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“*On prouve tout ce qu’on veut, et la vraie difficulté est de savoir ce qu’on veut prouver.*”

Alain, Système des Beaux-arts, 1920

À mes parents

Abstract

The three chapters of this thesis are independent. Each of them investigates how the design of allocation rules may shape the outcome of a market. The first chapter studies the consequences of restricting the mechanisms available to a monopsonist to uniform price posting mechanisms. I show that it doesn't always prevent him to extract meaningful information from the sellers before posting the price. I also show that conditioning this offer on the transaction achieving a minimal quantity facilitates this task. Finally, I address the welfare and the implementation issues and apply the results to takeover operations. The second chapter studies the allocation of houses to students, when students have preferences over the houses they receive and over their friends' allocation. I show that the random serial dictatorship can be modified to accommodate this new set-up. The two solutions proposed have weak incentive properties if students can cooperate. However, I show that this problem disappears for one of them if the market is large and competitive. Finally, the last chapter studies how the design of entrance university exams can be used to influence the composition of high schools and universities. It shows that if the test is noisy or if the peer effects for the good students are low, giving the university's slots to the best students of each high school selects better students than giving them to the best students overall and desegregates high schools.

Zusammenfassung

Diese Arbeit besteht aus drei unabhängigen Kapiteln. Jedes von ihnen untersucht, wie die Gestaltung von Allokationsregeln das Ergebnis eines Marktes beeinflussen kann. Das erste Kapitel untersucht die Folgen der Beschränkung der Mechanismen, die einem Monopsonisten zur Verfügung stehen, auf Mechanismen, die allen Verkäufern den selben Preis anbietet. Ich zeige, dass dies Beschränkung nicht immer verhindert, dass er Informationen der Verkäufer extrahiert und den Preis damit anpasst. Das zweite Kapitel befasst sich mit der Verteilung von Studentenwohnheimplätzen. Die Studenten dürfen eigene Präferenzen bezüglich eines Wohnheimplatzes sowie zur Zuteilung ihrer Freunde angeben. Ich zeige, dass der random serial dictatorship modifiziert werden kann, um diese neuen Präferenzen zu ermöglichen. Die beiden vorgeschlagenen Lösungen haben schwache Anreizeigenschaften, wenn die Studierenden kooperieren können. Ich zeige jedoch, dass dieses Problem für den ersten Lösungsvorschlag verschwindet, wenn der Markt groß und wettbewerbsfähig ist. Schließlich wird im letzten Kapitel untersucht, wie die Gestaltung von Aufnahmeprüfungen an Universitäten die Zusammensetzung von Gymnasien und Universitäten beeinflussen kann. Das Kapitel vergleicht zwei Aufnahmeprüfungen. In der ersten, werden die besten Schüler jedes Gymnasiums ausgewählt, während in der anderen die insgesamt besten Studenten ausgewählt werden. Wenn der Test verrauscht ist oder wenn die Peer-Effekte für die guten Schüler niedrig sind, schickt der erst Test bessere Studenten in der Universität und fordert Vielfalt in Gymnasien.

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Contents

List of Figures

Introduction

The three chapters of this thesis are independent. Each of them shows how the careful design of allocation rules can help to fix the problems encountered in a market. Despite the differences between the goods exchanged and the constraints imposed on these exchanges, all chapters rely on similar game theory concepts to model the interactions between the participants. Each player has preferences over the possible outcomes and these preferences are often partially unknown to other participants. The players act rationally and most of the time non cooperatively. These allocating mechanisms try to extract information from the agents and to induce actions in order to implement outcomes favored by the designer.

The most obvious difference between the three problems studied is the kind of goods that have to be allocated. In the first chapter, the shares of a firm have to be allocated between the original shareholders and a potential buyer. In the second chapter, rooms in different houses have to be assigned to students. In the last chapter, high schools' and university's slots have to be distributed among students. The differences between these goods have consequences on the type of tools available to design the allocation rule. In chapter one, because shareholders of the same firm hold a common good, a transaction involving only one of them affects the others. This led regulators to limit the use of price schemes and allocation rules discriminating between shareholders. In chapter two and three, the social norms about education forbid monetary transfers between the agents. In chapter two, the designer choice is reduced to algorithms that match the students to the rooms as a function of the information they voluntary disclose. In chapter three, the designer can use test scores and students' allocations in high school to extract information on the students. The form of the preferences is the last difference between the three problems. Contrary to the other chapters, the agents' preferences are not represented by a utility function in the second chapter.

The first chapter explores the problem of a buyer acquiring goods from many sellers with interdependent valuations under different constraints on price and quantity discrimination. The motivating example is a model of takeover of public firms under the European Union's regulation. This regulation forces the buyer to post a unique price to all shareholders but allows him to condition his offer on the transaction achieving a minimal quantity. First, I suppose that the buyer considers only mechanisms with a uniform posted price respecting ex-post constraints. I describe under which circumstances the buyer may elicit and use the private information held by the sellers. I compare it with two classical benchmarks, the optimal mechanism imposing a constant posted price and the optimal mechanism with no constraint. Then, I study the consequences of adding a lower bound on the demand. Two examples illustrate the mechanisms and provide insights on the welfare impacts. Finally, a simple implementation of the optimal mechanism confirms anecdotal evidences and sharpens the policy implications.

The second chapter of this thesis studies a house allocation problem in which students have preferences not only over the house to which they are allocated but also over the allocation of their friends. This is a special case of a matching problem with externalities. I first show that the random serial dictatorship is not efficient in such a set-up. Then, I study a mechanism, called the random serial group dictatorship, used at some colleges to allocate students to rooms. In this mechanism, students form groups that are treated as agents of a random serial dictatorship. I show that it is efficient and that truth-telling is a Nash equilibrium of the game induced. However, students may collectively lie about the composition of their groups to obtain better outcomes. I further show that if there is enough competition for the best slots, this incentive disappears as the market grows. Finally, I present a new efficient and strategy-proof mechanism, the random serial bossy mechanism. In this mechanism, the dictator is chosen among the non-allocated students and can allocate his entire group of friends. However, I show that it gives even stronger incentives to students to collectively lie about the composition of their group.

In the third chapter of this thesis, Renaud Foucard and I study how the design of entrance university exams can be used to influence the composition of high schools and universities. We set up a model in which peer effects display decreasing differences and the composition of high schools is the result of a decentralized matching. We show that while fully centralized exams lead to segregation through positive assortative matching, partially decentralized exams in which students enter university based on their relative performance within a school may desegregate secondary education. A social planer can always desegregate high schools by changing the net return of higher education, or sometimes by controlling the precision of the test. We then discuss different possible objectives for a social planner and the resulting optimal test. We show that the trade-off between the diversity of high schools and the performance of the university's selection disappears if the precision of the test or the peer effects for high ability students are weak.

Chapter I

Optimal Uniform Price Mechanisms with ex-post Constraints

Based on Frys (2018b).

1 Introduction

This paper studies the economic effects of restricting the mechanisms available to a monopolist to uniform price posting mechanisms. It also explores the consequences of conditioning this offer on the transaction achieving a minimal quantity. I analyze this problem in the form of the procurement problem of a potential buyer who wants to buy identical goods from multiple potential sellers. I suppose that the sellers have flat offer curves, interdependent reservation prices and correlated signals whereas the buyer has a fixed and flat demand curve. This set-up fits among others the problem of an offeror choosing a mechanism to buy shares of an offeree company.

This paper shows that the interdiction of price and quantity discrimination doesn't always prevent the buyer to extract meaningful information from the sellers before posting the price. When the valuations are asymmetric enough, the optimal mechanism uses the information of sellers with high valuations to buy at a lower price from others. Allowing the buyer to link the price offered with an acceptance threshold extends the set of signals and valuation functions where the buyer uses the sellers' reports. This lower bound is never violated in equilibrium. It enlarges the set of incentive compatible price functions by punishing reports inconsistent with the total quantity offered. As usual in problems of price discrimination, the profit of the buyer increases with discrimination but the expected total quantity sold, the sellers' surplus and the overall welfare might increase or decrease as a function of the valuation functions. Finally, the mechanism can be implemented through a series of sequentially decreasing posted prices and demand lower bounds. Between each step, the buyer collects the information of the sellers dropping out and uses it to post a lower price to the rest. This in turn shed a new light on takeovers regulation in the European Union's member states and challenges previous theoretical analysis of the question.

