:

$$D(I,\exp(A)) \le R(A). \tag{3.9}$$

With this knowledge we can show there exists a constant c_D such that:

$$\forall P,\hat{P} \in K \subset \mathrm{SL}(d)\colon D(P,\hat{P}) \le c_D, \qquad D(I,P) \le c_D|P-I|. \tag{3.10}$$

For the first inequality the continuity of D suffices, then the image of D over $K\times K$ is compact and thus bounded in $\mathbb{R}$. Having shown the first inequality, the second one needs only to be proved on a neighborhood of I. We may assume that on this neighborhood the matrix logarithm is well-defined. Taking $A = \log P$ in (3.9) and using $\log P \le c|I-P|$ then gives (3.10).

In the finite case we will install the constraint of global invertibility on the deformations $v_\varepsilon = \mathrm{id} + \varepsilon u$ by the GMS-condition:

$$\int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} \varphi(v_\varepsilon(x))\,\big|\det(\nabla v_\varepsilon(x))\big|\,\mathrm{d}x \le \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y \quad \text{for all } \varphi \in \mathrm{C_c}(\mathbb{R}^d,[0,\infty)). \tag{3.11}$$

The functionals of the stored energy with and without constraint, as well as the elastic and hardening parts we notate as

$$\mathcal{E}_\varepsilon\colon \mathcal{Q} \to [0,\infty], \quad \mathcal{E}_\varepsilon(u,z) := \begin{cases} \widetilde{\mathcal{E}}_\varepsilon(u,z) & \text{if } v_\varepsilon = \mathrm{id}+\varepsilon u \text{ satisfies (3.11)}, \\ \infty & \text{otherwise;} \end{cases}$$

$$\widetilde{\mathcal{E}}_\varepsilon(u,z) := \widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z) + \widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z);$$

$$\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z) := \frac{1}{\varepsilon^2}\int_{\Omega_{\mathrm{Cr}}} W_{\mathrm{el}}\big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\big)\,\mathrm{d}x \quad \text{and}$$

$$\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z) := \frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{h}}(I+\varepsilon z)\,\mathrm{d}x.$$

Note that in the definition of $\widetilde{\mathcal{E}}_\varepsilon$, if we have $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z) < \infty$, then for the plastic tensor we have $F_{\mathrm{pl}} = I + \varepsilon z \in K$ a.e., thus $(I+\varepsilon z)^{-1}$ exists and $\mathcal{E}_{\mathrm{el},\varepsilon}(u,z)$ is well-defined.

In the limit we will obtain the jump condition for local non-interpenetration:

$$[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0. \tag{3.12}$$

The corresponding limit functionals read as

$$\mathcal{E}_0\colon \mathcal{Q} \to [0,\infty], \qquad \mathcal{E}_0(u,z) = \begin{cases} \widetilde{\mathcal{E}}_0(u,z) & \text{if } u \text{ satisfies (3.12) } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}}, \\ \infty & \text{otherwise;} \end{cases}$$

$$\widetilde{\mathcal{E}}_0(u,z) := \widetilde{\mathcal{E}}_{\mathrm{el},0}(u,z) + \widetilde{\mathcal{E}}_{\mathrm{h},0}(z);$$

$$\widetilde{\mathcal{E}}_{\mathrm{el},0}(u,z) = \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}|\nabla u - z|^2_{\mathbb{C}}\,\mathrm{d}x \quad \text{and} \quad \widetilde{\mathcal{E}}_{\mathrm{h},0}(z) = \int_\Omega \frac{1}{2}|z|^2_{\mathbb{H}}\,\mathrm{d}x.$$

Note that this linearized stored energy displays the additive split of the linearized elastic tensor $A_{\mathrm{el}} = \nabla u - z$ instead of the multiplicative split $F_{\mathrm{el},\varepsilon} = (I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}$ in the finite case, which displays a regime change as discussed in Chapter 1.

The rescaled dissipation functions D_ε and functionals $\mathcal{D}_\varepsilon$ we define as:

$$D_\varepsilon(z_1,z_2) := \frac{1}{\varepsilon}D(I+\varepsilon z_1, I+\varepsilon z_2), \qquad D_0(z_1,z_2) = R(z_2 - z_1),$$

$$\mathcal{D}_\varepsilon(z_1,z_2) = \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} D_\varepsilon(z_1,z_2)\,\mathrm{d}x, \qquad \mathcal{D}_0(z_1,z_2) = \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} D_0(z_1,z_2)\,\mathrm{d}x.$$

Finally, for the fixed time step τ we fix a loading $\ell^{(\tau)} \in \mathcal{U}'$ and choose for the sake of simplicity the initial plastic strain to be $\tilde{z}^{(\tau)} = 0$. For different choices of $\tilde{z}^{(\tau)}$ see Remark 3.10. For the total energy defined as

$$\mathcal{G}_\varepsilon^{(\tau)}(u,z) = \mathcal{E}_\varepsilon(u,z) - \langle \ell^{(\tau)}, u\rangle + \mathcal{D}_\varepsilon(0,z) \qquad \text{for } \varepsilon \geq 0, \tag{3.13}$$

this chapter's main result reads as follows.

Theorem 3.1 (Mosco-convergence $\mathcal{G}_\varepsilon^{(\tau)} \xrightarrow{\mathrm{M}} \mathcal{G}_0^{(\tau)}$)**.** *Assume* (3.1)-(3.7) *and let* $\mathcal{G}_\varepsilon^{(\tau)}$ *and* $\mathcal{G}_0^{(\tau)}$ *be defined as in* (3.13) *above. Then* $\mathcal{G}_\varepsilon^{(\tau)}$ *Mosco-converges to* $\mathcal{G}_0^{(\tau)}$ *in* $\mathcal{Q}$.

The proof of Theorem 3.1 may build on some results from Chapter 2, for instance the rigidity of Ω_{Cr} from Corollary 2.6 or the density of the set of close-to-identity injective displacements with bounded gradient $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d)\cap\mathcal{I}$ in the set of displacements that satisfy the jump condition $\mathcal{J}$. Other results have to be adapted to the situation with plasticity, this is the content of Section 3.2. For example Lemma 3.2, which concerns the rotation matrices $R_{\mathrm{id}+\varepsilon u}$ in the rigidity estimate of close-to-identity deformations $\mathrm{id}+\varepsilon u$, and the linearization of the determinant in Lemma 3.4 are variants of Lemmas 2.7 and 2.11, respectively, with $\widetilde{\mathcal{F}}_\varepsilon$ substituted by $\widetilde{\mathcal{E}}_\varepsilon$ on the right-hand sides of the inequalities. Proposition 3.3 gives further a priori estimates in terms of $\widetilde{\mathcal{E}}_\varepsilon$, which include the equi-coercivity of $\widetilde{\mathcal{E}}_\varepsilon$ in $\mathcal{Q} \subset \mathrm{H}^1 \times \mathrm{L}^2$. Finally Theorem 3.5 gives the local non-interpenetration

(1.2) as a limit of the global GMS condition (1.1) in the plastic context, analogously as did Theorem 2.10 in the pure elastic case.

Subsequently Theorem 3.1 is proven in Section 3.3 which provides the Γ- lim inf-estimate on $\mathcal{G}_\varepsilon^{(\tau)}$ in Corollary 3.8 and the Γ- lim sup in Proposition 3.9. The constraints in the Γ- lim inf are taken care of by Theorem 3.5 and the integral quantities of the stored energy $\widetilde{\mathcal{E}}_\varepsilon$ are treated by the lower-semicontinuity tool from Lemma 2.12 where the pointwise estimates are obtained by Lemma 2.13 from the quadratic extensions (3.2d) and (3.4d), see Proposition 3.6. The proof of the Γ- lim inf of $\mathcal{D}_\varepsilon$ in Proposition 3.7 follows [MS13]. The Γ- lim sup is proven by the construction of a recovery sequence, where the results from section 2.5 enable us to restrict to the case of close-to-identity injective displacements with bounded gradients. Then ideas from [MS13] on the separate Γ-convergence of $\widetilde{\mathcal{E}}_\varepsilon$ and $\mathcal{D}_\varepsilon$ are used again to prove a *common recovery sequence* for the sum. The phrasing suites by both meanings of "common", on the one hand a common recovery sequence is a traditional recovery sequence for the sequence $\mathcal{G}_\varepsilon^{(\tau)}$ of functionals (in contrast to a mutual recovery sequence, see Chapter4), on the other hand it is a shared recovery for both the stored energy $\mathcal{E}_\varepsilon$ and the dissipation $\mathcal{D}_\varepsilon(0,\cdot)$. The Chapter is concluded by a remark on the choice of recovery sequence for more general cases of initial plastic strain.

3.2 Preliminaries

The cracked domain Ω_{Cr} is already proven to be a rigidity domain in Corollary 2.6, which will be as crucial for the coercivity of $\widetilde{\mathcal{E}}_\varepsilon$ as it was for $\widetilde{\mathcal{F}}_\varepsilon$ in Chapter 2. Other findings have to be adapted to the plasticity setting. In this section we collect results that have an obvious ananlogon in Chapter 2 and whose proofs base on the previous ideas. For instance in the proof of the next lemma in order to control the distance $\operatorname{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))$ of the deformation gradient $\nabla v_\varepsilon = I+\varepsilon\nabla u$ to $\mathrm{SO}(d)$ the elastic energy density is not enough as in the pure elastic setting in the proof of Lemma 2.7. Instead in (3.15) the combination of the coercivity assumptions (3.2c) on W_{el} and (3.4e) on W_{h} is used to control $\operatorname{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))$ in terms of the sum of elastic and hardening energy densities.

Lemma 3.2. *Let $\Omega, \Gamma_{\mathrm{Cr}}$, W_{el} and W_{h} satisfy the assumptions* (2.8)*,* (4.7) *and* (4.9) *and fix $g_{\mathrm{Dir}} \in \mathrm{W}^{1,\infty}(\Omega)$. Then, there exist constants $c_R, C_R > 0$ such that for all $\varepsilon \in\,]0,1[$ and all $(u,z) \in \mathcal{Q}$ the following holds:*

$$\int_{\Omega_{\mathrm{Cr}}} |I+\varepsilon\nabla u(x) - R_{\mathrm{id}+\varepsilon u}|^2 \,\mathrm{d}x \leq c_R\varepsilon^2 \widetilde{\mathcal{E}}_\varepsilon(u,z), \tag{3.14a}$$

$$|I-R_{\mathrm{id}+\varepsilon u}|^2 \leq C_R\varepsilon^2\Big(\, \widetilde{\mathcal{E}}_\varepsilon(u,z) \,+ \int_{\Gamma_{\mathrm{Dir}}} |g_{\mathrm{Dir}}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{3.14b}$$

where R_v denotes the minimizer $R \in \mathrm{SO}(d)$ in the rigidity estimate (2.16) *in Corollary 2.6 for a fixed deformation $v \in \mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$.*

Proof. We may assume $\widetilde{\mathcal{E}}_\varepsilon(u,z) < \infty$, otherwise the assertions would be satisfied in a trivial sense. In particular we have $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z) < \infty$ and thus $I + \varepsilon z \in K$ for a not relabeled subsequence.

For $v_\varepsilon = \mathrm{id} + \varepsilon u$ and $F_{\mathrm{el}} = \nabla v_\varepsilon (I + \varepsilon z)^{-1}$ we have for every $Q \in \mathrm{SO}(d)$:

$$|\nabla v_\varepsilon - Q|^2 = |\nabla v_\varepsilon - Q(I+\varepsilon z) + Q\varepsilon z|^2 = |(F_{\mathrm{el}} - Q)(I+\varepsilon z) + Q\varepsilon z|^2$$
$$\leq 2|F_{\mathrm{el}} - Q|^2\,|I+\varepsilon z|^2 + 2|Q\varepsilon z|^2 \overset{I+\varepsilon z\in K}{\leq} 2c_K|F_{\mathrm{el}} - Q|^2 + 2|\varepsilon z|^2.$$

We specialize to Q being the minimizer of the right-hand side and obtain:

$$\mathrm{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))^2 \leq |\nabla v_\varepsilon - Q|^2 \leq 2c_K\,\mathrm{dist}(F_{\mathrm{el}}, \mathrm{SO}(d))^2 + 2|\varepsilon z|^2$$
$$\overset{(3.2c),(3.4e)}{\leq} 2\frac{c_K}{c_{W_{\mathrm{el}}}} W_{\mathrm{el}}(F_{\mathrm{el}}) + 2\frac{1}{c_{\mathrm{h}}} W_{\mathrm{h}}(I+\varepsilon z). \tag{3.15}$$

Integrating over Ω_{Cr} and using corollary 2.6 with rigidity constant $\mathcal{R}(\Omega_{\mathrm{Cr}})$ gives the first assertion (3.14a) with $c_R := 2\mathcal{R}(\Omega_{\mathrm{Cr}})\max\{\frac{c_K}{c_{W_{\mathrm{el}}}}, \frac{1}{c_{\mathrm{h}}}\}$:

$$\|I+\varepsilon\nabla u - R_{\mathrm{id}+\varepsilon u}\|^2_{L^2(\Omega_{\mathrm{Cr}})} \leq \mathcal{R}(\Omega_{\mathrm{Cr}})\big\|\mathrm{dist}(\nabla v_\varepsilon, \mathrm{SO}(d))\big\|^2_{L^2(\Omega_{\mathrm{Cr}})}$$
$$\leq c_R \int_{\Omega_{\mathrm{Cr}}} W_{\mathrm{el}}(F_{\mathrm{el}}) + W_{\mathrm{h}}(I+\varepsilon z)\,\mathrm{d}x = c_R\varepsilon^2\widetilde{\mathcal{E}}_\varepsilon(u,z).$$

The second estimate (3.14b) follows from (3.14a) just like (2.17b) follows from (2.17a) in the proof of Lemma 2.7. We set $R_\varepsilon := R_{\mathrm{id}+\varepsilon u}$ and $\zeta_\varepsilon := \fint_{\Omega_{\mathrm{Cr}}} (x+\varepsilon u(x) - R_\varepsilon x)\,\mathrm{d}x$. By continuity of the traces and Poincaré's inequality we find

$$\int_{\Gamma_{\mathrm{Dir}}} |(x+\varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon|^2\,\mathrm{d}\mathcal{H}^{d-1} \leq C_1\big\|(x+\varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon\big\|_{H^1(\Omega_{\mathrm{Cr}},\mathbb{R}^d)}$$
$$\leq C_2\int_{\Omega_{\mathrm{Cr}}} |(I+\varepsilon\nabla u(x)) - R_\varepsilon|^2\,\mathrm{d}x \leq C_3\varepsilon^2\widetilde{\mathcal{E}}_\varepsilon(u,z)$$

with $C_3 = c_R C_2$. Exploiting $u|_{\Gamma_{\mathrm{Dir}}} = g_{\mathrm{Dir}}$ and the prefactor ε we obtain

$$\int_{\Gamma_{\mathrm{Dir}}} |(I-R_\varepsilon)x - \zeta_\varepsilon|^2\,\mathrm{d}\mathcal{H}^{d-1} \leq C_4\varepsilon^2\big(\,\widetilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\mathrm{Dir}}} |g_{\mathrm{Dir}}|^2\,\mathrm{d}\mathcal{H}^{d-1}\big).$$

Note that $R_\varepsilon - I$ is an element of the closed cone K generated by $\mathrm{SO}(d) - I$, on which Lemma 3.3 from [DMNP02] applies (see the derivation of (3.14) therein). Thus

$$|I-R_\varepsilon|^2 \leq C_5 \min_{\zeta\in\mathbb{R}^d}\int_{\Gamma_{\mathrm{Dir}}} |(I-R_\varepsilon)x - \zeta|^2\,\mathrm{d}\mathcal{H}^{d-1},$$

and we arrive at the estimate (2.17b) with $C_R = C_4C_5$. □

This Lemma will help proving the equi-coercivity of $\widetilde{\mathcal{E}}_\varepsilon$ in $\mathcal{Q}$ as it will be used in (3.16) to obtain an H^1-bound on u. For the L^2-bound on z the coercivity (3.4e) of the hardening energy density W_{h} is used. Furthermore compactness of the effective domain K of W_{h} and the inclusion $K \subset \mathrm{SL}(d)$ provide an L^∞-bound on z and an L^1-bound on the trace $\mathrm{tr}\,z$. Finally similarly to Proposition 2.9 by an interpolation argument an L^∞-bound on u is obtained, where the exponent

$$\beta = \frac{2p-2d}{2p-2d+pd}$$

as the convex coefficient of two Sobolev numbers appears. The latter will be much more important in Chapter 4 than in the current one.

Proposition 3.3 (A priori bounds)**.** *Assume that* $\Omega, \Gamma_{\mathrm{Cr}}$*,* W_{el} *and* W_{h} *satisfy the assumptions* (2.8)*,* (4.7) *and* (4.9) *. Then, there exists* $c_{\mathcal{E}}, C_{\mathcal{E}} > 0$ *such that*

$$\forall\, \varepsilon \in\,]0,1[\ \forall\,(u,z) \in \mathcal{Q}:$$
$$\|u\|^2_{\mathrm{H}^1} + \varepsilon^{1-\beta}\|u\|_{\mathrm{L}^\infty} + \|z\|^2_{\mathrm{L}^2} + \varepsilon\|z\|_{\mathrm{L}^\infty} + \varepsilon^{-1}\|\operatorname{tr} z\|_{\mathrm{L}^1} \le c_{\mathcal{E}}\Big(\widetilde{\mathcal{E}}_\varepsilon(u,z) + C_{\mathcal{E}}\Big).$$

Proof. The three bounds on z are proven first, subsequently the bounds on u will be treated. We may assume $\widetilde{\mathcal{E}}_\varepsilon(u,z) < \infty$, otherwise the assertions would be trivially satisfied. In particular we have $I + \varepsilon z \in K$ and thus by (3.5):

$$\|\varepsilon z\|_{\mathrm{L}^\infty} \le \|I + \varepsilon z\|_{\mathrm{L}^\infty} + \|I\|_{\mathrm{L}^\infty} \le c_K + \|I\|_{\mathrm{L}^\infty} \le C_1.$$

Moreover assumption (3.4e) gives

$$\|z\|^2_{\mathrm{L}^2} \le \frac{1}{c_{\mathrm{h}}}\int_{\Omega_{\mathrm{Cr}}} \frac{1}{\varepsilon^2} W_{\mathrm{h}}(I + \varepsilon z)\,\mathrm{d}x \le \frac{1}{c_{\mathrm{h}}}\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z) \le \frac{1}{c_{\mathrm{h}}}\widetilde{\mathcal{E}}_\varepsilon(u,z).$$

Furthermore we use $I{+}\varepsilon z \in K \subset \mathrm{SL}(d)$, which gives $\det(I{+}\varepsilon z) = 1$, and the estimate (2.20) on the linearization of the determinant to obtain:

$$\begin{aligned}
\|\varepsilon \operatorname{tr} z\|_{\mathrm{L}^1} &= \int_{\Omega_{\mathrm{Cr}}} |\operatorname{tr}(\varepsilon z)|\,\mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} |\det(I + \varepsilon z) - 1 - \operatorname{tr}(\varepsilon z)| \\
&\le C_d \int_{\Omega_{\mathrm{Cr}}} (|\varepsilon z|^2 + |\varepsilon z|^d)\,\mathrm{d}x \le C_d\varepsilon^2 \int_{\Omega_{\mathrm{Cr}}} (|z|^2 + \|\varepsilon z\|^{d-2}_{\mathrm{L}^\infty}|z|^2)\,\mathrm{d}x \\
&\le \varepsilon^2 C_d(1 + C_1^{d-2})\|z\|^2_{\mathrm{L}^2} \le \varepsilon^2 C_d(1 + C_1^{d-2})\frac{1}{c_{\mathrm{h}}}\widetilde{\mathcal{E}}_\varepsilon(u,z).
\end{aligned}$$

Hence, the asserted bounds on z are established.

To bound ∇u in L^2 we use both estimates from Lemma 3.2 to obtain

$$\begin{aligned}
\varepsilon^2\|\nabla u\|^2_{L^2} &\le 2\big(\|I{-}R_\varepsilon\|^2_{L^2} + \|I{+}\varepsilon\nabla u - R_{\mathrm{id}+\varepsilon u}\|^2_{L^2}\big) \\
&\le \varepsilon^2 c_1\Big(\widetilde{\mathcal{E}}_\varepsilon(u,z) + \int_{\Gamma_{\mathrm{Dir}}} |g_{\mathrm{Dir}}|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big)
\end{aligned} \tag{3.16}$$

with $c_1 = 2c_R + 2C_R|\Omega|$. Dividing by ε^2 and exploiting the boundary conditions of $u \in \mathcal{U}$ as well as Poincaré's inequality we arrive at

$$\|u\|^2_{\mathrm{H}^1} \le c_2\big(\widetilde{\mathcal{E}}_\varepsilon(u,z) + C_2\big) \tag{3.17}$$

with c_2 obtained from c_1 and the Poincaré constant and $C_2 = \int_{\Gamma_{\mathrm{Dir}}} |g_{\mathrm{Dir}}|^2\,\mathrm{d}\mathcal{H}^{d-1}$.

Finally we turn our attention to the L^∞-bound on u. By Gagliardo-Nirenberg interpolation we have

$$\|\varepsilon u\|_{\mathrm{L}^\infty} \le C_{\mathrm{GN}}\|\varepsilon\nabla u\|^{1-\beta}_{\mathrm{L}^p}\,\|\varepsilon u\|^\beta_{\mathrm{L}^2} + C_{\mathrm{GN}}\|\varepsilon u\|_{\mathrm{L}^2}, \tag{3.18}$$

where $\beta = \frac{2p-2d}{2p-2d+pd} \in (0,1)$ is given by $0 = (1-\beta)(\frac{1}{p} - \frac{1}{d}) + \beta\frac{1}{2}$.

Using the elementary estimate

$$|\varepsilon\nabla u| \le |I + \varepsilon\nabla u| + |I| \le 2\max\{|I + \varepsilon\nabla u|, |I|\} \le 2\big(|I + \varepsilon\nabla u|^p + |I|^p\big)^{\frac{1}{p}}$$

and (3.5), we obtain

$$\frac{1}{2}|\varepsilon\nabla u|^p - |I|^p \le |I + \varepsilon\nabla u|^p = |(I + \varepsilon\nabla u)(I + \varepsilon z)^{-1}(I + \varepsilon z)|^p$$

$$\leq c_K^p |(I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}|^p,$$

from which with assumption (3.2c) we conclude

$$\begin{aligned} W_{\mathrm{el}}\big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\big) &\geq c_{W_{\mathrm{el}}}\big(|(I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}|^p - C_{W_{\mathrm{el}}}\big) \\ &\geq \frac{c_{W_{\mathrm{el}}}}{2c_K^p}\big(|\varepsilon\nabla u|^p - 2|I|^p - C_{W_{\mathrm{el}}}\big). \end{aligned}$$

Integrating over Ω_{Cr} and reordering gives

$$\|\varepsilon\nabla u\|_{\mathrm{L}^p}^p \leq c_3(\varepsilon^2\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_3) \overset{\varepsilon\leq 1}{\leq} c_3(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_3) \tag{3.19}$$

with $c_3 = \frac{2c_K^p}{c_{W_{\mathrm{el}}}}$ and $C_3 = c_3^{-1}\int_{\Omega_{\mathrm{Cr}}}(2|I|^p - C_{W_{\mathrm{el}}})\,\mathrm{d}x$.

Inserting (3.17) and (3.19) into (3.18) finally gives:

$$\begin{aligned} \|\varepsilon u\|_{\mathrm{L}^\infty} &\leq C_{\mathrm{GN}}\big(c_3(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_3)\big)^{\frac{1-\beta}{p}}\big(c_2\varepsilon(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_2)\big)^\beta + C_{\mathrm{GN}}c_2(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_2) \\ &\leq c_4(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_4)\,\varepsilon^\beta \end{aligned}$$

with $c_1 = 2C_{\mathrm{GN}}\max\{c_2,c_3\}$ and $C_1 = 2C_{\mathrm{GN}}\max\{C_2,C_3\}$. □

The following Lemma gives a linearization result for the determinant $\det(I+\varepsilon\nabla u)$ of the deformation gradient. As its analogon Lemma 2.11 in Chapter 2 it contributes to the proof of infinitesimal non-interpenetration in Theorem 3.5.

Lemma 3.4. *There exists $C_{\mathrm{det}}{>}0$ depending on Ω, Γ_{Dir}, Γ_{Cr}, the exponent $p>d$, constants $c_{W_{\mathrm{el}}}, C_{W_{\mathrm{el}}}$ from assumption* (3.2c) *and c_K from* (3.5) *such that*

$$\begin{aligned} &\forall\,\varepsilon\in\,]0,1[\;\;\forall\,(u,z)\in\mathcal{Q}: \\ &\int_{\Omega_{\mathrm{Cr}}}\big|\det\big(I{+}\varepsilon\nabla u(x)\big) - 1{-}\varepsilon\,\mathrm{div}\,u(x)\big|\,\mathrm{d}x \leq \varepsilon^2\,C_{\mathrm{det}}\big(\widetilde{\mathcal{E}}_\varepsilon(u,z)+C_{\mathrm{det}}\big). \end{aligned} \tag{3.20}$$

Proof. We may assume $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z){<}\infty$ since for $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z){=}\infty$ the inequality (3.20) is satisfied trivially.

Recall (2.22) from the proof of Lemma 2.11 which holds for W_{el} as it did for W in Chapter 2 because W_{el} satisfies (3.2c) :

$$\big|\det(I{+}A) - (1{+}\,\mathrm{tr}\,A)\big| \leq C_1\big(|A|^2 + W_{\mathrm{el}}(I{+}A)\big).$$

Inserting

$$A = A_\varepsilon = \big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1} - I\big) = \varepsilon(\nabla u - z)(I+\varepsilon z)^{-1},$$

integrating over Ω_{Cr} and using Proposition 3.3 we arrive at:

$$\begin{aligned} &\int_{\Omega_{\mathrm{Cr}}}\big|\det\big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\big) - 1{-}\,\mathrm{tr}\,\big(\varepsilon(\nabla u - z)(I+\varepsilon z)^{-1}\big)\big|\,\mathrm{d}x \\ &\qquad\leq C_1\Big(\|\varepsilon(\nabla u - z)(I+\varepsilon z)^{-1}\|_{\mathrm{L}^2}^2 + \varepsilon^2\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z)\Big) \\ &\qquad\leq C_1\varepsilon^2\Big(c_K^2 2(\|u\|_{\mathrm{H}^1}^2 + \|z\|_{\mathrm{L}^2}^2) + \widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z)\Big) \\ &\qquad\leq C_2\varepsilon^2\big(\widetilde{\mathcal{E}}_\varepsilon(u,z) + C_{\mathcal{E}}\big). \end{aligned} \tag{3.21}$$

Regarding the left-hand side on one hand by $I+\varepsilon z \in K \subset \mathrm{SL}(d)$ we have

$$\det\left((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\right) = \det\left(I+\varepsilon\nabla u\right)\det\left((I+\varepsilon z)\right)^{-1} = \det\left(I+\varepsilon\nabla u\right), \tag{3.22}$$

on the other hand we can estimate

$$\begin{aligned}
&\left|\operatorname{tr}\left(\varepsilon(\nabla u - z)(I+\varepsilon z)^{-1}\right) - \operatorname{tr}(\varepsilon\nabla u - \varepsilon z)\right| \\
&\qquad = \left|\operatorname{tr}\left((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1} - I\right) - \operatorname{tr}(\varepsilon\nabla u - \varepsilon z)\right| \\
&\qquad \le \left|(I+\varepsilon\nabla u)(I+\varepsilon z)^{-1} - I - \varepsilon\nabla u + \varepsilon z\right| \\
&\qquad \le \left|I+\varepsilon\nabla u - (I+\varepsilon\nabla u - \varepsilon z)(I+\varepsilon z)\right| \left|(I+\varepsilon z)^{-1}\right| \\
&\qquad \le c_K\left|I+\varepsilon\nabla u - (I+\varepsilon\nabla u - \varepsilon z) - (I+\varepsilon\nabla u - \varepsilon z)\varepsilon z\right| \\
&\qquad = c_K\left|\varepsilon z - (I+\varepsilon\nabla u - \varepsilon z)\varepsilon z\right| \le c_K\varepsilon^2\left|\nabla u - z\right| |z|,
\end{aligned}$$

which integrated over Ω_{Cr} gives

$$\begin{aligned}
\left\|\operatorname{tr}\left((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1} - I\right) - \operatorname{tr}(\varepsilon\nabla u - \varepsilon z)\right\|_{\mathrm{L}^1} &\le c_K\varepsilon^2\|\nabla u - z\|_{\mathrm{L}^2}\|z\|_{\mathrm{L}^2} \\
&\le \varepsilon^2 c_K 2c_{\mathcal{E}}\left(\widetilde{\mathcal{E}}_\varepsilon(u,z) + C_{\mathcal{E}}\right).
\end{aligned} \tag{3.23}$$

Finally we arrive at the assertion by combining (3.21), (3.22), (3.23) and Proposition 3.3:

$$\begin{aligned}
&\int_{\Omega_{\mathrm{Cr}}} \left|\det\left(I+\varepsilon\nabla u(x)\right) - 1 - \varepsilon \operatorname{div} u(x)\right| \mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} \left|\det\left(I+\varepsilon\nabla u(x)\right) - 1 - \operatorname{tr}\left(\varepsilon\nabla u(x)\right)\right| \mathrm{d}x \\
&\quad \le \int_{\Omega_{\mathrm{Cr}}} \left|\det\left((I+\varepsilon\nabla u(x))(I+\varepsilon z)^{-1}\right) - 1 - \operatorname{tr}\left(\varepsilon(\nabla u - z)(I+\varepsilon z)^{-1}\right)\right| \mathrm{d}x \\
&\qquad\qquad + \int_{\Omega_{\mathrm{Cr}}} \left|\operatorname{tr}\left(\varepsilon(\nabla u - z)(I+\varepsilon z)^{-1}\right) - \operatorname{tr}(\varepsilon\nabla u - \varepsilon z)\right| \mathrm{d}x + \int_{\Omega_{\mathrm{Cr}}} \left|\operatorname{tr}(\varepsilon z)\right| \mathrm{d}x \\
&\quad \le \varepsilon^2(C_2 + c_K 2c_{\mathcal{E}} + c_{\mathcal{E}})\left(\widetilde{\mathcal{E}}_\varepsilon(u,z) + C_{\mathcal{E}}\right).
\end{aligned}$$

□

With Lemma 3.4 adapted to the situation with plastic strain the proof of the following Theorem 3.5 follows closely along the lines of that of Theorem 2.10.

Theorem 3.5 (Infinitesimal non-interpenetration for plasticity)**.** *Let* $(u_\varepsilon, z_\varepsilon), (u_0, z_0) \in \mathcal{Q}$ *with* $u_\varepsilon \xrightarrow{\mathcal{Q}} u_0$ *weakly and* $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$.

Then $[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0$ *holds.*

Proof. As $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$ there is a (not relabeled) subsequence u_ε such that $v_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ fulfills the GMS-condition (3.11) and $\det(I+\varepsilon\nabla u_\varepsilon) > 0$ a.e. on Ω. Hence, by rearranging (3.11) for every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $\varphi \ge 0$ we obtain:

$$\begin{aligned}
0 &\ge \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} \varphi(x+\varepsilon u_\varepsilon(x))\det\left(I+\varepsilon\nabla u_\varepsilon(x)\right) - \varphi(x)\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} \varphi(x+\varepsilon u_\varepsilon(x))\Big(\det\left(I+\varepsilon\nabla u_\varepsilon(x)\right) - \left(1+\varepsilon \operatorname{div} u_\varepsilon(x)\right)\Big)\,\mathrm{d}x
\end{aligned}$$

$$+\int_{\Omega_{\mathrm{Cr}}}\varphi(x+\varepsilon u_\varepsilon(x))\operatorname{div}u_\varepsilon(x)\,\mathrm{d}x+\int_{\Omega_{\mathrm{Cr}}}\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_\varepsilon(x))-\varphi(x)\Big)\,\mathrm{d}x$$
$$=: I_1+I_2+I_3.$$

The first summand I_1 on the right-hand side converges to 0 for $\varepsilon\to 0$ by Lemma 3.4 and boundedness of φ:

$$|I_1|\le\frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}}\Big|\varphi(x+\varepsilon u_\varepsilon(x))\Big(\det\big(I+\varepsilon\nabla u_\varepsilon(x)\big)-\big(1+\varepsilon\operatorname{div}u_\varepsilon(x)\big)\Big)\Big|\,\mathrm{d}x$$
$$\le\varepsilon\|\varphi\|_{\mathrm{L}^\infty}C_{\mathrm{det}}(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon,z_\varepsilon)+C_{\mathrm{det}}).$$

The remaining terms are treated as in the analogous proof of Proposition 2.10 in Chapter 2. The second summand $I_2=\int_{\Omega_{\mathrm{Cr}}}\varphi(x+\varepsilon u_\varepsilon(x))\operatorname{div}u_\varepsilon(x)\,\mathrm{d}x$ converges to

$$\int_{\Omega\setminus\Gamma_c}\varphi(x)\operatorname{div}u_0(x)\,\mathrm{d}x,$$

as $\operatorname{div}u_\varepsilon\overset{\mathrm{L}^2}{\rightharpoonup}\operatorname{div}u_0$ weakly and $\varphi(\mathrm{id}+\varepsilon u_\varepsilon)\overset{\mathrm{L}^2}{\to}\varphi$ strongly by Lipschitz continuity of φ:

$$\|\varphi(\mathrm{id}+\varepsilon u_\varepsilon)-\varphi\|_{\mathrm{L}^2}^2=\int_{\Omega_{\mathrm{Cr}}}\Big|\varphi(x-\varepsilon u_\varepsilon(x))-\varphi(x)\Big|^2\,\mathrm{d}x\le\varepsilon^2\|\nabla\varphi\|_{\mathrm{L}^\infty}^2\int_{\Omega_{\mathrm{Cr}}}|u_\varepsilon(x)|^2\,\mathrm{d}x.$$

Finally, for the integrand of third term $I_3=\int_{\Omega_{\mathrm{Cr}}}\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_\varepsilon(x))-\varphi(x)\Big)\,\mathrm{d}x$ we have pointwise convergence

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_\varepsilon(x))-\varphi(x)\Big)\\&\quad=\underbrace{\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_\varepsilon(x))-\varphi(x+\varepsilon u_0(x))\Big)}_{|\cdot|\le\|\nabla\varphi\|_{\mathrm{L}^\infty}|u_\varepsilon(x)-u_0(x)|\to 0}+\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_0(x)-\varphi(x)\Big)\\&\quad\to\nabla\varphi(x)u_0(x)\end{aligned}$$

as well as the dominating bound

$$\frac{1}{\varepsilon}\Big(\varphi(x+\varepsilon u_\varepsilon(x))-\varphi(x)\Big)\le\|\nabla\varphi\|_{\mathrm{L}^\infty}|u_\varepsilon(x)|=:g_\varepsilon(x).\tag{3.24}$$

By the convergence $u_\varepsilon\rightharpoonup u_0$ weakly in H^1 we have $u_\varepsilon\to u_0$ strongly in L^2, such that for the dominating bound g_ε we obtain $g_\varepsilon\to\|\nabla\varphi\|_{\mathrm{L}^\infty}|u_0(x)|$ strongly in L^1. Hence, by the generalized Lebesgue dominated convergence theorem we have the convergence of the integral $I_3\to\int_{\Omega_{\mathrm{Cr}}}\nabla\varphi(x)\cdot u_0(x)\,\mathrm{d}x$.