The European Union forbids indeed price and quantity discrimination between the firm's shareholders during a takeover but allows the offeror to withdraw his offer if the number of shares tendered doesn't reach a threshold. The regulation on takeovers is described by a directive of the European Parliament and Council (2004). Its transcription in the law of the different European countries as well as its economic consequences were evaluated for the European commission in a assessment report written by Marccus Partners and the Center for European Policy Studies (2012). This directive applies only to "companies [...] the securities of which are admitted to trading on a regulated market in a member state". Its first goal is "to protect the interest of holders of the securities". To achieve this goal, the European parliament voted two articles that are of particular relevance for a buyer choosing a buying mechanism. Article 3 states that "all holders of the securities of an offeree company of the same class must be afforded equivalent treatment". Article 5 states that if a person after buying securities of a company has the "control of that company, Member state should ensure that [he/she] makes a bid [...] to all the holders of those securities at [...] the highest price paid for the same securities by the offeror [...] over a period [...] no less than 6 month and no more than 12 month before the bid". The report confirms that these two articles force a buyer who wants to own more than 30% of a firm to restrict himself to mechanisms without price discrimination nor quantity discrimination. As resumed in this report, it can force "a potential offeror who may only wish to purchase 35% of the shares to be prepared to purchase 100% which includes people owning 34% and wishing to increase to 35%". Actually, the directive regulates the entire timing of the game. After a buyer declared his intention to own more than 30% of the firm, he has a deadline to submit a bid to all shareholders. Then, the shareholders have a deadline to accept or reject the bid. If the bid can be linked to an acceptance threshold and if in the end the percentage of shares tendered is below that threshold, the buyer can submit another bid and restart the whole procedure. As for the rest, the directive lets "the member states [...] lay down rules which govern the conduct of the bids at least as regards [...] the lapsing of the bid; the revision of the bid". In particular, "the offer document [...] shall state at least [...] the maximum and minimum percentage or quantities of securities which the offerer undertake to acquire". The rules implemented by the member states are summarized in the assessment report and a guide to public takeovers in Europe written by the law firms Bonellierede et al. (2016). In accordance to articles 3 and 5, a maximum on the overall quantity purchased can only be set if the offerer wishes to acquire less than 30% of the shares of the firm, which is the case this article focuses on. However, the directive lets the national regulators decide whether or not the buyer may set a lower bound on the total quantity of shares tendered under which the offer lapses. The requirements to set a so-called "level of acceptance" differ from one European country to another. In the six countries studied in Bonellierede et al. (2016): France, Germany, Italy, Spain, the Netherlands and the United Kingdom, the buyer can choose between at least a few different levels of acceptance.[1]

Modelling the takeover as a procurement problem with one buyer and many sellers find its justification in the assessment report written by Marccus Partners and the Center for European Policy Studies (2012). According to it, more than 90% of the takeover operations are the result of bilateral negotiations between one potential buyer and the owners of the firm. Moreover, it states that firms having a dispersed ownership structure are considered as easier targets and, therefore, more often concerned by takeover operations. Finally, the regulation of the European Union focuses almost exclusively on the case of one buyer and concerns only publicly traded firm, which supposes many owners. I opted for a setting with interdependent valuations and one-dimensional signals. The idea behind this hypothesis is that all shares have an underlying common value, namely the sum of the discounted future profits. Each seller gathers information to build an estimate of this value that is represented by the one-dimensional private signal he received. This information might partly come from sources common to all sellers and, therefore, might be correlated. Would a seller know the information of another seller, he would update his estimate of the value, making valuation functions interdependent. Note that even if a seller knows all signals, the underlying value remains uncertain. If sellers value their own information differently than the others', the valuations are not common. The second example presented in this article explains in detail this phenomenon. On the contrary, the buyer's valuation is exogenous. It is consistent with a buyer lending no credence to the information of the sellers or one having some intrinsic value from owning a share, independent of the future cash flows. Finally, the publicly traded price of the firm puts a lower bound on the possible valuations of the sellers.

The model studied is of greater relevance than its application to takeover bids in the

[1]This possibility may be conditional on the offer being voluntary, but a rational buyer would rather make a voluntary offer than buying shares that trigger a mandatory offer when the former is better for him.

EU suggests. First of all, a large number of countries have similar rules. The assessment report compares the legislation of nine non-EU countries and the same price must be offered to all shareholders in all cases. In the US only, the legislation allows the buyer to submit a first bid for the first $x\%$ of the shares and a second lower bid the remaining shares (two tier bids). More generally, in many monopoly markets the buyers might have interdependent valuations with private signals. The question of how to regulate price discrimination in such markets is an open question, just as the question about the consequences of stepping away from pure fixed price posting mechanisms. The paper highlights the differences between the optimal mechanisms with a constant and a uniform posted price. It shows that mechanisms partially regulating price discrimination might have interesting properties and in some cases outperform the strict forbidding as well as the laissez-faire cases.

The next subsection discusses the related literature. In Section 3, I present the basic model and definitions. Section 4 derives the two benchmarks with constant posted price and without regulation. Section 5 studies the case where the monopsonist must post a uniform price but cannot set an acceptance threshold. Section 6 takes this last option into account and derives the optimal mechanism under the regulation of the European Union. Section 7 presents a simple way to implement the mechanism and discusses policy implications as well as extensions to partial offers. Section 7 concludes.

2 Literature Overview

The literature on optimal selling mechanisms dates back to Myerson (1981). In this article, the seller of an object faces multiple potential buyers with independent valuations. It presents an optimal mechanism implementable in bayesian Nash equilibrium with interim individual rationality constraints. It provides some examples with interdependent values that will inspire succession of articles finishing with McAfee and Reny (1992). These articles finally show that under some conditions on the valuations the mechanism designer can extract almost the entire interim expected rent. The optimal mechanism offers to each seller a menu of random participation fees whose realization depends on the signals of the others. Inspired by the same article, Maskin et al. (1989) extended the initial set-up to one with objects to sell. Their mechanism with unit demand is very similar to the benchmark with no regulation. Later on, Ausubel and Cramton (2002) studied the efficiency and relative performance of the uniform price and pay-as-bid auctions to sell multiple objects. They show in my set-up that the uniform price auction has an efficient bayesian equilibrium if bidders are symmetric. However, the optimality of this auction is not addressed.

Wilson (1985) disputed the relevance of the interim constraints considered in these papers. He pointed out that the buyer may not know the beliefs of the sellers about each others' signals. In response, the literature developed mechanisms with ex-post constraints that are robust to this uncertainty. Perry and Reny (2002)[2] constructed an efficient multiple objects auction. They proved a revenue equivalence theorem for mechanisms with ex-post constraints when a single crossing condition holds. Segal (2003) showed that when valuations are independents, the optimal mechanism with ex-post constraints is a posting price mechanism. The price for player i is the price a monopolist would post if he knew all but player i's signals. The benchmark case, studied in section 4, is a simple generalization of this article. The theoretical justification for ex-post implementable mechanisms came only afterwards. Bergemann and Morris (2005) explored the relation between ex-post implementation and interim implementation. They showed that the mechanism designer can restrict to ex-post implementable mechanisms, if he requires the mechanism to work for all possible full support common prior of the agents. This result relies on the fact that the mechanism designer restricts himself to mechanisms where agents report only their payoff types and no higher order beliefs. However, Chung and Ely (2007) proved that if this is not the case and if the buyers have independent valuations, there exists a realistic belief of the sellers such that the optimal bayesian incentive compatible mechanism raises the same revenue as the optimal strategy mechanism. Finally, Jehiel et al. (2006) showed that in my set-up, the set of deterministic social choice functions that are non constant and ex-post implementable is non generic, if signals are multidimensional.

The last strand of literature relevant for this paper is the one focusing on third degree price discrimination and its effect on welfare. When all markets are covered, Schmalensee (1981) and Aguirre et al. (2010) provided a framework and techniques to derive the change in output and welfare due to price discrimination. It is particularly helpful in the first example presented in this paper. Finally, Cowan (2016) provided sufficient conditions on the distribution of the reservation prices such that the quantity sold increases. The proof of this proposition is used in the second example presented in this paper.

[2] More precisely in the working paper associated, Perry and Reny (1999).

3 The Model

There are n shareholders $i \in N = \{1, ..., n\}$, and one potential buyer, who designs the buying mechanism. I suppose that participants have quasilinear utilities, are risk neutral and have no budget constraint. Each shareholder i has $\lambda_i \in \mathbb{N}$ shares and values each share $v_i(x) \in \mathbb{R}$ where $x = (x_j)_{j=1}^n \in X$ and $x_i \in X_i \subseteq \mathbb{R}$ is a private signal received by i. I suppose that $v_i(x)$ is differentiable, strictly increasing in x_i and weakly increasing in x_{-i} a.e.. The buyer values each share $w \in [\min_{i,x \in X} v_i(x); +\infty)$ and believes that the signals are drawn according to the distribution function $f(x)$ positive a.e. with no mass point. Without loss of generality, I can normalise X_i to $[0; \bar{x}]$ and $\min_i v_i(0)$ to 0. Actually, for any n increasing bijections from X_i to $X_i' \subseteq \mathbb{R}$, $(g_i(\cdot))_{i=1}^n = g(\cdot)$, I can define a new vector of signals, $y = g(x)$, and new valuation functions, $u_i(y) = v(g^{-1}(y))$, without changing the distribution of the valuations. This new parameterization does not affect the properties defined above and the regularity, positivity and the weak symmetry on the axes' intersections defined hereafter.[3]

The game as it is regulated by the directive is the following. As soon as the buyer owns more than 30% of the firm, he has a month to submit a bid to all shareholders. This bid consists of a price p per share and an acceptance threshold Λ. Once this bid is submitted, each shareholder i has one month to decide how many shares he wants to tender $d_i \in [0; \lambda_i]$. If the shareholders tendered less shares than the threshold, i.e. if $\sum_i d_i = \Lambda_T < \Lambda$, the game stops and everybody receives a 0 utility. If the shareholders tendered more shares than the threshold, i.e. if $\Lambda_T \geq \Lambda$, the game ends, all the tendered shares Λ_T are bought at price p and the buyer's utility is $(w - p)\Lambda_T$ whereas seller i's utility is $d_i(p - v_i(x))$.

I suppose that the buyer can commit to the offer he does, that he can communicate with the sellers before submitting a bid and that this communication is contractible.[4] I also suppose that the buyer is restricted to use deterministic prices and acceptance thresholds. A general bidding mechanism can be then written as $(H, R, p(\cdot), \Lambda(\cdot), r(\cdot))$ where $H = \bigtimes_{i=1}^n H_i$ with H_i the set of possible messages that seller i can send to the buyer, $R = \bigtimes_{i=1}^n R_i$ with R_i the set of possible recommendations that the buyer can send to seller i, $p : H \to \mathbb{R}$ is the price per share, $\Lambda : H \to \mathbb{R}$ is the acceptance threshold and

[3] An intuitive normalization when the signal of a buyer increases his expected valuation, e.g. when signals are positive, is to set $y_i = E[v_i(x)|x_i]$. The signal is then simply interpreted as the interim valuation of the seller.

[4] The first assumption is verified in France and in the UK. See section 7 for a discussion on these assumptions and how they might be weaken.

$r : H \to R$ the vector of recommendations sent to the sellers. The timing of the game can be then described as follows:

1. The buyer commits to using a mechanism $(H, R, p(\cdot), \Lambda(\cdot), r(\cdot))$.

2. Each seller reports a message $h_i \in H_i$.

3. Each seller observes $p(h)$, $\Lambda(h)$ and $r_i(h)$ and decides how many shares he wants to tender $d_i \in [0; \lambda_i]$.

4. If $\Lambda_T < \Lambda$, the utilities are 0 for everybody. Otherwise, the shares tendered are bought, the buyer receives $(w - p(h))\Lambda_T$ and each seller receives $d_i(p(h) - v_i(x))$.

The strategy space of the sellers depends on how much information they have on the other sellers' signals in each step. There is no clear answer to this question for three reasons. First, the distribution of the signals may not be common knowledge. In particular, the shareholders may have a more precise estimate of each other's valuations than the buyer, due to their past interactions. Secondly, it seems difficult for the buyer to control how much information is released to the sellers during the course of the game. They may observe how many shares are tendered and may update their beliefs accordingly. Besides, a shareholder who doesn't tender his shares has an incentive to reveal his private information. Indeed, if he does, the others will update their beliefs and may change their decisions. In turn, observing these changes helps the first seller build more accurate beliefs and he may discover that he can sell his shares for a profit. Finally, if the mechanism is not constant, it releases automatically information about the signals of all sellers through the value of $p(m)$ and $\Lambda(m)$. If there are only two or three sellers, they may be able to inverse the price and threshold functions and pin down each other's signals[5].

To overcome this problem, I focus on ex-post equilibria. If the buyer restricts himself to mechanisms that depend only on the sellers' payoff types space and work for all sellers' common priors with full support, I can apply the revelation principle of the robust mechanism literature, Bergemann and Morris (2005) or Chung and Ely (2002). It implies that I can restrict with out loss of generality to direct mechanisms that are truthful and obedient ex-post[6].

[5] Green and Laffont (1987) studied an optimal mechanism where the sellers' strategies are robust to the information revealed through the mechanism. But the absence of revelation principle with this posterior equilibrium concept makes it impracticable in cases with more than two players.

[6] Chung and Ely (2007) proved that the buyer cannot improve the mechanism by asking the sellers to report higher order beliefs if he has a maxmin approach and the sellers have private valuations. But it could in principle happen with interdependent valuations.

In this ex-post set-up, the moral hazard constraint implies that $d_i(p, \Lambda, x) = \lambda_i \mathbb{1}_{\{p \geq v_i(x)\}}$ and, therefore, that $\Lambda_T(p, x) = \sum_{i=1}^n \lambda_i \mathbb{1}_{\{p \geq v_i(x)\}}$. Denoting $\mathcal{C}$ the set of functions from X to $\mathbb{R}$, the problem of the buyer reduces to the following:

$$\begin{gathered}\max_{(p(x),\Lambda(x))\in\mathcal{C}^2} \int_X \mathbb{1}_{\{\Lambda_T(p(x),x)\geq\Lambda(x)\}} \Lambda_T\left(p\left(x\right),x\right)\left[w-p(x)\right] f(x)dx \\ \text{s.t. for all } i \leq n,\ x\in X,\ z\in X_i\text{:} \\ \mathbb{1}_{\{\Lambda_T(p(x),x)\geq\Lambda(x)\}} \mathbb{1}_{\{p(x)\geq v_i(x)\}}[p(x)-v_i(x)] \geq \\ \mathbb{1}_{\{\Lambda_T(p(z,x_{-i}),x)\geq\Lambda(z,x_{-i})\}} \mathbb{1}_{\{p(z,x_{-i})\geq v_i(x)\}}[p(z,x_{-i})-v_i(x)]\end{gathered} \tag{1}$$

Besides, throughout the paper I use some of the following notations and definitions. I denote v_i the random variable $v_i(x)$. For all $J \subset N$ and $x_J \in X_J$, $v_i \mid x_J$ is distributed according to the c.d.f. $G(v_i \mid x_J)$ and the d.f. $g(v_i \mid x_J)$. Finally, I define a.e. the virtual valuation of i knowing x_J, $V_i(v_i, x_J) = v_i + \frac{G(v_i|x_J)}{g(v_i|x_J)}$[7]. To be able to easily separate high from low virtual valuations, I also need the following regularity condition:

Definition 1. *The problem is called* ***regular given*** $x_J \in X_J$ ***for seller*** i *if and only if the virtual valuation* $V_i(\cdot, x_J)$ *crosses* w *at most once and from below.*

In the rest of the article, I call the problem regular, if it is regular for all sellers with $J = \emptyset$. Similarly, I call the problem regular for seller i given X_J if it is regular given all $x_J \in X_J$. This regularity condition is the extension in this set-up of the classical regularity condition used in the literature. As in the set-up with independent valuations, I can find a simple sufficient condition for the problem to be regular given X_J.

Corollary 1. *The problem is regular for seller* i *given* X_J *for any* w*, if* $G(v_i \mid x_J)$ *is log-concave in* v_i *on its support for all* $x_J \in X_J$.