Altogether the limit $\varepsilon\to 0$ provides three limit values on the right-hand side, namely

$$\begin{aligned}0\ge\ 0\ +&\int_{\Omega_{\mathrm{Cr}}}\varphi(x)\operatorname{div}u_0(x)\,\mathrm{d}x+\int_{\Omega_{\mathrm{Cr}}}\nabla\varphi(x)\cdot u_0(x)\,\mathrm{d}x\\=&\int_{\Omega_{\mathrm{Cr}}}\operatorname{div}(\varphi u_0)(x)\,\mathrm{d}x=-\int_{\Gamma_{\mathrm{Cr}}}\varphi(x)\,[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}(x)\,\mathrm{d}a(x).\end{aligned}$$

For the last identity we now restricted to $\varphi\in\mathrm{C_c}(\Omega)$ such that no boundary terms on $\partial\Omega$ are present. Moreover, we have to recall that u lies in $\mathcal{U}\subset\mathrm{H}^1(\Omega_{\mathrm{Cr}};\mathbb{R}^d)$ such that the upper and lower traces at the crack Γ_{Cr} may be different. Applying the divergence theorem on the Lipschitz sets A_+ and $A_-\setminus A_+$ (see (2.9)) separately, all terms cancel except for the jump along Γ_{Cr}. As $\varphi\ge 0$ was arbitrary, we conclude $[\![u]\!]_{\Gamma_{\mathrm{Cr}}}\ge 0$ $\mathcal{H}^{d-1}$-a.e. on Γ_{Cr}. □

3.3 Lower and upper Gamma-limit

In this section we will prove the Mosco convergence $\mathcal{G}_\varepsilon^{(\tau)} \overset{\mathrm{M}}{\to} \mathcal{G}_0^{(\tau)}$. For the Γ-$\liminf$-inequality in the weak $\mathrm{H}^1\times\mathrm{L}^2$-topology on $\mathcal{Q}$ we can argue on the summands, namely stored energy $\mathcal{E}_\varepsilon$, dissipation $\mathcal{D}_\varepsilon$ and external work $-\langle \ell, u\rangle$ separately. The following proposition will deal with the of the stored energy. The constraint part in Γ-$\liminf \mathcal{E}_\varepsilon \geq \mathcal{E}_0$ is already treated by Theorem 3.5, thus it remains to prove the $\liminf$ estimate for the integral part $\widetilde{\mathcal{E}}_\varepsilon$, which is formulated explicitly in the following proposition for future reference in Chapter 4, where $\widetilde{\mathcal{E}}_\varepsilon$ will be combined with a different constraint. The proof uses Lemma 2.12 on elastic $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}$ and hardening part $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}$ separately, where the pointwise estimates on the densities are provided by Lemma 2.13 by the quadratic expansions (3.2d) and (3.4d).

Proposition 3.6 ($\Gamma-\liminf$ for stored energy)**.** *Assume* (3.2) *and* (3.4)*. Then for every sequence* $(u_\varepsilon, z_\varepsilon) \rightharpoonup (u_0, z_0)$ *weakly in* $\mathcal{Q}$ *we have*

(a) $\widetilde{\mathcal{E}}_0(u_0, z_0) \leq \liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon)$;

(b) $\mathcal{E}_0(u_0, z_0) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon)$.

Proof. Let us first consider the $\liminf$-estimate from (b). We may assume without loss of generality $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$, otherwise the inequality holds true trivially. Thus a subsequence $v_\varepsilon = \mathrm{id}+\varepsilon u_\varepsilon$ fulfills the GMS-condition (3.11) and by Theorem 3.5 we get $[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$. Hence, the inequality (b) reduces to the one without the constraints:

$$\mathcal{E}_0(u_0, z_0) = \widetilde{\mathcal{E}}_0(u_0, z_0) \leq \liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon).$$

We are left to show (a). Again we may assume without loss of generality that

$$\liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty.$$

We will employ the lower-semicontinuity tool Lemma 2.12 for the elastic and the hardening terms separately, hence we need a pointwise estimate for the respective choice of functions f_ε, convexity of the limit function f_0 and weak L^1 convergence of tensors w_ε to conclude the $\liminf$ estimate for the functionals $\int_{\Omega_{\mathrm{Cr}}} f_\varepsilon(w_\varepsilon)\,\mathrm{d}x$. For $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}$ we take $f_\varepsilon = \frac{1}{\varepsilon^2} W_{\mathrm{el}}(I+\varepsilon\cdot)$ and by assumption (3.2d) Lemma 2.13 gives the pointwise limit

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W_{\mathrm{el}}(I+\varepsilon A) = \frac{1}{2}|A|_{\mathbb{C}}^2, \tag{3.25}$$

where $f_0 = \frac{1}{2}|\cdot|_{\mathbb{C}}^2$ is convex. The role of w_ε from Lemma 2.12 is taken over by the linearized elastic tensor

$$A_\varepsilon = \frac{1}{\varepsilon}\Big((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon z_\varepsilon)^{-1} - I\Big).$$

To investigate the convergence of A_ε let us consider the inverse of the plastic part $(I+\varepsilon z_\varepsilon)^{-1}$ first. On one hand, by (4.10) we have the L^∞-bound

$$\|(I+\varepsilon z_\varepsilon)^{-1} - (I-\varepsilon z_\varepsilon)\|_{\mathrm{L}^\infty} \leq c_K$$

on the other hand rewriting

$$\begin{aligned}(I+\varepsilon z_\varepsilon)^{-1}-(I-\varepsilon z_\varepsilon) &= (I+\varepsilon z_\varepsilon)^{-1}(I-(I+\varepsilon z_\varepsilon)(I-\varepsilon z_\varepsilon)) \\ &= \varepsilon^2(I+\varepsilon z_\varepsilon)^{-1}z_\varepsilon^2\end{aligned}$$

gives an L^1-bound

$$\|(I+\varepsilon z_\varepsilon)^{-1}-(I-\varepsilon z_\varepsilon)\|_{\mathrm{L}^1} \leq \varepsilon^2 c_K \|z_\varepsilon\|_{\mathrm{L}^2}^2.$$

Together, this gives a bound on the L^2-norm of $d_\varepsilon := \frac{1}{\varepsilon}\big((I+\varepsilon z_\varepsilon)^{-1}-(I-\varepsilon z_\varepsilon)\big)$:

$$\|d_\varepsilon\|_{\mathrm{L}^2}^2 \leq \|d_\varepsilon\|_{\mathrm{L}^1}\|d_\varepsilon\|_{\mathrm{L}^\infty} \leq \frac{1}{\varepsilon^2}\varepsilon^2 c_K^2 \|z_\varepsilon\|_{\mathrm{L}^2}^2 \leq C.$$

In particular a subsequence of d_ε converges weakly in L^2 to some limit and since the above L^1-bound means $d_\varepsilon \xrightarrow{\mathrm{L}^1} 0$, the limits have to coincide and we get:

$$\frac{1}{\varepsilon}\big((I+\varepsilon z_\varepsilon)^{-1}-I\big) \rightharpoonup -z_0 \qquad \text{in } \mathrm{L}^2(\Omega,\mathbb{R}^{d\times d}).$$

Using d_ε we can rewrite

$$\begin{aligned}A_\varepsilon - \nabla u_\varepsilon + z_\varepsilon &= \frac{1}{\varepsilon}\big((I+\varepsilon\nabla u_\varepsilon)(I-\varepsilon z_\varepsilon+\varepsilon d_\varepsilon)-I\big) - \nabla u_\varepsilon + z_\varepsilon \\ &= d_\varepsilon + \varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon).\end{aligned}$$

The weak L^2-convergence of the first summand $d_\varepsilon \rightharpoonup 0$ on the right-hand side is already established. Furthermore on the one hand the established L^∞-bound on $\varepsilon d_\varepsilon$ and (3.5) give weak L^2-convergence of the second summand by the bound

$$\|\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon)\|_{\mathrm{L}^2} \leq \|\nabla u_\varepsilon\|_{\mathrm{L}^2}(\|\varepsilon d_\varepsilon\|_{\mathrm{L}^\infty} + \|\varepsilon z_\varepsilon\|_{\mathrm{L}^\infty}) \leq C,$$

and on the other hand we can estimate

$$\|\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon)\|_{\mathrm{L}^1} \leq \varepsilon\|\nabla u_\varepsilon\|_{L^2}(\|d_\varepsilon\|_{L^2} + \|z_\varepsilon\|_{L^2}) \leq c\varepsilon,$$

which gives L^1-convergence to 0. Hence again the limits have to coincide, such that $\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon) \overset{\mathrm{L}^2}{\rightharpoonup} 0$ follows and we get:

$$A_\varepsilon \rightharpoonup \nabla u_0 - z_0 \text{ in } \mathrm{L}^2(\Omega_{\mathrm{Cr}},\mathbb{R}^{d\times d}).$$

Finally for the elastic parts of the stored energy Lemma 2.12 gives:

$$\widetilde{\mathcal{E}}_{\mathrm{el},0}(u_0,z_0) = \int_{\Omega_{\mathrm{Cr}}} f_0(w_0)\,\mathrm{d}x \leq \liminf_{\varepsilon\to0} f_\varepsilon(w_\varepsilon)\,\mathrm{d}x = \liminf_{\varepsilon\to0} \widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u_\varepsilon,z_\varepsilon).$$

Let us come to the hardening part of the stored energy. Since we assumed the finiteness $\liminf_{\varepsilon\to0}\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z_\varepsilon) < \infty$, we have

$$\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z_\varepsilon) = \int_\Omega \frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\,\mathrm{d}x = \int_\Omega \frac{1}{\varepsilon^2}\widetilde{W}_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\,\mathrm{d}x.$$

Thus, we take $f_\varepsilon = \frac{1}{\varepsilon^2}\widetilde{W}_\mathrm{h}(I + \varepsilon\cdot)$ and again the assumption of quadratic extension (3.4d) gives the pointwise equality

$$\lim_{\varepsilon\to 0} f_\varepsilon(A) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\widetilde{W}_\mathrm{h}(I + \varepsilon A) = \frac{1}{2}|A|^2_{\mathbb{H}} = f_0(A), \tag{3.26}$$

with convex limit $f_0 = \frac{1}{2}|\cdot|^2_{\mathbb{C}}$. For $w_\varepsilon = z_\varepsilon$ the weak L^1-convergence of w_ε follows from weak L^2 convergence $z_\varepsilon \rightharpoonup z_0$ and we get by the lower-semicontinuity tool Lemma 2.12:

$$\widetilde{\mathcal{E}}_{\mathrm{h},0}(z_0) = \int_{\Omega_\mathrm{Cr}} f_0(w_0)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} f_\varepsilon(w_\varepsilon)\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z_\varepsilon).$$

□

The Γ-$\liminf$-inequality of $z \mapsto \mathcal{D}_\varepsilon(0, z)$ in L^2 obviously follows from the Γ-$\liminf$-inequality of $\mathcal{D}_\varepsilon$ on $\mathrm{L}^2\times\mathrm{L}^2$. The more general version is proven in the following proposition. This way we can reference it in Chapter 4. The prove follows ideas from [MS13].

Proposition 3.7 ($\Gamma-\liminf$ for dissipation)**.** *Let assumption* (3.7) *hold and let* $(z_\varepsilon, \hat{z}_\varepsilon) \rightharpoonup (z_0, \hat{z}_0)$ *converge weakly in* $\big(\mathrm{L}^2(\Omega_\mathrm{Cr}, \mathbb{R}^{d\times d})\big)^2$. *Assume additionally* $\operatorname{tr}\hat{z}_0 = \operatorname{tr} z_0 = 0$ *and*

$$\sup_\varepsilon \big\|(I + \varepsilon z_\varepsilon)^{-1}\big\|_{\mathrm{L}^\infty} < \infty. \tag{3.27}$$

Then we have

$$\mathcal{D}_0(z_0, \hat{z}_0) \le \liminf_{\varepsilon\to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon).$$

Proof. Assume we had shown the pointwise $\liminf$ estimate on the functions

$$D_0(z_0, \hat{z}_0) = R(\hat{z}_0 - z_0) \le \liminf_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon). \tag{3.28}$$

Then we can use the lower semicontinuity tool Lemma 2.12 with $f_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon)$, $f_0(z_\varepsilon, \hat{z}_\varepsilon) = R(\hat{z}_\varepsilon - z_\varepsilon)$, which is convex, and $w_\varepsilon = (z_\varepsilon, \hat{z}_\varepsilon)$ weakly L^1-converging by weak L^2-convergence $(z_\varepsilon, \hat{z}_\varepsilon) \rightharpoonup (z_0, \hat{z}_0)$ to conclude the $\liminf$ estimate of the functionals:

$$\mathcal{D}_0(z_0, \hat{z}_0) = \int_{\Omega_\mathrm{Cr}} R(\hat{z}_0, z_0)\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} \int_{\Omega_\mathrm{Cr}} D_\varepsilon(z_0, \hat{z}_0)\,\mathrm{d}x = \liminf_{\varepsilon\to 0} \mathcal{D}_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon).$$

Thus we are left to show (3.28). For the remainder of the proof let under slight abuse of notation $z_\varepsilon, \hat{z}_\varepsilon \in \mathbb{R}^{d\times d}$ be converging matrices not weakly converging functions.

We may assume $\liminf\limits_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = \lim\limits_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) < \infty$, otherwise the inequality would be satisfied trivially. In particular, by passing to a subsequence we can assume for every $\varepsilon > 0$

$$D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = D\big(I, (I + \varepsilon\hat{z}_\varepsilon)(I + \varepsilon z_0)^{-1}\big) < \infty$$

and thus $(I + \varepsilon\hat{z}_\varepsilon)(I + \varepsilon z_0)^{-1} \in \mathrm{SL}(d)$. Set

$$\zeta_\varepsilon := \frac{1}{\varepsilon}\big((I + \varepsilon\hat{z}_\varepsilon)(I + \varepsilon z_0)^{-1} - I\big),$$

which converges to $\hat{z}_0 - z_0$

$$\big|\zeta_\varepsilon - (\hat{z}_0 - z_0)\big| = \Big|\frac{1}{\varepsilon}(I + \varepsilon\hat{z}_\varepsilon)(I + \varepsilon z_0)^{-1} - I) - (\hat{z}_0 - z_0)\Big|$$

$$\le \big|(\hat z_\varepsilon - z_\varepsilon) - (\hat z_0 - z_0) + \varepsilon(\hat z_0 - z_0) z_\varepsilon\big|\,|(I+\varepsilon z_0)^{-1}|$$
$$\le \Big(|\hat z_\varepsilon - \hat z_0| + |z_\varepsilon - z_0| + \varepsilon|\hat z_0 - z_0|\,|z_\varepsilon|\Big)|(I+\varepsilon z_0)^{-1}| \to 0,$$

where we used $|\hat z_\varepsilon - \hat z_0| + |z_\varepsilon - z_0| \to 0$ and $\varepsilon|\hat z_0 - z_0|\,|z_\varepsilon| < C\varepsilon$ by convergence $(\hat z_\varepsilon, z_\varepsilon) \to (\hat z_0, z_0)$ as well as the boundedness of $(I+\varepsilon z_0)^{-1}$ by the additional assumption (3.27).

Furthermore, we get $\hat z_0 - z_0 \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ from $\det(I+\varepsilon\zeta_\varepsilon) = 1$, since by linearizing the determinant as in (2.20) we see:

$$\big|\operatorname{tr}(\hat z_0 - z_0)\big| = \lim_{\varepsilon\to 0}\big|\operatorname{tr}\zeta_\varepsilon\big| = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big|\det(I+\varepsilon\zeta_\varepsilon) - 1 - \operatorname{tr}(\varepsilon\zeta_\varepsilon)\big|$$
$$\le \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}C_d(\varepsilon^2|\zeta|^2 + \varepsilon^d|\zeta|^d) \le \lim_{\varepsilon\to 0}\varepsilon C = 0.$$

By the definition of D in terms of an infimum there exists $P_\varepsilon \in \mathrm{C}^1(]0,1[,\mathbb{R}^{d\times d})$ with $P_\varepsilon(0) = I$ and $P_\varepsilon(1) = I + \varepsilon\zeta_\varepsilon$, such that

$$D(I, I+\varepsilon\zeta_\varepsilon) \ge (1-\varepsilon)\int_0^1 R(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t = (1-\varepsilon)\int_0^1 R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t. \tag{3.29}$$

Since the right-hand side is invariant under reparametrization of P_ε by the 1-homogeneity (3.7a) of R^{dev} we will pass to a suitable reparametrization. Consider

$$s(t) = \frac{1}{\int_0^1 R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t' + D(I, I+\varepsilon\zeta_\varepsilon)}\Big(\int_0^t R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t' + tD(I, I+\varepsilon\zeta_\varepsilon)\Big),$$

where $s(0) = 0$ and $s(1) = 1$. On the one hand, if $D(I, I+\varepsilon\zeta_\varepsilon) > 0$, we have

$$s'(t) = \frac{R^{\mathrm{dev}}(\dot P_\varepsilon(t)P_\varepsilon^{-1}(t)) + D(I, I+\varepsilon\zeta_\varepsilon)}{\int_0^1 R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t' + D(I, I+\varepsilon\zeta_\varepsilon)} > 0,$$

such that $s^{-1} \in \mathrm{C}^1([0,1])$ with $s^{-1}(0) = 0$ and $s^{-1}(1) = 1$ is a valid parametrization and

$$Q_\varepsilon\colon t \mapsto P_\varepsilon(s^{-1}(t))$$

gives by 1-homogeneity:

$$R^{\mathrm{dev}}(\dot Q_\varepsilon(t)Q_\varepsilon^{-1}(t)) = R^{\mathrm{dev}}\Big((s^{-1})'(t)\dot P_\varepsilon(s^{-1}(t))P_\varepsilon^{-1}(s^{-1}(t))\Big)$$
$$= \frac{1}{s'(s^{-1}(t))}R^{\mathrm{dev}}\Big(\dot P_\varepsilon(s^{-1}(t))P_\varepsilon^{-1}(s^{-1}(t))\Big)$$
$$= \frac{\Big(\int_0^1 R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t' + D(I, I+\varepsilon\zeta_\varepsilon)\Big)R^{\mathrm{dev}}\Big(\dot P_\varepsilon(s^{-1}(t))P_\varepsilon^{-1}(s^{-1}(t))\Big)}{R^{\mathrm{dev}}\Big(\dot P_\varepsilon(s^{-1}(t))P_\varepsilon^{-1}(s^{-1}(t))\Big) + D(I, I+\varepsilon\zeta_\varepsilon)}$$
$$\le \int_0^1 R^{\mathrm{dev}}(\dot P_\varepsilon P_\varepsilon^{-1})\,\mathrm{d}t' + D(I, I+\varepsilon\zeta_\varepsilon)$$
$$\overset{(3.29)}{\le} \frac{2-\varepsilon}{1-\varepsilon}D(I, I+\varepsilon\zeta_\varepsilon) \overset{(3.10)}{\le} 3c_D|\varepsilon\zeta_\varepsilon| \le c_1\varepsilon.$$

If on the other hand $D(I, I+\varepsilon\zeta_\varepsilon) = 0$, from $R \ge 0$ we would get $R(\dot P_\varepsilon P_\varepsilon^{-1}) = 0$ for all $t \in [0,1]$ and the above inequality would still hold by $0 \le c_1\varepsilon$.

Hence, by either reparametrizing $Q_\varepsilon := P_\varepsilon \circ s^{-1}$ in the case $D(I, I+\varepsilon\zeta_\varepsilon) > 0$ or by renaming $Q_\varepsilon := P_\varepsilon$ in the case $D(I, I+\varepsilon\zeta_\varepsilon) = 0$, we have

$$\|\dot{Q}_\varepsilon Q_\varepsilon^{-1}\|_{\mathrm{L}^\infty(0,1)} \leq \frac{1}{\underline{c}_R}\|R^{\mathrm{dev}}(\dot{Q}_\varepsilon Q_\varepsilon^{-1})\|_{\mathrm{L}^\infty(0,1)} \leq c_1\varepsilon, \tag{3.30}$$

and thus

$$\begin{aligned}|Q_\varepsilon(t) - I| = \Big|\int_0^t \dot{Q}_\varepsilon \,\mathrm{d}t'\Big| &\leq \int_0^t |\dot{Q}_\varepsilon Q_\varepsilon^{-1}|\,|Q_\varepsilon|\,\mathrm{d}t' \leq c_1\varepsilon \int_0^t |Q_\varepsilon|\,\mathrm{d}t' \\ &\leq c_1\varepsilon(1 + \int_0^t |Q_\varepsilon - I|\,\mathrm{d}t'),\end{aligned}$$

from which by the Gronwall Lemma we get the uniform convergence $Q_\varepsilon \to I$:

$$|Q_\varepsilon(t) - I| \leq c_1\varepsilon\exp(c_1\varepsilon t) \leq c_1\varepsilon\exp(c_1\varepsilon) \leq c_2\varepsilon. \tag{3.31}$$

Let us further consider the rescaled trajectory

$$\hat{Q}_\varepsilon := I + \frac{1}{\varepsilon}(Q_\varepsilon - I),$$

with $\hat{Q}_\varepsilon(0) = I$ and $\hat{Q}_\varepsilon(1) = I + \zeta_\varepsilon$. On one hand by $\dot{\hat{Q}}_\varepsilon = \frac{1}{\varepsilon}\dot{Q}_\varepsilon$ and the 1-homogeneity of R^{dev} from (3.29) we get

$$\frac{1}{\varepsilon}D(I, I+\varepsilon\zeta_\varepsilon) \geq (1-\varepsilon)\int_0^1 R^{\mathrm{dev}}(\dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1})\,\mathrm{d}t, \tag{3.32}$$

on the other hand by (3.30) we have $\dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1}$ bounded in L^∞, so it converges weakly-* in L^∞ and in particular weakly in L^1 to some limit Q and the lower-semicontinuity tool Lemma 2.12 gives

$$\liminf_{\varepsilon\to 0}\int_0^1 R^{\mathrm{dev}}(\dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1})\,\mathrm{d}t \geq \int_0^1 R^{\mathrm{dev}}(Q)\,\mathrm{d}t \geq R^{\mathrm{dev}}\Big(\int_0^1 Q\,\mathrm{d}t\Big), \tag{3.33}$$

where in the last step we used Jensen's inequality.

By integrating we have

$$\int_0^1 Q\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^1 \dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1}\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_0^1 \dot{\hat{Q}}_\varepsilon\,\mathrm{d}t + \lim_{\varepsilon\to 0}\int_0^1 \dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1}(I - Q_\varepsilon)\,\mathrm{d}t,$$

where on one hand the integral in the second limit is bounded by (3.30) and (3.31) by $c_1c_2\varepsilon$, thus converges to 0, and on the other hand the integral in the first limit equals $\hat{Q}_\varepsilon(1) - \hat{Q}_\varepsilon(0) = \zeta_\varepsilon$ and thus converges to $\hat{z}_0 - z_0$. Hence we have $\int_0^1 Q\,\mathrm{d}t = \hat{z}_0 - z_0$, which we combine with (3.32) and (3.33) to get

$$\begin{aligned}\liminf_{\varepsilon\to 0} D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}D(I, I+\varepsilon\zeta_\varepsilon) &\geq \liminf_{\varepsilon\to 0}(1-\varepsilon)\int_0^1 R^{\mathrm{dev}}(\dot{\hat{Q}}_\varepsilon Q_\varepsilon^{-1})\,\mathrm{d}t \\ &\geq R^{\mathrm{dev}}\Big(\int_0^1 Q\,\mathrm{d}t\Big) = R^{\mathrm{dev}}(\hat{z}_0 - z_0) = R(\hat{z}_0 - z_0) \\ &= D_0(z_0, \hat{z}_0).\end{aligned}$$

□

With the Γ- lim inf-inequalities of $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$ proven, the Γ- lim inf-inequality of $\mathcal{G}_\varepsilon^{(\tau)}$ is an easy corollary.

Corollary 3.8. *The total energy $\mathcal{G}_\varepsilon^{(\tau)}$ satisfies the Γ- lim inf-inequality:*

$$\forall (u_\varepsilon, z_\varepsilon) \xrightarrow{\mathcal{Q}} (u_0, z_0): \qquad \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(u_\varepsilon, z_\varepsilon) \geq \mathcal{G}_0^{(\tau)}(u_0, z_0). \tag{3.34}$$

Proof. The second summand of

$$\mathcal{G}_\varepsilon^{(\tau)}(u_\varepsilon, z_\varepsilon) := \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) - \langle \ell^{(\tau)}, u_\varepsilon\rangle + \mathcal{D}_\varepsilon(0, z_\varepsilon).$$

is a linear functional, thus the external work converges:

$$\liminf_{\varepsilon\to 0} \langle u_\varepsilon, \ell^{(\tau)}\rangle = \lim_{\varepsilon\to 0} \langle \ell^{(\tau)}, u_\varepsilon\rangle = \langle u_0, \ell^{(\tau)}\rangle.$$

The proof now follows mainly by applying Propositions 3.6 and 3.7 to the first and third summands, except we are left to prove the additional assumption (3.27) in the latter.

We may assume $\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(u_\varepsilon, z_\varepsilon) < \infty$, since otherwise the inequality (3.34) holds trivially. From the convergence of the external work and $\mathcal{D}_\varepsilon \geq 0$ we then get boundedness of the stored energies $\sup \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$. Hence the a priori bounds from Proposition 3.3 hold and give (3.27). □

In Propositions 2.17 and 2.19 we showed that one can approximate $u \in \mathcal{U}$ with $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ by $u_k \in \mathcal{U} \cap \mathrm{W}^{1,\infty}$ that are near-identity invertible. When proving the Γ- lim sup-inequality for $\mathcal{G}_\varepsilon^{(\tau)}$ we will thus be able to restrict to the set

$$\begin{aligned}\tilde{\mathcal{Q}} = \{(u, z) \in \mathcal{Q} \mid (u, z) \in & \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \times \mathrm{L}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d}),\\ & \operatorname{tr} z = 0, [\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0 \text{ and } u \text{ is near-identity invertible}\},\end{aligned} \tag{3.35}$$

and conclude the full Γ- lim sup-inequality by density arguments. For $(\bar{u}_0, \bar{z}_0) \in \tilde{\mathcal{Q}}$ we prove the Γ- lim sup-inequality by proving

$$(\bar{u}_\varepsilon, \bar{z}_\varepsilon) := \Big(\bar{u}_0, \frac{1}{\varepsilon}\big(\exp(\varepsilon \bar{z}_0) - I\big)\Big) \tag{3.36}$$

to be a recovery sequence.

Proposition 3.9. *The total energy $\mathcal{G}_\varepsilon^{(\tau)}$ satisfies the Γ- lim sup-inequality strongly in $\mathcal{Q}$:*

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)} \leq \mathcal{G}_0^{(\tau)}. \tag{3.37}$$

In particular for $(\bar{u}_0, \bar{z}_0) \in \tilde{\mathcal{Q}}$ there exists a strongly converging recovery sequence $(\bar{u}_\varepsilon, \bar{z}_\varepsilon)$, i.e:

$$(\bar{u}_\varepsilon, \bar{z}_\varepsilon) \to (\bar{u}_0, \bar{z}_0) \text{ strongly in } \mathcal{Q} \quad \text{ and } \quad \limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(\bar{u}_\varepsilon, \bar{z}_\varepsilon) \leq \mathcal{G}_0^{(\tau)}(\bar{u}_0, \bar{z}_0).$$

Proof. For $\mathcal{G}_0^{(\tau)}(\bar{u}_0, \bar{z}_0) = \infty$ there is nothing to show, so we may restrict to $\mathcal{G}_0^{(\tau)}(\bar{u}_0, \bar{z}_0) < \infty$ which implies $\operatorname{tr} \bar{z}_0 = 0$ and $[\![\bar{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$. Thus the constraint in the limit functional is satisfied

$$\mathcal{G}_0^{(\tau)}(\bar{u}_0, \bar{z}_0) = \widetilde{\mathcal{E}}_0(\bar{u}_0, \bar{z}_0) - \langle \ell^{(\tau)}, \bar{u}_0\rangle + \mathcal{D}_0(0, \bar{z}_0) =: \widetilde{\mathcal{G}}_0^{(\tau)}(\bar{u}_0, \bar{z}_0),$$

where the right hand side is strongly continuous in $(\bar{u}_0, \bar{z}_0) \in \mathcal{Q}$, because it is the sum of three continuous functionals: the external work $u \mapsto \langle \ell^{(\tau)}, u\rangle$ is linear, $\mathcal{E}_0$ is quadratic in $\mathrm{H}^1 \times \mathrm{L}^2$ and $z \mapsto \mathcal{D}_0(0, z) = \int_{\Omega_{\mathrm{Cr}}} R(z)\,\mathrm{d}x$ is strongly continuous by assumption (3.7b) and the triangle inequality(3.8). Furthermore $(u, z) \mapsto \Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(u, z)$ is lower semicontinuous as every lower and upper Γ-limit is lower semicontinuous. Thus, if we have an approximating sequence $(u_k, z_k) \overset{\mathcal{Q}}{\to} (u, z)$, such that the Γ- lim sup-inequality for (u_k, z_k) holds, then the Γ- lim sup-inequality for (u, z) follows:

$$\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(u, z) \le \liminf_{k\to\infty} \Big(\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(u_k, z_k)\Big) \le \liminf_{k\to\infty} \Big(\mathcal{G}_0^{(\tau)}(u_k, z_k)\Big) = \mathcal{G}_0^{(\tau)}(u, z).$$

Since $\mathrm{L}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$ is dense in $\mathrm{L}^2(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$ and by Propositions 2.17 and 2.19 for $u \in \mathcal{U}$ with $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0$, there is an approximating sequence $u_k \in \mathcal{U} \cap \mathrm{W}^{1,\infty}$ that is near-identity invertible, we may now restrict to constructing a recovery sequence for $(\bar{u}_0, \bar{z}_0) \in \tilde{\mathcal{Q}}$ and the lim sup inequality on the whole $\mathcal{Q}$ would follow. We define $(\bar{u}_\varepsilon, \bar{z}_\varepsilon)$ as in (3.36) and prove it to be a recovery sequence, i.e. $(\bar{u}_\varepsilon, \bar{z}_\varepsilon) \to (\bar{u}_0, \bar{z}_0)$ strongly in $\mathcal{Q}$ and

$$\limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\tau)}(\bar{u}_\varepsilon, \bar{z}_\varepsilon) \le \mathcal{G}_0^{(\tau)}(\bar{u}_0, \bar{z}_0), \tag{3.38}$$

by considering the three summands separately.

<u>External work:</u> Since we chose $\bar{u}_\varepsilon = \bar{u}_0$, the external work is even constant:

$$\lim_{\varepsilon\to 0} \langle \ell^{(\tau)}, \bar{u}_\varepsilon\rangle = \lim_{\varepsilon\to 0} \langle \ell^{(\tau)}, \bar{u}_0\rangle = \langle \ell^{(\tau)}, \bar{u}_0\rangle. \tag{3.39}$$

Furthermore obviously we have the strong convergence $\bar{u}_\varepsilon \to \bar{u}_0$.

<u>Stored energy:</u> By the near-identity invertibility of $u_\varepsilon = u_0$ the constraint in $\mathcal{E}_\varepsilon$ is satisfied for ε small enough, thus we have:

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon) = \lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon). \tag{3.40}$$

In $\widetilde{\mathcal{E}}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon)$ the elastic tensor takes the form

$$I + \varepsilon \bar{A}_\varepsilon := (I + \varepsilon\nabla\bar{u}_\varepsilon)(I + \varepsilon\bar{z}_\varepsilon)^{-1} = (I + \varepsilon\nabla\bar{u}_0)\exp(-\varepsilon\bar{z}_0)$$

and by the smoothness of $(\bar{u}_0, \bar{z}_0)$ we have uniform convergence for both the elastic and plastic tensors appearing in $\widetilde{\mathcal{E}}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon) = \frac{1}{\varepsilon^2}\int_{\Omega_{\mathrm{Cr}}} W_{\mathrm{el}}(I + \varepsilon\bar{A}_\varepsilon) + W_{\mathrm{h}}(I + \varepsilon\bar{z}_\varepsilon)\,\mathrm{d}x$

$$\begin{aligned} |\bar{A}_\varepsilon - (\nabla\bar{u}_0 - \bar{z}_0)| &= \frac{1}{\varepsilon}\Big|(I{+}\varepsilon\nabla\bar{u}_0)\exp(-\varepsilon\bar{z}_0) - I - \varepsilon(\nabla\bar{u}_0 - \bar{z}_0)\Big| \\ &= \frac{1}{\varepsilon}\Big|\big(\exp(-\varepsilon\bar{z}_0) - (I - \varepsilon\bar{z}_0)\big) + \big(\varepsilon\nabla\bar{u}_0\exp(-\varepsilon\bar{z}_0) - \varepsilon\nabla\bar{u}_0\big)\Big| \\ &\le \frac{1}{\varepsilon}\Big|\exp(-\varepsilon\bar{z}_0) - (I - \varepsilon\bar{z}_0)\Big| + |\nabla\bar{u}_0|\,|\exp(-\varepsilon\bar{z}_0) - I| \\ &\le \varepsilon\|\nabla\bar{u}_0\|_{\mathrm{L}^\infty}\|\exp\|_{\mathrm{C}^2(B,\mathbb{R}^{d\times d})}, \\ |\bar{z}_\varepsilon - \bar{z}_0| &= \frac{1}{\varepsilon}|\exp(\varepsilon\bar{z}_0) - I - \varepsilon\bar{z}_0| \le \varepsilon\|\exp\|_{\mathrm{C}^2(B,\mathbb{R}^{d\times d})}, \end{aligned}$$

where in the respective last steps we used Taylor estimates on the function exp on the bounded domain $B := B_{\|z_0\|_{\mathrm{L}^\infty}}(0) \subset \mathbb{R}^{d\times d}$. In particular this gives the convergence $\bar{z}_\varepsilon \to \bar{z}_0$ strongly in $\mathcal{U}$ and we obtain

$$(\bar{u}_\varepsilon, \bar{z}_\varepsilon) \to (\bar{u}_0, \bar{z}_0) \text{ strongly in } \mathcal{Q}.$$

Furthermore using assumptions (3.2d) and (3.4e) by Lemma 2.13 we thus have the pointwise limit:

$$\frac{1}{\varepsilon^2} W_{\mathrm{el}}\big((I+\varepsilon\nabla\bar{u}_\varepsilon)(I+\varepsilon\bar{z}_\varepsilon)^{-1}\big) + \frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{h}}(I+\varepsilon\bar{z}_\varepsilon) \to |\nabla\bar{u}_0 - \bar{z}_0|_{\mathbb{C}} + |\bar{z}_0|_{\mathbb{H}}.$$

Moreover, for ε small enough we have $|\varepsilon\bar{A}_\varepsilon| < r_{\mathrm{el}}(\frac{1}{2})$ and $|\varepsilon\bar{z}_\varepsilon| < r_{\mathrm{h}}(\frac{1}{2})$, where $r_{\mathrm{el}}(\delta)$ and $r_{\mathrm{h}}(\delta)$ are taken from (3.2d) and (3.4e) respectively for $\delta = \frac{1}{2}$, hence we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2} W_{\mathrm{el}}(I+\varepsilon\bar{A}_\varepsilon) + \frac{1}{\varepsilon^2} W_{\mathrm{h}}(I+\varepsilon\bar{z}_\varepsilon) &\le (\frac{1}{2}+\frac{1}{2})(|\bar{A}_\varepsilon|^2_{\mathbb{C}} + |\bar{z}_\varepsilon|^2_{\mathbb{H}}) \le |\mathbb{C}|\,\|\bar{A}_\varepsilon\|^2_{\mathrm{L}^\infty} + |\mathbb{H}|\,\|\bar{z}_\varepsilon\|^2_{\mathrm{L}^\infty}\\ &\le 2|\mathbb{C}|\,\|\bar{A}_0\|^2_{\mathrm{L}^\infty} + 2|\mathbb{H}|\,\|\bar{z}_0\|^2_{\mathrm{L}^\infty}\end{aligned}$$

Lebesgue's dominated convergence thus provides convergence of the integrals:

$$\begin{aligned}\lim_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon) &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int_{\Omega_{\mathrm{Cr}}} W_{\mathrm{el}}(I+\varepsilon\bar{A}_\varepsilon) + W_{\mathrm{h}}(I+\varepsilon\bar{z}_\varepsilon)\,\mathrm{d}x\\ &= \int_{\Omega_{\mathrm{Cr}}} |\nabla\bar{u}_0 - \bar{z}_0|_{\mathbb{C}} + |\bar{z}_0|_{\mathbb{H}}\,\mathrm{d}x = \widetilde{\mathcal{E}}_0(\bar{u}_0, \bar{z}_0). \end{aligned} \tag{3.41}$$

In particular, this also gives $\sup_\varepsilon \widetilde{\mathcal{E}}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon) < \infty$, which together with the near-identity invertibility of u_0 by Theorem 3.5 implies $[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0$. Thus we have $\widetilde{\mathcal{E}}_0(\bar{u}_0, \bar{z}_0) = \mathcal{E}_0(\bar{u}_0, \bar{z}_0)$, which combined with (3.40) and (3.41) finally gives

$$\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(\bar{u}_\varepsilon, \bar{z}_\varepsilon) = \mathcal{E}_0(\bar{u}_0, \bar{z}_0). \tag{3.42}$$

Dissipation: In the definition of $D(I, \hat{P})$ in terms of an infimum we can for any $\zeta \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ estimate from above by specializing to chosen trajectory P. For any $\zeta \in \mathbb{R}^{d\times d}_{\mathrm{dev}}$ we take $P(t) = \exp(t\zeta)$, which satisfies $P(0) = I$ and $P(1) = \exp(\zeta)$ and obtain:

$$D(I, \exp(\zeta)) \le \int_0^1 R(\dot{P}P^{-1})\,\mathrm{d}t = \int_0^1 R(\zeta\exp(t\zeta)\exp(-t\zeta))\,\mathrm{d}t = R(\zeta). \tag{3.43}$$

Inserting $\zeta := \varepsilon\bar{z}_0$, using $\operatorname{tr}\bar{z}_0$ as well as the 1-homogeneity of R^{dev} and integrating over Ω_{Cr} we arrive at:

$$\begin{aligned}\mathcal{D}_\varepsilon(0, \bar{z}_\varepsilon) &= \int_{\Omega_{\mathrm{Cr}}} D_\varepsilon(0, \bar{z}_\varepsilon)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} D(I, I+\varepsilon\bar{z}_\varepsilon)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} D(I, \exp(\varepsilon\bar{z}_0))\,\mathrm{d}x\\ &\le \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} R(\varepsilon\bar{z}_0)\,\mathrm{d}x = \int_{\Omega_{\mathrm{Cr}}} R^{\mathrm{dev}}(\bar{z}_0)\,\mathrm{d}x = \mathcal{D}_0(0, \bar{z}_0).\end{aligned}$$

Combining the latter with (3.39) and (3.42) we arrive at (3.38), which concludes the proof. □

The Γ-lim sup inequality for $\mathcal{G}_\varepsilon^{(\tau)}$ concludes the proof of Theorem 3.1. In the definition (3.13) of $\mathcal{G}_\varepsilon^{(\tau)}$ the initial plastic strain $\tilde{z}^{(\tau)}$ for the fixed time step τ was chosen to be zero. The following remark comments on more general choices of $\tilde{z}^{(\tau)}$.