The log-concavity of the c.d.f. in a procurement problem is the equivalent of the increasing hazard rate condition in selling problems. Both are a consequence of log-concavity of the distribution function. Bagnoli and Bergstrom (2005) show that this condition is verified for a large range of classical distribution functions. The log-concavity conditions on x_J and not v_{-i} because the valuations are interdependent. Note that these two properties are not affected by a reparameterization of the model. In the special case where $J = N \setminus \{i\}$, I denote $F(x_i \mid x_{-i})$ the cumulative distribution function of x_i knowing x_{-i}, and $f(x_i \mid x_{-i})$ its distribution function. For all v in the support of $v_i \mid x_{-i}$,

[7] I extend $V_i(\cdot, x_J)$ outside the support of $v_i \mid x_J$ by its value in the lowest or the highest v_i on the support.

I define $x_i(v, x_{-i}) = \{x_i \mid v_i(x) = v\}$, the signal of i such that his valuation given x_{-i} is equal to v[8] and $m_i(x) = v_i(x) + \frac{dv_i}{dx_i}(x)\frac{F(x_i|x_{-i})}{f(x_i|x_{-i})}$ the virtual valuation function[9]. The regularity condition can be translated into conditions on $F(\cdot)$ and $v(\cdot)$.

Corollary 2. *The problem is regular given x_{-i} for seller i if and only if $m_i(\cdot, x_{-i})$ crosses w at most once and from below*

The proof is in the appendix. The regularity condition implies that m_i is well defined on the indifference curve going through w. In the same way, the regularity condition given x_J can be translated using the generalized Leibniz integral rule.[10] The next definition describes how the signal of a seller j helps to infer the virtual valuation of a seller i. The introduction claims that in the takeover case signals are liable to be positively correlated. So a higher signal for j might mean a higher virtual valuation for i. I actually use a notion more local than correlation.

Definition 2. *The signal of seller $j \in J$ **is positive for seller** i **given** $x_J \in X_J$ if and only if the virtual valuation of i knowing x_J, $V_i(w, x_J)$, increases in x_j.*

If this condition is required for all valuations and all vectors in X_J, it extends to this set-up the classical dominance hazard rate condition used in the literature. Indeed:

Corollary 3. *The signal of j is positive for i given X_J for all w if and only if for all $x_J \in X_J$ and $x'_j > x_j$, $v_i \mid x'_{-i}$ dominates in terms of the reverse hazard rate $v_i \mid x_{-i}$.*

It implies that the distribution of seller i's valuation knowing the others' signals is unambiguously improved when the signal of j rises. A sufficient condition for this property is that the two variables $v_i, x_j \mid x_{-ij}$ are affiliated for all x_{-ij}[11]. As for the previous definition, if $J = N \setminus \{i\}$, this condition can be transposed to conditions on $F(\cdot)$ and $v(\cdot)$ that are developed in the Appendix. The two examples presented hereafter are used throughout the paper. In this section, they help illustrates the implications of the properties defined above.

Example 1. *The first example explores the case where the sellers have linear valuations. For any sellers i and $j \in N \cup \{0\}$ there are $\alpha_j^i \geq 0$ such that: $v_i(x) = \alpha_0^1 + \sum_{j=1}^n \alpha_j^i x_j$.*

[8] I extend $x_i(\cdot)$ outside of the support of $v_i \mid x_{-i}$ by 0 and $\bar{x}$.

[9] m_i is well defined a.e. which is sufficient here.

[10] Denoting $x_{i,J}(v, x_J) = \{x_{N\setminus J} \mid v_i(x_{N\setminus J}, x_J) = v\}$, $G(v_i \mid x_J) = P(x_{N\setminus J} \leq x_{i,J}(v, x_J) \mid x_J) = \int_{x_{N\setminus J} \leq x_{i,J}(v,x_J)} f(x_{N\setminus J} \mid x_J) dx_{N\setminus J}$, and $g(v_i \mid x_J) = \int_{s \in x_{i,J}(v,x_J)} f(s \mid x_J)\vec{u}(s) \cdot \vec{n}(s) ds$.

[11] $x_i, x_j \mid x_{-\{ij\}}$ affiliated for all $i, j, x_{-\{ij\}}$ is equivalent to x affiliated, so what I require here is a bit different from affiliation of the signals. The properties of affiliated random variables are extensively studied in Milgrom and Weber (1982) and Karlin and Rinott (1980). Karlin and Rinott (1980) showed that a large range of the classical correlated multivariate distributions are affiliated.

I will focus on the symmetric two sellers case where $\alpha_0^i = 0$, $\alpha_i^i = \alpha_1$ *and* $\alpha_j^i = 1 - \alpha_1$. *Besides, I suppose that the joint cumulative distribution of the buyer's belief about the sellers' signals can be written as follow* $F(x_1, x_2) = \frac{x_1 x_2}{1-\theta(1-x_1)(1-x_2)}$ *where* θ *is a dependance parameter belonging to* $[-1, 1]$. *This random variable, derived from the copula of Ali et al. (1978), has uniform marginals on* $[0, 1]$ *and verifies the regularity condition. Finally, the information of the sellers is positive for all* $\theta > 0$.

Example 2. *The second example details the micro foundations presented in the introduction to justify the set-up with correlated valuation and interdependent signals. I supposes that seller* i *believes that the signals are drawn from a multi-dimensional log-normal distribution* $\ln \mathcal{N}(lnV, \Sigma_i)$. V *is the* n *dimensional vector with the true valuation* v *on each coordinate.* Σ_i *is the matrix of seller* i*'s beliefs about the covariance matrix of the log-errors of the signals. I suppose that his valuation of one share given the vector of signals is the maximum likelihood estimator of* v.[12] *In the case with two sellers the valuation functions can be written as:*

$$\ln\left(v_i\left(x_1, x_2\right)\right) = \beta_{i,1} \ln(x_1) + (1 - \beta_{i,1}) \ln(x_2), \text{ with } \beta_{i,1} = \frac{\sigma_{i,2} - \rho_i \sigma_{i,1}}{\sigma_{i,1}^2 + \sigma_{i,2}^2 - 2\rho_i \sigma_{i,1}\sigma_{i,2}} \sigma_{i,2}$$

$\beta_{i,1}$ *increases with the relative precision of the seller* i*'s signal and approaches* $1/2$ *when the correlation increases. Moreover, I suppose that* $\beta_{i,1} \in [0, 1]$ *and focus on the case of two symmetric sellers. Reparameterizing the model and denoting* $y_i = \ln(x_i)$ *and* $u_i(y_1, y_2) = v_i(e^{y_1}, e^{y_2})$ *shows that the log-concavity of the c.d.f. of the buyer's belief about* $y_i \mid y_{-i}$ *implies the log-concavity of the c.d.f. of* $v_i \mid x_{-i}$[13]. *In the same way, the dominance of the reverse hazard rate of the buyer's belief about* $y_i \mid y_{-i}$ *implies the same property on* $v_i \mid x_{-i}$. *Finally, to ensure that the problem is regular and the information of the sellers is positive, I suppose that the buyer also believes that the signals follow a symmetric log-normal distribution with a positive correlation parameter.*

Having now clearly defined the problem, I study three simpler set-ups. I first solve the classical case of a monopsonist who has to post a constant price. It also represents the case where communication with the sellers before setting the price is impossible.

[12] A closely related bayesian version is one where the sellers have an improper log-normal prior about v with a infinite variance parameter and where his valuation function is E[v|x], the min square error estimator.

[13] There is no contradiction between the fact that the buyer doesn't know the beliefs of the sellers and the fact that he knows their valuation functions. The belief that pins down the valuation function of seller i is the belief of seller i about how the other sellers' private information is produced and its relationship with his private information and the true valuation. On the contrary, the belief that enters his bayesian incentive constraint is the belief about the realization of the others' private information. Indeed, when seller i learns about the realization of the other players' signals, it doesn't alter his own valuation function.

In the same section, I also solve the case where the buyer can communicate with the sellers and freely discriminate between them through prices and quantities. The section afterwards studies the case where the sellers can communicate with the buyer but price and quantity discriminations are forbidden. This corresponds to problem 1 with the additional condition that $\Lambda(x) = 0$. The complete problem presented in this section is studied in section 6. The results of the first three parts are used as building blocks and benchmarks to understand how these restrictions change the optimal mechanism. Finally, the last section discusses the implementation of the optimal mechanism, its robustness to weaker commitment hypotheses as well as its extension to the case where the buyer can set an upper bound on the quantity bought.

4 The Two Benchmarks

4.1 Optimal Constant Posted Price

In this subsection, I suppose that the buyer has to post a uniform price and cannot communicate with the sellers beforehand. The buyer is then restricted to mechanisms where $p(x)$ is a constant function and $\Lambda(x)$ is equal to 0. This benchmark is feasible under the restrictions described in section 3. It constitutes, therefore, a lower bound on the profit that the buyer can make in problem 1. With these additional restrictions, the problem reduces to the following choice of $p \in \mathbb{R}$:

$$\max_{p\in\mathbb{R}} \int_X \Lambda_T(p, x)[w - p]f(x)dx$$

Denoting $G_i(p)$ the cumulative marginal distribution function of v_i and $Q(p) = \sum_i G_i(p)$ the expected number of shares tendered at price p, I can rewrite the problem as:

$$\max_{p\in\mathbb{R}} [w - p]Q(p)$$

The buyer finds himself in the classical problem of a monopsonist. The buyer's belief about the distribution of the sellers' valuations sets his belief about the offer curve $Q(p)$ and the optimal price. The optimal posted price p^* exists and belongs to $(0; \max_i\{v_i(\bar{x})\}]$. If the solution is interior, it verifies the following equation:

$$-Q(p^*) + [w - p^*]Q'(p^*) = 0$$

Denoting p_i^* the optimal price for player i if the buyer can discriminate between the sellers, I can show that:

Proposition 1. *If the problem is regular, p^* belongs to $[\min_i\{p_i^*\}, \max_i\{p_i^*\}]$. Moreover, if $p_i^* \geq p^*$ and $G_i'(p)$ dominates $G_i(p)$, p^* increases if the buyer changes his belief from $G_i(p)$ to $G_i'(p)$*[14].

The proof of the general case is in the appendix. But the case where the marginal distributions of the v_i are symmetric provides a good intuition. Indeed, in that case, $Q(p) = \left(\sum_{i=1}^N \lambda_i\right) G(p)$ and the properties of the marginals are passed onto the offer function. Therefore, if the problem is regular the p_i^* are unique and all equal to p^*. They increase if the buyer switches his belief to a new belief that dominates the original one.

To give a better understanding of the different mechanisms, I use the examples presented in the previous section. I draw for each of them the regions of the signals space where the sale takes place. Seller 2 sells his shares in the region filled with vertical lines whereas seller 1 in the one filled with horizontal lines. Both sellers sell their shares in the gridded region. The shaded areas represent the regions where efficiency requires a seller to sell, but where the monopsonist proposes a too low price for the transaction to take place.

Example 3. *In the example with linear utility functions, I can derive with mathematica the cumulative distributions of the v_i for a large range of parameters*[15] *and show that they are log-concave functions. As the signals, the valuations are distributed on $[0,1]^2$ and have a mean of 0.5. In Figure 1 represents the selling mechanism for $\alpha_1 = 0.6$, in the case where the buyer's valuation is $w = 0.4$. The other cases can easily be pictured. If w increases, all curves move up. We zoom in the graph, cropping out the part outside the unit square. If w decreases, all curves move down. With $\theta = 0.8$ the optimal price is $p^* \approx 0.23$ and the regions where the sale takes place are delimited by the solid lines. The dashed lines delimit these regions in the two limit cases where the signals are independent ($\theta = 0$)*[16] *and where they are equal* [17].

Example 4. *In the second example, with log-linear valuations, the distribution of the valuations is log-concave. Indeed, the buyer's belief follows a normal distribution and, therefore, $u_1(y_1, y_2) = \exp\left(\beta_1 y_1 + (1-\beta_1) y_2\right)$ follows a log-Normal distribution. Figure 2 represents the second example when $\beta_1 = 0.6$. It supposes that the sellers believe the signals are correlated with a $\rho_i = 0.5$ and that the standard deviation of their own*

[14] The converse property holds if $p_i^* \leq p^*$. Nahata et al. (1990) have a similar statement.

[15] for all $\theta \leq 2/3$ or $\alpha_1 \in [1/3, 2/3]$ or $\theta + \alpha_1 \leq 1.5$ or $\theta - \alpha_1 \leq 0.5$

[16] $\alpha = \min\{\alpha_1, 1-\alpha_1\}$. If $w \leq 3/2(1-\alpha)$, $p^* = 2/3w$. If $w \in [3/2(1-\alpha), -1/2 + 5/2\alpha]$, $p^* = w/2 + 1/4 * (1-\alpha)$.

[17] $p^* = w/2$ if $w \leq 1$. It cannot be described by A.M.H. copula. $\theta = 1$ implies a correlation of 0.48 only.

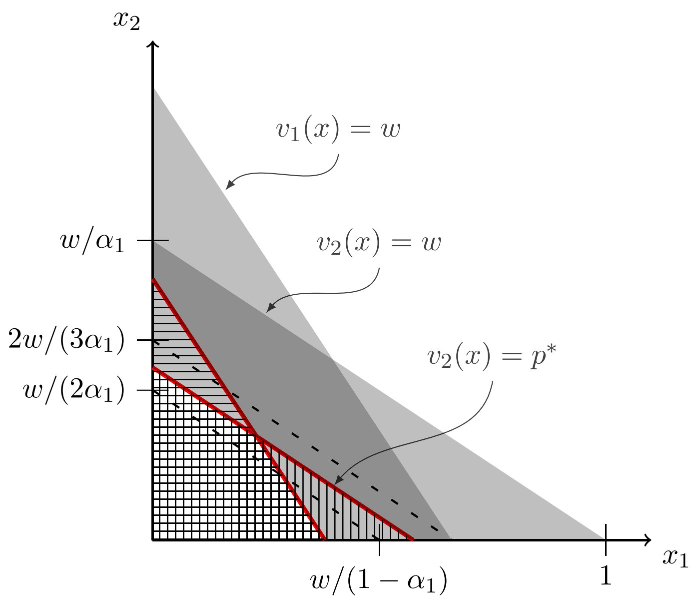


FIGURE 1: Linear utilities with a constant posted price

signals is 10% *smaller than the other's one. The figure is drawn for a buyer's valuation* $w = 1$*. His belief about the sellers' signals follows log-normal distribution with parameters* $\mu = (0, 0)$*,* $\sigma = 1$ *and* $\rho = 0.5$*. The mode of his belief is* 1 *and its mean* 1.64*. The mean of the valuations is* 1.46 *and the optimal price* $p^* \approx 0.58$*. The dashed curves represent the cases where* $\rho = 0$ *and* $\rho = 1$*. As in the first example, if* w *increases all curves move up and we zoom in the graph. If* σ *decreases, the buyer is more confident in his belief and the price rises towards* 1*.*

In the two examples, changing the parameter regulating the correlation of the signals does not dramatically change the mechanism. Indeed, the degree of dependence of the signals influences the price chosen only through the movements of probability masses between the shaded and the hashed area that it triggers.

In this subsection, I presented how a set-up with interdependent valuations and correlated signals changes the classical problem of a monopsonist posting a constant price. In the next subsection, I study the other extreme case where neither the pricing nor the communication between the buyer and the sellers is restricted.

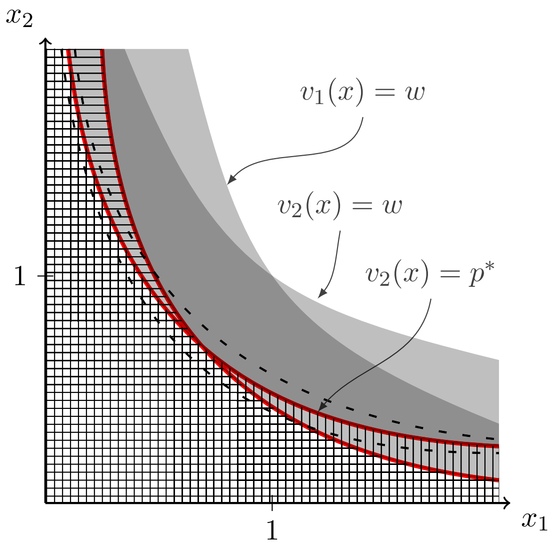


FIGURE 2: Log-linear utilities with a constant posted price

4.2 Optimal Mechanism Without Regulation

In this section, I derive the optimal ex-post mechanism allowing for price and quantity discrimination. Any solution of the problem 1 is feasible under the constraints imposed in this subsection. Therefore, the resulting profit constitutes a upper bound of the profit that can be achieved in the original problem. This subsection is an extension of Segal (2003) to cases where the valuation functions are interdependent. Denoting $\mathcal{Q}_i$ the set of functions from X to $\{0, 1, ..., \lambda_i\}$ and $\mathcal{C}$ the set of functions from X to $\mathbb{R}$, I can write the problem of the buyer as solving the following maximization problem:

$$\max_{(q_i,c_i)^n \in (\mathcal{Q}_i \times \mathcal{C})^n} \int_X \sum_{i=1}^{N} [q_i(x)w - c_i(x)] f(x) \mathrm{d}x$$

$$\text{s.t. for all i, } x \in X,\ z \in X_i\text{:} \quad \begin{cases} c_i(x) - q_i(x)v_i(x) \geq c_i(z, x_{-i}) - q_i(z, x_{-i})v_i(x) \\ c_i(x) - q_i(x)v_i(x) \geq 0 \end{cases}$$

If the problem is regular for seller i given X_{-i}, I define $s_i(w, x_{-i}) = \max\{x_i \mid m_i(x) = w\}$, the signal of i such that his virtual valuation is equal to w[18].