Remark 3.10. *There are two possibilities for the choice of initial plastic strain we want to remark on. The first possibility is to fix some $\tilde{z}^{(\tau)} \in \mathrm{L}^2(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$ independent of ε. Then the finiteness $\mathcal{G}_\varepsilon^{(\tau)}(u_\varepsilon, z_\varepsilon)$ of the total energy for some sequence $(u_\varepsilon, z_\varepsilon) \in \mathcal{Q}$, where now*

$$\mathcal{G}_\varepsilon^{(\tau)}(u, z) = \mathcal{E}_\varepsilon(u, z) - \langle \ell^{(\tau)}, u\rangle + \mathcal{D}_\varepsilon(\tilde{z}^{(\tau)}, z),$$

would have implications restricting the choice of $\tilde{z}^{(\tau)}$.

From $\mathcal{G}_\varepsilon^{(\tau)}(u_\varepsilon, z_\varepsilon) < \infty$ follows $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(z_\varepsilon) < \infty$ and $\mathcal{D}_\varepsilon(\tilde{z}^{(\tau)}, z_\varepsilon) < \infty$, the former giving $I + \varepsilon z_\varepsilon \in K \subset \mathrm{SL}(d)$, in particular $\det(I + \varepsilon z_\varepsilon) = 1$, the latter giving $\det(I + \varepsilon z_\varepsilon) = \det(I+\varepsilon\tilde{z}^{(\tau)})$. Together we have $\det(I+\varepsilon\tilde{z}^{(\tau)}) = 1$ for the sequence ε and since $\det(I+\varepsilon\tilde{z}^{(\tau)})$ is a polynomial in ε, it has to be a constant and all coefficients of positive order have to vanish. This is d scalar conditions on $\tilde{z}^{(\tau)}$. In 3d for instance it means:

$$\operatorname{tr} \tilde{z}^{(\tau)} = \det \tilde{z}^{(\tau)} = \operatorname{tr}(\operatorname{adj}(\tilde{z}^{(\tau)})) = 0.$$

To avoid these restrictions the second possibility is to prescribe initial plastic strains $\tilde{z}_\varepsilon^{(\tau)}$ dependent on ε. The Γ-convergence of the total energies

$$\mathcal{G}_\varepsilon^{(\tau)}(u, z) = \mathcal{E}_\varepsilon(u, z) - \langle \ell^{(\tau)}, u\rangle + \mathcal{D}_\varepsilon(\tilde{z}_\varepsilon^{(\tau)}, z),$$

where $\mathcal{D}_0(\tilde{z}_0^{(\tau)}, z) = R(z - \tilde{z}_0^{(\tau)})$, would require certain convergence $\tilde{z}_\varepsilon^{(\tau)} \to \tilde{z}_0^{(\tau)}$.

For the Γ-lim inf inequality of $\mathcal{G}_\varepsilon^{(\tau)}$ we can use that the Γ-lim inf inequality of $z \mapsto \mathcal{D}_\varepsilon(\tilde{z}_\varepsilon^{(\tau)}, z)$ in the weak L^2-topology is implied by the Γ-lim inf inequality of $\mathcal{D}_\varepsilon$ in weak $\mathrm{L}^2\times\mathrm{L}^2$-topology from Proposition 3.7, if $\tilde{z}_\varepsilon^{(\tau)} \rightharpoonup \tilde{z}_0^{(\tau)}$ weakly converges in $\mathrm{L}^2(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$.

For the Γ-lim sup of $\mathcal{G}_\varepsilon^{(\tau)}$ much stronger assumptions on $\tilde{z}_\varepsilon^{(\tau)}$ would be needed. A candidate for the common recovery sequence for $(\bar{u}_0, \bar{z}_0) \in \tilde{\mathcal{Q}}$ would be

$$(\bar{u}_\varepsilon, \bar{z}_\varepsilon) = \Big(\bar{u}_0, \frac{1}{\varepsilon}\Big(\exp\big(\varepsilon(\bar{z}_0 - \tilde{z}_\varepsilon^{(\tau)})\big)(I + \varepsilon\tilde{z}_\varepsilon^{(\tau)}) - I\Big)\Big).$$

Thus in both cases $\tilde{z}_\varepsilon^{(\tau)} \in \mathrm{L}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$ would be needed for the exponential term to be well-defined in L^2. Furthermore in Proposition 3.9 the convergence of the elastic tensor was needed. Without initial plastic strain, i.e. $\tilde{z}_\varepsilon^{(\tau)} = 0$, the convergence of the elastic tensor was shown:

$$\bar{A}_\varepsilon^{(=0)} \frac{1}{\varepsilon}((I + \varepsilon\nabla\bar{u}_0)\exp(-\varepsilon\bar{z}_0) - I) \to \nabla\bar{u}_0 - \bar{z}_0.$$

With $\tilde{z}_\varepsilon^{(\tau)} \neq 0$ the elastic tensor takes the form

$$\bar{A}_\varepsilon^{(\neq 0)} = \frac{1}{\varepsilon}((I + \varepsilon\nabla\bar{u}_0)(I + \varepsilon\tilde{z}_\varepsilon^{(\tau)})^{-1}\exp\big(-\varepsilon(\bar{z}_0 - \tilde{z}_\varepsilon^{(\tau)})\big) - I)$$

and we want $\bar{A}_\varepsilon^{(\neq 0)}$ to have the same limit as $\bar{A}_\varepsilon^{(=0)}$:

$$|\bar{A}_\varepsilon^{(=0)} - \bar{A}_\varepsilon^{(\neq 0)}| = \frac{1}{\varepsilon}\Big|(I + \varepsilon\bar{u}_0)\Big((I + \varepsilon\tilde{z}_\varepsilon^{(\tau)})^{-1}\exp(\varepsilon\tilde{z}_\varepsilon^{(\tau)}) - I\Big)\exp(-\varepsilon\bar{z}_0)\Big|$$

$$\leq \big|I + \varepsilon \bar{u}_0\big| \Big|(I + \varepsilon \tilde{z}_\varepsilon^{(\tau)})^{-1}\Big| \frac{1}{\varepsilon}\Big| \exp(\varepsilon \tilde{z}_\varepsilon^{(\tau)}) - (I + \varepsilon \tilde{z}_\varepsilon^{(\tau)})\Big| \big|\exp(-\varepsilon \bar{z}_0)\big|.$$

The first and last factors on the right-hand side are bounded by $(\bar{u}_0, \bar{z}_0) \in \tilde{\mathcal{Q}}$. *If additionally we assume* $\|\tilde{z}_\varepsilon^{(\tau)}\|_{\mathrm{L}^\infty} \leq C$, *then* $|(I+\varepsilon\tilde{z}_\varepsilon^{(\tau)})^{-1}|$ *is also bounded and* $\frac{1}{\varepsilon}|\exp(\varepsilon\tilde{z}_\varepsilon^{(\tau)}) - (I+\varepsilon\tilde{z}_\varepsilon^{(\tau)})|$ *converges to* 0 *by a first-order Taylor estimate. Thus for the* Γ-$\limsup$ *of* $\mathcal{G}_\varepsilon^{(\tau)}$ *the weak-* convergence* $\tilde{z}_\varepsilon^{(\tau)} \to \tilde{z}_0^{(\tau)}$ *in* $\mathrm{L}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^{d\times d})$ *is needed.*

4 Evolutionary Gamma-convergence in Elastoplasticity

4.1 Introduction

The goal of this chapter is the small-deformation limit of the rate-independent full evolution of elastoplasticity on the cracked domain Ω_{Cr}. Although the GMS condition passed the proof of concept for plasticity in Chapter 3, where it proved to be appropriate for the small-deformation limit in (one-step) deformation plasticity, it seems to be still too difficult to apply the GMS condition in the full evolution, see Remark 4.15. Thus we will investigate two stages of the generalization of the evolutionary Γ-convergence proven in [MS13] in the case without self contact. On one hand we will prove the evolutionary Γ-convergence in Section 4.5 for the non-Lipschitz domain Ω_{Cr} but without constraint following the abstract theory from [MRS08]. For that we use separate $\liminf$ estimates on energy and dissipation, that were already shown in the previous Chapter 3 as well as an $\limsup$ estimate on the transition cost given by the mutual recovery sequence constructed in Section 4.3. On the other hand a strategy is presented how these $\liminf$ and $\limsup$ estimates could be approached in the case with constraints. Instead of the full GMS-condition from Chapters 2 and 3, the δ-GMS condition is proposed, that allows for interpenetration in a small neighborhood of Γ_{Cr}:

$$\int\limits_{\Omega\backslash U_\delta(\Gamma_c)} \varphi(v(x))\,\left|\det\nabla v(x)\right|\mathrm{d}x \leq \int\limits_{\mathbb{R}^d}\varphi(y)\,\mathrm{d}y \qquad \text{for all } \varphi\in \mathrm{C}_0(\mathbb{R}^d,\mathbb{R}) \text{ with } \varphi\geq 0. \quad (4.1)$$

For $\delta(\varepsilon)=\varepsilon^\alpha$ with $\alpha>1$ in Proposition 4.10 this weaker condition is proven to still imply local non-interpenetration

$$0\leq [\![u(x)]\!]_{\Gamma_{\mathrm{Cr}}} := \left(u^+(x)-u^-(x)\right)\cdot\nu(x) \qquad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}} \quad (4.2)$$

in the limit $\varepsilon\to 0$, which is needed for the $\liminf$ estimate with constraint. For the $\limsup$ estimate with constraint the injectivity of the so-called crack-respecting composition defined in Lemma 4.8 is needed. Proposition 4.13 proves this for $\delta(\varepsilon)=\varepsilon^\alpha$ with $\alpha<\beta$, where $\beta=\frac{2p-2d}{2p-2d+pd}<1$ is the exponent that emerged proving a priori estimates in L^∞ on u in Propositions 2.9 and 3.3.

The assumptions on the cracked domain Ω_{Cr} in this chapter differ from the previous chapters slightly in multiple aspects. On one hand for the sake of simplicity the crack Γ_{Cr} is assumed to coincide with the model crack $\hat\Gamma_{\mathrm{Cr}}$, i.e. the transformation $T\colon\Omega_{\mathrm{Cr}}\to\hat\Omega_{\mathrm{Cr}}$ from Section 2.2 is assumed to be the identity $T=\mathrm{id}$. Furthermore additionally the Lipschitz boundary of certain sets is assumed. In (4.3d) it is sets used in the proof of Lemma 4.4 about extensions of smooth functions on Ω_{Cr} and in (4.3e) it is the domains appearing on the left-hand side of (4.1). On the other hand the transversality of Γ_{Cr} and $\partial\Omega$ in the fashion of (2.7d) and (2.7e) may be dropped, because this chapter's density result in Proposition 4.5 does not involve the jump condition (4.2) and hence does not need transversality in contrast to Proposition 2.19. Thus the collected assumptions on $\Omega_{\mathrm{Cr}} = \Omega \setminus \Gamma_{\mathrm{Cr}}$ read as follows:

$$\Omega \subset \mathbb{R}^d \text{ is a bounded Lipschitz domain;} \tag{4.3a}$$

$$\left.\begin{aligned} &\Gamma_{\mathrm{Cr}} := \Big(([0,1]\times\{0\}\times\mathbb{R}^{d-2}) \cup (\{0\}\times[0,\infty]\times\mathbb{R}^{d-2})\Big),\\ &\Gamma_{\mathrm{edge}} := \{(1,0)\} \times \mathbb{R}^{d-2},\\ &\Gamma_{\mathrm{kink}} := \{(0,0)\} \times \mathbb{R}^{d-2}; \end{aligned}\right\} \tag{4.3b}$$

$$\left.\begin{aligned} &\text{the sets } A_+ := \{\, x \in \Omega \mid (x_1 > 0 \text{ and } x_2 > 0) \;\text{ or }\; x_1 > 1 \,\}\\ &\text{and } A_- := \{\, x \in \Omega \mid x_1 < 0 \;\text{ or }\; x_1 > 1 \;\text{ or }\; x_2 < 0 \,\} \text{ as well as}\\ &A_+ \cap A_- \text{ and } A_- \setminus A_+ \text{ have Lipschitz boundary;} \end{aligned}\right\} \tag{4.3c}$$

$$\left.\begin{aligned} &\exists \delta_B{>}0\, \exists R_B{>}0\, \forall \delta{<}\delta_B\, \forall R{>}R_B :\\ &\text{for } B_0^{(\delta)} := \{x \in \mathbb{R}^d \mid x_1 > 1{-}\delta\} \text{ the sets } (A_+ \cup B_0^{(\delta)}) \cap B_R(0) \text{ and}\\ &(A_- \cup B_0^{(\delta)}) \cap B_R(0) \text{ have Lipschitz boundary} \end{aligned}\right\} \tag{4.3d}$$

$$\exists \delta_{\mathrm{Cr}}\, \forall \delta{<}\delta_{\mathrm{Cr}}: \quad \Omega \setminus U_\delta(\Gamma_{\mathrm{Cr}}) \text{ has Lipschitz boundary.} \tag{4.3e}$$

The set $B_0^{(\delta)}$ from (4.3d) appears in Lemma 4.4 as the intersection of

$$\begin{aligned} B_+^{(\delta)} &:= \{(x_1 > 0 \text{ and } x_2 > 0) \text{ or } x_1 > 1{-}\delta\} \quad \text{ and}\\ B_-^{(\delta)} &:= \{x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_1 > 1{-}\delta\}. \end{aligned} \tag{4.4}$$

Note that $B_\pm^{(\delta)}$ are defined such that they are extensions of $A_\pm$ in the sense that $A_\pm \subset B_\pm^{(\delta)}$.

We restrict to homogenous boundary conditions for the state space

$$\mathcal{Q} = \mathcal{U} \times \mathcal{Z} = \operatorname{clos}_{\mathrm{H}^1(\Omega_{\mathrm{Cr}})}\Big(\{u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{Cr}};\mathbb{R}^d) \mid u|_{\Gamma_{\mathrm{Dir}}} = 0\}\Big) \times \mathrm{L}^2(\Omega, \mathbb{R}^{d\times d}), \tag{4.5}$$

where the assumptions on the Dirichlet boundary Γ_{Dir} still read as in Chapter 3:

$$\overline{\Gamma_{\mathrm{Dir}}} \cap \overline{\Gamma_{\mathrm{Cr}}} = \emptyset \quad \text{ and } \quad \mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0. \tag{4.6}$$

The assumptions on the elastic energy density W_{el} include those from the previous chapter as well as the additional assumption (4.7e):

$$\forall\, F \in \mathbb{R}^{d\times d} \setminus \mathrm{GL}_+(d) \colon W_{\mathrm{el}}(F) = \infty; \tag{4.7a}$$

$$\forall\, F \in \mathbb{R}^{d\times d},\; R \in \mathrm{SO}(d): \; W_{\mathrm{el}}(RF) = W(F); \tag{4.7b}$$

$$\left.\begin{aligned} &\exists\, p > d, c_{W_{\mathrm{el}}}, C_{W_{\mathrm{el}}} > 0 \forall\, F \in \mathbb{R}^{d\times d} :\\ &\qquad W_{\mathrm{el}}(F) \geq c_{W_{\mathrm{el}}} \max\big\{\operatorname{dist}(F,\mathrm{SO}(d))^2, |F|^p {-} C_{W_{\mathrm{el}}}\big\}; \end{aligned}\right\} \tag{4.7c}$$

$$\left.\begin{array}{l}\exists\,\mathbb{C}\geq 0 \text{ with } \mathbb{C}^\top=\mathbb{C}\ \forall\,\delta>0\ \exists\, r_{\mathrm{el}}(\delta)>0\ \forall A\in B_{r_{\mathrm{el}}(\delta)}(0)\subset\mathbb{R}^{d\times d}:\\ \qquad\Big|W_{\mathrm{el}}(I{+}A)-\dfrac{1}{2}\langle A,\mathbb{C}A\rangle\Big|\ \leq\ \delta\,\langle A,\mathbb{C}A\rangle.\end{array}\right\}\tag{4.7d}$$

$$\exists\, c_{\mathrm{M}}{>}0\ \forall\, F{\in}\,\mathrm{GL}_+(d):\ |F^{\mathrm{T}}\partial_F W_{\mathrm{el}}(F)|\leq c_{\mathrm{M}}(W_{\mathrm{el}}(F)+1).\tag{4.7e}$$

For a discussion of assumptions (4.7a)-(4.7d) see (2.1). Assumption (4.7e) provides control of the Mandel tensor $F^{\mathrm{T}}\partial_F W_{\mathrm{el}}(F)$, which is a crucial condition in finite elastoplasticity (see[Bal84, Bal02]) and has been used in the context of rate-independent processes in [FM06, MM09, MS13]. In particular in [MS13] the combination of conditions (4.7e) and (4.7a) enables the authors to use [MS13, Lemma 4.1] to obtain an estimate combining left and right multiplication, which will be used in this chapter in the proof of Proposition 4.7:

$$\begin{aligned}&\exists C_{\mathrm{M}}, r_{\mathrm{M}}>0\quad \forall G_1,G_2\in B_{r_{\mathrm{M}}}(I)\quad \forall F\in\mathrm{GL}_+(d):\\ &\qquad |W_{\mathrm{el}}(G_1FG_2)-W_{\mathrm{el}}(F)|\leq C_{\mathrm{M}}(W_{\mathrm{el}}(F)+1)(|G_1-I|+|G_2-I|).\end{aligned}\tag{4.8}$$

For the assumptions and definitions of the hardening part $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}$ of the stored energy, the dissipation potential R and the dissipation distance $\mathcal{D}_\varepsilon$, the current chapter follows the previous one to the letter. We restrict to state them briefly, for a discussion see the analogs in Chapter 3. On the hardening energy density W_{h} the identical assumption as in (3.4) are posed:

$$W_{\mathrm{h}}(P):=\begin{cases}\widetilde{W}_{\mathrm{h}}(P) & \text{if } P\in K,\\ \infty & \text{otherwise;}\end{cases}\tag{4.9a}$$

$$K \text{ is compact in } \mathrm{SL}(d) \text{ and contains a neighborhood of } I;\tag{4.9b}$$

$$\widetilde{W}_{\mathrm{h}}\colon \mathbb{R}^d\to\mathbb{R} \text{ is locally Lipschitz continuous;}\tag{4.9c}$$

$$\left.\begin{array}{c}\exists\mathbb{H}{\geq}0,\mathbb{H}^{\mathrm{T}}{=}\mathbb{H}\quad\forall\delta>0\quad\exists r_{\mathrm{h}}(\delta){>}0\quad\forall A{\in}B_{r_{\mathrm{h}}(\delta)}(0):\\ |\widetilde{W}_{\mathrm{h}}(I+A)-\tfrac{1}{2}\langle A,\mathbb{H}A\rangle|\leq\delta|\tfrac{1}{2}\langle A,\mathbb{H}A\rangle|;\end{array}\right\}\tag{4.9d}$$

$$\exists c_{\mathrm{h}}{>}0\quad\forall A{\in}\mathbb{R}^{d\times d}\colon W_{\mathrm{h}}(I+A)\geq c_{\mathrm{h}}|A|^2.\tag{4.9e}$$

Recall from the previous chapter that by compactness of K there is a constant c_K with:

$$P\in K\Rightarrow\quad |P|+|P^{-1}|\leq c_K,\tag{4.10}$$

$$P\in\mathrm{SL}(d)\setminus K\Rightarrow\quad |P-I|\geq\frac{1}{c_K}.\tag{4.11}$$

As in (3.7) on the dissipation potential the following is assumed:

$$R^{\mathrm{dev}}\colon\mathbb{R}^{d\times d}_{\mathrm{dev}}=\{A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\mid \mathrm{tr}\,A=0\}\to[0,\infty) \text{ convex and 1-homogenous;}\tag{4.12a}$$

$$\exists c_R,C_R{>}0\,\forall P\in\mathbb{R}^{d\times d}_{\mathrm{dev}}\colon c_R|P|\leq R^{\mathrm{dev}}(P)\leq C_R|P|;\tag{4.12b}$$

$$R\colon\mathbb{R}^{d\times d}\to[0,\infty];\, R(z):=\begin{cases}R^{\mathrm{dev}}(z) & \text{if } z\in\mathbb{R}^{d\times d}_{\mathrm{dev}},\\ \infty & \text{otherwise.}\end{cases}\tag{4.12c}$$

With this the dissipation distance function

$$D\colon\mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d}\to[0,\infty],\, D(P,\hat P):=\begin{cases}D(I,\hat PP^{-1}) & \text{if } P \text{ is invertible,}\\ \infty & \text{otherwise,}\end{cases}$$

where

$$D(I,\hat{P}) := \inf\Big\{ \int_0^1 R(\dot{P}P^{-1})\,\mathrm{d}t \mid P \in \mathrm{C}^1([0,1],\mathbb{R}^{d\times d}), P(0)=I, P(1)+\hat{P} \Big\}. \tag{4.13}$$

Recall from the discussion following (3.7) in the previous Chapter 3 that $D(I,\hat{P}) < \infty$ implies $\det \hat{P} = 1$ by Jacobi's formula and that

$$D(I,\exp(A)) \le R(A) \tag{4.14}$$

by inserting $P(t) = \exp(At)$ into the infimum in the definition of D. Moreover, D satisfies the triangle inequality

$$D(P_1,P_3) \le D(P_1,P_2) + D(P_2,P_3), \tag{4.15}$$

and there exists a constant c_D such that

$$\forall P,\hat{P} \in K \subset \mathrm{SL}(d)\colon D(P,\hat{P}) \le c_D, \qquad D(I,P) \le c_D|P-I|. \tag{4.16}$$

The dissipation functions $D_\varepsilon, D_0\colon \mathbb{R}^{d\times d}\times\mathbb{R}^{d\times d} \to [0,\infty]$ and dissipation functionals $\mathcal{D}_\varepsilon,\mathcal{D}_0\colon \big(\mathrm{L}^1(\Omega\setminus\Gamma_{\mathrm{Cr}},\mathbb{R}^{d\times d})\big)^2 \to [0,\infty]$ are defined by:

$$\begin{aligned} D_\varepsilon(z_1,z_2) &:= \frac{1}{\varepsilon}D(I+\varepsilon z_1, I+\varepsilon z_2), \qquad & D_0(z_1,z_2) &= R(z_2-z_1),\\ \mathcal{D}_\varepsilon(z_1,z_2) &= \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} D_\varepsilon(z_1,z_2)\,\mathrm{d}x, \qquad & \mathcal{D}_0(z_1,z_2) &= \int_{\Omega\setminus\Gamma_{\mathrm{Cr}}} D_0(z_1,z_2)\,\mathrm{d}x. \end{aligned}$$

Finally, on a time subinterval $[s,t]\subset[0,T]$ the total dissipation of a trajectory $z\colon[0,T]\to\mathcal{Z}$ is defined as:

$$\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z,[s,t]) := \sup\Big\{ \sum_{i=1}^N \mathcal{D}_\varepsilon(z(t^{i-1}),z(t^i)) \mid s=t^0<\ldots<t^N=t \Big\}.$$

For each fixed exponent $\alpha > 0$ we define the sequence of the rescaled stored energy functionals

$$\mathcal{E}_\varepsilon^{(\alpha)}\colon \mathcal{Q}\to\mathbb{R}\cup\{\infty\}, \quad (u,z)\mapsto \begin{cases} \widetilde{\mathcal{E}}_\varepsilon(u,z) & \text{if } v=\mathrm{id}+\varepsilon u \text{ fulfills } \varepsilon^\alpha\text{-GMS-condition (4.1)},\\ \infty & \text{otherwise,} \end{cases}$$

by imposing the δ-GMS condition for $\delta(\varepsilon)=\varepsilon^\alpha$ on the integral quantity $\widetilde{\mathcal{E}}_\varepsilon$, that is defined as in Chapter 3:

$$\begin{aligned} \widetilde{\mathcal{E}}_\varepsilon(u,z) &:= \widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z) + \widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z), \text{ where}\\ \widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}(u,z) &:= \frac{1}{\varepsilon^2}\int_{\Omega_{\mathrm{Cr}}} W_{\mathrm{el}}\big((I+\varepsilon\nabla u)(I+\varepsilon z)^{-1}\big)\,\mathrm{d}x \quad \text{and}\\ \widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}(z) &:= \frac{1}{\varepsilon^2}\int_\Omega W_{\mathrm{h}}(I+\varepsilon z)\,\mathrm{d}x. \end{aligned}$$

Note that if the second summand $\widetilde{\mathcal{E}}_{\mathrm{h},\varepsilon}$ in $\widetilde{\mathcal{E}}_\varepsilon$ is finite, then $I+\varepsilon z$ is invertible by (4.10) and the first summand $\widetilde{\mathcal{E}}_{\mathrm{el},\varepsilon}$ is well-defined.

The (α)-notation for the limiting stored energy $\mathcal{E}_0^{(\alpha)} = \mathcal{E}_0$ is dropped to emphasize that the constraint is the original one from the previous chapters:

$$\mathcal{E}_0 \colon \mathcal{Q} \to [0,\infty], \qquad \mathcal{E}_0(u,z) := \begin{cases} \widetilde{\mathcal{E}}_0(u,z) & \text{if } u \text{ satisfies (4.2) } \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{\mathrm{Cr}}, \\ \infty & \text{otherwise;} \end{cases}$$

$$\widetilde{\mathcal{E}}_0(u,z) := \widetilde{\mathcal{E}}_{\mathrm{el},0}(u,z) + \widetilde{\mathcal{E}}_{\mathrm{h},0}(z),$$

$$\widetilde{\mathcal{E}}_{\mathrm{el},0}(u,z) := \int_{\Omega_{\mathrm{Cr}}} \frac{1}{2}|\nabla u - z|_{\mathbb{C}}^2 \,\mathrm{d}x \quad \text{and} \quad \widetilde{\mathcal{E}}_{\mathrm{h},0}(z) = \int_\Omega \frac{1}{2}|z|_{\mathbb{H}}^2 \,\mathrm{d}x.$$

The evolution is driven by the generalized loading

$$\ell \in \mathrm{W}^{1,1}([0,T],\mathcal{U}') \tag{4.17}$$

such that for the small loadings $\ell_\varepsilon = \varepsilon\ell$ we arrive at the total energies:

$$\begin{aligned} \mathcal{G}_\varepsilon^{(\alpha)} \colon [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}, \qquad \mathcal{G}_\varepsilon^{(\alpha)}(t,u,z) &:= \mathcal{E}_\varepsilon^{(\alpha)}(u,z) - \frac{1}{\varepsilon^2}\langle \ell_\varepsilon(t), \varepsilon u\rangle \\ &= \mathcal{E}_\varepsilon^{(\alpha)}(u,z) - \langle \ell(t), u\rangle \quad \text{and} \\ \mathcal{G}_0 \colon [0,T] \times \mathcal{Q} \to \mathbb{R} \cup \{\infty\}, \qquad \mathcal{G}_0(t,u,z) &:= \mathcal{E}_0(t,u,z) - \langle \ell(t), u\rangle. \end{aligned}$$

The total energies without constraints we will notate analogously to the stored energies with a tilde:

$$\widetilde{\mathcal{G}}_\varepsilon(t,u,z) = \widetilde{\mathcal{E}}_\varepsilon(u,z) - \langle \ell(t), u\rangle, \qquad \widetilde{\mathcal{G}}_0(t,u,z) := \widetilde{\mathcal{E}}_0(t,u,z) - \langle \ell(t), u\rangle.$$

Note that in contrast to the total energy $\mathcal{G}_\varepsilon^{(\tau)}$ in Chapter 3 the dissipation is not considered a part of the total energies.

The triples $(\mathcal{Q}, \widetilde{\mathcal{G}}_\varepsilon, \mathcal{D}_\varepsilon)$ and $(\mathcal{Q}, \mathcal{G}_\varepsilon^{(\alpha)}, \mathcal{D}_\varepsilon)$ given by the state space $\mathcal{Q}$, the total energies $\widetilde{\mathcal{G}}_\varepsilon$ and $\mathcal{G}_\varepsilon^{(\alpha)}$ and the dissipation $\mathcal{D}_\varepsilon$ each form a *Rate-Independent System* (RIS). A crucial structure for a RIS $(\mathcal{Q}, \mathcal{G}, \mathcal{D})$ is the set $S(t)$ of stable states at time $t \in [0,T]$ defined by:

$$\begin{aligned} S(t) := \{q \in \mathcal{Q} \mid \mathcal{G}(t,q) < \infty & \text{ and} \\ & \mathcal{G}(t,q) \le \mathcal{G}(t,\hat{q}) + \mathcal{D}(q,\hat{q}) \quad \forall \hat{q} \in \mathcal{Q}\}. \end{aligned}$$

The stable sets corresponding to the RIS $(\mathcal{Q}, \widetilde{\mathcal{G}}_\varepsilon, \mathcal{D}_\varepsilon)$ and $(\mathcal{Q}, \mathcal{G}_\varepsilon^{(\alpha)}, \mathcal{D}_\varepsilon)$ we notate as $\widetilde{S}_\varepsilon(t)$ and $S_\varepsilon^{(\alpha)}(t)$ respectively.

Introducing a notation for the *transition cost* $\widetilde{\mathcal{T}}_\varepsilon$ and $\mathcal{T}_\varepsilon^{(\alpha)}$ for a time $t \in [0,T]$, a state $(u,z) \in \mathcal{Q}$ and a competitor $(\hat{u},\hat{z}) \in \mathcal{Q}$

$$\begin{aligned} \widetilde{\mathcal{T}}_\varepsilon(t,u,z,\hat{u},\hat{z}) &:= \widetilde{\mathcal{G}}_\varepsilon(t,\hat{u},\hat{z}) - \widetilde{\mathcal{G}}_\varepsilon(t,u,z) + \mathcal{D}_\varepsilon(z,\hat{z}) \\ \mathcal{T}_\varepsilon^{(\alpha)}(t,u,z,\hat{u},\hat{z}) &:= \mathcal{G}_\varepsilon^{(\alpha)}(t,\hat{u},\hat{z}) - \mathcal{G}_\varepsilon^{(\alpha)}(t,u,z) + \mathcal{D}_\varepsilon(z,\hat{z}) \end{aligned} \tag{4.18}$$

one can say, that stable states are states $(u,z) \in \mathcal{Q}$ with finite total energy such that for every competitor $(\hat{u},\hat{z}) \in \mathcal{Q}$ the transition cost is nonnegative:

$$\begin{aligned} & (u,z) \in \widetilde{S}_\varepsilon(t) \\ \Leftrightarrow \quad & \widetilde{\mathcal{G}}_\varepsilon(t,u,z) < \infty \text{ and } \forall (\hat{u},\hat{z}) \in \mathcal{Q}\colon\ 0 \le \widetilde{\mathcal{T}}_\varepsilon(t,u,z,\hat{u},\hat{z}); \end{aligned}$$

$$
\begin{aligned}
&(u,z)\in S_\varepsilon^{(\alpha)}(t)\\
\Leftrightarrow\quad &\mathcal{G}_\varepsilon^{(\alpha)}(t,u,z)<\infty \text{ and } \forall(\hat u,\hat z)\in\mathcal{Q}\colon\ 0\le\mathcal{T}_\varepsilon^{(\alpha)}(t,u,z,\hat u,\hat z).
\end{aligned}
$$

By assuming for the initial data of $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha)},\mathcal{D}_\varepsilon)$

$$
\left.\begin{aligned}
&(u_\varepsilon^0,z_\varepsilon^0)\rightharpoonup(u_0^0,z_0^0) \text{ weakly in } \mathcal{Q}, \text{ with}\\
&(u_\varepsilon^0,z_\varepsilon^0)\in S_\varepsilon(t),\quad z_0^0\in \mathrm{L}^2(\Omega_{\mathrm{Cr}},\mathbb{R}^{d\times d}_{\mathrm{dev}})\quad\text{and}\\
&\mathcal{G}_\varepsilon^{(\alpha)}(0,u_\varepsilon^0,z_\varepsilon^0)\to\mathcal{G}_0(0,u_0^0,z_0^0)
\end{aligned}\right\}\tag{4.19}
$$

and analogously for the initial data of $(\mathcal{Q},\widetilde{\mathcal{G}}_\varepsilon,\mathcal{D}_\varepsilon)$

$$
\left.\begin{aligned}
&(u_\varepsilon^0,z_\varepsilon^0)\rightharpoonup(u_0^0,z_0^0) \text{ weakly in } \mathcal{Q}, \text{ with}\\
&(u_\varepsilon^0,z_\varepsilon^0)\in S_\varepsilon(t),\quad z_0^0\in \mathrm{L}^2(\Omega_{\mathrm{Cr}},\mathbb{R}^{d\times d}_{\mathrm{dev}})\quad\text{and}\\
&\widetilde{\mathcal{G}}_\varepsilon(0,u_\varepsilon^0,z_\varepsilon^0)\to\widetilde{\mathcal{G}}_0(0,u_0^0,z_0^0)
\end{aligned}\right\}\tag{4.20}
$$

we can come to the definition of the solution concept considered in this chapter, which is that of energetic solutions.