[18]I extend s_i outside the support of $m_i|x_{-i}$ by 0 and $\bar{x}$.

Proposition 2. *If the problem is regular for any seller i given X_{-i}*[19]*, the optimal mechanism with no regulation is the following:*

$$q_i^*(x) = \begin{cases} \lambda_i & \text{if } m_i(x) \leq w \\ 0 & \text{otherwise} \end{cases} \quad c_i^*(x) = \begin{cases} \lambda_i p_i^0(w, x_{-i}) & \text{if } m_i(x) \leq w \\ 0 & \text{otherwise} \end{cases}$$

Where the price per share offered $p_i^0(w, x_{-i}) = v_i\left(s_i\left(w, x_{-i}\right), x_{-i}\right)$.

This proposition follows from the taxation principle stated in the first theorem of Chung and Ely (2002) or from a adaptation of Myerson (1981). The proof is presented in the appendix. Compared to Segal (2003), $F(v_i \mid v_{-i})$ is simply replaced by $G(v_i \mid x_{-i})$. Note that $s_i(w, \cdot)$, respectively $x_i(w, \cdot)$, draws the indifference hypermanifold of $m_i(x)$, respectively $v_i(x)$, going through w.

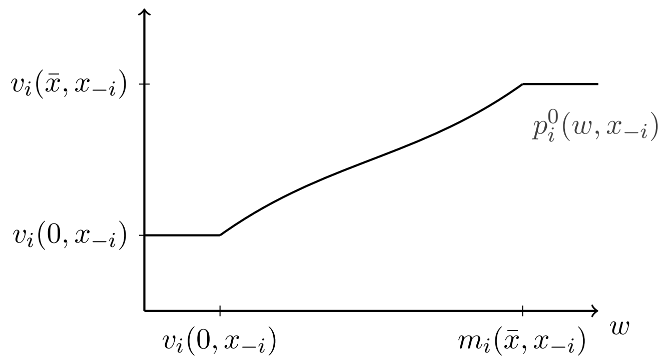


FIGURE 3: Posted price as a function of w

As in other set-ups with bayesian constraints or independent valuations, the optimal mechanism is a posting price mechanism. The buyer solves the problem of a monopsonist who can discriminate between the different sellers as well as for each seller between the different realizations of the others' signals.

Proposition 3. *If the problem is regular i knowing X_{-i}, $p_i^0(w, x_{-i})$ solves the problem of a monopsonist posting the optimal price per share for seller i knowing the others' signals.*

$$p_i^0(w, x_{-i}) = \arg\max_{p \in \mathbb{R}} \{(w - p) G_i(p \mid x_{-i})\}$$

The formal proof is in the appendix. The buyer is facing as many markets as there are vectors (i, x_{-i}) with a demand function $G_i(p \mid x_{-i}) f(x_{-i})$ on each of these markets. It implies that comparing the efficiency of the two benchmarks is equivalent to the question

[19]If it is not the case something similar as Maskin et al. (1989) can be constructed.

of the efficiency of third degree price discrimination in a monopoly setting. Unfortunately, I cannot draw any result for the general case from this literature. The sufficient conditions presented in Aguirre et al. (2010) for an increase or decrease in efficiency are not verified aside from the trivial case where the distribution of v_i is independent of x_{-i}.

Corollary 4.
- *For all i, the optimal price posted to i with discrimination, $p_i^0(w, x_{-i})$, crosses the optimal constant price posted to i, p_i^*, on a x_{-i} inside the support of $x_{-i} \mid v_i = p_i^*$.*
- *If the signal of j is positive for i in $p_i^0(w, x_{-i})$, $p_i^0(w, x_{-i})$ is increasing in x_j.*

The formal proofs are in the appendix. In the terminology of the third degree price discrimination literature, all x_{-i} such that $p_i^0(w.x_{-i}) < p_i^*$ are the strong markets, whereas the rest of the signals are the weak markets. Therefore, if the signals are positive, the strong markets are the one with low x_{-i}, and the weak markets are those with high x_{-i}. In the previous sections, I showed that in the two examples the problems were regular and the signals were positive for all positive correlation parameters. Therefore, all the previous propositions apply.

Example 5. *Figure 4 represents the first example with the same parameters as in the previous subsection. If the signals are independent and $w < (1-\alpha_1)$, I can show that the mechanism without regulation is more efficient than the mechanism with a constant posted price. The demand for each market is linear. Allowing discrimination opens all the markets where $x_1 \in [2/3.w/(1-\alpha_1), w/(1-\alpha_1)]$* [20]*. On the contrary, if $w \geq 3/2.(1-\alpha_1)$, all markets are served with a constant posted price. Schmalensee (1981) showed that discrimination reduces welfare, in this case.*

Example 6. *Figure 5 represents the second example with the same parameters as in the previous subsection. Adapting the proof of Cowan (2016), I can show that the expected sold quantity rises with discrimination for all parameters. However, I cannot conclude on the overall efficiency gain because sellers with low valuations sell less often and sellers with high valuations more often.*

In this section, I presented how the optimal mechanism looks when the price must be independent from the sellers' identity and reports and when the regulator allows for price discrimination. I transposed in my set-up results of the literature and use them as building blocks and benchmarks in the remaining of the paper. In next section, I present

[20] $p_i^0(w, x_j) = \min\{\frac{w+(1-\alpha_1)x_j}{2}, (1-\alpha_1)x_j + \alpha_1\}$ and the welfare increases by a factor $w^3/(648.\alpha_1.(1-\alpha_1))$

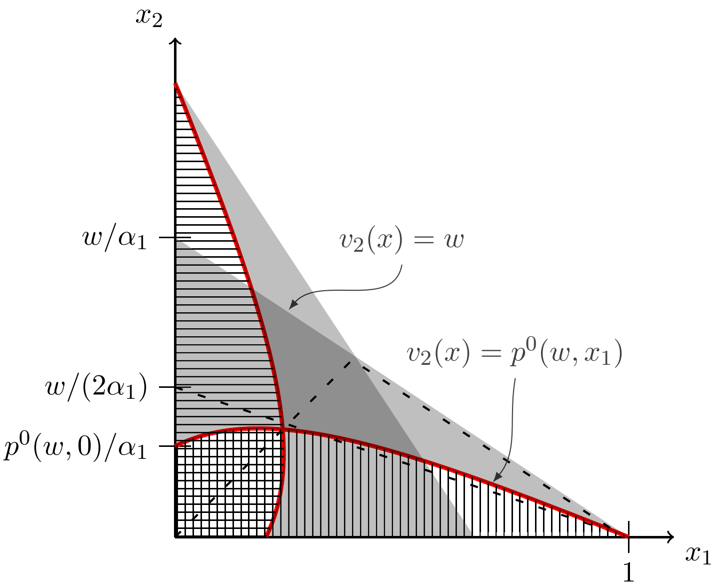


FIGURE 4: Linear utilities without regulation

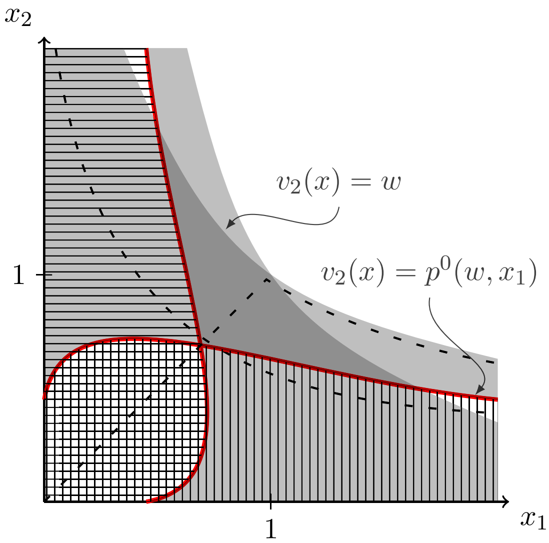


FIGURE 5: Log-linear utilities without regulation

new results and study a case closer to the motivating example. I construct the optimal mechanism when the regulator forbids price and quantity discrimination, but allows the posted price to change with the sellers' reports.

5 Optimal Uniform Posted Price

Before studying the problem presented in section 3, I explore the implications of limiting the buyer's choice to mechanisms with a uniform posted price. Contrary to the problem with constant posted price presented in subsection 4.1, the players can communicate before the announcement of the price. Therefore, the price posted may depend on the sellers' private information. However, contrary to the problem without regulation presented in subsection 4.2, the price offered must be the same for all sellers. In this case, the buyer solves the problem presented in problem 1 with the additional constraint that $\Lambda(x) = 0$. It amounts to finding the function $p(x)$ belonging to $\mathcal{C}$ that solves the following maximization program:

$$\max_{p(x)\in\mathcal{C}} \int_X \Lambda_T(p(x), x)[w - p(x)]f(x)dx$$

$$\text{s.t. for all } i \leq N,\ x \in X,\ z \in X_i:$$

$$\mathbb{1}_{\{p(x)\geq v_i(x)\}}[p(x) - v_i(x)] \geq \mathbb{1}_{\{p(z,x_{-i})\geq v_i(x)\}}[p(z, x_{-i}) - v_i(x)]$$

I cannot derive the solution using the classical techniques of the mechanism design literature as in the previous section. The buyer's objective cannot be split in as many independent problems as there are sellers as in the proof of proposition 2. Indeed, the sellers' incentive compatibility constraints are linked through the common price function $p(x)$. However, considering separately the incentive constraint of a seller given the others' signals, I can infer some simple properties that the price function must verify.

Proposition 4. *The price offered, $p(x)$, satisfies seller i's incentive constraint given x_{-i} if and only if:*

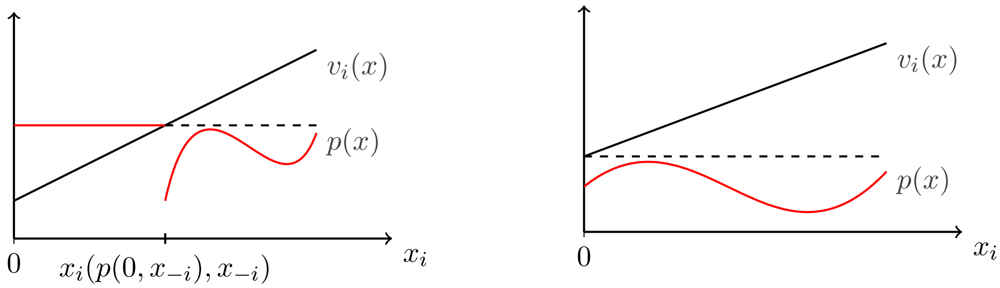
- $p(x) \leq \max\{p(0, x_{-i}), v_i(0, x_{-i})\}$ *for all* x_i
- $p(x) = p(0, x_{-i})$ *for all* x_i *such that* $v_i(x) \leq p(0, x_{-i})$

Proof. I will work only with the incentive constraint of seller i and keep the private information of the others x_{-i} fixed. I denote $p_0 = p(0, x_{-i})$ and will prove first the necessity part.

Suppose $p_0 > v_i(0, x_{-i})$. For all x_i such that $v_i(x) < p_0$, the incentive constraint of seller i implies that $p(x) = p_0$. Otherwise, if $p(x) < p_0$, when seller i's information is x_i, he would lie, report 0 and sell for a strictly higher profit than if he would have reported truthfully. On the contrary, if $p(x) > p_0$, seller i would lie in 0, report x_i and make a strictly higher profit. For the same reason, for all x_i such that $v_i(x) \geq p_0$, $p(x) \leq p_0$.

Suppose $p_0 \leq v_i(0, x_{-i})$. For all x_i, the constraint implies that $p(x) \leq v_i(0, x_{-i})$. Otherwise, seller i would lie in 0 and report x_i.

Suppose now that the two conditions hold. If seller i sells in x, he can only have a weakly lower price by changing his report. If he is not selling in x, $v_i(x) \leq p_0$ and he cannot make a strictly positive profit because the price is always smaller that p_0. □



(A) $p(0, x_{-i}) > v_i(0, x_{-i})$ (B) $p(0, x_{-i}) \leq v_i(0, x_{-i})$

FIGURE 6: individual IC with a uniform price

In the plane going through $(0, x_{-i})$ parallel to x_i axis, the proposition implies that the price and seller i's valuation must belong to one of the two cases presented in figure 6. This simple restriction on $p(x)$ becomes very strong when applied for all sellers and signals. Moving figure 6 in the set of the possible signals narrows down a lot the possible price functions. In general, the price must be constant on every segment of the signal space where only the signals of sellers tendering shares change. The next lemma uses this property recursively to show that the prices in two points are the same if I can go from one to the other by following only such segments.

Lemma 1. *For all $x \in X$, $J \subset N$, if there is $k(\cdot)$, an ordering of the sellers in J, such that for all $1 \leq j \leq |J|$, $v_{k(j)}(x_{N-J}, x_{k(1)}, .., x_{k(j)}, 0) < \max\{p(x), p(x_{N-J}, 0)\}$[21], then $p(x) = p(x_{N-J}, 0)$.*

Proof. If $p(x) \leq p(x_{N-J}, 0)$, the proof consists of iteratively using figure 6a of proposition 4 for seller $k(1)$ until $k(|J|)$ along the path:

$$(x_{N-J}, 0), (x_{N-J}, x_{k(1)}, 0), ..., (x_{N-J}, x_{k(1)}, .., x_{k(|J-1|)}, 0), (x)$$

[21]with weak inequality if $p(x) \geq p(x_{N-J}, 0)$

Because $v_{k(1)}(x_{N-J}, x_{k(1)}, 0) < p(x_J, 0)$ and $v_{k(1)}$ is increasing, $v_{k(1)}(x_{N-J}, 0, 0) < p(x_J, 0)$ and figure 6a of proposition 4 implies that $p(x_{N-J}, 0) = p(x_{N-J}, x_{k(1)}0)$. The following inequalities together with figure 6a of proposition 4 and the fact that $v_{k(2)}$ is increasing implies that $p(x_{N-J}, 0) = p(x_{N-J}, x_{k(1)}, 0) = p(x_{N-J}, x_{k(1)}, x_{k(2)}, 0)$.

$$v_{k(2)}(x_{N-J}, x_{k(1)}, x_{k(2)}, 0) < p(x_{N-J}, 0) = p(x_{N-J}, x_{k(1)}0)$$
$$v_{k(2)}(x_{N-J}, x_{k(1)}, 0) < p(x_{N-J}, x_{k(1)}0)$$

Repeating this on until $k(|J|)$ proves that $p(x_{N-J}, 0) = p(x)$.

If $p(x_{N-J}, 0) \leq p(x)$, iteratively using figure 6a starting in point x with seller $k(|J|)$ and price $p(x)$ until point $(x_{N-J}, 0)$ with seller $k(1)$ proves that $p(x_{N-J}, 0) = p(x)$. □

The maximum in the formulation is to be understood as an " or ". The path can start either from $(x_{N-J}, 0)$ with price $p(x_{N-J}, 0)$ or from x with price $p(x)$. The following corollary illustrates more precisely how it restricts the possible price functions. The first bullet point states that if all sellers tender their shares in the origin, the price is constant on all points where all sellers tender their shares. Indeed, in such a point, setting sequentially all coordinates to 0 is a path verifying the condition of lemma 1. The second bullet point states that if a seller is more pessimistic than the sellers with a signal equal to 0, the price posted will be independent of this seller's information.