Definition 4.1 (Energetic solutions)**.** *A trajectory* $q\colon[0,T]\to\mathcal{Q}, t\mapsto(u(t),z(t))$ *is said to be an* energetic solution *to the RIS* $(\mathcal{Q},\mathcal{G},\mathcal{D})$ *with initial data* $(u^0,z^0)\in\mathcal{Q}$, *if* $(u(0),z(0))=(u^0,z^0)$, *the map* $t\mapsto\langle\dot\ell(t),u(t)\rangle$ *is integrable and for all* $t\in[0,T]$ *the following holds:*

$$(u(t),z(t))\in S(t)\quad\text{and}\tag{4.21}$$

$$\mathcal{G}(t,u(t),z(t))+\mathrm{Diss}_{\mathcal{D}}(z;[0,t])=\mathcal{G}(0,u^0,z^0)-\int_0^t\langle\dot\ell(t'),u(t')\rangle\,\mathrm{d}t'.\tag{4.22}$$

For $\varepsilon>0$ *energetic solutions to* $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha)},\mathcal{D}_\varepsilon)$ *or to* $(\mathcal{Q},\widetilde{\mathcal{G}}_\varepsilon,\mathcal{D}_\varepsilon)$ *will be called* finite-plasticity solution with constraint *or* finite-plasticity solution without constraint *respectively.*

Energetic solutions to $(\mathcal{Q},\mathcal{G}_0,\mathcal{D}_0)$ *or* $(\mathcal{Q},\widetilde{\mathcal{G}}_0,\mathcal{D}_0)$ *will be called* linearized-plasticity solutions with constraint *or* without constraint *respectively.*

The condition (4.21) is called stability and (4.22) is the energy balance. In the stability one can see the quasi-static nature of RIS and the distinction into u as the fast variable and z the slow variable in the following sense. Comparing for any time $t\in[0,T]$ the state $(u_\varepsilon(t),z_\varepsilon(t))$ to a competitor $(\hat u,z_\varepsilon(t))$ with the same plastic strain by the stability gives $\mathcal{G}_\varepsilon^{(\alpha)}(t,u_\varepsilon(t),z_\varepsilon(t))\le\mathcal{G}_\varepsilon^{(\alpha)}(t,\hat u,z_\varepsilon(t))$ for all $\hat u\in\mathcal{U}$. Hence u is much faster than z in the sense that for given time t and plastic variable $z_\varepsilon(t)$ the displacement $u_\varepsilon(t)$ is the minimizer of the total energy $\mathcal{G}_\varepsilon^{(\alpha)}(t,\cdot,z_\varepsilon(t))$, which is a *static* minimization problem.

The aim of this chapter is the evolutionary Γ-convergence of finite-plasticity solutions to linearized plasticity solutions both with and without constraints respectively. According to the abstract theory of evolutionary Γ-convergence for energetic solutions of RIS developed in [MRS08], the proof relies on showing the two separate $\Gamma\text{-}\liminf$-inequalities for energy and dissipation, and on constructing a mutual recovery sequence. In the case without constraint the lower bounds on the $\Gamma\text{-}\liminf$ of energy and dissipation were already covered in Chapter 3. After Section 4.2 dealt with some technical subtleties concerning spaces of smooth functions on cracked domains and in particular their density in Sobolev spaces, in the subsequent Section 4.3 the mutual recovery sequence for the case without constraints is constructed. This is done generalizing the ideas from [MS13] via the *crack-respecting composition* defined in Lemma 4.8. The lower bound on energy and dissipation as well as the mutual recovery sequence in the case with constraints are dealt with in Section 4.4. For $\alpha > 1$ the lim inf estimate on the stored energy $\mathcal{E}_\varepsilon^{(\alpha)}$ is shown by proving a version of the infinitesimal non-interpenetration for the weaker ε^α-GMS condition in Theorem 4.10. For $\alpha < 1$ the ε^α-GMS condition of the crack-respecting composition is recovered, which gives the lim sup estimate on the transition cost $\mathcal{T}_\varepsilon^{(\alpha)}$ by the mutual recovery sequence. Furthermore the existence of $\alpha_{\mathrm{Con}} > 1$ is conjectured for which he crack-respecting composition satisfies the $\varepsilon^{\alpha_{\mathrm{Con}}}$-GMS condition. The difficulties in proving this Conjecture 4.14 are discussed in Remark 4.15. Finally in Section 4.5 the evolutionary Γ-convergences of $(\mathcal{Q}, \widetilde{\mathcal{G}}_\varepsilon, \mathcal{D}_\varepsilon)$ and $(\mathcal{Q}, \mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})}, \mathcal{D}_\varepsilon)$ are proven in Theorem 4.19 and 4.20 respectively.

4.2 Smooth functions on cracked domains

The proof of evolutionary Γ-convergence in Section 4.5 uses a density argument to extend positivity of the transition cost for smooth competitors to all competitors in $\mathcal{Q}$. On Lipschitz domains O one usually relies on the density of $\mathrm{C}^\infty(\overline{O})$ in $\mathrm{H}^1(O)$, i.e. one uses the extension of smooth function from the interior O to the closure $\overline{O}$. For the non-Lipschitz domain $\Omega_{\mathrm{Cr}} = \Omega \setminus \Gamma_{\mathrm{Cr}}$ however, we have $\overline{\Omega_{\mathrm{Cr}}} = \overline{\Omega}$, such that smooth functions on the closure $\mathrm{C}^\infty(\overline{O})$ have no jump. Hence in our context of functions on Ω_{Cr} with jumps on the crack Γ_{Cr} the set $\mathrm{C}^\infty(\overline{\Omega_{\mathrm{Cr}}})$ is not dense and thus it is not suitable for cracked domains.

The aim of this section is to introduce a suitable notion of smooth functions on Ω_{Cr}, that still allows for jumps and gives density in the Sobolev spaces. For that it is crucial to understand the extending of smooth functions to the outside of domains. We use the following classical theorem, which we cite without proof.

Theorem 4.2 (Whitney extension theorem)**.** *For multi-indices μ let f_μ be a collection of scalar functions on a closed subset $A \subset \mathbb{R}^d$. Suppose for all $|\mu| \leq m$ and all $x, y \in A$ it holds*

$$f_\mu(x) = \sum_{\nu \leq m-|\mu|} \frac{f_{\mu+\nu}(y)}{\nu!}(x-y)^\nu + R_{m,\mu}(x,y)$$

with $R_{m,\mu} \in o(|x-y|^{m-|\mu|})$.

Then there exists an extension $F \in C^m(\mathbb{R}^d)$ of f_0, such that:

- *$F = f_0$ on A,*
- *$D^\mu F = f_\mu$ on A,*
- *F is real analytic in a neighborhood of every point in $\mathbb{R}^d \setminus A$.*

With that we are able to prove the following proposition on the extension of a smooth function, that will be used in this section for the proof of the density in Proposition 4.5 as well as in the following section for the definition of the crack-respecting composition in Lemma 4.8. It shows that the extension of a smooth function, as in the definition of $\mathrm{C}^\infty(\overline{O})$, is connected to the boundedness of the derivatives.

Lemma 4.3. *Let $O \in \mathbb{R}^d$ be a domain with Lipschitz boundary and let $f \in \mathrm{C}^\infty(O)$. Suppose for all multi-indices μ, that $\|D^\mu f\|_{L^\infty} < \infty$.*

Then there exists $F \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $F|_O = f$.

Proof. By assumption every derivative $D^\mu f \in W^{1,\infty}(O)$ and since O has Lipschitz boundary by Sobolev embedding $D^\mu f$ is Lipschitz continuous. In particular $D^\mu f$ is uniformly continuous on O and thus admits a continuous extension to $\bar{O}$.

To show that this extension is m times differentiable on $\bar{O}$ we want to apply Whitney extension theorem. Consider for fixed m at any point $y \in O$ the m-th Taylor expansion of f at y and for each μ with $|\mu| \leq m$ the $(m-|\mu|)$-th Taylor expansion of $D^\mu f$ at y:

$$f(x) = \sum_{|\nu| \leq m} \frac{D^\nu f(y)}{\nu!}(x-y)^\nu + R_{m,0}(x,y),$$

$$D^\mu f(x) = \sum_{|\nu| \leq m-|\mu|} \frac{D^{\mu+\nu} f(y)}{\nu!}(x-y)^\nu + R_{m,\mu}(x,y).$$

By Lagrange representation of the remainder we have:

$$|R_{m,\mu(x,y)}| \leq \frac{\|D^{m+1} f\|_{L^\infty}}{(m+1)!}|x-y|^{m-|\mu|+1} \in o(|x-y|^{m-|\mu|}).$$

Both the derivatives of f in the Taylor expansions and the bounds on the remainder terms continuously extend to $\bar{O}$. Thus the assumptions of Whitney extension theorem are fulfilled, which gives that the continuous extension of f lies in $C^m(\bar{O})$ for every m and there exists an extension to $\mathbb{R}^d$ that is real analytic on $\mathbb{R}^d \setminus \bar{O}$. Multiplying with a smooth cut-off function that is 1 on $\bar{O}$ with compact support concludes the construction of $F \in C_c^\infty(\mathbb{R}^d)$. □

Motivated by above proposition, for cracked domains the derivatives of smooth functions will not be required to admit continuous extensions to the closure but instead their boundedness is demanded. This way jumps are still allowed. The set of smooth functions on a domain U is introduced by

$$\mathrm{C}_b^\infty(U, \mathbb{R}^d) := \bigcap_{k \in \mathbb{N}} \mathrm{W}^{k,\infty}(U, \mathbb{R}^d).$$

In fact the differentiability of $\mathrm{C}_b^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ in the interior Ω_{Cr} will not be enough for our arguments. Additionally we need regularity in a neighborhood of Γ_{edge}, thus we write

$$\Gamma_{\mathrm{Cr}}^{(\delta)} := \Gamma_{\mathrm{Cr}} \setminus U_\delta(\Gamma_{\mathrm{edge}}) = \left(\{0\} \times [0,\infty) \times \mathbb{R}^{d-2}\right) \cup \left((0, 1-\delta] \times \{0\} \times \mathbb{R}^d\right)$$

and define

$$\mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) := \bigcup_{\delta \in (0,1)} \mathrm{C}_b^\infty(\Omega \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}, \mathbb{R}^d).$$

In the following lemma we show an extension result analogous to Lemma 4.3 that in particular will provide existence of smooth extensions above and below the crack as premised in Lemma 4.8.

Lemma 4.4. *[Lower and upper extensions on cracked domains] Let* $f \in \mathrm{C}_b^\infty(\Omega \setminus \Gamma_{\mathrm{Cr}}^{(\delta)})$.

There exists a function

$$F \in \mathrm{C}_c^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}) := \{F \in \mathrm{C}_b^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}) \mid \operatorname{supp} F \text{ compact in } \mathbb{R}^d\}$$

with $F|_{\Omega_{\mathrm{Cr}}} = f$ *and for the sets* $B_+^{(\delta)}$ *and* $B_-^{(\delta)}$ *from* (4.4) *there exist two functions*

$$F_\pm \in \mathrm{C}_c(\mathbb{R}^d, \mathbb{R}) \text{ with } F_\pm|_{B_\pm^{(\delta)}} = F|_{B_\pm^{(\delta)}}.$$

Proof. We construct F by using Lemma 4.3 three times: one time in the first step, where we extend f from Ω_{Cr} to $\Omega_{\mathrm{Cr}} \cup B_0^{(\delta)} = \Omega_{\mathrm{Cr}} \cup (B_+^{(\delta)} \cap B_-^{(\delta)})$ by some $F_0 \in \mathrm{C}_c^\infty(\Omega_{\mathrm{Cr}} \cup B_0^{(\delta)})$, then twice in the second step when extending F_0 by two functions F_+ and F_- to $B_+^{(\delta)}$ and $B_-^{(\delta)}$ respectively. Finally F can be defined piecewise on $B_+^{(\delta)}$ and $B_-^{(\delta)}$.

For the first step consider the restriction $f|_{A_+ \cap A_-}$. By assumption (4.3c) this is a bounded Lipschitz domain. Thus we can use Lemma 4.3 and obtain $\tilde{F}_0 \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $\tilde{F}_0|_{A_+ \cap A_-} = f$, such that

$$F_0 \in \mathrm{C}_b^\infty(\Omega_{\mathrm{Cr}} \cup B_0^{(\delta)}), \qquad x \mapsto \begin{cases} f(x) & \text{if } x \in \Omega_{\mathrm{Cr}}, \\ \tilde{F}_0(x) & \text{if } x \in B_0^{(\delta)}, \end{cases}$$

is well-defined. Recall $R_B > 0$ from assumption (4.3d) and let $R > R_B$ with $\operatorname{supp} \tilde{F}_0 \subset B_R(0)$, such that $\operatorname{supp} F_0 \subset B_R(0)$.

For the second step consider the two restrictions $f_\pm = F_0|_{(A_\pm \cup B_0^{(\delta)}) \cap B_R(0)}$ on the domains $(A_\pm \cup B_0^{(\delta)}) \cap B_R(0)$, which have Lipschitz boundary by (4.3d). The intersection with $B_R(0)$ cuts away only points x with values $f_1(x) = 0$ but is necessary to make the domain $(A_\pm \cup B_0^{(\delta)}) \cap B_R(0)$ bounded, thus enabling us to apply Lemma 4.3 again. We obtain $\tilde{F}_\pm \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ that coincide with $f_\pm$ on $(A_\pm \cup B_0^{(\delta)}) \cap B_R(0)$ respectively and in particular

$$\tilde{F}_\pm|_{A_\pm} = f|_{A_\pm}.$$

Choosing a cut-off function $\rho \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $\rho = 1$ on $\operatorname{supp} F_0$ and $\rho = 0$ outside of $B_R(0)$ we have that:

$$\rho \tilde{F}_\pm = F_0 \text{ on } A_\pm \cup B_0^{(\delta)} \quad \text{and}$$
$$\rho \tilde{F}_\pm = \tilde{F}_\pm = f \text{ on } A_\pm.$$

Thus $F \in C_c^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}})$ is well-defined by

$$F \colon \mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}} \to \mathbb{R}, \qquad x \mapsto \begin{cases} \rho \tilde{F}_+(x) & \text{if } x \in B_+^{(\delta)}, \\ \rho \tilde{F}_-(x) & \text{if } x \in B_-^{(\delta)}; \end{cases}$$

and the assertion is proven with $F_\pm := \rho \tilde{F}_\pm$. □

The following proposition shows the density of $\mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ in $\mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$.

Proposition 4.5 (Density of C^∞ on non-Lipschitz domains)**.** *For every* $u \in H^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ *there exist sequences* $\delta_k > 0$ *and* $u_k \in C_b^\infty(\Omega \setminus {\Gamma_{\mathrm{Cr}}}^{(\delta_k)}, \mathbb{R}^d)$ *with* $u_k \to u$ *in* $H^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$.

In particular $\mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ *is dense in* $H^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$.

Proof. It suffices to show locally in a neighborhood $U^{(x^*)}$ of each point $x^* \in \overline{\Omega}$ the existence of $\delta_k^{(x^*)}>0$ and $u_k^{(x^*)} \in C_b^\infty(U^{(x^*)} \setminus \Gamma_{\mathrm{Cr}}{}^{(\delta_k^{(x^*)})}, \mathbb{R}^d)$ with $u_k^{(x^*)} \overset{k\to\infty}{\to} u$ in $H^1(U^{(x^*)} \setminus \Gamma_{\mathrm{Cr}}, \mathbb{R}^d)$, because then by compactness of $\overline{\Omega}$ there is a finite cover of such neighborhoods and we can obtain $\delta_k > 0$ as the minimum of $\delta_k^{(x^*)}$ and u_k as recombination by partition of unity from $u_k^{(x^*)}$.

Away from the crack: For $x^* \in \overline{\Omega} \setminus \Gamma_{\mathrm{Cr}}$ take a neighborhood $U^{(x^*)}$ of x^* such that $U^{(x^*)} \cap (\Omega_{\mathrm{Cr}})$ has Lipschitz boundary, i.e. $U^{(x^*)}$ does not touch the crack Γ_{Cr}. On such Lipschitz domains the density of $C^\infty(\overline{U^{(x^*)}}, \mathbb{R}^d) = C_b^\infty(U^{(x^*)}, \mathbb{R}^d)$ in H^1 is well-known. As $U^{(x^*)}$ does not touch Γ_{Cr}, $\delta_k^{(x^*)}$ is arbitrary.

On crack , away from the edge: For $x^* \in \Gamma_{\mathrm{Cr}} \setminus \Gamma_{\mathrm{edge}}$ there exists a neighborhood $U^{(x^*)}$ of x^* with $\delta_k^{(x^*)} := \mathrm{dist}(U^{(x^*)}, \Gamma_{\mathrm{edge}}) > 0$, such that $U^{(x^*)} \setminus \Gamma_{\mathrm{Cr}}$ consists of the two connected components with Lipschitz boundary $U^{(x^*)} \cap \Omega_+$ and $U^{(x^*)} \cap \Omega_-$. An approximating sequence $u_k^{(x^*)} \in C_b^\infty(U^{(x^*)} \setminus \Gamma_{\mathrm{Cr}}{}^{(\delta_k^{(x^*)})}, \mathbb{R}^d)$ can be defined piecewise from respective approximating sequences on $U^{(x^*)} \cap \Omega_+$ and $U^{(x^*)} \cap \Omega_-$ from $C^\infty(\overline{U^{(x^*)} \cap \Omega_+}, \mathbb{R}^d)$ and $C^\infty(\overline{U^{(x^*)} \cap \Omega_-}, \mathbb{R}^d)$.

Crack edge: For $x^* \in \Gamma_{\mathrm{edge}}$ take a neighborhood $U^{(x^*)}$ of x^* that does not touch the crack kink Γ_{kink}, such that $U^{(x^*)} \setminus \Gamma_{\mathrm{Cr}} = U^{(x^*)} \setminus (-\infty, 1] \times \{0\} \times \mathbb{R}^{d-2}$.

The basic idea is again to take separate approximating sequences above and below the crack and recombine, however we have to refine the procedure to make the recombination possible.

Extend $u \in H^1((U \cap \Omega) \setminus (-\infty, 1] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$ to $u \in H^1(\mathbb{R}^d \setminus (-\infty, 1] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$ and consider the translations $u_k \colon x \mapsto u(x + \frac{1}{k}e_1)$, where $e_1 = (1, 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^d$. On the Lipschitz domains $\mathbb{R} \times (-\infty, 0) \times \mathbb{R}^{d-1}$ and $\mathbb{R} \times (0, \infty) \times \mathbb{R}^{d-1}$ the strong H^1 convergence of the translations is well known, thus $u_k \to u$ in $H^1(\mathbb{R}^d \setminus (-\infty, 1] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$ follows. Furthermore u_k has no jump in a $\frac{1}{k}$-neighborhood of Γ_{edge}.

We are left to find an approximating sequence in $C_{b,*}^\infty(\mathbb{R}^d \setminus (-\infty, 1] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$ for each $u_k \in H^1(\mathbb{R}^d \setminus (-\infty, 1 - \frac{1}{k}] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$, as a diagonal sequence then gives the desired approximating sequence in $C_b^\infty(U \setminus \Gamma_{\mathrm{Cr}}, \mathbb{R}^d)$ for u. We will find such sequences $u_{k,m} \overset{m\to\infty}{\to} u_k$ by taking to separate sequences above and below the crack and recombine, where in the recombination the neighborhood, in which $u_{k,m}$ has no jump, is a little smaller then $\frac{1}{k}$.

For each fixed u_k consider approximating sequences $u_{k,m}^+ \in C^\infty(\overline{A_k^+}, \mathbb{R}^d)$ and $u_{k,m}^- \in C^\infty(\overline{A_k^-}, \mathbb{R}^d)$ on the Lipschitz domains $A_k^+ := \{x_1 \geq 1 - \frac{1}{k} \text{ or } x_2 \geq 0\}$ and $A_k^- := \{x_1 \geq 1 - \frac{1}{k} \text{ or } x_2 \leq 0\}$. Further choose a cut-off function $\phi_k \in C_b^\infty(\mathbb{R}^d \setminus (-\infty, 1 - \frac{1}{2k}] \times \{0\} \times \mathbb{R}^{d-2}, [0, 1])$ that is uncracked in a $\frac{1}{2k}$-neighborhood of Γ_{edge} with $\phi_k = 1$ on $A_k^+ \setminus A_k^-$ and $\phi_k = 0$ on $A_k^- \setminus A_k^+$, then $u_{k,m} := \phi_k u_{k,m}^+ + (1 - \phi_k) u_{k,m}^- \in C_b^\infty(\mathbb{R}^d \setminus (-\infty, 1 - \frac{1}{2k}] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)$ has no jump in a $\frac{1}{2k}$-neighborhood of Γ_{edge} and is an approximating sequence for u_k because

$$
\begin{aligned}
&\|u_k - u_{k,m}\|_{H^1(\mathbb{R}^d \setminus (-\infty,1] \times \{0\} \times \mathbb{R}^{d-2}, \mathbb{R}^d)} \\
&\leq \|\phi_k\|_{W^{1,\infty}(\mathbb{R}^d \setminus (-\infty,1] \times \{0\} \times \mathbb{R}^{d-2}, [0,1])} \big(\|u_k - u_{k,m}^+\|_{H^1(A_k^+, \mathbb{R}^d)} + \|u_k - u_{k,m}^-\|_{H^1(A_k^-, \mathbb{R}^d)}\big),
\end{aligned}
$$

thus the assertion follows. □

We conclude this section with the following lemma, that generalizes Lipschitz-type estimates $|f(a) - f(b)| \leq \|\nabla f\|_{\mathrm{L}^\infty} |a - b|$ to higher order Taylor-type estimates for non-scalar functions f.

Lemma 4.6 (Taylor estimate of order k)**.** *Let U be open in a normed vector space X containing the line segment $[a, x] := \{a + t(x - a) \mid t \in [0, 1] \subset \mathbb{R}\}$ from $a \in U$ to $x \in U$. Let $f : U \to Y$ be $k+1$ times Fréchet differentiable.*

Then there exists $\xi \in [a, x]$, such that:

$$\Big\| f(x) - \sum_{|\nu| \leq k} \mathrm{D}^\nu f(a)(x - a)^\nu \Big\| \leq \Big\| \sum_{|\nu| = k} \frac{\mathrm{D}^\nu f(\xi)}{(k+1!)} (x - a)^\nu \Big\|.$$

Proof. Let $\phi \in Y^*$ be arbitrary but fixed, then $\mathrm{D}^\nu(\phi \circ f) = \phi \circ \mathrm{D}^\nu f$. Applying the scalar Taylor Theorem to the function $\phi \circ f \colon A \to \mathbb{R}$ we get

$$\begin{aligned} \phi\Big(f(x) - \sum_{|\nu| \leq k} \frac{\mathrm{D}^\nu f(a)}{\nu!} (x - a)^\nu\Big) &= (\phi \circ f)(x) - \sum_{|\nu| \leq k} \frac{\mathrm{D}^\nu (\phi \circ f)(a)}{\nu!} (x - a)^\nu \\ &= \sum_{|\nu| = k+1} \frac{\mathrm{D}^\nu (\phi \circ f)(\xi)}{\nu!} (x - a)^\nu \\ &= \phi\Big(\sum_{|\nu| = k+1} \frac{\mathrm{D}^\nu f(\xi)}{\nu!} (x - a)^\nu \Big) \end{aligned}$$

Let $v := f(x) - \sum_{|\nu| \leq k} \mathrm{D}^\nu f(a)(x - a)^\nu$ and assume w.l.o.g. $v \neq 0$, otherwise the assertion is trivial. By Hahn-Banach Theorem, there exists $\phi \in Y^*$ such that $\phi(v) = \|v\|$ and $\|\phi\| = 1$. Insert this particular ϕ into above equality to conclude:

$$\|v\| = |\phi(v)| = \Big|\phi\Big(\sum_{|\nu| = k+1} \frac{\mathrm{D}^\nu f(\xi)}{\nu!} (x - a)^\nu \Big)\Big| \leq \|\phi\| \Big\| \sum_{|\nu| = k+1} \frac{\mathrm{D}^\nu f(\xi)}{\nu!} (x - a)^\nu \Big\|. \quad \square$$

4.3 Mutual recovery sequence without constraint

The mutual recovery sequence is needed in the proof of the evolutionary Γ-convergence in Section 4.5 to give a lim sup-estimate on the transition cost $\widetilde{\mathcal{T}}_\varepsilon$ and $\mathcal{T}_\varepsilon^{(\alpha)}$ respectively. In this section we restrict to considering the transition cost $\widetilde{\mathcal{T}}_\varepsilon$ without the constraints in the stored energies:

$$\begin{aligned} \widetilde{\mathcal{T}}_\varepsilon(t, u, z, \hat{u}, \hat{z}) &= \widetilde{\mathcal{G}}_\varepsilon(t, \hat{u}, \hat{z}) - \widetilde{\mathcal{G}}_\varepsilon(t, u, z) + \mathcal{D}_\varepsilon(z, \hat{z}) \\ &= \widetilde{\mathcal{E}}_\varepsilon(\hat{u}, \hat{z}) - \widetilde{\mathcal{E}}_\varepsilon(u, z) - \langle \ell(t), \hat{u} - u \rangle + \mathcal{D}_\varepsilon(z, \hat{z}). \end{aligned} \tag{4.23}$$

Thus $\widetilde{\mathcal{T}}_\varepsilon$ consists of a sum of integrals and for the proof of the mutual recovery sequence we can exploit the additivity of $\widetilde{\mathcal{T}}_\varepsilon$ in the domain. Therefor we will notate the domain of the integrals as an additional argument. Introducing for a domain $O \subset \mathbb{R}^d$ the restricted energy and dissipation

$$\begin{aligned} \widetilde{\mathcal{E}}_\varepsilon(O, u, z) &= \frac{1}{\varepsilon^2} \int_O W_{\mathrm{el}}\big((I + \varepsilon \nabla u)(I + \varepsilon z)^{-1}\big) \, \mathrm{d}x + \frac{1}{\varepsilon^2} \int_O W_{\mathrm{h}}(I + \varepsilon z) \, \mathrm{d}x, \\ \widetilde{\mathcal{E}}_0(O, u, z) &= \int_O \frac{1}{2} |\nabla u - z|_{\mathbb{C}}^2 \, \mathrm{d}x + \int_O \frac{1}{2} |z|_{\mathbb{H}}^2 \, \mathrm{d}x, \end{aligned}$$

$$\mathcal{D}_\varepsilon(O, z_1, z_2) = \int_O D_\varepsilon(z_1, z_2)\,\mathrm{d}x \quad \text{and} \quad \mathcal{D}_0(O, z_1, z_2) = \int_O D_0(z_1, z_2)\,\mathrm{d}x$$

we define for $\varepsilon \geq 0$ the restricted transition cost:

$$\widetilde{\mathcal{T}}_\varepsilon(O, t, u, z, \hat{u}, \hat{z}) := \widetilde{\mathcal{E}}_\varepsilon(O, \hat{u}, \hat{z}) - \widetilde{\mathcal{E}}_\varepsilon(O, u, z) - \int_O \ell(t) \cdot (\hat{u} - u)\,\mathrm{d}x + \mathcal{D}_\varepsilon(O, z, \hat{z}). \quad (4.24)$$

Using the disjoint cover $\Omega_{\mathrm{Cr}} = A_+ \cup (A_- \setminus A_+)$ into Lipschitz domains from (4.3c) for $\varepsilon \geq 0$ we have the equality

$$\begin{aligned}\widetilde{\mathcal{T}}_\varepsilon(t, u, z, \hat{u}, \hat{z}) &= \widetilde{\mathcal{T}}_\varepsilon(\Omega_{\mathrm{Cr}}, t, u, z, \hat{u}, \hat{z}) \\ &= \widetilde{\mathcal{T}}_\varepsilon(A_+, t, u, z, \hat{u}, \hat{z}) + \widetilde{\mathcal{T}}_\varepsilon(A_- \setminus A_+, t, u, z, \hat{u}, \hat{z}), \end{aligned} \quad (4.25)$$

which we will use to reduce the proof of the mutual recovery sequence on the non-Lipschitz Ω_{Cr} to the case of Lipschitz domains. This is exactly the contend of [MS13, Lemma 3.6], which we will reenact in the following.

Proposition 4.7 (Mutual recovery sequence on Lipschitz domain)**.** *Let $O \subset \mathbb{R}^d$ be a Lipschitz domain, $(u_\varepsilon, z_\varepsilon) \rightharpoonup (u_0, z_0)$ weakly in $\mathrm{H}^1(O, \mathbb{R}^d) \times \mathrm{L}^2(O, \mathbb{R}^{d\times d})$ with*

$$\sup \widetilde{\mathcal{E}}_\varepsilon(O, u_\varepsilon, z_\varepsilon) < \infty.$$

Moreover, let $(\hat{u}_0, \hat{z}_0) := (u_0, z_0) + (\tilde{u}, \tilde{z})$ with $(\tilde{u}, \tilde{z}) \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d, \mathbb{R}^d) \times \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^d, \mathbb{R}^d)$.

Then the sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)$ given by

$$\hat{u}_\varepsilon := \frac{1}{\varepsilon}\big((\mathrm{id} + \varepsilon\tilde{u}) \circ (\mathrm{id} + \varepsilon u_\varepsilon) - \mathrm{id}\big) \qquad \textit{and}$$

$$\hat{z}_\varepsilon := \begin{cases} \frac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I\big) & \textit{on } O_\varepsilon, \\ z_\varepsilon & \textit{otherwise,} \end{cases}$$

where $O_\varepsilon := \{x \in O \mid \exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) \in K\}$, fulfills

(a) $\|\hat{u}_\varepsilon - u_\varepsilon - \tilde{u}\|_{\mathrm{H}^1(O,\mathbb{R}^d)} \leq c\varepsilon$,

(b) $\|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(O,\mathbb{R}^d)} \leq c\varepsilon^2$ *and*

(c) $\limsup_{\varepsilon\to 0} \widetilde{\mathcal{T}}_\varepsilon(O, t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \leq \widetilde{\mathcal{T}}_0(O, t, u_0, z_0, \hat{u}_0, \hat{z}_0)$.

Proof. We decompose the proof into four steps:

- weak convergence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \to (\hat{u}_0, \hat{z}_0)$,
- convergence of dissipation: $\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(O, z_\varepsilon, \hat{z}_\varepsilon) \leq \mathcal{D}_0(O, z_0, \hat{z}_0) = \mathcal{R}(O, \tilde{z})$,
- cancellation of jumps in stored energy:
$\limsup_{\varepsilon\to 0}\Big(\widetilde{\mathcal{E}}_\varepsilon(O, \hat{u}_\varepsilon, \hat{z}_\varepsilon) - \widetilde{\mathcal{E}}_\varepsilon(O, u_\varepsilon, z_\varepsilon)\Big) \leq \widetilde{\mathcal{E}}_0(O, u_0, z_0) - \widetilde{\mathcal{E}}_0(O, u_0, z_0)$ and
- final conclusion of proof.

Convergence of $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)$: For the convergence of $\hat{u}_\varepsilon$ in L^2 let us consider

$$(\hat{u}_\varepsilon - u_\varepsilon - \tilde{u})(x) = \Big(\frac{1}{\varepsilon}((\mathrm{id} + \varepsilon\tilde{u}) \circ (\mathrm{id} + \varepsilon u_\varepsilon) - \mathrm{id}) - u_\varepsilon - \tilde{u}\Big)(x)$$
$$= \tilde{u}(x + \varepsilon u_\varepsilon(x)) - \tilde{u}(x),$$

thus by Lipschitz continuity of $\tilde{u}$ we have:

$$\|\hat{u}_\varepsilon - u_\varepsilon - \tilde{u}\|_{\mathrm{L}^2} = \|\tilde{u}(\mathrm{id} + \varepsilon u_\varepsilon) - \tilde{u}\|_{\mathrm{L}^2} \leq \varepsilon \|\nabla\tilde{u}\|_{\mathrm{L}^\infty} \|u_\varepsilon\|_{\mathrm{L}^2}.$$

For the gradient of $\hat{u}_\varepsilon$ we have

$$\begin{aligned}\nabla(\hat{u}_\varepsilon - u_\varepsilon - \tilde{u})(x) &= \nabla\Big(\tilde{u}(x + \varepsilon u_\varepsilon(x))\Big) - \nabla\tilde{u}(x) \\ &= \nabla\tilde{u}(x + \varepsilon u_\varepsilon(x))(I + \varepsilon\nabla u_\varepsilon(x)) - \nabla\tilde{u}(x) \\ &= \Big(\nabla\tilde{u}(x + \varepsilon u_\varepsilon(x)) - \nabla\tilde{u}(x)\Big) + \varepsilon\nabla\tilde{u}(x + \varepsilon u_\varepsilon(x)) \cdot \nabla u_\varepsilon(x).\end{aligned}$$

The first summand on the right hand side is bounded by Lipschitz continuity of $\nabla\tilde{u}$ by $\varepsilon\|\nabla^2\tilde{u}\|_{\mathrm{L}^\infty}|u_\varepsilon(x)|$ and the second is bounded by $\varepsilon\|\nabla\tilde{u}\|_{\mathrm{L}^\infty}|\nabla u_\varepsilon(x)|$, thus we get

$$\|\nabla\hat{u}_\varepsilon - \nabla u_\varepsilon - \nabla\tilde{u}\|_{\mathrm{L}^2} \leq \varepsilon(\|\nabla^2\tilde{u}\|_{\mathrm{L}^\infty}\|u_\varepsilon\|_{\mathrm{L}^2} + \|\nabla\tilde{u}\|_{\mathrm{L}^\infty}\|\nabla u_\varepsilon\|_{\mathrm{L}^2}).$$

Since $\tilde{u} \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and u_ε is H^1-bounded by $u_\varepsilon \stackrel{\mathrm{H}^1}{\rightharpoonup} u_0$, we finally get:

$$\begin{aligned}\|\hat{u}_\varepsilon - u_\varepsilon - \tilde{u}\|_{\mathrm{H}^1} &\leq \varepsilon(\|\nabla\tilde{u}\|_{\mathrm{L}^\infty}\|u_\varepsilon\|_{\mathrm{L}^2} + \|\nabla^2\tilde{u}\|_{\mathrm{L}^\infty}\|u_\varepsilon\|_{\mathrm{L}^2} + \|\nabla\tilde{u}\|_{\mathrm{L}^\infty}\|\nabla u_\varepsilon\|_{\mathrm{L}^2}) \\ &\leq \varepsilon\|\nabla\tilde{u}\|_{\mathrm{W}^{1,\infty}}\|u_\varepsilon\|_{\mathrm{H}^1} \leq \varepsilon\tilde{c}_0.\end{aligned}$$

This concludes the proof of (a) and gives the strong and weak convergences

$$\nabla\hat{u}_\varepsilon - \nabla u_\varepsilon \to \nabla\tilde{u} \qquad \text{in } \mathrm{L}^2(O, \mathbb{R}^d) \text{ and} \tag{4.26}$$
$$\nabla\hat{u}_\varepsilon \rightharpoonup \nabla\tilde{u} - \nabla u_0 = \nabla\hat{u}_0 \qquad \text{in } \mathrm{L}^2(O, \mathbb{R}^d). \tag{4.27}$$

Let us come to the convergence of $\hat{z}_\varepsilon$. By assumption on W_h we have $I + \varepsilon z_\varepsilon \in K$ a. e., thus $I + \varepsilon\hat{z}_\varepsilon \in K$ a. e. and by (4.10) we get $\|I + \varepsilon\hat{z}_\varepsilon\|_{\mathrm{L}^\infty} \leq c_K$. Furthermore, from $\det\exp(\varepsilon\tilde{z}) = \exp(\varepsilon \operatorname{tr}\tilde{z}) = 1$, which is easy to see by Jordan normal form, we get $\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) \in \mathrm{SL}(d)$ a. e.

Since $\hat{z}_\varepsilon$ is defined piecewise, we want to estimate the measure of $O \setminus O_\varepsilon$

$$\begin{aligned}|O \setminus O_\varepsilon| &= \int_{O\setminus O_\varepsilon} 1 \,\mathrm{d}x \stackrel{(4.11)}{\leq} c_K^2 \int_{O\setminus O_\varepsilon} |\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I|^2 \,\mathrm{d}x \\ &\leq c_K^2 \int_O |\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I|^2 \,\mathrm{d}x \\ &\leq c_K^2 \int_O |\exp(\varepsilon\tilde{z})\varepsilon z_\varepsilon + \exp(\varepsilon\tilde{z}) - I|^2 \,\mathrm{d}x \\ &\leq 2c_K^2 \int_O |\exp(\varepsilon\tilde{z})\varepsilon z_\varepsilon|^2 + |\exp(\varepsilon\tilde{z}) - I|^2 \,\mathrm{d}x \\ &\leq \tilde{c}_1\varepsilon^2(\int_O |z_\varepsilon|^2 \,\mathrm{d}x + C),\end{aligned}$$

where in the last inequality for the first summand we used boundedness of $\exp(\varepsilon\tilde{z})$ for bounded ε and for the second summand we used local Lipschitz continuity of exp to get for some $\tau_\varepsilon \in (0,1)$

$$|\exp(\varepsilon\tilde{z}) - I| = |\exp(\varepsilon\tilde{z}) - \exp(0)| \leq \exp(\tau_\varepsilon\varepsilon\tilde{z})|\varepsilon\tilde{z}|,$$

which is bounded by $c|\varepsilon\tilde{z}|$ for again bounded ε. Together with the weak convergence $z_\varepsilon \overset{\mathrm{L}^2}{\rightharpoonup} z_0$ and thus L^2-boundedness of z_ε we get:

$$|O \setminus O_\varepsilon| \leq \tilde{c}_2\varepsilon^2. \tag{4.28}$$

To check convergence of $\hat{z}_\varepsilon$ let us consider $\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z})$ on O_ε and $O \setminus O_\varepsilon$ separately. On $O \setminus O_\varepsilon$ we have $\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z}) = -\tilde{z}$ and since $\tilde{z}$ is bounded we get

$$\|\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z})\|^2_{\mathrm{L}^2(O\setminus O_\varepsilon,\mathbb{R}^{d\times d})} \leq \big|O \setminus O_\varepsilon\big| \, \|\tilde{z}\|^2_{\mathrm{L}^\infty} \leq \tilde{c}_3\varepsilon^2.$$

On O_ε we have

$$\begin{aligned}
\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z}) &= \frac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I\big) - (z_\varepsilon + \tilde{z}) \\
&= \frac{1}{\varepsilon}\Big(\exp(\varepsilon\tilde{z}) + \exp(\varepsilon\tilde{z})\varepsilon z_\varepsilon - I - \varepsilon(z_\varepsilon + \tilde{z})\Big) \\
&= \frac{1}{\varepsilon}\Big(\exp(\varepsilon\tilde{z}) - I - \varepsilon\tilde{z}\Big) + z_\varepsilon\big(\exp(\varepsilon\tilde{z}) - I\big).
\end{aligned}$$

We can use on one hand a first order Taylor estimate on the first summand to get for some $\tau_\varepsilon \in (0,1)$

$$\big|\exp(\varepsilon\tilde{z}) - I + \varepsilon\tilde{z}\big| = \big|\exp(\varepsilon\tilde{z}) - \exp(0) + \exp'(0)\varepsilon\tilde{z}\big| \leq \big|\exp''(\tau_\varepsilon\varepsilon\tilde{z})\big|\,|\varepsilon\tilde{z}|^2,$$

which is bounded by $\tilde{c}_4\varepsilon^2$ if ε is bounded and on the other hand a Lipschitz estimate on the second summand to get

$$\big|z_\varepsilon\big(\exp(\varepsilon\tilde{z}) - I\big)\big| \leq |z_\varepsilon|\,\big|\exp'(\mu_\varepsilon\varepsilon\tilde{z})\big|\,|\varepsilon\tilde{z}|$$

for some $\mu_\varepsilon \in (0,1)$ which is bounded by $|z_\varepsilon|\tilde{c}_5\varepsilon$ for bounded ε. Altogether we arrive at

$$\big\|\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z})\big\|^2_{\mathrm{L}^2(O_\varepsilon,\mathbb{R}^{d\times d})} \leq \tilde{c}_4^2\varepsilon^2|O| + \|z_\varepsilon\|^2_{\mathrm{L}^2}\tilde{c}_5^2\varepsilon^2 \leq \tilde{c}_6\varepsilon^2.$$

Thus we have $\big\|\hat{z}_\varepsilon - (z_\varepsilon + \tilde{z})\big\|^2_{\mathrm{L}^2(O,\mathbb{R}^{d\times d})} \leq \varepsilon^2(\tilde{c}_3 + \tilde{c}_6)$ and we conclude the strong convergence

$$\hat{z}_\varepsilon - z_\varepsilon \to \tilde{z} \text{ in } \mathrm{L}^2(O, \mathbb{R}^{d\times d}) \tag{4.29}$$

as well as the weak convergence

$$\hat{z}_\varepsilon \rightharpoonup \tilde{z} - z_0 = \hat{z}_0 \text{ in } \mathrm{L}^2(O, \mathbb{R}^{d\times d}). \tag{4.30}$$

In addition to the convergences of the displacements and plastic strains we want to investigate the convergence of the linearized elastic strains $A_\varepsilon = \frac{1}{\varepsilon}(F_{\mathrm{el}} - I)$ and $\widehat{A}_\varepsilon = \frac{1}{\varepsilon}(\widehat{F}_{\mathrm{el}} - I)$ which in terms of the linearized quantities read

$$A_\varepsilon = \frac{1}{\varepsilon}((I + \varepsilon\nabla u_\varepsilon)(I + \varepsilon z_\varepsilon)^{-1} - I) \qquad \text{and}$$

$$\widehat{A}_\varepsilon = \frac{1}{\varepsilon}((I + \varepsilon\nabla\hat{u}_\varepsilon)(I + \varepsilon\hat{z}_\varepsilon)^{-1} - I).$$

For the convergence of A_ε we can proceed as in the proof of Proposition 3.6. Let us consider the inverse of the plastic part $(I + \varepsilon z_\varepsilon)^{-1}$. On one hand, by (4.10) we have the L^∞-bound

$$\|(I + \varepsilon z_\varepsilon)^{-1} - (I - \varepsilon z_\varepsilon)\|_{\mathrm{L}^\infty} \leq c_K$$

on the other hand rewriting

$$\begin{aligned}(I + \varepsilon z_\varepsilon)^{-1} - (I - \varepsilon z_\varepsilon) &= (I + \varepsilon z_\varepsilon)^{-1}(I - (I + \varepsilon z_\varepsilon)(I - \varepsilon z_\varepsilon)) \\ &= \varepsilon^2 (I + \varepsilon z_\varepsilon)^{-1} z_\varepsilon^2\end{aligned}$$

gives an L^1-bound

$$\|(I + \varepsilon z_\varepsilon)^{-1} - (I - \varepsilon z_\varepsilon)\|_{\mathrm{L}^1} \leq \varepsilon^2 c_K \|z_\varepsilon\|_{\mathrm{L}^2}^2.$$

Together, this gives a bound on the L^2-norm of $d_\varepsilon := \frac{1}{\varepsilon}((I + \varepsilon z_\varepsilon)^{-1} - (I - \varepsilon z_\varepsilon))$:

$$\|d_\varepsilon\|_{\mathrm{L}^2}^2 \leq \|d_\varepsilon\|_{\mathrm{L}^1}\|d_\varepsilon\|_{\mathrm{L}^\infty} \leq \frac{1}{\varepsilon^2}\varepsilon^2 c_K^2 \|z_\varepsilon\|_{\mathrm{L}^2}^2 \leq C.$$

In particular a subsequence of d_ε converges weakly in L^2 to some limit and since the above L^1-bound means $d_\varepsilon \xrightarrow{\mathrm{L}^1} 0$, the limits have to coincide and we get:

$$\frac{1}{\varepsilon}((I + \varepsilon z_\varepsilon)^{-1} - I) \rightharpoonup -z_0 \qquad \text{in } \mathrm{L}^2(O, \mathbb{R}^d). \tag{4.31}$$

Using d_ε we can rewrite

$$\begin{aligned}A_\varepsilon - \nabla u_\varepsilon + z_\varepsilon &= \frac{1}{\varepsilon}((I + \varepsilon\nabla u_\varepsilon)(I - \varepsilon z_\varepsilon + \varepsilon d_\varepsilon) - I) - \nabla u_\varepsilon + z_\varepsilon \\ &= d_\varepsilon + \varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon).\end{aligned}$$

The weak L^2-convergence $d_\varepsilon \rightharpoonup 0$ we saw above. Furthermore on one hand the established L^∞-bound on $\varepsilon d_\varepsilon$ and (4.10) give an L^2-bound on the second summand

$$\|\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon)\|_{\mathrm{L}^2} \leq \|\nabla u_\varepsilon\|_{\mathrm{L}^2}(\|\varepsilon d_\varepsilon\|_{\mathrm{L}^\infty} + \|\varepsilon z_\varepsilon\|_{\mathrm{L}^\infty}) \leq C,$$

and on the other hand we can estimate

$$\|\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon)\|_{\mathrm{L}^1} \leq \varepsilon\|\nabla u_\varepsilon\|_{L^2}(\|d_\varepsilon\|_{L^2} + \|z_\varepsilon\|_{L^2}) \leq c\varepsilon,$$

such that $\varepsilon\nabla u_\varepsilon(d_\varepsilon - z_\varepsilon) \overset{\mathrm{L}^2}{\rightharpoonup} 0$ follows and we get:

$$A_\varepsilon \rightharpoonup \nabla u_0 - z_0 \text{ in } \mathrm{L}^2(O, \mathbb{R}^{d\times d}). \tag{4.32}$$

Finally we want to show strong convergence of the difference $\hat{A}_\varepsilon - A_\varepsilon$. Using the indicator function $\mathbb{1}_{O_\varepsilon}$ we have $(I + \varepsilon\hat{z}_\varepsilon)^{-1} = (I + \varepsilon z_\varepsilon)^{-1}\exp(-\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z})$ and we can write:

$$\hat{A}_\varepsilon - A_\varepsilon - \nabla\tilde{u} + \mathbb{1}_{O_\varepsilon}\tilde{z} = \frac{1}{\varepsilon}((I + \varepsilon\nabla\hat{u}_\varepsilon)(I + \varepsilon z_\varepsilon)^{-1}\exp(-\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z}) - I)$$

$$
\begin{aligned}
&- \frac{1}{\varepsilon}((I + \varepsilon \nabla \hat{u}_\varepsilon)(I + \varepsilon z_\varepsilon)^{-1} - I) - \nabla \tilde{u} + \mathbb{1}_{O_\varepsilon} \tilde{z} \\
&= (I + \varepsilon z_\varepsilon)^{-1} \frac{1}{\varepsilon}(\exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - I) + \mathbb{1}_{O_\varepsilon} \tilde{z} \\
&\quad + (\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon)(I + \varepsilon z_\varepsilon)^{-1} - \nabla \tilde{u} \\
&\quad + \nabla \hat{u}_\varepsilon (I + \varepsilon z_\varepsilon)^{-1}(\exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - I) \\
&= (\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon - \nabla \tilde{u})(I + \varepsilon z_\varepsilon)^{-1} \\
&\quad + (I + \varepsilon z_\varepsilon)^{-1} \frac{1}{\varepsilon}\big(\exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - (I - \varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z})\big) \\
&\quad + \nabla \tilde{u}((I + \varepsilon z_\varepsilon)^{-1} - I) - ((I + \varepsilon z_\varepsilon)^{-1} - I)\tilde{z} \\
&\quad + \nabla \hat{u}_\varepsilon (I + \varepsilon z_\varepsilon)^{-1}(\exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - I).
\end{aligned}
$$

Consequently we can bound the L^2-norm:

$$
\begin{aligned}
\|\hat{A}_\varepsilon - A_\varepsilon - \nabla \tilde{u} + \mathbb{1}_{O_\varepsilon} \tilde{z}\|_{\mathrm{L}^2} = {} & c_K \|\nabla \hat{u}_\varepsilon - \nabla u_\varepsilon - \nabla \tilde{u}\|_{\mathrm{L}^2} \\
& + c_K \frac{1}{\varepsilon} \big\| \exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - (I - \varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) \big\|_{\mathrm{L}^2} \\
& + \big\|(I + \varepsilon z_\varepsilon)^{-1} - I\big\|_{\mathrm{L}^2} (\|\nabla \tilde{u}\|_{\mathrm{L}^\infty} + \|\tilde{z}\|_{\mathrm{L}^\infty}) \\
& + c_K \|\nabla \hat{u}_\varepsilon\|_{\mathrm{L}^2} \big\| \exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - I \big\|_{\mathrm{L}^\infty}.
\end{aligned}
$$

The first summand on the right hand side vanishes in the limit $\varepsilon \to 0$ by (4.26), the second one vanishes since a first order Taylor estimate gives a factor ε^2. By the weak convergence (4.31) the factor $\|(I + \varepsilon z_\varepsilon)^{-1} - I\|_{\mathrm{L}^2}$ multiplied with $\frac{1}{\varepsilon}$ is still bounded, hence the third summand vanishes as $\varepsilon \to 0$. Finally the fourth summand vanishes as by (4.27) $\nabla \hat{u}$ is bounded in L^2 and a Lipschitz estimate gives $\big\| \exp(-\varepsilon \mathbb{1}_{O_\varepsilon} \tilde{z}) - I \big\|_{\mathrm{L}^\infty} \leq \varepsilon \|\nabla \exp\|_{\mathrm{L}^\infty} \|\tilde{z}\|_{\mathrm{L}^\infty}$.

Since obviously $\mathbb{1}_{O_\varepsilon} \tilde{z} \to \tilde{z}$ strongly in L^2 we have on one hand

$$
\hat{A}_\varepsilon - A_\varepsilon \to \nabla \tilde{u} - \tilde{z} \qquad \text{strongly in } \mathrm{L}^2(O, \mathbb{R}^{d \times d}) \tag{4.33}
$$

and on the other hand by (4.32)

$$
\hat{A}_\varepsilon + A_\varepsilon \rightharpoonup \nabla(\hat{u}_0 - \hat{z}_0) + (u_0 - z_0) \qquad \text{weakly in } \mathrm{L}^2(O, \mathbb{R}^{d \times d}). \tag{4.34}
$$

Limsup on the dissipation: By definition it is $\hat{z}_\varepsilon = z_\varepsilon$ on $O \setminus O_\varepsilon$, thus $D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = 0$ on $O \setminus O_\varepsilon$ and we get

$$
\begin{aligned}
\mathcal{D}_\varepsilon(O, z_\varepsilon, \hat{z}_\varepsilon) &= \frac{1}{\varepsilon} \int_{O_\varepsilon} D(I + \varepsilon z_\varepsilon, I + \varepsilon \hat{z}_\varepsilon) \, \mathrm{d}x = \frac{1}{\varepsilon} \int_{O_\varepsilon} D(I + \varepsilon z_\varepsilon, \exp(\varepsilon \tilde{z})(I + \varepsilon z_\varepsilon)) \, \mathrm{d}x \\
&= \frac{1}{\varepsilon} \int_{O_\varepsilon} D(I, \exp(\varepsilon \tilde{z})) \, \mathrm{d}x.
\end{aligned} \tag{4.35}
$$

Recall the definition (4.13) of $D(I, \hat{P})$ in terms of an infimum. Inserting $P(t) = \exp(t\varepsilon\tilde{z})$ into it we get:

$$
D(I, \exp(\varepsilon \tilde{z})) \leq \int_0^1 R(\dot{P}P^{-1}) \, \mathrm{d}t = \int_0^1 R(\varepsilon \tilde{z}) \, \mathrm{d}t = \varepsilon R(\tilde{z}). \tag{4.36}
$$

Since by $\tilde{z}_\varepsilon \in \mathrm{C}^\infty(O, \mathbb{R}^{d\times d})$ and assumption (4.12b) this gives the dominating function

$$D_\varepsilon(z_\varepsilon, \hat{z}_\varepsilon) = \frac{1}{\varepsilon} D(I, \exp(\varepsilon\tilde{z})) \leq R(\tilde{z}) \leq C_R \|\tilde{z}_\varepsilon\|_{\mathrm{L}^\infty},$$

we may apply the lim sup version of Fatou's Lemma on (4.35) to obtain the desired estimate:

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(O, z_\varepsilon, \hat{z}_\varepsilon) &= \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{O_\varepsilon} D(I, \exp(\varepsilon\tilde{z}))\,\mathrm{d}x \\ &\leq \frac{1}{\varepsilon} \int_O \limsup_{\varepsilon\to 0} \mathbb{1}_{O_\varepsilon} D(I, \exp(\varepsilon\tilde{z}))\,\mathrm{d}x \leq \int_O R(\tilde{z})\,\mathrm{d}x \\ &= \mathcal{D}_0(O, z_0, \hat{z}_0). \end{aligned} \tag{4.37}$$

<u>Limsup on the difference of stored energies:</u> Fix an arbitrary $\delta > 0$ and let $r_{\mathrm{el}}(\delta)$ and $r_{\mathrm{h}}(\delta)$ be from assumptions (4.7d) and (4.9d) on W_{el} and W_{h}, respectively. For every $\varepsilon > 0$ we define the *good sets*

$$\left.\begin{aligned} U_\varepsilon^{(\delta)} &:= \{x \in O \mid |\varepsilon A_\varepsilon(x)| + |\varepsilon \widehat{A}_\varepsilon(x)| \leq r_{\mathrm{el}}(\delta)\} \\ Z_\varepsilon^{(\delta)} &:= \{x \in O \mid |\varepsilon z_\varepsilon(x)| + |\varepsilon \hat{z}_\varepsilon(x)| \leq r_{\mathrm{h}}(\delta)\} \end{aligned}\right\} \tag{4.38}$$

on which we can replace the nonlinear densities W_{el} and W_{h} by the above mentioned quadratic extensions (4.7d) and (4.9d):

$$\begin{aligned} &\frac{1}{\varepsilon^2} W_{\mathrm{el}}((I + \varepsilon\nabla\hat{u}_\varepsilon)(I + \varepsilon\hat{z}_\varepsilon)^{-1}) - \frac{1}{\varepsilon^2} W_{\mathrm{el}}((I + \varepsilon\nabla u_\varepsilon)(I + \varepsilon z_\varepsilon)^{-1}) \\ &\quad = \frac{1}{\varepsilon^2} W_{\mathrm{el}}(I + \varepsilon\hat{A}_\varepsilon) - \frac{1}{\varepsilon^2} W_{\mathrm{el}}(I + \varepsilon A_\varepsilon) \\ &\quad \overset{(4.7\mathrm{d})}{\leq} \frac{1}{2}\langle \hat{A}_\varepsilon, \mathbb{C}\hat{A}_\varepsilon\rangle - \frac{1}{2}\langle A_\varepsilon, \mathbb{C}A_\varepsilon\rangle + \delta\frac{1}{2}\langle \hat{A}_\varepsilon, \mathbb{C}\hat{A}_\varepsilon\rangle + \delta\frac{1}{2}\langle A_\varepsilon, \mathbb{C}A_\varepsilon\rangle \\ &\quad = \frac{1}{2}\langle \hat{A}_\varepsilon - A_\varepsilon, \mathbb{C}\hat{A}_\varepsilon + A_\varepsilon\rangle + \delta\frac{1}{2}\langle \hat{A}_\varepsilon, \mathbb{C}\hat{A}_\varepsilon\rangle + \delta\frac{1}{2}\langle A_\varepsilon, \mathbb{C}A_\varepsilon\rangle \qquad \text{on } U_\varepsilon^{(\delta)}, \end{aligned} \tag{4.39}$$

$$\begin{aligned} &\frac{1}{\varepsilon^2} W_{\mathrm{h}}(I + \varepsilon\hat{z}_\varepsilon) - \frac{1}{\varepsilon^2} W_{\mathrm{h}}(I + \varepsilon z_\varepsilon) \\ &\quad \overset{(4.9\mathrm{d})}{\leq} \frac{1}{2}\langle \hat{z}_\varepsilon, \mathbb{H}\hat{z}_\varepsilon\rangle - \frac{1}{2}\langle z_\varepsilon, \mathbb{H}z_\varepsilon\rangle + \delta\frac{1}{2}\langle \hat{z}_\varepsilon, \mathbb{H}\hat{z}_\varepsilon\rangle + \delta\frac{1}{2}\langle z_\varepsilon, \mathbb{H}z_\varepsilon\rangle \\ &\quad = \frac{1}{2}\langle \hat{z}_\varepsilon - z_\varepsilon, \mathbb{H}\hat{z}_\varepsilon + z_\varepsilon\rangle + \delta\frac{1}{2}\langle \hat{z}_\varepsilon, \mathbb{H}\hat{z}_\varepsilon\rangle + \delta\frac{1}{2}\langle z_\varepsilon, \mathbb{H}z_\varepsilon\rangle \qquad \text{on } Z_\varepsilon^{(\delta)}. \end{aligned} \tag{4.40}$$

On the complements $O \setminus U_\varepsilon^{(\delta)}$ and $O \setminus Z_\varepsilon^{(\delta)}$, which we will naturally call *bad sets*, we will counter the big strains with the smallness of the sets:

$$\begin{aligned} |O \setminus U_\varepsilon^{(\delta)}| &= \int_{O\setminus U_\varepsilon^{(\delta)}} 1\,\mathrm{d}x \overset{(4.38)}{\leq} \frac{\varepsilon^2}{r_{\mathrm{el}}(\delta)^2} \int_{O\setminus U_\varepsilon^{(\delta)}} |A_\varepsilon(x)|^2 + |\widehat{A}_\varepsilon(x)|^2\,\mathrm{d}x \\ &\leq \varepsilon^2 \frac{c_K}{r_{\mathrm{el}}(\delta)^2}(\|\nabla u_\varepsilon\|_{L^2}^2 + \|\nabla\hat{u}_\varepsilon\|_{L^2}^2 + \|z_\varepsilon\|_{L^2}^2 + \|\hat{z}_\varepsilon\|_{L^2}^2), \\ |O \setminus Z_\varepsilon^{(\delta)}| &\leq \int_{O\setminus Z_\varepsilon^{(\delta)}} 1\,\mathrm{d}x \overset{(4.38)}{\leq} \frac{\varepsilon^2}{r_{\mathrm{h}}(\delta)^2} \int_{O\setminus Z_\varepsilon^{(\delta)}} |z_\varepsilon(x)|^2 + |\hat{z}_\varepsilon(x)|^2\,\mathrm{d}x \\ &\leq \varepsilon^2 r_{\mathrm{h}}(\delta)^{-2}(\|z_\varepsilon\|_{L^2}^2 + \|\hat{z}_\varepsilon\|_{L^2}^2). \end{aligned}$$

Let us introduce

$$G_{1,\varepsilon} := (I+\varepsilon\nabla\hat{u}_\varepsilon)(I+\varepsilon\nabla u_\varepsilon)^{-1} \quad \text{and}$$
$$G_{2,\varepsilon} := (I+\varepsilon z\varepsilon)(I+\varepsilon\hat{z}_\varepsilon)^{-1},$$

such that $I+\varepsilon\hat{A}_\varepsilon = G_{1,\varepsilon}(I+\varepsilon A_\varepsilon)G_{2,\varepsilon}$. Note that $\det(I+\varepsilon\nabla u_\varepsilon)>0$ a.e., hence $G_{1,\varepsilon}$ is well-defined. With these we want to utilize the estimate on left an right multiplication from (4.8):

$$\begin{aligned}
&\frac{1}{\varepsilon^2}W_{\mathrm{el}}\big((I+\varepsilon\nabla\hat{u}_\varepsilon)(I+\varepsilon\hat{z}_\varepsilon)^{-1}\big)-\frac{1}{\varepsilon^2}W_{\mathrm{el}}\big((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon z_\varepsilon)^{-1}\big)\\
&\quad=\frac{1}{\varepsilon^2}W_{\mathrm{el}}(I+\varepsilon\hat{A}_\varepsilon)-\frac{1}{\varepsilon^2}W_{\mathrm{el}}(I+\varepsilon A_\varepsilon)\\
&\quad=\frac{1}{\varepsilon^2}W_{\mathrm{el}}(G_{1,\varepsilon}(I+\varepsilon A_\varepsilon)G_{2,\varepsilon})-\frac{1}{\varepsilon^2}W_{\mathrm{el}}(I+\varepsilon A_\varepsilon)\\
&\quad\overset{(4.8)}{\le}\frac{1}{\varepsilon^2}C_{\mathrm{M}}(W_{\mathrm{el}}(I+\varepsilon A_\varepsilon)+1)(|G_{1,\varepsilon}-I|+|G_{2,\varepsilon}-I|).
\end{aligned}$$

Together with the estimates

$$\|G_{1,\varepsilon}-I\|_{\mathrm{L}^\infty}=\|(I+\varepsilon\nabla\tilde{u})(I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon\nabla u_\varepsilon)^{-1}-I\|_{\mathrm{L}^\infty}\le\varepsilon\|\nabla\tilde{u}\|_{\mathrm{L}^\infty} \quad \text{and}$$
$$\|G_{2,\varepsilon}-I\|_{\mathrm{L}^\infty}=\|\exp(\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z})-I\|_{\mathrm{L}^\infty}\le\varepsilon\|\nabla\exp\|_{\mathrm{L}^\infty}\|\tilde{z}\|_{\mathrm{L}^\infty}$$

we thus get:

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_{O\setminus U_\varepsilon^{(\delta)}}W_{\mathrm{el}}\big((I+\varepsilon\nabla\hat{u}_\varepsilon)(I+\varepsilon\hat{z}_\varepsilon)^{-1}\big)-\frac{1}{\varepsilon^2}W_{\mathrm{el}}\big((I+\varepsilon\nabla u_\varepsilon)(I+\varepsilon z_\varepsilon)^{-1}\big)\,\mathrm{d}x\\
&\quad\le C_{\mathrm{M}}(\|G_{1,\varepsilon}-I\|_{\mathrm{L}^\infty}+\|G_{2,\varepsilon}-I\|_{\mathrm{L}^\infty})\Big(\frac{1}{\varepsilon^2}\int_O W_{\mathrm{el}}(I+\varepsilon A_\varepsilon)\,\mathrm{d}x+\frac{1}{\varepsilon^2}|O\setminus U_\varepsilon^{(\delta)}|\Big)\\
&\quad\le c(1+r_{\mathrm{el}}(\delta)^{-2})\varepsilon. \qquad (4.41)
\end{aligned}$$

For controlling the hardening parts on the bad set $O\setminus Z_\varepsilon^{(\delta)}$ we use the local Lipschitz continuity of $\widetilde{W}_{\mathrm{h}}$ from (4.9c):

$$\begin{aligned}
\Big|\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon\hat{z}_\varepsilon)-\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\Big|&=\Big|\frac{1}{\varepsilon^2}\widetilde{W}_{\mathrm{h}}(I+\varepsilon\hat{z}_\varepsilon)-\frac{1}{\varepsilon^2}\widetilde{W}_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\Big|\\
&=\Big|\frac{1}{\varepsilon^2}\widetilde{W}_{\mathrm{h}}\big(\exp(\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z})(I+\varepsilon z_\varepsilon)\big)-\frac{1}{\varepsilon^2}\widetilde{W}_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\Big|\\
&\le\frac{1}{\varepsilon^2}\big|\exp(\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z})(I+\varepsilon z_\varepsilon)-(I+\varepsilon z_\varepsilon)\big|\\
&\le\frac{1}{\varepsilon^2}c_K\big|\exp(\varepsilon\mathbb{1}_{O_\varepsilon}\tilde{z})-I\big|.
\end{aligned}$$

Local Lipschitz continuity of exp thus gives the bound

$$\Big\|\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon\hat{z}_\varepsilon)-\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\Big\|_{\mathrm{L}^\infty}\le c\frac{1}{\varepsilon}$$

and we can estimate:

$$\int_{O\setminus Z_\varepsilon^{(\delta)}}\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon\hat{z}_\varepsilon)-\frac{1}{\varepsilon^2}W_{\mathrm{h}}(I+\varepsilon z_\varepsilon)\,\mathrm{d}x$$

$$\leq |O \setminus Z_\varepsilon^{(\delta)}| \, \big\| \frac{1}{\varepsilon^2} W_\mathrm{h}(I + \varepsilon \hat{z}_\varepsilon) - \frac{1}{\varepsilon^2} W_\mathrm{h}(I + \varepsilon z_\varepsilon) \big\|_{\mathrm{L}^\infty} \leq c r_\mathrm{h}(\delta)^{-2} \varepsilon. \tag{4.42}$$

Collecting (4.39), (4.40), (4.41) and (4.42) and using the strong convergences (4.29) and (4.33) as well as the weak convergences (4.30) and (4.34) we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} (\widetilde{\mathcal{E}}_\varepsilon(O, \hat{u}_\varepsilon, \hat{z}_\varepsilon) - \widetilde{\mathcal{E}}_\varepsilon(O, u_\varepsilon, z_\varepsilon)) \\
&\quad = \limsup_{\varepsilon\to 0} \Big(\frac{1}{\varepsilon^2} \int_{U_\varepsilon^{(\delta)}} W_\mathrm{el}(I + \varepsilon \hat{A}_\varepsilon) - W_\mathrm{el}(I + \varepsilon A_\varepsilon) \,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{\varepsilon^2} \int_{O \setminus U_\varepsilon^{(\delta)}} W_\mathrm{el}(I + \varepsilon \hat{A}_\varepsilon) - W_\mathrm{el}(I + \varepsilon A_\varepsilon) \,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{\varepsilon^2} \int_{Z_\varepsilon^{(\delta)}} W_\mathrm{h}(I + \varepsilon \hat{z}_\varepsilon) - W_\mathrm{h}(I + \varepsilon z_\varepsilon) \,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{\varepsilon^2} \int_{O \setminus Z_\varepsilon^{(\delta)}} W_\mathrm{h}(I + \varepsilon \hat{z}_\varepsilon) - W_\mathrm{h}(I + \varepsilon z_\varepsilon) \,\mathrm{d}x \Big) \\
&\quad \leq \limsup_{\varepsilon\to 0} \Big(\int_{U_\varepsilon^{(\delta)}} \Big(\frac{1}{2} \langle \hat{A}_\varepsilon - A_\varepsilon, \mathbb{C}(\hat{A}_\varepsilon + A_\varepsilon) \rangle + \delta \frac{1}{2} \langle \hat{A}_\varepsilon, \mathbb{C} \hat{A}_\varepsilon \rangle + \delta \frac{1}{2} \langle A_\varepsilon, \mathbb{C} A_\varepsilon \rangle \Big) \,\mathrm{d}x \\
&\qquad\qquad + \int_{Z_\varepsilon^{(\delta)}} \Big(\frac{1}{2} \langle \hat{z}_\varepsilon - z_\varepsilon, \mathbb{H} \hat{z}_\varepsilon + z_\varepsilon \rangle + \delta \frac{1}{2} \langle \hat{z}_\varepsilon, \mathbb{H} \hat{z}_\varepsilon \rangle + \delta \frac{1}{2} \langle z_\varepsilon, \mathbb{H} z_\varepsilon \rangle \Big) \,\mathrm{d}x \\
&\qquad\qquad + c \big(1 + r_\mathrm{el}(\delta)^{-2}\big) \varepsilon + c r_\mathrm{h}(\delta)^{-2} \varepsilon \Big) \\
&\quad \leq \int_O \frac{1}{2} \langle \nabla(\hat{u}_0 - u_0) - (\hat{z}_0 - z_0), \mathbb{C}(\nabla(\hat{u}_0 + u_0) - (\hat{z}_0 + z_0)) \rangle \,\mathrm{d}x \\
&\qquad + \int_O \frac{1}{2} \langle \hat{z}_0 - z_0, \mathbb{H}(\hat{z}_0 + z_0) \rangle \,\mathrm{d}x + c\delta \\
&\quad = \int_O \frac{1}{2} \langle \nabla \hat{u}_0 - \hat{z}_0, \mathbb{C}(\nabla \hat{u}_0 - \hat{z}_0) \rangle \,\mathrm{d}x - \int_O \frac{1}{2} \langle \nabla u_0 - z_0, \mathbb{C}(\nabla u_0 - z_0) \rangle \,\mathrm{d}x \\
&\qquad + \int_O \frac{1}{2} \langle \hat{z}_0, \mathbb{H} \hat{z}_0 \rangle \,\mathrm{d}x - \int_O \frac{1}{2} \langle z_0, \mathbb{H} z_0 \rangle \,\mathrm{d}x + c\delta \\
&\quad = \widetilde{\mathcal{E}}_0(O, \hat{u}_0, \hat{z}_0) - \widetilde{\mathcal{E}}_0(O, u_0, z_0) + c\delta.
\end{aligned}$$

Since δ is arbitrary, this gives the cancellation of jumps in the stored energy:

$$\limsup_{\varepsilon\to 0} \Big(\widetilde{\mathcal{E}}_\varepsilon(O, t, \hat{u}_\varepsilon, \hat{z}_\varepsilon) - \widetilde{\mathcal{E}}_\varepsilon(O, t, u_\varepsilon, z_\varepsilon) \Big) \leq \widetilde{\mathcal{E}}_0(O, t, u_0, z_0) - \widetilde{\mathcal{E}}_0(O, t, u_0, z_0). \tag{4.43}$$

Conclusion of proof: The external work converges since $\ell(t) \in \mathcal{U}'$ for every $t \in [0, T]$ and by (4.26) and (4.27) we have the weak convergences $u_\varepsilon \rightharpoonup u_0$ and $\hat{u}_\varepsilon \rightharpoonup \hat{u}_0$ in $\mathcal{U}$:

$$\int_O \ell(t)(\hat{u}_\varepsilon - u_\varepsilon) \,\mathrm{d}x \to \int_O \ell(t)(\hat{u}_0 - u_0) \,\mathrm{d}x.$$

Together with (4.37) and (4.43) this concludes the proof:

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \widetilde{\mathcal{T}}_\varepsilon(O, t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) &\leq \limsup_{\varepsilon\to 0} \Big(\widetilde{\mathcal{E}}_\varepsilon(O, \hat{u}_\varepsilon, \hat{z}_\varepsilon) - \widetilde{\mathcal{E}}_\varepsilon(O, u_\varepsilon, z_\varepsilon) \Big) \\
&\quad + \limsup_{\varepsilon\to 0} \Big(- \int_O \ell(t) \cdot (\hat{u}_\varepsilon - u_\varepsilon) \,\mathrm{d}x \Big) + \limsup_{\varepsilon\to 0} \mathcal{D}_\varepsilon(O, z_\varepsilon, \hat{z}_\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&\leq \widetilde{\mathcal{E}}_0(O,u_0,z_0) - \widetilde{\mathcal{E}}_0(O,u_0,z_0) \\
&\quad - \int_O \ell(t)\cdot(\hat{u}_0 - u_0)\,\mathrm{d}x + \mathcal{D}_\varepsilon(O,z_0,\hat{z}_0) \\
&= \widetilde{\mathcal{T}}_0(O,t,u_0,z_0,\hat{u}_0,\hat{z}_0).
\end{aligned}
$$

□

In the case of a Lipschitz domain as in [MS13] the mutual recovery sequence contains a composition. This is necessary due to the multiplicative split displayed by the stored energy $\widetilde{\mathcal{E}}_\varepsilon$ in the finite case. For a cracked domain the classical composition is not adequate. For example the composition $v_2 \circ v_1$ of two deformations $v_1, v_2 \in \mathrm{H}^1(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}, \mathbb{R}^d)$ would create another crack, if the image $v_1(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}})$ touches the crack Γ_{Cr}. Thus in the following Lemma we refine the notion of a composition $v_2 \tilde{\circ} v_1$ of two deformations.

Lemma 4.8 (Crack-respecting composition)**.** *Assume the deformation* $v \in \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ *and the smooth displacement* $\tilde{u} \in C_b^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}, \mathbb{R}^d)$ *for some* $\delta{>}0$*. For*

$$
\begin{aligned}
B_+^{(\delta)} &= \{(x_1 > 0 \text{ and } x_2 > 0) \text{ or } x_1 > 1{-}\delta\} \quad \text{and} \\
B_-^{(\delta)} &= \{x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_1 > 1{-}\delta\}
\end{aligned}
$$

let $\tilde{u}_+, \tilde{u}_- \in \mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *with* $\tilde{u}_+|_{B_+^{(\delta)}} = \tilde{u}|_{B_+^{(\delta)}}$ *and* $\tilde{u}_-|_{B_-^{(\delta)}} = \tilde{u}|_{B_-^{(\delta)}}$.

If additionally $r := \|v - \mathrm{id}\|_{\mathrm{L}^\infty} < \delta$, *then* $\tilde{v}_\varepsilon \tilde{\circ} v$ *given piecewise on* A_+ *and* A_- *from* (4.3c) *by*

$$
\tilde{v}_\varepsilon \tilde{\circ} v := \begin{cases} (\mathrm{id} + \varepsilon \tilde{u}_+) \circ v & \text{on } A_+, \\ (\mathrm{id} + \varepsilon \tilde{u}_-) \circ v & \text{on } A_-, \end{cases} \tag{4.44}
$$

is well-defined, lies in $\mathrm{H}^1(\Omega \setminus \Gamma_{\mathrm{Cr}}, \mathbb{R}^d)$ *and satisfies:*

$$
\tilde{v}_\varepsilon \tilde{\circ} v = (\mathrm{id} + \varepsilon \tilde{u}) \circ v \text{ on } \Omega \setminus U_r(\Gamma_{\mathrm{Cr}}). \tag{4.45}
$$

We call $\tilde{v}_\varepsilon \tilde{\circ} v$ *a* crack-respecting composition *of* $\tilde{v}_\varepsilon := \mathrm{id} + \varepsilon \tilde{u}$ *and* v.

Proof. Note that $\tilde{u}_+$ and $\tilde{u}_-$ coincide on $B_0^{(\delta)} := B_+^{(\delta)} \cap B_-^{(\delta)} = \{x_1 > 1{-}\delta\}$. Moreover by $\|v - \mathrm{id}\|_{\mathrm{L}^\infty} < \delta$ we have $v(A_+ \cap A_-) \subset B_0^{(\delta)}$, thus $\tilde{v}_\varepsilon \tilde{\circ} v$ is well-defined.

Furthermore $\tilde{v}_\varepsilon \tilde{\circ} v \in \mathrm{H}^1(\Omega \setminus \Gamma_{\mathrm{Cr}}, \mathbb{R}^d)$ follows from it being a composition of functions from $\mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathrm{H}^1(A_\pm, \mathbb{R}^d)$ piecewise on the finite open cover $\Omega_{\mathrm{Cr}} = A_+ \cup A_-$ of Lipschitz domains $A_\pm$.

Finally, for $x \in A_\pm \setminus U_r(\Gamma_{\mathrm{Cr}})$ we have $(\tilde{v}_\varepsilon \tilde{\circ} v)(x) = (\mathrm{id} + \varepsilon \tilde{u}_\pm)(v(x))$ because $v(x) \in B_\pm^{(\delta)}$ by $\|v - \mathrm{id}\|_{\mathrm{L}^\infty} < \delta$. Since $\tilde{u}$ coincides with $\tilde{u}_\pm$ on $B_\pm^{(\delta)}$ either way we have $(\tilde{v}_\varepsilon \tilde{\circ} v)(x) = (\mathrm{id} + \varepsilon \tilde{u})(v(x))$, which shows (4.45). □

Finally in the following proposition the mutual recovery sequence for the transition cost $\widetilde{\mathcal{T}}_\varepsilon$ without constraints on the non-Lipschitz domain Ω_{Cr} is proven. It is defined similarly to the Lipschitz case with the classical composition substituted by a crack-respecting composition.

Proposition 4.9 (Mutual recovery sequence without constraint)**.** *Let* $t \in [0, T]$, $(u_\varepsilon, z_\varepsilon) \to (u_0, z_0)$ *weakly in* $\mathcal{Q}$ *with*

$$\sup \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$$

and $(\hat{u}_0, \hat{z}_0) \in \mathcal{Q}$ *with* $(\tilde{u}, \tilde{z}) := (\hat{u}_0, \hat{z}_0) - (u_0, z_0) \in \mathrm{C}_b^\infty(\Omega \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}, \mathbb{R}^d) \times \mathrm{C}_c^\infty(\Omega, \mathbb{R}^d)$ *for some* $\delta > 0$.

Consider a smooth extension of $\tilde{u}$

$$\tilde{U} \in \mathrm{C}_c^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}, \mathbb{R}^d) \quad \text{with} \quad \tilde{U}|_{\Omega_{\mathrm{Cr}}} = \tilde{u}$$

as well as upper and lower extensions

$$\tilde{u}_\pm \in \mathrm{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^d) \quad \text{with} \quad \tilde{u}_\pm|_{B_\pm^{(\delta)}} = \tilde{U}|_{B_\pm^{(\delta)}}.$$

Then there exists $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$ *the corresponding crack-respecting compositions* $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$ *of the deformations* $\tilde{v}_\varepsilon = \mathrm{id} + \varepsilon\tilde{u}$ *and* $v_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ *are well-defined and the sequence* $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \in \mathcal{Q}$ *given by*

$$\left.\begin{aligned} \hat{u}_\varepsilon &:= \tfrac{1}{\varepsilon}(\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon - \mathrm{id}), \\ \hat{z}_\varepsilon &:= \begin{cases} \tfrac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I\big) & \text{on } \Omega_\varepsilon, \\ z_\varepsilon & \text{otherwise,} \end{cases} \end{aligned}\right\} \tag{4.46}$$

where $\Omega_\varepsilon := \{x \in \Omega \setminus \Gamma_{\mathrm{Cr}} \mid \exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) \in K\}$, *is a mutual recovery sequence, i.e.:*

$$\begin{aligned} &(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \to (\hat{u}_0, \hat{z}_0) \text{ weakly in } \mathcal{Q} \text{ and} \\ &\limsup_{\varepsilon\to 0} \widetilde{\mathcal{T}}_\varepsilon(t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \le \widetilde{\mathcal{T}}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0). \end{aligned}$$

Proof. For the well-definedness of the crack-respecting composition $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon \in \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ by Lemma 4.8 we need $r_\varepsilon := \|v_\varepsilon - \mathrm{id}\|_{\mathrm{L}^\infty} < \delta$. That we obtain by the a priori bound in Proposition 3.3, which gives:

$$\varepsilon^{1-\beta}\|u\|_{\mathrm{L}^\infty} \le c_{\mathcal{E}}\Big(\widetilde{\mathcal{E}}_\varepsilon(u, z) + C_{\mathcal{E}}\Big).$$

Thus for $\varepsilon < \Big(\frac{\delta}{c_{\mathcal{E}}(E + C_{\mathcal{E}})}\Big)^{\frac{1}{\beta}}$ by the boundedness of the stored energy $\sup \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) := E < \infty$ we get:

$$\|v_\varepsilon - \mathrm{id}\|_{\mathrm{L}^\infty} = \|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} \le \varepsilon^\beta c_{\mathcal{E}}(E + C_{\mathcal{E}}) < \delta.$$

For the well-definedness of $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \in \mathcal{Q}$ we still need to ensure the boundary condition $\hat{u}_\varepsilon|_{\Gamma_{\mathrm{Dir}}} = 0$. For that we use $u_\varepsilon \in \mathcal{U}$, $\hat{u}_0 \in \mathcal{U}$ and (4.45) in Lemma 4.8. The former two give $u_\varepsilon|_{\Gamma_{\mathrm{Dir}}} = \hat{u}_0|_{\Gamma_{\mathrm{Dir}}} = \tilde{u}|_{\Gamma_{\mathrm{Dir}}} = 0$. To use the latter, consider assumption (4.6) on the Dirichlet boundary Γ_{Dir} by which there exists $r_{\mathrm{Dir}} > 0$ such that $\Gamma_{\mathrm{Dir}} \subset \mathbb{R}^d \setminus U_{r_{\mathrm{Dir}}}(\Gamma_{\mathrm{Cr}})$. Thus, by Proposition 3.3 for $\varepsilon < \Big(\frac{r_{\mathrm{Dir}}}{c_{\mathcal{E}}(E + C_{\mathcal{E}})}\Big)^{\frac{1}{\beta}}$ we have $r_\varepsilon = \|v_\varepsilon - \mathrm{id}\|_{\mathrm{L}^\infty} < r_{\mathrm{Dir}}$ such that by (4.45) we have:

$$(\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)|_{\Gamma_{\mathrm{Dir}}} = (\mathrm{id} + \varepsilon\tilde{u}) \circ (\mathrm{id}|_{\Gamma_{\mathrm{Dir}}} + \varepsilon u_\varepsilon|_{\Gamma_{\mathrm{Dir}}}) = (\mathrm{id} + \varepsilon\tilde{u}) \circ \mathrm{id}|_{\Gamma_{\mathrm{Dir}}}$$

$$= \mathrm{id}|_{\Gamma_{\mathrm{Dir}}} + \varepsilon\tilde{u}|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}|_{\Gamma_{\mathrm{Dir}}}$$

$$\Rightarrow \qquad \hat{u}_\varepsilon|_{\Gamma_{\mathrm{Dir}}} = 0.$$

To summarize, for $\varepsilon < \varepsilon_0 := \left(\frac{\min\{\delta, r_{\mathrm{Dir}}\}}{c\varepsilon(E+C\varepsilon)}\right)^{\frac{1}{\beta}}$ both $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon \in \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ and $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \in \mathcal{Q}$ are well-defined.

To prove $(\hat{u}_\varepsilon, \hat{z}_\varepsilon)$ being a mutual recovery sequence, recall the disjoint decomposition of Ω_{Cr} into Lipschitz domains A_+ and $A_- \setminus A_+$ from (4.3c). The strategy is to use Proposition 4.7 two times, for each $O \in \{A_+, A_- \setminus A_+\}$ and $\tilde{u} \in \{\tilde{u}_+, \tilde{u}_-\}$ respectively, and then combine both by the sum (4.25).

In Proposition 4.7 we take one time $O = A_+$, $(u_\varepsilon, z_\varepsilon) = (u_\varepsilon|_{A_+}, z_\varepsilon|_{A_+})$ and $(\tilde{u}, \tilde{z}) = (\tilde{u}_+, \tilde{z})$ and one time we take $O = A_- \setminus A_+$, $(u_\varepsilon, z_\varepsilon) = (u_\varepsilon|_{A_- \setminus A_+}, z_\varepsilon|_{A_- \setminus A_+})$ and $(\tilde{u}, \tilde{z}) = (\tilde{u}_-, \tilde{z})$. Then for

$$\hat{u}_{+,\varepsilon} := \frac{1}{\varepsilon}\big((\mathrm{id}+\varepsilon\tilde{u}_+)\circ(\mathrm{id}+\varepsilon u_\varepsilon|_{A_+})-\mathrm{id}\big) \quad \text{and} \quad \hat{u}_{-,\varepsilon} := \frac{1}{\varepsilon}\big((\mathrm{id}+\varepsilon\tilde{u}_-)\circ(\mathrm{id}+\varepsilon u_\varepsilon|_{A_- \setminus A_+})-\mathrm{id}\big)$$

we obtain the respective assertions (a),(b) and (c) in Proposition 4.7 twice:

(a) $\|\hat{u}_{+,\varepsilon} - u_\varepsilon - \tilde{u}_+\|_{\mathrm{H}^1(A_+,\mathbb{R}^d)} \leq c\varepsilon$ and $\|\hat{u}_{-,\varepsilon} - u_\varepsilon - \tilde{u}_-\|_{\mathrm{H}^1(A_- \setminus A_+,\mathbb{R}^d)} \leq c\varepsilon$,

(b) $\|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(A_+,\mathbb{R}^d)} \leq c\varepsilon^2$ and $\|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(A_- \setminus A_+,\mathbb{R}^d)} \leq c\varepsilon^2$ as well as

(c)
$$\limsup_{\varepsilon \to 0} \widetilde{\mathcal{T}}_\varepsilon(A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{+,\varepsilon}, \hat{z}_\varepsilon) \leq \widetilde{\mathcal{T}}_0(A_+, t, u_0, z_0, u_0 + \tilde{u}_+, \hat{z}_0) \qquad \text{and}$$
$$\limsup_{\varepsilon \to 0} \widetilde{\mathcal{T}}_\varepsilon(A_- \setminus A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{-,\varepsilon}, \hat{z}_\varepsilon) \leq \widetilde{\mathcal{T}}_0(A_- \setminus A_+, t, u_0, z_0, u_0 + \tilde{u}_-, \hat{z}_0).$$

On Ω_{Cr} the displacements $\hat{u}_\varepsilon$ are given in terms of the crack-respecting composition $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$, but by (4.44) on each of the two parts $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$ is given in terms of a classical composition, such that we have:

$$\hat{u}_\varepsilon = \begin{cases} \hat{u}_{+,\varepsilon} & \text{on } A_+, \\ \hat{u}_{-,\varepsilon} & \text{on } A_- \setminus A_+; \end{cases}$$

where $\hat{u}_{\pm,\varepsilon} = \frac{1}{\varepsilon}\big((\mathrm{id} + \varepsilon\tilde{u}_\pm) \circ v_\varepsilon - \mathrm{id}\big)$ as above. Since $\tilde{u} = \tilde{u}_\pm$ on $A_\pm$, from (a) we have

$$\begin{aligned}&\|\hat{u}_\varepsilon - u_\varepsilon - \tilde{u}\|_{\mathrm{H}^1(\Omega_{\mathrm{Cr}},\mathbb{R}^d)} \\ &\quad = \|\hat{u}_{+,\varepsilon} - u_\varepsilon - \tilde{u}_+\|_{\mathrm{H}^1(A_+,\mathbb{R}^d)} + \|\hat{u}_{-,\varepsilon} - u_\varepsilon - \tilde{u}_-\|_{\mathrm{H}^1(A_- \setminus A_+,\mathbb{R}^d)} < 2c\varepsilon,\end{aligned}$$

and the convergence $\hat{u}_\varepsilon \rightharpoonup \hat{u}_0$ weakly in H^1 follows by $u_\varepsilon \overset{\mathrm{H}^1}{\rightharpoonup} u_0$.

Concerning the plastic variable the definition of $\hat{z}_\varepsilon$ on the non-Lipschitz domain Ω_{Cr} and on the Lipschitz domain $O \in \{A_+, A_- \setminus A_+\}$ coincide, thus from (b) we directly get

$$\|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d})} = \|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(A_+,\mathbb{R}^{d\times d})} + \|\hat{z}_\varepsilon - z_\varepsilon - \tilde{z}\|_{\mathrm{L}^2(A_- \setminus A_+,\mathbb{R}^{d\times d})} < 2c\varepsilon,$$

and the weak convergence $\hat{z}_0 \rightharpoonup \hat{z}_0$ in L^2 follows from $z_\varepsilon \overset{\mathrm{L}^2}{\rightharpoonup} z_0$.

Finally, the asserted lim sup estimate on the transition cost we obtain from (c) via the sum rule (4.25):

$$\limsup_{\varepsilon \to 0} \widetilde{\mathcal{T}}_\varepsilon(t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) = \limsup_{\varepsilon \to 0} \widetilde{\mathcal{T}}_\varepsilon(\Omega_{\mathrm{Cr}}, t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon)$$

$$
\begin{aligned}
&= \limsup_{\varepsilon\to0} \Big(\widetilde{\mathcal{T}}_\varepsilon(A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{+,\varepsilon}, \hat{z}_\varepsilon) + \widetilde{\mathcal{T}}_\varepsilon(A_- \setminus A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{-,\varepsilon}, \hat{z}_\varepsilon)\Big) \\
&\le \limsup_{\varepsilon\to0} \widetilde{\mathcal{T}}_\varepsilon\Big(A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{+,\varepsilon}, \hat{z}_\varepsilon\Big) + \limsup_{\varepsilon\to0} \widetilde{\mathcal{T}}_\varepsilon\Big(A_- \setminus A_+, t, u_\varepsilon, z_\varepsilon, \hat{u}_{-,\varepsilon}, \hat{z}_\varepsilon\Big) \\
&\le \widetilde{\mathcal{T}}_0\Big(A_+, t, u_0, z_0, u_0+\tilde{u}_+, \hat{z}_0\Big) + \widetilde{\mathcal{T}}_0\Big(A_- \setminus A_+, t, u_0, z_0, u_0+\tilde{u}_-, \hat{z}_0\Big) \\
&= \widetilde{\mathcal{T}}_0\Big(t, u_0, z_0, u_0+\tilde{u}, \hat{z}_0\Big).
\end{aligned}
$$

□

4.4 Lower and upper bounds with constraints

Having shown the lim inf estimate on the stored energy $\widetilde{\mathcal{E}}_\varepsilon$ without constraint in Proposition 3.6 (a) and the lim sup estimate on the transition cost $\widetilde{\mathcal{T}}_\varepsilon$ without constraint in Proposition 4.9, in this section we will discuss how to add the ε^α-GMS condition (4.1) and local non-interpenetration (4.2) into the picture. For the lim inf estimate we need that limits of sequences satisfying the former necessarily satisfy the latter. This slight generalization of Theorem 2.10 is shown in the following

Theorem 4.10 (Infinitesimal relaxed non-interpenetration for $\alpha > 0$)**.** *Let $\alpha > 1$ and consider $(u_\varepsilon, z_\varepsilon), (u_0, z_0) \in \mathcal{Q}$ with $u_\varepsilon \overset{\mathcal{U}}{\rightharpoonup} u$ and $\liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) < \infty$.*

Then $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \ge 0$ holds.

Proof. As $\liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) < \infty$ there is a (not relabeled) subsequence u_ε such that $\mathrm{id} + \varepsilon u_\varepsilon$ fulfills the ε^α-GMS-condition (4.1) and $\det(I + \varepsilon\nabla u_\varepsilon) > 0$ a.e. on Ω. Hence, taking an arbitrary $\varphi \in \mathrm{C}_0^\infty(\Omega)$ with $\varphi \ge 0$ we have:

$$
\begin{aligned}
\int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})} \varphi(y)\,\mathrm{d}y + \int_{U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})} \varphi(y)\,\mathrm{d}y &= \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y \\
&\le \int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})} \varphi(x + \varepsilon u_\varepsilon(x))\big|\det\big(I + \varepsilon\nabla u_\varepsilon(x)\big)\big|\,\mathrm{d}x.
\end{aligned}
$$

Rearranging and dividing by ε we arrive at the following inequality that displays the same three integrals as in Theorem 3.5 now with the smaller domain on the right-hand side and an additional term on the left-hand side:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})} \varphi(y)\,\mathrm{d}y \\
&\quad\ge \frac{1}{\varepsilon}\int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \varphi(x + \varepsilon u_\varepsilon(x))\det\big(I + \varepsilon\nabla u_\varepsilon(x)\big) - \varphi(x)\,\mathrm{d}x \\
&\quad= \frac{1}{\varepsilon}\int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \varphi(x + \varepsilon u_\varepsilon(x))\Big(\det\big(I + \varepsilon\nabla u_\varepsilon(x)\big) - \big(1 + \varepsilon\,\mathrm{div}\,u_\varepsilon(x)\big)\Big)\,\mathrm{d}x \\
&\qquad+ \int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \varphi(x + \varepsilon u_\varepsilon(x))\,\mathrm{div}\,u_\varepsilon(x)\,\mathrm{d}x \\
&\qquad+ \int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \frac{1}{\varepsilon}\Big(\varphi(x + \varepsilon u_\varepsilon(x)) - \varphi(x)\Big)\,\mathrm{d}x \\
&\quad:= I_1 + I_2 + I_3.
\end{aligned}
$$

Since $\operatorname{Vol}\big(U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})\big) \in O(\varepsilon^\alpha)$, the integral on the left-hand side is bounded by $C_d\varepsilon^{\alpha-1}$, such that for $\alpha > 1$ it vanishes in the limit $\varepsilon \to 0$.

The three integrals I_1, I_2 and I_3 can be treated as in the proof of Theorem 3.5. By Lemma 2.11 and boundedness of φ the first summand I_1 on the right-hand side converges to 0 for $\varepsilon \to 0$:

$$\begin{aligned} |I_1| &\leq \frac{1}{\varepsilon}\int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \Big|\varphi(x+\varepsilon u_\varepsilon(x))\Big(\det\big(I+\varepsilon\nabla u_\varepsilon(x)\big) - \big(1+\varepsilon \operatorname{div} u_\varepsilon(x)\big)\Big)\Big|\,\mathrm{d}x \\ &\leq \frac{1}{\varepsilon}\int_{\Omega_{\mathrm{Cr}}} \Big|\varphi(x+\varepsilon u_\varepsilon(x))\Big(\det\big(I+\varepsilon\nabla u_\varepsilon(x)\big) - \big(1+\varepsilon \operatorname{div} u_\varepsilon(x)\big)\Big)\Big|\,\mathrm{d}x \\ &\leq \varepsilon\|\varphi\|_{\mathrm{L}^\infty} C_{\det}(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) + C_{\det}). \end{aligned}$$

The second summand

$$\begin{aligned} I_2 &= \int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \varphi(x+\varepsilon u_\varepsilon(x)) \operatorname{div} u_\varepsilon(x)\,\mathrm{d}x \\ &= \int_{\Omega_{\mathrm{Cr}}} \mathbb{1}_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)}\varphi(x+\varepsilon u_\varepsilon(x)) \operatorname{div} u_\varepsilon(x)\,\mathrm{d}x \end{aligned}$$

converges to $\int_{\Omega_{\mathrm{Cr}}} \varphi(x) \operatorname{div} u(x)\ \mathrm{d}x$, as in $\mathrm{L}^2(\Omega_{\mathrm{Cr}})$ we have $\operatorname{div} u_\varepsilon \rightharpoonup \operatorname{div} u$ weakly and $\mathbb{1}_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)}\varphi(\mathrm{id}+\varepsilon u_\varepsilon) \to \varphi$ strongly by Lipschitz continuity of φ:

$$\begin{aligned} \|\mathbb{1}_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)}\varphi(\mathrm{id}+\varepsilon u_\varepsilon) - \varphi\|^2_{\mathrm{L}^2(\Omega_{\mathrm{Cr}})} &= \int_{\Omega_{\mathrm{Cr}}} \Big|\mathbb{1}_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)}\varphi(x-\varepsilon u_\varepsilon(x)) - \varphi(x)\Big|^2\,\mathrm{d}x \\ &= \int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \Big|\varphi(x-\varepsilon u_\varepsilon(x)) - \varphi(x)\Big|^2\,\mathrm{d}x + \int_{U_{\varepsilon^\alpha}(\Gamma_c)} |\varphi(x)|^2\,\mathrm{d}x \\ &\leq \varepsilon^2\|\nabla\varphi\|^2_{\mathrm{L}^\infty}\|u_\varepsilon(x)\|^2_{\mathrm{L}^2} + \|\nabla\varphi\|^2_{\mathrm{L}^\infty} C_d\varepsilon^\alpha. \end{aligned}$$

Finally the generalized Lebesgue dominated convergence with the same dominating sequence (3.24) as in the proof of Theorem 3.5 can be used to show that the third summand I_3 converges to $\int_{\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)} \nabla\varphi(x)u(x)\,\mathrm{d}x$.

Altogether, the limit $\varepsilon \to 0$ gives three values:

$$\begin{aligned} 0 &\geq 0 + \int_{\Omega_{\mathrm{Cr}}} \varphi(x) \operatorname{div} u(x)\,\mathrm{d}x + \int_{\Omega_{\mathrm{Cr}}} \nabla\varphi(x)u(x)\,\mathrm{d}x \\ &= \int_{\Omega_{\mathrm{Cr}}} \operatorname{div}(\varphi u)(x)\,\mathrm{d}x = -\int_{\Gamma_{\mathrm{Cr}}} \varphi(x)\,[\![u]\!]_{\Gamma_{\mathrm{Cr}}}(x)\,\mathrm{d}a. \end{aligned}$$

For the last identity as in Theorem 3.5 the divergence theorem is applied on the Lipschitz sets A_+ and $A_- \setminus A_+$ (see (4.3c)) separately and all terms cancel except for the jump along Γ_{Cr}. As $\varphi \geq 0$ was arbitrary, we conclude $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ $\mathcal{H}^{d-1}$-a.e. on Γ_{Cr}. □

This theorem assures the right behavior of the constraints in the lim inf estimate for the stored energy $\mathcal{E}_\varepsilon^{(\alpha)}$ if $\alpha > 1$ and enables us to prove the following lim inf inequality on the total energy $\mathcal{G}_\varepsilon^{(\alpha)}$.

Corollary 4.11. *Let $\alpha > 1$ and $t \in [0,T]$. Then for every sequence $(u_\varepsilon, z_\varepsilon) \rightharpoonup (u_0, z_0)$ weakly in $\mathcal{Q}$ the* lim inf *inequality on the total energy with the ε^α-GMS condition holds:*

$$\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon) \geq \mathcal{G}_0(t, u_0, z_0). \tag{4.47}$$

Proof. The external work is a continuous linear functional $u \mapsto -\langle \ell(t), u\rangle$ and thus converges. Hence it is left to show the lim inf inequality on the stored energy:

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) \geq \mathcal{E}_0^{(\alpha)}(u_0, z_0).$$

We may assume $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) < \infty$, otherwise this inequality holds trivially. Thus on one hand for at least a subsequence u_ε satisfies the ε^α-GMS condition and we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) = \liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon).$$

On the other hand by $\alpha > 1$ Theorem 4.10 provides $\mathcal{E}_0^{(\alpha)}(u_0, z_0) = \widetilde{\mathcal{E}}_0(u_0, z_0)$ and the above lim inf inequality on the stored energy with constraints reduces to the case without constraints

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon, z_\varepsilon) = \liminf_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon, z_\varepsilon) \geq \widetilde{\mathcal{E}}_0(u_0, z_0) = \mathcal{E}_0^{(\alpha)}(u_0, z_0),$$

which was proven in Proposition 3.6 (a). □

For the mutual recovery sequence from Proposition 4.9 to also work for the transition cost $\mathcal{T}_\varepsilon^{(\alpha)}$ with constraints we need the crack-respecting composition to satisfy the ε^α-GMS condition. For the proof of that we will want to test the ε^α-GMS condition with functions that have a jump. The following lemma enables us to do so.

Lemma 4.12. *Let $v \in W^{1,1}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ fulfill*

$$\int_{\Omega\setminus B_\delta(\Gamma_c)} \varphi(v(x))\,\big|\det \nabla v(x)\big|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} \varphi(y)\,\mathrm{d}y \tag{4.48}$$

for all $\varphi \in \mathrm{C}^0(\mathbb{R}^d, \mathbb{R})$ with $\varphi \geq 0$.

Then (4.48) *is also fulfilled for*

$$\bar{\varphi} = \begin{cases} \varphi & \text{on } \mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}, \\ 0 & \text{on } \Gamma_{\mathrm{Cr}}; \end{cases} \tag{4.49}$$

where $\varphi \in \mathrm{C}^0_{b,c}(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}) := \{\varphi \in \mathrm{C}^0(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}) \cap \mathrm{L}^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}) \mid \operatorname{supp}\varphi \text{ compact in } \mathbb{R}^d\}$ with $\varphi \geq 0$.

Proof. Consider a sequence $\varphi_k \in \mathrm{C}_0(\mathbb{R}^d, \mathbb{R})$ with $\varphi_k \geq 0$ and suppose there is a limit $\bar{\varphi}$ such that $\varphi_k \xrightarrow{L^1} \bar{\varphi}$ and $\varphi_k(x) \to \bar{\varphi}(x)$ for every $x \in \mathbb{R}^d$. Then (4.48) hold with $\varphi = \bar{\varphi}$ by Fatou's Lemma:

$$\begin{aligned} \int_{\Omega\setminus B_\delta(\Gamma_c)} \bar{\varphi}(v(x))\,\big|\det \nabla v(x)\big|\,\mathrm{d}x &\leq \liminf_k \int_{\Omega\setminus B_\delta(\Gamma_c)} \varphi_k(v(x))\,\big|\det \nabla v(x)\big|\,\mathrm{d}x \\ &\leq \liminf_k \int_{\mathbb{R}^d} \varphi_k(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \bar{\varphi}(y)\,\mathrm{d}y. \end{aligned}$$

To prove the assertion we are left to give such a sequence $\varphi_k \in \mathrm{C_c}(\mathbb{R}^d, \mathbb{R})$ for $\bar{\varphi}$ from (4.49). Let $\varphi \in \mathrm{C}^0_{b,c}(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}})$ and define $\varphi_k : \mathbb{R}^d \to \mathbb{R}$ by

$$\varphi_k(x) := \begin{cases} \varphi(x)\rho_k\big(\operatorname{dist}(x, \Gamma_{\mathrm{Cr}})\big) & \text{on } \mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}, \\ 0 & \text{on } \Gamma_{\mathrm{Cr}}; \end{cases}$$

using the continuous cut-off function $\rho_k \in \mathrm{C}^0([0,\infty))$ given by:

$$\rho_k(s) := \begin{cases} 1 & \text{for } s \geq \frac{1}{k}, \\ k \cdot s & \text{for } s \in [0, \frac{1}{k}], \\ 0 & \text{for } s = 0. \end{cases}$$

On one hand ρ_k is non-negative and has compact support by $\operatorname{supp} \varphi_k \subset \operatorname{supp} \varphi$, on the other hand φ_k is continuous on $\mathbb{R}^d$: away from the crack on $\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}$ φ_k is a product of continuous functions, on the crack Γ_{Cr} by $\varphi \in \mathrm{L}^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}})$ and $\rho_k = 0$ on Γ_{Cr} we have $\lim \varphi_k = 0$. Thus $\varphi_k \in \mathrm{C}^0_c(\mathbb{R}^d)$ with $\varphi_k \geq 0$ is an admissible test function in (4.48). The convergence

$$\varphi_k \to \bar{\varphi} = \begin{cases} \varphi & \text{on } \mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}, \\ 0 & \text{on } \Gamma_{\mathrm{Cr}}, \end{cases}$$

pointwise and in L^1 follows from $\rho_k(\operatorname{dist}(x, \Gamma_{\mathrm{Cr}})) \to \mathbb{1}_{\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}}(x)$ and $\mathcal{L}^d(U_{\frac{1}{k}}(\operatorname{supp} \varphi \cap \Gamma_{\mathrm{Cr}})) \in O(\frac{1}{k})$ respectively. □

To prove the mutual recovery sequence for $\mathcal{T}^{(\alpha)}_\varepsilon$ we would like to use that the mutual recovery sequence for $\widetilde{\mathcal{T}}_\varepsilon$ from (4.46) actually satisfies the constraint, i.e. that the crack-respecting composition fulfills the ε^α-GMS condition. For exponents $\alpha < \beta < 1$ this is the content of the following

Proposition 4.13 (Relaxed global injectivity of crack-respecting composition)**.** *Let a sequence $(u_\varepsilon, z_\varepsilon) \rightharpoonup (u_0, z_0)$ weakly in $\mathcal{Q}$ with*

$$\sup \mathcal{E}^{(\alpha)}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$$

and $(\hat{u}_0, \hat{z}_0) \in \mathcal{Q}$ with $(\tilde{u}, \tilde{z}) := (\hat{u}_0, \hat{z}_0) - (u_0, z_0) \in \mathrm{C}^\infty_b(\Omega \setminus \Gamma^{(\delta)}_{\mathrm{Cr}}, \mathbb{R}^d) \times \mathrm{C}^\infty_c(\Omega, \mathbb{R}^d)$ for some $\delta{>}0$.

Consider a smooth extension of $\tilde{u}$

$$\tilde{U} \in \mathrm{C}^\infty_c(\mathbb{R}^d \setminus \Gamma^{(\delta)}_{\mathrm{Cr}}, \mathbb{R}^d) \quad \text{with} \quad \tilde{U}|_{\Omega_{\mathrm{Cr}}} = \tilde{u}$$

as well as upper and lower extensions

$$\tilde{u}_\pm \in \mathrm{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d) \quad \text{with} \quad \tilde{u}_\pm|_{B^{(\delta)}_\pm} = \tilde{U}|_{B^{(\delta)}_\pm}.$$

Then for every $\alpha < \beta = \frac{2p-2d}{2p-2d+pd}$ there exists $\tilde{\varepsilon}(\alpha) > 0$ such that for $\varepsilon < \tilde{\varepsilon}$ the crack-respecting composition

$$\hat{v}_\varepsilon := \tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon := (\mathrm{id} + \varepsilon\tilde{u}) \tilde{\circ} (\mathrm{id} + \varepsilon u_\varepsilon)$$

given by Lemma 4.8 for $\tilde{U}$ and $\tilde{u}_\pm$ fulfills the ε^α-GMS condition.

Proof. From $\sup_\varepsilon \mathcal{E}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty$ by definition $v_\varepsilon := \mathrm{id} + \varepsilon u_\varepsilon$ fulfills the ε^α-GMS-condition and by Proposition 3.3 we have $r_\varepsilon := \|\varepsilon u\|_{\mathrm{L}^\infty} \leq C\varepsilon^\beta$. Since $\alpha < \beta$, for ε small enough we thus have $r_\varepsilon < \varepsilon^\alpha$ for ε small enough and conclude by (4.45) from Lemma 4.8:

$$\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon = (\mathrm{id} + \varepsilon\tilde{u}) \tilde{\circ} v_\varepsilon = (\mathrm{id} + \varepsilon\tilde{U}) \circ v_\varepsilon \quad \text{on} \quad \Omega \setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}).$$

The aim of the proof is to show, that $\tilde{V}_\varepsilon := \mathrm{id}+\varepsilon\tilde{U}$ is injective on the image $v_\varepsilon(\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}))$ in the classical sense. Then the ε^α-GMS-condition for the crack-respecting composition $\tilde{v}_\varepsilon\tilde{\circ}v_\varepsilon$ would follow by using integral transformation for the diffeomorphism

$$\tilde{V}_\varepsilon = \mathrm{id} + \varepsilon\tilde{U} \in \mathrm{C}^\infty(\mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}}^{(\delta)}, \mathbb{R}^d)$$

and then testing the ε^α-GMS condition of v_ε by

$$\tilde{\varphi}\colon \mathbb{R}^d \setminus \Gamma_{\mathrm{Cr}} \to [0,\infty), \qquad x \mapsto \varphi(\tilde{V}_\varepsilon(x))|\det \nabla\tilde{V}_\varepsilon(x)|,$$

which is an admissible test function by Lemma 4.12 (4.49):

$$\begin{aligned}
\int_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} \varphi(\hat{v}_\varepsilon(x))\big|\det\nabla\hat{v}_\varepsilon\big|\,\mathrm{d}x &= \int_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} \varphi((\tilde{v}_\varepsilon\tilde{\circ}v_\varepsilon)(x))\big|\det\nabla(\tilde{v}_\varepsilon\tilde{\circ}v_\varepsilon)(x)\big|\,\mathrm{d}x \\
&= \int_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} \varphi((\tilde{V}_\varepsilon\circ v_\varepsilon)(x))\big|\det\nabla(\tilde{V}_\varepsilon\circ v_\varepsilon)(x)\big|\,\mathrm{d}x \\
&= \int_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} (\varphi\circ\tilde{V}_\varepsilon)(v_\varepsilon(x))\,\big|\det\nabla\tilde{V}_\varepsilon(v_\varepsilon(x))\big|\,|\det\nabla v_\varepsilon(x)|\,\mathrm{d}x \\
&= \int_{\Omega\setminus U_\delta(\Gamma_{\mathrm{Cr}})} \tilde{\varphi}(v_\varepsilon(x))\,|\det\nabla v_\varepsilon(x)|\,\mathrm{d}x \overset{\varepsilon^\alpha\text{-GMS}}{\le} \int_{\mathbb{R}^d\setminus\Gamma_{\mathrm{Cr}}} \tilde{\varphi}(\tilde{x})\,\mathrm{d}\tilde{x} \\
&= \int_{\mathbb{R}^d\setminus\Gamma_{\mathrm{Cr}}} \varphi(\tilde{V}_\varepsilon(\tilde{x}))\,\big|\det\nabla V_\varepsilon(\tilde{x})\big|\,\mathrm{d}\tilde{x} \overset{\hat{x}=\tilde{V}_\varepsilon(\tilde{x})}{=} \int_{\tilde{V}_\varepsilon(\mathbb{R}^d\setminus\Gamma_{\mathrm{Cr}})} \varphi(\hat{x})\,\mathrm{d}\hat{x} \\
&\le \int_{\mathbb{R}^d} \varphi(\hat{x})\,\mathrm{d}\hat{x}.
\end{aligned}$$

Note that we cannot expect $\tilde{V}_\varepsilon = \mathrm{id}+\varepsilon\tilde{U}$ or $\tilde{v}_\varepsilon = \mathrm{id}+\varepsilon\tilde{u}$ to be globally injective on $\mathbb{R}^d\setminus\Gamma_{\mathrm{Cr}}$ or $\Omega_{\mathrm{Cr}} = \Omega\setminus\Gamma_{\mathrm{Cr}}$ respectively. As seen in Proposition 2.17 close-to-identity injectivity is related to the positivity of the jump, but $[\![\tilde{u}]\!]_{\Gamma_{\mathrm{Cr}}} = [\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} - [\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}$ may be negative.

For the proof of the injectivity of $\tilde{V}_\varepsilon := \mathrm{id} + \varepsilon\tilde{U}$ on the image $v_\varepsilon(\Omega\setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}))$ consider the following overlapping open cover $C_+\cup C_-\cup C_0 = \mathbb{R}^d\setminus\Gamma_{\mathrm{Cr}}$

$$\begin{aligned}
C_+ &:= \{x_1 > 0 \text{ and } x_2 > \max(0, x_1{-}1)\},\\
C_- &:= \{x_1 < 0 \text{ or } x_2 < \min(0, -x_1{+}1)\} \text{ and} \qquad (4.50)\\
C_0 &:= \{x_1 > 1\}.
\end{aligned}$$

Furthermore let $\tilde{x}_\varepsilon, \tilde{y}_\varepsilon \in v_\varepsilon(\Omega\setminus B_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}))$, $\tilde{x}_\varepsilon \ne \tilde{y}_\varepsilon$ with preimages $x_\varepsilon, y_\varepsilon \in \Omega\setminus B_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})$:

$$\tilde{x}_\varepsilon = v_\varepsilon(x_\varepsilon), \qquad \tilde{y}_\varepsilon = v_\varepsilon(y_\varepsilon).$$

To show $\tilde{v}_\varepsilon(\tilde{x}_\varepsilon) \ne \tilde{v}_\varepsilon(\tilde{y}_\varepsilon)$ for ε small enough we distinguish two cases:

- either the pair $x_\varepsilon, y_\varepsilon$ lies together in $\Omega\setminus C_+$ or in $\Omega\setminus C_-$,
- or, up to interchanging, we have $x_\varepsilon \in \Omega\cap C_+$ and $y_\varepsilon \in \Omega\cap C_-$.

Preimages in common Lipschitz domain: We have either $\{x_\varepsilon, y_\varepsilon\} \subset \mathbb{R}^d\setminus C_+$ or $\{x_\varepsilon, y_\varepsilon\} \subset \mathbb{R}^d\setminus C_-$. Combined with $\{x_\varepsilon, y_\varepsilon\} \subset \Omega\setminus U_{\varepsilon^\alpha}(\Gamma_c)$ that gives us

$$\{x_\varepsilon, y_\varepsilon\} \subset \Omega\setminus(C_+\cup U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})) \quad \text{or} \quad \{x_\varepsilon, y_\varepsilon\} \subset \Omega\setminus(C_-\cup U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})).$$

By $\alpha < \beta$ and Proposition 3.3 for ε small enough we have

$$r_\varepsilon = \|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} \leq C\varepsilon^\beta < \frac{1}{4}\varepsilon^\alpha.$$

Thus by $v_\varepsilon(x) \in U_{r_\varepsilon}(x)$ in both cases we get either

$$\begin{aligned}\tilde{x}_\varepsilon, \tilde{y}_\varepsilon \in U_{r_\varepsilon}\Big(\Omega \setminus (C_+ \cup U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}))\Big) &\subset U_{r_\varepsilon}\Big(\Omega \setminus (C_+ \cup U_{4r_\varepsilon}(\Gamma_{\mathrm{Cr}}))\Big)\\ \subset (C_+ \cup C_0) \setminus U_{2r_\varepsilon}(\Gamma_{\mathrm{Cr}}) \qquad \text{or}&\\ \tilde{x}_\varepsilon, \tilde{y}_\varepsilon \in U_{r_\varepsilon}\Big(\Omega \setminus (C_- \cup U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}))\Big) &\subset U_{r_\varepsilon}\Big(\Omega \setminus (C_- \cup U_{4r_\varepsilon}(\Gamma_{\mathrm{Cr}}))\Big)\\ \subset (C_- \cup C_0) \setminus U_{2r_\varepsilon}(\Gamma_{\mathrm{Cr}}).&\end{aligned}$$

On both sets $C_+ \cup C_0$ and $C_- \cup C_0$ the function $\tilde{U}$ is Lipschitz continuous with some constant L and we obtain:

$$\begin{aligned}|\tilde{v}_\varepsilon(\tilde{x}_\varepsilon) - \tilde{v}_\varepsilon(\tilde{y}_\varepsilon)| &= |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon + \varepsilon(\tilde{u}(\tilde{x}_\varepsilon) - \tilde{u}(\tilde{y}_\varepsilon))|\\ &\geq |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon| - \varepsilon|\tilde{u}(\tilde{x}_\varepsilon) - \tilde{u}(\tilde{y}_\varepsilon)|\\ &\geq |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon| - \varepsilon L|\tilde{x}_\varepsilon - \tilde{y}_\varepsilon| = (1 - \varepsilon L)\,|\tilde{x}_\varepsilon - \tilde{y}_\varepsilon|.\end{aligned}$$

Since $\tilde{x}_\varepsilon \neq \tilde{y}_\varepsilon$, the right-hand side is positive for ε small enough and $\tilde{v}_\varepsilon(\tilde{x}_\varepsilon) \neq \tilde{v}_\varepsilon(\tilde{y}_\varepsilon)$ follows.

Preimages on opposite sides of Γ_{Cr}: If we do not have $\{x_\varepsilon, y_\varepsilon\} \subset \mathbb{R}^d \setminus C_+$ or $\{x_\varepsilon, y_\varepsilon\} \subset \mathbb{R}^d \setminus C_-$, that means $x_\varepsilon \in C_+ \setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})$ and $y_\varepsilon \in C_- \setminus U_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})$ or vice versa. Either way, $\{x_\varepsilon, y_\varepsilon\} \subset \Omega \setminus U_{\varepsilon^\alpha}(\Gamma_c)$ gives us

$$|x_\varepsilon - y_\varepsilon| \geq \mathrm{dist}\left(\Omega_+ \setminus B_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}}), \Omega_- \setminus B_{\varepsilon^\alpha}(\Gamma_{\mathrm{Cr}})\right) \geq \frac{\sqrt{2}}{2}\varepsilon^\alpha, \tag{4.51}$$

thus we can estimate

$$\begin{aligned}|\tilde{v}_\varepsilon(\tilde{x}_\varepsilon) - \tilde{v}_\varepsilon(\tilde{y}_\varepsilon)| &\geq |\tilde{x}_\varepsilon - \tilde{y}_\varepsilon| - 2\|\varepsilon\tilde{u}_\varepsilon\|_{\mathrm{L}^\infty} \geq |x_\varepsilon - y_\varepsilon| - 2\|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} - 2\|\varepsilon\tilde{u}_\varepsilon\|_{\mathrm{L}^\infty}\\ &\geq \frac{\sqrt{2}}{2}\varepsilon^\alpha - C\varepsilon^\beta - C\varepsilon,\end{aligned}$$

which is positive for ε small enough, since $\alpha < \beta < 1$, and $\tilde{v}_\varepsilon(\tilde{x}_\varepsilon) \neq \tilde{v}_\varepsilon(\tilde{y}_\varepsilon)$ follows. □

Note that above Proposition 4.13 does not require the jump condition of the competitor $[\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$. Since by Theorem 4.10 for $\alpha > 1$ the ε^α-GMS condition of $\mathrm{id} + \varepsilon\hat{u}_\varepsilon$ implies $[\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$, Proposition 4.13 cannot hold for $\alpha > 1$ in the current form. We still expect the crack-respecting composition to satisfy the ε^α-GMS condition under the additional assumption $[\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ and after adding a forcing apart φ_k as in Proposition 2.17. For three small parameters $\delta, \eta, \mu > 0$ we set

$$\varphi_{\delta,\eta,\mu}(x) = \mu\lambda_{\delta,\eta}(x)n \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma^{(\delta)}_{\mathrm{Cr}}, \mathbb{R}^d)$$

with $n = (1, 1, 0, ..., 0)^\top \in \mathbb{R}^d$ and the scalar function $\lambda_\eta \in \mathrm{W}^{1,\infty}(\mathbb{R}^d \setminus \Gamma^{(\delta)}_{\mathrm{Cr}})$ given by

$$\lambda_{\delta,\eta}(x_1, x_2, \ldots, x_d) = \begin{cases} 0 & \text{if } x_1 > 1{-}\delta,\\ \min\{1, \frac{1}{\eta}(1{-}\delta{-}x_1)\} & \text{for } x_1 \in\,]0, 1{-}\delta] \text{ and } x_2 > 0,\\ -\min\{1, \frac{1}{\eta}(1{-}\delta{-}x_1)\} & \text{for } x_1 \in\,]0, 1{-}\delta] \text{ and } x_2 < 0,\\ -1 & \text{for } x_1 \leq 0.\end{cases}$$

Hence the jump of $\lambda_{\delta,\eta}$ grows linearly with slope $1/\eta$ with the distance from the edge $\Gamma^{(\delta)}_{\text{edge}} = \{\delta, 0\} \times \mathbb{R}^{d-2}$ of the smaller crack $\Gamma^{(\delta)}_{\text{Cr}}$ and then saturates at the values ± 1. For $\tilde{u} \in \mathrm{C}^\infty_{b,*}(\Omega_{\text{Cr}}, \mathbb{R}^d)$, such that $\tilde{u} \in \mathrm{C}^\infty_b(\Omega \setminus \Gamma^{(\delta)}_{\text{Cr}}, \mathbb{R}^d)$ for some $\delta > 0$, we propose to take

$$\varphi_k = \varphi_{\delta,\mu_k^\gamma,\mu_k} \tag{4.52}$$

with an exponent $\gamma \in (1, 2)$. Then for $\mu_k \to 0$ we have $\varphi_k \overset{\mathrm{H}^1}{\to} 0$ by $\gamma < 2$ and the slope of φ_k at the crack edge $\mu_k^{1-\gamma} \to \infty$ is expected to ensure the injectivity as in Proposition 2.17. More precisely we even expect that instead of $\varphi_k \in \mathrm{H}^1_{\text{loc}}(\mathbb{R}^d \setminus \Gamma^{(\delta)}_{\text{Cr}}, \mathbb{R}^d)$ a smooth approximation $\widetilde{\varphi}_k \in \mathrm{C}^\infty_c(\mathbb{R}^d \setminus \Gamma_{\text{Cr}}(\tfrac{1}{2}), \mathbb{R}^d)$ with $[\![\widetilde{\varphi}_k]\!]_{\Gamma_{\text{Cr}}} \geq [\![\varphi_k]\!]_{\Gamma_{\text{Cr}}}$ and $\widetilde{\varphi}_k \overset{\mathrm{H}^1}{\to} 0$ should be suitable, thus we formulate the following

Conjecture 4.14. *There exists $\alpha_{\text{Con}} \in (1, \infty]$ such that for*

- *every sequence of states $(u_\varepsilon, z_\varepsilon) \to (u_0, z_0)$ weakly in $\mathcal{Q}$ with*

$$\sup \mathcal{E}^{(\alpha)}_\varepsilon(u_\varepsilon, z_\varepsilon) < \infty,$$

- *every competitor $(\hat{u}_0, \hat{z}_0) \in \mathcal{Q}$ with $(\tilde{u}, \tilde{z}) := (\hat{u}_0, \hat{z}_0) - (u_0, z_0) \in \mathrm{C}^\infty_{b,*}(\Omega \setminus \Gamma^{(\delta)}_{\text{Cr}}, \mathbb{R}^d) \times \mathrm{C}^\infty_c(\Omega, \mathbb{R}^d)$ and*

$$[\![\hat{u}_0]\!]_{\Gamma_{\text{Cr}}} \geq 0,$$

there exists a sequence $\widetilde{\varphi}_k \in \mathrm{C}^\infty_{b,*}(\mathbb{R}^d \setminus \Gamma_{\text{Cr}}, \mathbb{R}^d)$ with $\widetilde{\varphi}_k \overset{\mathrm{H}^1}{\to} 0$ as well as upper and lower extensions $\tilde{u}_\pm \in \mathrm{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ of $\tilde{u}$ and $\tilde{\varepsilon}(\alpha_{\text{Con}}, k) > 0$ such that for every $\varepsilon < \tilde{\varepsilon}$ the crack-respecting compositions

$$\hat{v}_{k,\varepsilon} := \big(\mathrm{id} + \varepsilon(\tilde{u} + \varphi_k)\big) \tilde{\circ} (\mathrm{id} + \varepsilon u_\varepsilon)$$

given by Lemma 4.8 fulfill the $\varepsilon^{\alpha_{\text{Con}}}$-GMS condition.

We want to comment on the difficulties when approaching the extension of the results from Proposition 4.13 to the statement of the previous Conjecture 4.14.

Remark 4.15. *In the proof of Proposition 4.13 the exclusion of $U_{\varepsilon^\alpha}(\Gamma_{\text{Cr}})$ in the weaker ε^α-GMS condition helped in two ways : (i) by (4.45) in Lemma 4.8 we were able to substitute the crack-respecting composition $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$ by a single classical composition $\tilde{V}_\varepsilon \circ v_\varepsilon = (\mathrm{id} + \varepsilon \tilde{U}) \circ v_\varepsilon$, which enabled us to reduce the proof to showing injectivity of $\mathrm{id} + \varepsilon \tilde{U}$ on the image $v_\varepsilon(\Omega \setminus U_{\varepsilon^\alpha}(\Gamma_{\text{Cr}}))$; (ii) it gave the lower bound $|x_\varepsilon - y_\varepsilon| \geq \frac{\sqrt{2}}{2}\varepsilon^\alpha$ for the preimages $x_\varepsilon \in C_\pm$ and $y_\varepsilon \in C_\mp$ on opposite sides of the crack Γ_{Cr}, which then lead to a positive lower bound on the distance of the images.*

Even in the case of full the GMS condition, i.e. for $\alpha = \infty$, without the exclusion of $U_{\varepsilon^\alpha}(\Gamma_{\text{Cr}})$ the crack-respecting composition $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$ still is a classical composition piecewise on A_+ and A_- by the definition (4.44): $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon = (\mathrm{id} + \varepsilon \tilde{u}_\pm) \circ v_\varepsilon$ on $A_\pm$. Thus in the spirit of (i) the GMS condition (and the ε^α-GMS condition for any $\alpha < \infty$) of the crack-respecting composition restricted to $A_\pm$ respectively would follow from the injectivity of $\mathrm{id} + \varepsilon \tilde{u}_\pm \in \mathrm{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$, which are diffeomorphisms for $\varepsilon < \|\tilde{u}_\pm\|^{-1}_{\mathrm{C}^1}$.

Hence the only non-injectivity of $\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon$ to be expected would be an overlap of the two sides $A_+ \setminus A_- \subset C_+$ and $A_- \setminus A_+ \subset C_-$ opposite of the crack. At this point the lower bound from (ii) is too strong as it gives the injectivity without actually using the jump condition $[\![\hat{u}]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ of the competitor. Our idea to employ the jump conditions of u_0 and $\hat{u}_0$ relies on considering the following estimate for $x_\varepsilon \in A_+ \setminus A_-$ and $y_\varepsilon \in A_- \setminus A_+$ with $x_\varepsilon \to z_0 \in \Gamma_{\mathrm{Cr}}$ and $y_\varepsilon \to z_0 \in \Gamma_{\mathrm{Cr}}$:

$$
\begin{aligned}
\big|(\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(x_\varepsilon) - (\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(y_\varepsilon)\big| &\geq \Big((\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(x_\varepsilon) - (\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(y_\varepsilon)\Big) \cdot \nu(z_0) \\
&= \Big(((\mathrm{id} + \varepsilon \tilde{u}_+) \circ v_\varepsilon)(x_\varepsilon) - ((\mathrm{id} + \varepsilon \tilde{u}_-) \circ v_\varepsilon)(y_\varepsilon)\Big) \cdot \nu(z_0) \\
&= (x_\varepsilon - y_\varepsilon) \cdot \nu(z_0) + \big(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\big) \cdot \nu(z_0) + \Big(\tilde{u}_+(v_\varepsilon(x_\varepsilon)) - \tilde{u}_-(v_\varepsilon(y_\varepsilon))\Big) \cdot \nu(z_0) \\
&\geq \big(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\big) \cdot \nu(z_0) + \Big(\tilde{u}_+(v_\varepsilon(x_\varepsilon)) - \tilde{u}_-(v_\varepsilon(y_\varepsilon))\Big) \cdot \nu(z_0).
\end{aligned}
$$

Since, the deformation v_ε converges uniformly by Proposition 3.3, by continuity of $\tilde{u}_\pm$ the second summand converges to $[\![\tilde{u}]\!]_{\Gamma_{\mathrm{Cr}}}(z_0)$. Hence, if the limit

$$
\big(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)\big) \cdot \nu(z_0) \stackrel{!}{\to} \big(u_0^+(z_0) - u_0^-(z_0)\big) \cdot \nu(z_0) = [\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}(z_0) \tag{4.53}
$$

held, we could ensure $[\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq \delta > 0$ by the forcing-apart φ_k from Proposition 2.17 as mentioned in Conjecture 4.14 and obtain:

$$
\lim_{\varepsilon \to 0} \big|(\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(x_\varepsilon) - (\tilde{v}_\varepsilon \tilde{\circ} v_\varepsilon)(y_\varepsilon)\big| \geq [\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}(z_0) + [\![\tilde{u}]\!]_{\Gamma_{\mathrm{Cr}}}(z_0) = [\![\hat{u}_0]\!]_{\Gamma_{\mathrm{Cr}}}(z_0) \geq \delta > 0. \tag{4.54}
$$

Let us further investigate the hypothetical limit (4.53) by rewriting the pointwise expressions as integrals using the Hölder continuity of u_ε given by the coercivity (4.7c):

$$
\|u_\varepsilon\|_{\mathrm{C}^\gamma} \leq \|u_\varepsilon\|_{\mathrm{W}^{1,p}} \leq \frac{1}{\varepsilon} C_\gamma.
$$

Consider some mollifier $\rho \in \mathrm{C}_c^\infty(B_1(0), [0, \infty))$ and for the half balls

$$
\begin{aligned}
B_r^+(x) &= \{y \in B_r(x) \mid (y - x) \cdot \nu(z_0) > 0\} \qquad \textit{and} \\
B_r^-(x) &= \{y \in B_r(x) \mid (y - x) \cdot \nu(z_0) < 0\}
\end{aligned}
$$

define the rescaled and translated kernels

$$
\rho_{x,r} \colon y \mapsto \frac{C}{r^{d-1}} \rho\Big(\frac{1}{r}(y - x)\Big),
$$

as well as the restrictions

$$
\rho_{x,r}^\pm := \rho_{x,r}|_{B_r^\pm(x)},
$$

where the constant C and the scaling by r^{1-d} are chosen such that for the surface integrals independently of x and r we have

$$
\int_{\partial B_r^\pm(x)} \rho_{x,r}^\pm(y) \,\mathrm{d}y = 1.
$$

Since in (4.53) we may make an error of $\frac{\delta}{2}$ and still get positivity in (4.54), we choose radii

$$
r_\varepsilon := \Big(\frac{\varepsilon \delta}{4 C_\gamma}\Big)^{\frac{1}{\gamma}}
$$

and rewrite the pointwise terms in (4.53) *as surface integrals:*

$$
\begin{aligned}
&(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)) \cdot \nu(z_0) \\
&\quad = \Big(u_\varepsilon(x_\varepsilon) \int_{\partial B^+_{r_\varepsilon}(x_\varepsilon)} \rho^+_{x_\varepsilon, r_\varepsilon}(y)\,\mathrm{d}y - u_\varepsilon(x_\varepsilon) \int_{\partial B^+_{r_\varepsilon}(y_\varepsilon)} \rho^-_{y_\varepsilon, r_\varepsilon}(y)\,\mathrm{d}y\Big) \cdot \nu(z_0) \\
&\quad \geq -\frac{\delta}{2} + \int_{\partial B^+_{r_\varepsilon}(x_\varepsilon)} (\rho^+_{x_\varepsilon, r_\varepsilon}(y) u_\varepsilon(y)) \cdot \nu(z_0)\,\mathrm{d}y - \int_{\partial B^+_{r_\varepsilon}(y_\varepsilon)} (\rho^-_{y_\varepsilon, r_\varepsilon}(y) u_\varepsilon(y)) \cdot \nu(z_0)\,\mathrm{d}y
\end{aligned}
$$

Using the translation $\bar{u}_\varepsilon$ *of* u_ε *given by*

$$
\bar{u}_\varepsilon \colon x \mapsto \begin{cases} u_\varepsilon(x - (x_\varepsilon - z_0)) & \text{for } (x - z_0) \cdot \nu(z_0) > 0, \\ u_\varepsilon(x - (y_\varepsilon - z_0)) & \text{for } (x - z_0) \cdot \nu(z_0) < 0, \end{cases}
$$

and doing the derivation from Theorem 4.10 with the divergence theorem and product rule for the divergence backwards we arrive at:

$$
\begin{aligned}
&(u_\varepsilon(x_\varepsilon) - u_\varepsilon(y_\varepsilon)) \cdot \nu(z_0) \\
&\quad \geq -\frac{\delta}{2} + \int_{\partial B^+_{r_\varepsilon}(z_0)} (\bar{u}_\varepsilon(y) \rho^+_{z_0, r_\varepsilon}(y)) \cdot \nu(z_0)\,\mathrm{d}y - \int_{\partial B^-_{r_\varepsilon}(z_0)} (\bar{u}_\varepsilon(y) \rho^-_{z_0, r_\varepsilon}(y)) \cdot \nu(z_0)\,\mathrm{d}y \\
&\quad = -\frac{\delta}{2} + \int_{B^+_{r_\varepsilon}(z_0)} \operatorname{div}\,(\bar{u}_\varepsilon\, \rho^+_{z_0, r_\varepsilon})(y)\,\mathrm{d}y + \int_{B^-_{r_\varepsilon}(z_0)} \operatorname{div}(\bar{u}_\varepsilon\, \rho^-_{z_0, r_\varepsilon})(y)\,\mathrm{d}y \\
&\quad = -\frac{\delta}{2} + \int_{B_{r_\varepsilon}(z_0) \setminus \Gamma_{\mathrm{Cr}}} \operatorname{div}\,(\bar{u}_\varepsilon\, \rho_{z_0, r_\varepsilon})(y)\,\mathrm{d}y \\
&\quad = -\frac{\delta}{2} + \int_{B_{r_\varepsilon}(z_0) \setminus \Gamma_{\mathrm{Cr}}} \bar{u}_\varepsilon(y)\, \nabla \rho_{z_0, r_\varepsilon}(y)\,\mathrm{d}y + \int_{B_{r_\varepsilon}(z_0) \setminus \Gamma_{\mathrm{Cr}}} \operatorname{div} \bar{u}_\varepsilon(y)\, \rho_{z_0, r_\varepsilon}(y)\,\mathrm{d}y \\
&\quad = -\frac{\delta}{2} + \int_{B_1(z_0) \setminus \Gamma_{\mathrm{Cr}}} \bar{u}_\varepsilon(y)\, \nabla \rho_{z_0, r_\varepsilon}(y)\,\mathrm{d}y + \int_{B_1(z_0) \setminus \Gamma_{\mathrm{Cr}}} \operatorname{div} \bar{u}_\varepsilon(y)\, \rho_{z_0, r_\varepsilon}(y)\,\mathrm{d}y
\end{aligned}
$$

In the case of space dimension $d = 2$ *and under the additional assumption of strong convergence*

$$
\operatorname{div} u_\varepsilon \to \operatorname{div} u_0 \qquad \text{strongly in } \mathrm{L}^2 \tag{4.55}
$$

we could further investigate. By the scaling of $\rho_{z_0, r_\varepsilon}$ *by* $r^{1-d} = r^{-1}$ *it converges to* 0 *in* L^1 *and for* $d = 2$ *it is bounded in* L^2 *to, thus* $\rho_{z_0, r_\varepsilon}$ *converges weakly to* 0 *in* L^2 *and with* (4.55) *the second integral vanishes in the limit. To treat the first integral observe, that* $\nabla \rho_{z_0, r_\varepsilon}$ *displays a scaling by* r^{-2}*, such that it weak-$*$ converges in* L^1*. In particular one can even show* $\nabla \rho_{z_0, r_\varepsilon} \overset{\varepsilon \to 0}{\rightharpoonup} \pm \nu(z_0) \delta_{z_0}$ *on* $B^\pm_1(z_0)$*, such that the version of the div-curl lemma cited below in Theorem 4.16, which requires strong convergence of the divergence, would give*

$$
\int_{B_1(z_0) \setminus \Gamma_{\mathrm{Cr}}} \bar{u}_\varepsilon(y)\, \nabla \rho_{z_0, r_\varepsilon}(y)\,\mathrm{d}y \to [\![u_0]\!]_{\Gamma_{\mathrm{Cr}}}
$$

and as desired (4.54) *would follow with* $\frac{\delta}{2}$ *on the right-hand side.*

Unfortunately it seems there is no method available in evolutionary problems to improve the weak convergence of $\operatorname{div} u_\varepsilon$ *to strong convergence in* L^2 *as it does in stationary problems (, e.g. see [MR15, p.149-150] or the so-called Visintin trick [Vis84]).*

We will cite the following div-curl lemma from [BCDM09]. It is a more rigorous version of the classical div-curl lemma allowing integrability exponents not dual to each other.

Theorem 4.16. *Let Ω be an open set of $\mathbb{R}^d$, $d \geq 2$. Consider two sequences $v_k \in \mathrm{L}^d(\Omega)^d$ and $w_k \in \mathcal{M}(\Omega)^d$, that satisfy the following conditions:*

- $\begin{cases} v_k \rightharpoonup v \text{ weakly in } \mathrm{L}^d(\Omega)^d, \\ w_k \rightharpoonup w \text{ weakly-}* \text{ in } \mathcal{M}(\Omega)^d, \end{cases}$
- $\begin{cases} |v_k - v|^d \rightharpoonup \mu \text{ weakly-}* \text{ in } \mathcal{M}(\Omega), \\ |w_k - w| \rightharpoonup \nu \text{ weakly-}* \text{ in } \mathcal{M}(\Omega), \end{cases}$
- $\begin{cases} \operatorname{div} v_k \to \operatorname{div} v \text{ strongly in } \mathrm{L}^d(\Omega), \\ \operatorname{curl} w_k \to \operatorname{curl} w \text{ weakly-}* \text{ in } \mathcal{M}(\Omega)^d. \end{cases}$

Then, up to a subsequence, there exist two sequences $x_j \in \Omega$ and $r_j \in \mathbb{R}^d$, such that

$$v_k \cdot w_k \to v \cdot w + \sum_{j=1}^{\infty} \operatorname{div}(r_j \delta_{x_j}) \qquad \text{in } \mathcal{D}'(\Omega) \tag{4.56}$$

where $v_k \cdot w_k$ is understood in the sense of [BCDM09, Remark 4.2] and

$$\forall\, j \geq 1 \colon |r_j| \leq c\mu(\{x_j\})^{\frac{1}{d}} \nu(\{x_j\})$$

with c only depending on the dimension d.

Assuming Conjecture 4.14 in the following corollary we are able to prove the mutual recovery sequence for $\mathcal{T}_\varepsilon^{(\alpha_{\mathrm{Con}})}$ with constraint using the mutual recovery sequence for $\widetilde{\mathcal{T}}_\varepsilon$ without constraint constructed in Proposition 4.9.

Corollary 4.17 (Mutual recovery sequence with constraint)**.** *Assume Conjecture 4.14 and let $t \in [0,T]$, $(u_\varepsilon, z_\varepsilon) \to (u_0, z_0)$ weakly in $\mathcal{Q}$ with*

$$\sup \mathcal{E}_\varepsilon^{(\alpha_{\mathrm{Con}})}(u_\varepsilon, z_\varepsilon) < \infty$$

as well as $(\hat{u}_0, \hat{z}_0) \in \mathcal{Q}$ with $(\tilde{u}, \tilde{z}) := (\hat{u}_0, \hat{z}_0) - (u_0, z_0) \in \mathrm{C}^\infty_{b,*}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \times \mathrm{C}^\infty_c(\Omega, \mathbb{R}^d)$.

Then there exists a mutual recovery sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \rightharpoonup (\hat{u}_0, \hat{z}_0)$ weakly in $\mathcal{Q}$, i.e the $\limsup$ *inequality on the transition cost holds:*

$$\limsup_{\varepsilon \to 0} \mathcal{T}_\varepsilon^{(\alpha_{\mathrm{Con}})}(t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \leq \mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0). \tag{4.57}$$

Proof. From the boundedness $\sup \mathcal{E}_\varepsilon^{(\alpha_{\mathrm{Con}})}(u_\varepsilon, z_\varepsilon) =: E < \infty$ by the $\liminf$ inequality from Corollary 4.11 we obtain $\mathcal{E}_0(u_0, z_0) \leq E < \infty$, which also gives $\mathcal{G}_0(u_0, z_0) =: G < \infty$. This together with $\mathcal{D}_0 \geq 0$ gives $\mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0) \geq \mathcal{G}_0(\hat{u}_0, \hat{z}_0) - G$ on the right-hand side of (4.57), thus for $\mathcal{G}_0(\hat{u}_0, \hat{z}_0) = \infty$ the inequality would be satisfied trivially and for the rest of the proof we may assume $\mathcal{G}_0(u_0, z_0) < \infty$, which in particular gives:

$$[\![u_0]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0.$$

With this jump condition ensured, the Conjecture 4.14 provides a sequence $\widetilde{\varphi}_k \overset{\mathrm{H}^1}{\to} 0$ of forcing-aparts as discussed above Conecture 4.14 as well as upper and lower extensions $\tilde{u}_\pm \in \mathrm{C}^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ of $\tilde{u}$ such that the crack-respecting compositions

$$\hat{v}_{k,\varepsilon} := \big(\mathrm{id} + \varepsilon(\tilde{u} + \varphi_k)\big)\tilde{\circ}(\mathrm{id} + \varepsilon u_\varepsilon) \quad \text{ satisfy the } \varepsilon^{\alpha_{\mathrm{Con}}}\text{-GMS condition}$$

for every $\varepsilon < \tilde{\varepsilon}(\alpha_{\mathrm{Con}}, k)$. Defining now

$$\hat{u}_{k,\varepsilon} := \frac{1}{\varepsilon}(\hat{v}_{k,\varepsilon} - \mathrm{id}) \quad \text{and}$$
$$\hat{z}_\varepsilon := \begin{cases} \frac{1}{\varepsilon}\big(\exp(\varepsilon\tilde{z})(I + \varepsilon z_\varepsilon) - I\big) & \text{on } \Omega_\varepsilon, \\ z_\varepsilon & \text{otherwise,} \end{cases}$$

the constraints in the transition costs in both the finite for $\varepsilon < \tilde{\varepsilon}(\alpha_{\mathrm{Con}}, k)$ as well as in the linearized case are satisfied and we have the equalities:

$$\mathcal{T}_\varepsilon^{(\alpha_{\mathrm{Con}})}(t, u_\varepsilon, z_\varepsilon, \hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) = \widetilde{\mathcal{T}}_\varepsilon(t, u_\varepsilon, z_\varepsilon, \hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) \quad \text{and} \tag{4.58}$$
$$\mathcal{T}_0(t, u_0, z_0, \hat{u}_0 + \varphi_k, \hat{z}_0) = \widetilde{\mathcal{T}}_0(t, u_0, z_0, \hat{u}_0 + \varphi_k, \hat{z}_0). \tag{4.59}$$

Furthermore Proposition 4.9 shows that $(\hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) \overset{\mathcal{Q}}{\rightharpoonup} (\hat{u}_0 + \tilde{\varphi}_k, \hat{z}_0)$ is a mutual recovery for the transition cost $\widetilde{\mathcal{T}}_\varepsilon$ without constraints and we arrive for each k at

$$\begin{aligned} \limsup_{\varepsilon\to0} \mathcal{T}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon, \hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) &= \limsup_{\varepsilon\to0} \widetilde{\mathcal{T}}_\varepsilon(t, u_\varepsilon, z_\varepsilon, \hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) \\ &\le \widetilde{\mathcal{T}}_0(t, u_0, z_0, \hat{u}_0 + \tilde{\varphi}_k, \hat{z}_0) \le \mathcal{T}_0(t, u_0, z_0, \hat{u}_0 + \tilde{\varphi}_k, \hat{z}_0). \end{aligned} \tag{4.60}$$

The asserted mutual recovery sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \overset{\mathcal{Q}}{\rightharpoonup} (\hat{u}_0, \hat{z}_0)$ we will now construct from $(\hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon)$ by a diagonal sequence where we use the continuity of $\hat{u} \mapsto \widetilde{\mathcal{T}}_0(t, u_0, z_0, \hat{u}, \hat{z}_0)$ and separability of the dual space of $\mathcal{U}$, i.e. there exists a sequence e_m that is dense in $\mathcal{U}'$.

For each fixed k we can find $\varepsilon_k > 0$ such that for every $\varepsilon < \varepsilon_k$ we have on one hand by above lim sup inequality (4.60)

$$\mathcal{T}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon, \hat{u}_{k,\varepsilon}, \hat{z}_\varepsilon) \le \frac{1}{2^k} + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0 + \varphi_k, \hat{z}_0) \tag{4.61}$$

and on the other hand by the weak convergence $\hat{u}_{k,\varepsilon} \overset{\varepsilon\to0}{\rightharpoonup} \hat{u}_0 + \varphi_k$ for every $m \le k$

$$\big|\langle e_m, \hat{u}_{k,\varepsilon} - (\hat{u}_0 + \varphi_k)\rangle\big| < \frac{1}{2^k}. \tag{4.62}$$

Furthermore by (4.59), the convergence $\tilde{\varphi}_k \overset{\mathrm{H}^1}{\to} 0$ and the continuity of $\hat{u} \mapsto \widetilde{\mathcal{T}}_0(t, u_0, z_0, \hat{u}, \hat{z}_0)$ for every k we find ℓ_k such that:

$$\mathcal{T}_0(t, u_0, z_0, \hat{u}_0 + \tilde{\varphi}_\ell, \hat{z}_0) \le \frac{1}{2^k} + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0) \quad \text{for every } \ell \ge \ell_k. \tag{4.63}$$

Then defining $\hat{\varepsilon}(k) := \min_{j\le k}\{\varepsilon_j\}$ and setting

$$\hat{u}_\varepsilon := \hat{u}_{k,\varepsilon} \qquad \text{for } \hat{\varepsilon}(k) > \varepsilon \ge \hat{\varepsilon}(k+1)$$

gives the desired mutual recovery sequence. The convergence $\hat{u}_\varepsilon \rightharpoonup \hat{u}_0$ weakly in $\mathcal{U}$ follows from $\varphi_k \overset{\mathrm{H}^1}{\to} 0$ and (4.62) as for each fixed e_m and $\delta > 0$ you may take k_δ such that $\frac{1}{2^k} < \frac{\delta}{2}$ and $\|\varphi_k\|_{\mathrm{H}^1} < \frac{\delta}{2}\|e_m\|_{\mathrm{H}^{-1}}$ for all $k \ge k_\delta$, then for every $\varepsilon < \hat{\varepsilon}\big(\max\{m, k_\delta\}\big)$, i.e. $\hat{\varepsilon}(\ell) > \varepsilon \ge \hat{\varepsilon}(\ell+1)$ with some $\ell \ge \max\{m, k_\delta\}$, we have

$$\big|\langle e_m, \hat{u}_\varepsilon - \hat{u}_0\rangle\big| \le \big|\langle e_m, \hat{u}_{\ell,\varepsilon} - (\hat{u}_0 + \varphi_\ell)\rangle\big| + \big|\langle e_m, \varphi_\ell\rangle\big| < \frac{1}{2^\ell} + \frac{\delta}{2} \le \delta.$$

The lim sup inequality follows combining (4.61) and (4.63) as for every $\delta > 0$ we may take k_δ with $2^{k_\delta} < \frac{\delta}{2}$, then consider ℓ_{k_δ} and obtain that for $\varepsilon < \hat{\varepsilon}\big(\max\{k_\delta, \ell_{k_\delta}\}\big)$, i.e. $\hat{\varepsilon}(\ell) > \varepsilon \geq \hat{\varepsilon}(\ell+1)$ with some $\ell \geq \max\{k_\delta, \ell_{k_\delta}\}$ the following:

$$\begin{aligned}\mathcal{T}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) = \mathcal{T}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon, \hat{u}_{\ell,\varepsilon}, \hat{z}_\varepsilon) &\leq \frac{1}{2^\ell} + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0 + \varphi_\ell, \hat{z}_0)\\ &\leq \frac{1}{2^\ell} + \frac{1}{2^{k_\delta}} + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0) \leq \delta + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0).\end{aligned}$$

Taking the lim sup this gives

$$\limsup_{\varepsilon \to 0} \mathcal{T}_\varepsilon^{(\alpha)}(t, u_\varepsilon, z_\varepsilon, \hat{u}_\varepsilon, \hat{z}_\varepsilon) \leq \delta + \mathcal{T}_0(t, u_0, z_0, \hat{u}_0, \hat{z}_0)$$

and the assertion follows from $\delta > 0$ being arbitrary. □

4.5 Evolutionary Gamma-convergence

Before we come to the proof of evolutionary Γ-convergence we note the following result that specializes the a priori bounds from Proposition 3.3 to the case of an energetic solution.

Corollary 4.18. *Let $(u_\varepsilon, z_\varepsilon)\colon [0,T] \to \mathcal{Q}$ be finite plasticity solutions with or without constraints. There exists a constant $C_{ES} > 0$ such that:*

$$\forall t \in [0,T]\colon \quad \|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) \leq C_{ES}. \tag{4.64}$$

Proof. Let us first consider the case of $(u_\varepsilon, z_\varepsilon)$ being an energetic solution of $(\mathcal{Q}, \mathcal{G}_\varepsilon^{(\alpha)}, \mathcal{D}_\varepsilon)$. By Proposition 3.3, the energy balance (4.22) and the boundedness of the initial total energy $\mathcal{G}_\varepsilon^{(\alpha)}(0, u_\varepsilon^0, z_\varepsilon^0) \leq G^{(\alpha)} < \infty$, which is implied by the convergence in (4.19), we have:

$$\begin{aligned}&\|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])\\
&\quad\leq (c_{\mathcal{E}} + 1)\Big(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(t), z_\varepsilon(t)) + C_{\mathcal{E}} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])\Big)\\
&\quad= (c_{\mathcal{E}} + 1)(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(t), z_\varepsilon(t)) + C_{\mathcal{E}}\\
&\qquad\qquad + \mathcal{G}_\varepsilon^{(\alpha)}(0, u_\varepsilon^0, z_\varepsilon^0) - \mathcal{G}_\varepsilon^{(\alpha)}(t, u_\varepsilon(t), z_\varepsilon(t)) - \int_0^t \langle \dot{\ell}(t'), u_\varepsilon(t')) \rangle \,\mathrm{d}t'\\
&\quad\leq (c_{\mathcal{E}} + 1)\Big(\mathcal{E}_\varepsilon^{(\alpha)}(u_\varepsilon(t), z_\varepsilon(t)) - \mathcal{G}_\varepsilon^{(\alpha)}(t, u_\varepsilon(t), z_\varepsilon(t)) + C_{\mathcal{E}} + G^{(\alpha)} - \int_0^t \langle \dot{\ell}(t'), u_\varepsilon(t')) \rangle \,\mathrm{d}t'\\
&\quad\leq (c_{\mathcal{E}} + 1)\Big(G^{(\alpha)} + \langle \ell(t), u_\varepsilon(t) \rangle + C_{\mathcal{E}}\Big) - \int_0^t \langle \dot{\ell}(t'), u_\varepsilon(t') \rangle \,\mathrm{d}t'\\
&\quad\leq (c_{\mathcal{E}} + 1)(G^{(\alpha)} + \|\ell(t)\|_{\mathrm{H}^{-1}} \|u_\varepsilon(t)\|_{\mathrm{H}^1} + C_{\mathcal{E}}) + \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}} \|u_\varepsilon(t')\|_{\mathrm{H}^1} \,\mathrm{d}t'\\
&\quad\leq (2c_{\mathcal{E}} + 2)(G^{(\alpha)} + \|\ell(t)\|_{\mathrm{H}^{-1}}(G^{(\alpha)} + C_{\mathcal{E}}) + C_{\mathcal{E}}) + \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}} \|u_\varepsilon(t')\|_{\mathrm{H}^1} \,\mathrm{d}t'.\end{aligned} \tag{4.65}$$

In the case of $(u_\varepsilon, z_\varepsilon)$ being an energetic solution to $(\mathcal{Q}, \widetilde{\mathcal{G}}_\varepsilon, \mathcal{D}_\varepsilon)$ analogously by (4.20) we have the boundedness $\widetilde{\mathcal{G}}_\varepsilon(t, u_\varepsilon^0, z_\varepsilon^0) =: \widetilde{G} < \infty$ and obtain:

$$\|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t])$$

$$
\begin{aligned}
&\le (c_{\mathcal{E}}+1)\Big(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(t),z_\varepsilon(t)) + C_{\mathcal{E}} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])\Big)\\
&= (c_{\mathcal{E}}+1)\Big(\widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon(t),z_\varepsilon(t)) - \widetilde{\mathcal{G}}_\varepsilon(t,u_\varepsilon(t),z_\varepsilon(t)) + C_{\mathcal{E}}\\
&\qquad + \widetilde{\mathcal{G}}_\varepsilon(0,u_\varepsilon^0,z_\varepsilon^0) - \int_0^t \langle \dot{\ell}(t'),u_\varepsilon(t')\rangle\,\mathrm{d}t'\Big)\\
&\le (2c_{\mathcal{E}}+2)(\widetilde{G} + \|\ell(t)\|_{\mathrm{H}^{-1}}(\widetilde{G}+C_{\mathcal{E}}) + C_{\mathcal{E}}) + \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}}\|u_\varepsilon(t')\|_{\mathrm{H}^1}\,\mathrm{d}t'. \qquad (4.66)
\end{aligned}
$$

In both cases with $G \in \{G^{(\alpha)}, \widetilde{G}\}$ we thus have

$$
\begin{aligned}
&\|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t])\\
&\quad\le (2c_{\mathcal{E}}+2)(G + \|\ell(t)\|_{\mathrm{H}^{-1}}(G+C_{\mathcal{E}}) + C_{\mathcal{E}}) + \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}}\|u_\varepsilon(t')\|_{\mathrm{H}^1}\,\mathrm{d}t', \qquad (4.67)
\end{aligned}
$$

such that to the end we argument for both cases simultaneously.

In particular, as every term on the left-hand side of (4.67) is nonnegative, the bound holds on each summand separately. We want to use Gronwall Lemma to obtain a uniform bound on $\|u_\varepsilon(t)\|_{\mathrm{H}^1}$, therefor we want to have equal powers on both sides. Thus we introduce

$$
\begin{aligned}
f(t) &:= \max\{1, \|u_\varepsilon(t)\|_{\mathrm{H}^1}^2\}\\
A(t) &:= \max\Big\{1, (2c_{\mathcal{E}}+2)(G + \|\ell(t)\|_{\mathrm{H}^{-1}}(G+C_{\mathcal{E}}) + C_{\mathcal{E}})\Big\} \quad \text{and}\\
B(t) &:= \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}},
\end{aligned}
$$

and by the estimate $s \le \max\{1, s^2\}$ above inequality (4.67) gives

$$
f(t) \le A(t) + \int_0^t B(t')f(t')\,\mathrm{d}t'.
$$

Recall assumption (4.17) by which we have on one hand by continuity a uniform bound

$$
A(t) \le A_*
$$

on the other hand we have that $\|\dot{\ell}(t')\|_{\mathrm{H}^{-1}}$ is integrable, which by non-negativity of B gives a bound

$$
\int_s^t B(t)\,\mathrm{d}t \le \int_0^T B(t)\,\mathrm{d}t \le B_* \qquad \text{for all } s,t \in [0,T].
$$

Thus using the Gronwall Lemma provides us with the uniform bound

$$
\begin{aligned}
\|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 \le f(t) &\le A(t) + \int_0^t A(s)B(s)\exp\Big(\int_s^t B(r)\,\mathrm{d}r\Big)\,\mathrm{d}s\\
&\le A_* + A_*\exp(B_*)B_* =: F_*.
\end{aligned}
$$

Inserting this back into (4.67) finally gives the asserted bound

$$
\|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \le A_* + F_*B_* =: C_{ES}.
$$

□

We now come to the evolutionary Γ-convergence in the case without constraint. Using the Γ-lim inf inequalities on the stored energy $\widetilde{\mathcal{E}}_\varepsilon$ and dissipation $\mathcal{D}_\varepsilon$ from Proposition 3.6 (a) and Proposition 3.7 respectively as well as the mutual recovery sequence from Proposition 4.9 the proof follows the abstract theory of RIS presented in [MR15]. Note that this is a mere convergence result, in particular the existence of finite-plasticity solutions is *assumed.*

Theorem 4.19 (Evolutionary Γ-convergence)**.** *Assume* (4.7)*,* (4.9)*,* (4.12)*,* (4.17) *and* (4.20)*. Let* $(u_\varepsilon, z_\varepsilon)$ *be a finite-plasticity solution without constraint, i.e. an energetic solution of* $(\mathcal{Q}, \widetilde{\mathcal{G}}_\varepsilon, \mathcal{D}_\varepsilon)$ *with initial data* $(u_\varepsilon^0, z_\varepsilon^0)$*.*

Then there exists a subsequence of ε*, such that* $(u_\varepsilon(t), z_\varepsilon(t)) \rightharpoonup (u_0(t), z_0(t))$ *weakly in* $\mathcal{Q}$ *for every* $t \in [0, T]$*, where* (u_0, z_0) *is a linearized-plasticity solution without constraint, i.e. an energetic solution of* $(\mathcal{Q}, \widetilde{\mathcal{G}}_0, \mathcal{D}_0)$ *with initial data* (u_0^0, z_0^0)*.*

Proof. We adapt the same splitting of the proof into five steps as suggested in [MR15] as a general strategy for dealing with energetic solutions.

<u>Step 1: A priori estimates.</u> From Corollary 4.18 we have the estimate

$$\forall t \in [0,T]\colon \quad \|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon; [0,t]) \leq C_{ES}.$$

<u>Step 2: Selection of convergent subsequences.</u> For the pointwise convergence of $z_\varepsilon(t)$ we will employ the generalized version of Helly's Selection Principle [MRS08, Theorem A.1], whose assumptions on $\mathcal{D}_\varepsilon$ include the triangle inequality (4.15), its definiteness, which follows from (3.7b) and the lower-semicontinuity shown in Proposition 3.7. Since Step 1 gives the uniform precompactness of $z_\varepsilon(t)$ weakly in L^2 and the uniform bound on $\mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon, [0,t])$ we are thus provided functions $z_0\colon [0,T] \to \mathcal{Z}$ and $\delta\colon [0,T] \to [0,\infty]$, such that for a non-relabled subsequence:

(a) $\forall t \in [0,T]\colon \delta(t) = \lim_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon, [0,t])$;

(b) $\forall t \in [0,T]\colon z_\varepsilon(t) \rightharpoonup z_0(t)$ weakly in L^2;

(c) $\forall s,t \in [0,T]$ with $s < t$: $\mathrm{Diss}_{\mathcal{D}_0}(z_0, [s,t]) < \delta(t) - \delta(s)$.

For the pointwise convergence of u_ε let $t \in [0,T]$ be fixed. By the a priori estimates from step 1 there is a subsequence (again not relabled), such that $u_\varepsilon(t) \rightharpoonup u_*$ weakly in H^1. We will now show that u_* is actually independent of the selection of a subsequence. For that we use the transition cost $\widetilde{\mathcal{T}}_\varepsilon$. Consider a competitor of the form $(\hat{u}_0, \hat{z}_0) = (u_*, z_0(t)) + (\tilde{u}, \tilde{z}) \in \mathcal{Q}$ with $(\tilde{u}, \tilde{z}) \in \mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \times \mathrm{C}_c^\infty(\Omega, \mathbb{R}^{d\times d})$ and use the mutual recovery sequence $(\hat{u}_\varepsilon, \hat{z}_\varepsilon) \rightharpoonup (\hat{u}_0, \hat{z}_0)$ from Proposition 4.9 to conclude from the stability (4.21) of $(u_\varepsilon, z_\varepsilon)$:

$$0 \leq \limsup_{\varepsilon\to 0} \widetilde{\mathcal{T}}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t), \hat{u}_\varepsilon, \hat{z}_\varepsilon) \leq \widetilde{\mathcal{T}}_0(t, u_*, z_0(t), \hat{u}_0, \hat{z}_0)$$

$$\text{for } (\hat{u}_0, \hat{z}_0) - (u_*, z_0(t)) \in \mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \times \mathrm{C}_c^\infty(\Omega, \mathbb{R}^{d\times d}).$$

Since $(\hat{u}, \hat{z}) \mapsto \widetilde{\mathcal{T}}_0(t, u, z, \hat{u}, \hat{z})$ is lower-semicontinuous and $\mathrm{C}^\infty_{b,*}(\Omega_{\mathrm{Cr}}, \mathbb{R}^d) \subset \mathrm{H}^1(\Omega_{\mathrm{Cr}}, \mathbb{R}^d)$ is dense (see Proposition 4.5) we actually have:

$$0 \leq \widetilde{\mathcal{T}}_0(t, u_*, z_0(t), \hat{u}_0, \hat{z}_0) \quad \text{for all } (\hat{u}_0, \hat{z}_0) \in \mathcal{Q}. \tag{4.68}$$

Specializing to $\hat{z}_0 = z_0(t)$ this gives

$$u_* \in \operatorname{Argmin}\{u \mapsto \widetilde{\mathcal{G}}_0(t, u, z_0(t))\}.$$

Note that by (2.2) and Korn inequality the minimizer $u_0(t)$ of $\widetilde{\mathcal{G}}_0(t, \cdot, z_0(t))$ is unique, thus we conclude that $u_* = u_0(t)$ independently of the choice of subsequence and that the whole sequence converges:

$$u_\varepsilon(t) \rightharpoonup u_0(t) \text{ weakly in } \mathrm{H}^1 \text{ for every } t \in [0, T]. \tag{4.69}$$

Step 3: Upper energy estimate. Consider an arbitrary partition $0 = t_1 < \ldots < t_N = t$ and use the lim inf estimates on $\widetilde{\mathcal{G}}_\varepsilon$ and $\mathcal{D}_\varepsilon$ from Proposition 3.6 (a) and Proposition 3.7 respectively as well as the energy balance (4.22) of $(u_\varepsilon, z_\varepsilon)$:

$$\begin{aligned}
\widetilde{\mathcal{G}}_0(t,&u_0(t), z_0(t)) + \sum_{i=1}^N \mathcal{D}_0(z_0(t_{i-1}), z_0(t_i)) \\
&\leq \liminf_{\varepsilon\to 0} \Big(\widetilde{\mathcal{G}}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \sum_{i=1}^N \mathcal{D}_\varepsilon(z_\varepsilon(t_{i-1}), z_\varepsilon(t_i))\Big) \\
&\leq \liminf_{\varepsilon\to 0} \Big(\widetilde{\mathcal{G}}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon, [0, t])\Big) \\
&= \liminf_{\varepsilon\to 0} \Big(\widetilde{\mathcal{G}}_\varepsilon(0, u^0_\varepsilon, z^0_\varepsilon) - \int_0^t \langle \dot{\ell}(t'), u_\varepsilon(t')\rangle \,\mathrm{d}t'\Big) \\
&= \widetilde{\mathcal{G}}_0(0, u^0_0, z^0_0) - \int_0^t \langle \dot{\ell}(t'), u_0(t')\rangle \,\mathrm{d}t'.
\end{aligned}$$

In the last step we used the assumption on the initial data (4.20) for the first summand and dominated convergence for the second one, where pointwise convergence follows from the weak convergence (4.69) and from Corollary 4.18 we have the dominating function $|\langle \dot{\ell}(t'), u_0(t')\rangle| \leq C_{ES}\|\dot{\ell}(t')\|_{\mathrm{H}^{-1}}$, which is integrable by (4.17). Since the partition t_i was arbitrary we may take the supremum and obtain the upper energy estimate:

$$\widetilde{\mathcal{G}}_0(t, u_0(t), z_0(t)) + \mathrm{Diss}_{\mathcal{D}_0}(z_0, [0, t]) \leq \widetilde{\mathcal{G}}_0(0, u^0_0, z^0_0) - \int_0^t \langle \dot{\ell}(t'), u_0(t')\rangle \,\mathrm{d}t'. \tag{4.70}$$

Step 4: Stability of the limit. To conclude $(u_0(t), z_0(t)) \in \widetilde{S}_0(t)$, apart from (4.68), which was already shown and used in Step 2, we still need the finiteness of the total energy $\widetilde{\mathcal{G}}_0(t, u_0(t), z_0(t)) < \infty$. We can obtain that from above upper energy balance using $\mathrm{Diss}_{\mathcal{D}_0}(z_0, [0, t]) \geq 0$, the a priori bounds from step 1 and assumptions (4.17) on the loading ℓ and (4.20) on the initial data:

$$\widetilde{\mathcal{G}}_0(t, u_0(t), z_0(t)) \leq |\widetilde{\mathcal{G}}_0(0, u^0_0, z^0_0)| + C_{ES} \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}} \,\mathrm{d}t' < \infty.$$

Thus with (4.68) the stability follows:

$$(u_0(t)z_0(t)) \in \widetilde{S}_0(t) \quad \text{for all } t \in [0,T]. \tag{4.71}$$

Step 5: Lower energy estimate: The lower energy estimate can in turn be classically recovered from stability. Since above stability (4.71) is already a statement in the linearized setting $\varepsilon = 0$ the argument in the current step happens solely on that level and is thus very easy. We use the abstract theorem [MR15, Proposition 2.1.23] whose assumptions include

(a) the triangle inequality, definiteness and lower-semicontinuity weakly in $\mathcal{Z} \times \mathcal{Z}$ of $\mathcal{D}_0$,

(b) the compactness of sublevels of $(u,z) \mapsto \widetilde{\mathcal{G}}_0(t,u,z)$ weakly in $\mathcal{Q}$ for each $t \in [0,T]$,

(c) energetic control of the power $\partial_t \widetilde{\mathcal{G}}_0$ and

(d) the so called compatibility between $\widetilde{\mathcal{G}}_0$ and $\mathcal{D}_0$.

For (a) the triangle inequality of $\mathcal{D}_0$ follows from the convexity and 1-homogeneity (4.12a) of R, the definiteness

$$\mathcal{D}_0(z_1,z_2) = 0 \quad \Leftrightarrow \quad z_1 = z_2$$

follows from (4.12b) and the weak lower-semicontinuity of $\mathcal{D}_0$ follows from the lower-semicontinuity of the integrand D_0 by the lower-semicontinuity tool Lemma 2.12 as $D_0 = R$ is assumed to be convex in (4.12a).

The weak compactness (b) follows by $\mathrm{H}^1 \times \mathrm{L}^2$-boundedness from the coercivity

$$\|(u,z)\|_{\mathcal{Q}}^2 \le c\,\widetilde{\mathcal{E}}_0(u,z) \le c(\widetilde{\mathcal{G}}_0(t,u,z) + C),$$

which follows using (4.7c) and (4.7d) to obtain (2.2) and combining that with Korn inequality and assumption (4.9e). Concerning the power in fact we have

$$\partial_t \widetilde{\mathcal{G}}_0(t,u,z) = -\langle \dot{\ell}(t), u \rangle,$$

such that together with above coercivity we obtain the energetic control of the power required in (c)

$$|\partial_t \widetilde{\mathcal{G}}_0(t,u,z)| \le \|\dot{\ell}(t)\|_{\mathrm{H}^{-1}} \|u\|_{\mathrm{H}^1} \le \|\dot{\ell}(t)\|_{\mathrm{H}^{-1}} c_{\mathcal{E}}(\widetilde{\mathcal{E}}_0(u,z) + C_{\mathcal{E}}) \le \|\dot{\ell}(t)\|_{\mathrm{H}^{-1}} c(\widetilde{\mathcal{G}}_0(t,u,z) + C),$$

where $t \mapsto \|\dot{\ell}(t)\|_{\mathrm{H}^{-1}} \in \mathrm{L}^1(0,T)$ by (4.17). The compatibility required in (d) contains two statements on stable sequences $(t_k, u_k, z_k) \in [0,T] \times \mathcal{Q}$, i.e. $(u_k, z_k) \in \widetilde{S}_0(t_k)$ and $\sup_k \widetilde{\mathcal{G}}_0(t_k, u_k, z_k) < \infty$. In particular by coercivity stable sequences satisfy for at least a subsequence $t_k \to t$ and $(u_k, z_k) \rightharpoonup (u,z)$ weakly in $\mathcal{Q}$. This already gives the first statement of the compatibility, namely the convergence of the power:

$$\forall s \in [0,T]\colon \quad \lim_{k\to\infty} \partial_t \widetilde{\mathcal{G}}_0(s,u_k,z_k) = \lim_{k\to\infty} -\langle \dot{\ell}(s), u_k \rangle = -\langle \dot{\ell}(s), u \rangle = \partial_t \widetilde{\mathcal{G}}_0(s,u,z).$$

The second statement is the stability of the limit:

$$(u,z) \in \widetilde{S}_0(t).$$

On the one hand the finiteness $\widetilde{\mathcal{G}}_0(t,u,z) < \infty$ follows from $\sup_k \widetilde{\mathcal{G}}_0(t_k,u_k,z_k) < \infty$ by lower semi-continuity. On the other hand for any competitor $(\hat{u},\hat{z}) \in \mathcal{Q}$ we may insert the sequence of competitiors $(\hat{u}_k,\hat{z}_k) := (\hat{u}+u_k-u, \hat{z}+z_k-z)$ into the transition cost $\widetilde{\mathcal{T}}_0(t_k,u_k,z_k,\hat{u},\hat{z})$ and from $(u_k,z_k) \in \widetilde{S}_0(t_k)$ via the quadratic trick in the limit we obtain:

$$0 \leq \liminf_{k\to\infty} \widetilde{\mathcal{T}}_0(t_k,u_k,z_k,\hat{u}_k,\hat{z}_k) = \widetilde{\mathcal{T}}_0(t,u,z,\hat{u},\hat{z}).$$

Thus [MR15, Proposition 2.1.23] provides us with the lower energy balance

$$\widetilde{\mathcal{G}}_0(t,u_0(t),z_0(t)) + \mathrm{Diss}_{\mathcal{D}_0}(z_0,[0,t]) \leq \widetilde{\mathcal{G}}_0(0,u_0^0,z_0^0) - \int_0^t \langle \dot{\ell}(t'),u_0(t')\rangle \,\mathrm{d}t',$$

and (u_0,z_0) is finally proven to be an energetic solution of $(\mathcal{Q},\widetilde{\mathcal{G}}_0,\mathcal{D}_0)$. □

We now come to the proof of the evolutionary Γ-convergence of $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})},\mathcal{D}_0)$. Assuming Conjecture 4.14 the proof of the following Theorem 4.20 resembles the one of above Theorem 4.19 with the lim inf estimate from Proposition 3.6 (a) and the mutual recovery sequence from Proposition 4.9 replaced by Corollary 4.11 and Corollary 4.17 respectively.

Theorem 4.20 (Evolutionary Γ-convergence)**.** *Assume* (4.7)*,* (4.9)*,* (4.12)*,* (4.17) *and* (4.19) *and suppose Conjecture 4.14. Let* $(u_\varepsilon,z_\varepsilon)$ *be a finite-plasticity solution with the constraint of* $\varepsilon^{\alpha_{\mathrm{Con}}}$*-GMS condition, i.e. an energetic solution of* $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})},\mathcal{D}_\varepsilon)$ *with initial data* $(u_\varepsilon^0,z_\varepsilon^0)$*.*

Then $(u_\varepsilon(t),z_\varepsilon(t)) \rightharpoonup (u_0(t),z_0(t))$ *weakly in* $\mathcal{Q}$ *for every* $t\in[0,T]$*, where* (u_0,z_0) *is a linearized-plasticity solution with constraint, i.e. an energetic solution of* $(\mathcal{Q},\mathcal{G}_0,\mathcal{D}_0)$ *with initial data* (u_0^0,z_0^0)*.*

Proof. As in the proof of Theorem 4.19 we use the splitting of the proof into five steps.

Step 1: A priori estimates. Again Corollary 4.18 provides the estimate

$$\forall t\in[0,T]\colon \quad \|u_\varepsilon(t)\|_{\mathrm{H}^1}^2 + \|z_\varepsilon(t)\|_{\mathrm{L}^2}^2 + \|\varepsilon z_\varepsilon(t)\|_{\mathrm{L}^\infty} + \mathrm{Diss}_{\mathcal{D}_\varepsilon}(z_\varepsilon;[0,t]) \leq C_{ES}.$$

Step 2: Selection of convergent subsequences. As $(\mathcal{Q},\mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})},\mathcal{D}_\varepsilon)$ and $(\mathcal{Q},\widetilde{\mathcal{G}}_\varepsilon,\mathcal{D}_\varepsilon)$ share the same dissipation functional we may use the identical argument using Helly's Selection Principle to obtain the pointwise convergence of the plastic variable

$$z_\varepsilon(t) \rightharpoonup z_0(t) \text{ weakly in } \mathrm{L}^2.$$

For the pointwise convergence of u_ε by the a priori estimates from Step 1 there is a subsequence, such that $u_\varepsilon(t) \rightharpoonup u_*$ weakly in H^1. As in Theorem 4.19 the transition cost $\mathcal{T}_\varepsilon^{(\alpha_{\mathrm{Con}})}$ is used to show that u_* is actually independent of the selection of a subsequence. Owing to Conjecture 4.14 the mutual recovery sequence from Corollary 4.17 provides by the stability (4.21) of $(u_\varepsilon,z_\varepsilon)$:

$$0 \leq \limsup_{\varepsilon\to 0} \mathcal{T}_\varepsilon^{(\alpha_{\mathrm{Con}})}(t,u_\varepsilon(t),z_\varepsilon(t),\hat{u}_\varepsilon,\hat{z}_\varepsilon) \leq \mathcal{T}_0(t,u_*,z_0(t),\hat{u}_0,\hat{z}_0)$$
$$\text{for } (\hat{u}_0,\hat{z}_0) - (u_*,z_0(t)) \in \mathrm{C}_{b,*}^\infty(\Omega_{\mathrm{Cr}},\mathbb{R}^d)\times \mathrm{C}_c^\infty(\Omega,\mathbb{R}^{d\times d}).$$

Since the local non-interpenetration constraint $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ is convex, we still have the linearized total energy $\mathcal{G}_0$ lower-semicontinuous and uniformly convex. Thus using the identical argumentation as in the case without constraint by density we obtain

$$0 \leq \mathcal{T}_0(t, u_*, z_0(t), \hat{u}_0, \hat{z}_0) \quad \text{for all } (\hat{u}_0, \hat{z}_0) \in \mathcal{Q}, \tag{4.72}$$

from which u_* turns out to be uniquely determined by

$$u_* \in \operatorname{Argmin}\{u \mapsto \widetilde{\mathcal{G}}_0(t, u, z_0(t))\},$$

independently of the choice of subsequence. Thus the whole sequence converges:

$$u_\varepsilon(t) \rightharpoonup u_0(t) \text{ weakly in } \mathrm{H}^1 \text{ for every } t \in [0, T].$$

Step 3: Upper energy estimate. This step reads identical to Step 3 in Theorem 4.19 with the lim inf estimate on $\widetilde{\mathcal{G}}_\varepsilon$ from Proposition 3.6 (a) replaced by the lim inf estimate on $\mathcal{G}_\varepsilon^{(\alpha_{\mathrm{Con}})}$ from Corollary 4.11.

Step 4: Stability of the limit. Apart from (4.72) the missing ingredient to conclude

$$(u_0(t), z_0(t)) \in S_0^{(\alpha)}(t)$$

is the finiteness of the total energy $\mathcal{G}_0(t, u_0(t), z_0(t)) < \infty$. As in the case without constraints this follows from the upper energy balance using $\mathrm{Diss}_{\mathcal{D}_0}(z_0, [0, t]) \geq 0$, the a priori bounds from Step 1 as well as the assumptions on the loading (4.17) and on the initial data (4.19):

$$\mathcal{G}_0(t, u_0(t), z_0(t)) \leq |\mathcal{G}_0(0, u_0^0, z_0^0)| + C_{ES} \int_0^t \|\dot{\ell}(t')\|_{\mathrm{H}^{-1}} \,\mathrm{d}t' < \infty.$$

Step 5: Lower energy estimate: Exactly as in the proof of Theorem 4.19 the lower energy estimate is recovered from stability using the abstract theorem [MR15, Proposition 2.1.23], for which the assumptions read:

(a) the triangle inequality, definiteness and lower-semicontinuity weakly in $\mathcal{Z} \times \mathcal{Z}$ of $\mathcal{D}_0$,

(b) the compactness of sublevels of $(u, z) \mapsto \mathcal{G}_0(t, u, z)$ weakly in $\mathcal{Q}$ for each $t \in [0, T]$,

(c) energetic control of the power $\partial_t \mathcal{G}_0$ and

(d) the so called compatibility between $\mathcal{G}_0$ and $\mathcal{D}_0$.

Since the dissipation $\mathcal{D}_0$ is the same as in the case without constraint the point (a) follows identically as in the previous theorem. We get (b) by the a priori bounds from Step 1. The points (c) and (d) in the current case with constraint both reduce to the case without constraint because the involved displacements satisfy the constraint of local non-interpenetration and can be thus shown identically as in above Theorem 4.19. For the energetic control of the power

$$|\partial_t \mathcal{G}_0(t, u, z)| \leq \lambda(t)(\mathcal{G}_0(t, u, z) + C)$$

the constraint $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ may be assumed because it has to be proven for $(t, u, z) \in \mathrm{Dom}\, \mathcal{G}_0 = [0, T] \times \mathrm{Dom}\, \mathcal{G}_0(0, \cdot, \cdot)$. For the compatibility stable sequences $(t_k, u_k, z_k) \in [0, T] \times \mathcal{Q}$ are considered, i.e. $(u_k, z_k) \in S_0(t_k)$ and $\sup_k \mathcal{G}_0(t_k, u_k, z_k) < \infty$. Not only does the latter imply $[\![u_k]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ but for the weak limit $u_k \rightharpoonup u$ by convexity of the condition $[\![\cdot]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$ and continuity of the traces we also get $[\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0$. Hence the convergence of the power and the finiteness of the total energy needed for the stability of the limit (u, z) follow identically as in the case without constraint. The nonnegativity of the transition cost we see by noting

$$\widetilde{\mathcal{T}}_0(t, u, z, \hat{u}, \hat{z}) \leq \mathcal{T}_0(t, u, z, \hat{u}, \hat{z}) \qquad \text{for } [\![u]\!]_{\Gamma_{\mathrm{Cr}}} \geq 0.$$

Thus [MR15, Proposition 2.1.23] provides us with the lower energy balance and (u_0, z_0) is finally proven to be an energetic solution of $(\mathcal{Q}, \mathcal{G}_0, \mathcal{D}_0)$. □

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Selbstständigkeits- und Einverständniserklärung

Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 21. 06. 2018 Pascal Gussmann