Corollary 5.
- *For all $x \in X$ such that $\max_i v_i(x) < p(0)$, $p(x) = p(0)$.*
- *If $v_i(0, x_{-j}) \leq v_j(0, x_{-j})$ for all $j \in N$ and $x_{-j} \in X_j$, the information of i is useless.*

The second item follows from the fact that the information of i may be useful only in a point where somebody sells, i.e. in x such that $p(x) \geq \min_j v_j(x)$. Denoting $j_0 = \arg\min_j\{v_j(x)\}$, lemma 1, starting in x with $k(2) = j_0$ and $k(1) = i$, implies that $p(x) = p(x_{-\{i,j_0\}}, 0)$. If this condition is verified for all sellers, allowing the buyer to communicate with the sellers before announcing the price will be useless. In this case, he would simply act as the classical monopsonist of subsection 4.1 and post a price solely according to his prior belief about the sellers' valuations. This is actually the case in the first example if $\alpha_i^i \leq \alpha_j^i$ and in the second example for all parameters.

Example 7. *In the example with linear utilities, if $\alpha_1 \leq 1/2$, i.e. when the sellers value the other's signal more than their own, corollary 5 applies and communication is useless. The optimal uniform posted price mechanism can be represented by Figure 1 where the two axes have been swapped.*

Example 8. *Corollary 5 always applies in the example with log-linear valuations and communication is useless. Figure 2 represents also the optimal mechanism when the buyer is allowed to communicate with the sellers before setting the price.*

If the valuation functions are close enough one to another, the connected paths presented in lemma 1 appear easily. The price in one point may then restrict the price on entire regions of the signals' space. When $n = 2$, it will be the case if the valuations at the origin are equal. For $n > 2$, I extend this assumption and suppose the following:

Definition 3. *The valuations are* ***symmetric on the axes' intersections*** *if and only if for all sellers* (i, j) *and* $x_{-\{i,j\}} \in X_{-\{i,j\}}$, $v_j(0, 0, x_{-\{i,j\}}) = v_i(0, 0, x_{-\{i,j\}})$.

The symmetry is only required on a set of probability zero. Outside the axes where two sellers' signals are equal to 0, their valuations can be arbitrary asymmetric. The symmetry in these boundary points concerns how two sellers value the others' signals. It doesn't require that the others' signals be permutable or that sellers value their own signals in a similar way. It requires rather that the sellers with the lowest possible signal have a common valuation. It is a weaker requirement than the classical symmetry hypothesis where for every agent i, $v_i(x) = v(x_i, x_{-i})$. As in the second example, it will be verified if receiving the lowest signal triggers the lowest possible valuation. The next proposition shows that the symmetry on the axes' intersections implies that the price and the number of sellers increase as the number of signals equal to 0 increases.

Lemma 2. *If the valuations are symmetric on the axes' intersections and seller* i *sells in* $x \in X$*, then every seller* j *sells in* $(0, 0, x_{-\{i,j\}})$ *and, therefore,* $p(x) = p(0, x_{-i}) \leq p(0, 0, x_{-\{i,j\}})$.

Proof. Figure 6a and the monotonicity of v_i implies that: $p(x) = p(0, x_{-i}) \geq v_i(x) \geq v_i(0, 0, x_{-\{i,j\}})$. Therefore, the symmetry assumption implies that:

$$p(0, x_{-i}) = p(0, x_j, x_{-\{i,j\}}) \geq v_j(0, 0, x_{-\{i,j\}})$$

Together with figure 6a drawn in $(0, 0, x_{-\{i,j\}})$ along the axis j, it implies that j must sell in $(0, 0, x_{-\{i,j\}})$ and, therefore, that $p(0, 0, x_{-\{i,j\}}) \geq p(0, x_{-i})$. □

The proof of this lemma shows why the symmetry assumption is important. It ensures that a price independent of two sellers' information cannot separate the support of their valuations. The symmetry assumption excludes the cases where some sellers' valuations are so high that the buyer can exclude them and use their signals to set a more precise price. In a sense, it rules out the cases where the buyer can use the information of some sellers for free. It implies for instance that $w > \max_i v_i(0)$. The results of this section are robust to mild violations of this hypothesis. When $n = 2$, proposition 6 shows that the results hold for a larger class of valuation functions and details what happens outside this extended set. In the first example, this extension includes a large set of parameters. If $n > 2$, the problem is not fundamentally different and the intuition stays the same. The following corollary illustrates more precisely how lemma 2 restricts the possible price functions. The first bullet point follows from repeatedly using this lemma. It states that the price in the origin is an upper bound on all relevant prices. The second bullet point is a direct consequence of the first one. It states that a sale can only take place if a seller has a lower valuation than the price at the origin.

Corollary 6.

- *For all $x \in X$ such that a sale takes place, $p(x) \leq p(0)$.*
- *For all $x \in X$ such that $\min_i v_i(x) > p(0)$, no sale takes place.*

To ease the exposition, I present in the core of the paper only the case where $n = 2$. The generalization to cases where $n > 2$ can be found in the appendix. For all $p \in \mathbb{R}$, I denote $U_j(p) = \{x \in X : v_j(x) \leq p\}$, the half space where the valuation of j is below p. I also denote $D_j(p) = \{x \in X : v_i(x_i, 0) > p \text{ and } v_j(x) \leq p\}$, the subset of $U_j(p)$ such that seller i doesn't sell for all signals of j. The next proposition shows that the mechanism imposes different price structures on these different regions. Figure 7 represents these regions for the first example. The gridded area depicts $\cup_i U_i(p^*(0)) \setminus \cup_i D_i(p^*(0))$, the region where the price posted must be equal to the price in the origin. In $D_1(p^*(0))$, hashed horizontally, the buyer posts the minimum between the price at the origin and the optimal discriminatory price for 1 given x_2 derived in the previous subsection. Symmetrically, in $D_2(p^*(0))$, hashed vertically, the buyer posts the minimum between the price at the origin and the optimal discriminatory price for 2 given x_1. In the remaining area no sale takes place.

Proposition 5. *If the problem is regular for every seller i given X_{-i} and the valuations are symmetric on the axes' intersections, the optimal mechanism is such that $p^*(0) > v(0)$ and:*

- *In $x \in \cup_i U_i(p^*(0)) \setminus \cup_i D_i(p^*(0))$, the buyer posts a price $p^*(x) = p^*(0)$, and both sellers may sell.*
- *In $x \in D_j(p^*(0))$, the buyer posts a price $p^*(x) = \min\{p^*(0), p_j^0(w, x_i)\}$, and only j may sell.*

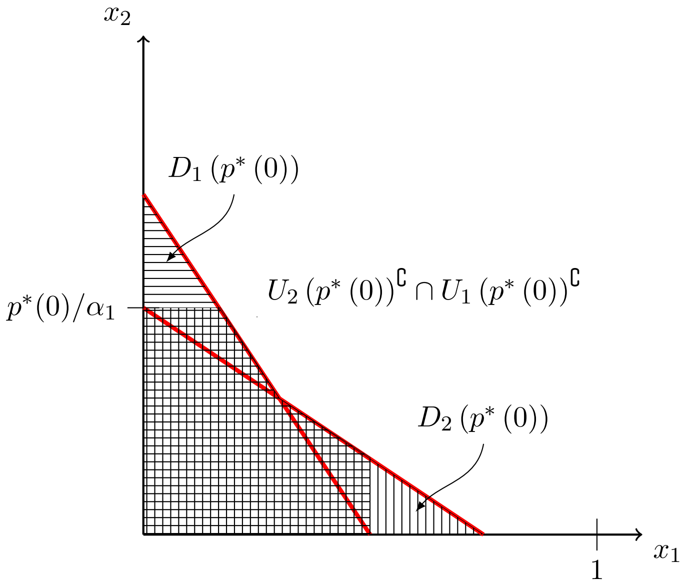


FIGURE 7: Linear utility with a uniform price

- *In* $x \notin U_1(p^*(0)) \cup U_2(p^*(0))$, *the buyer posts a price* $p^*(x) = \min\{p^*(x_1, 0), p^*(0, x_2)\}$ *and nobody sells.*

Proof. The second item of corollary 6 implies that for any incentive compatible price function $p(x)$, nobody sells in $U_2(p(0))^{\complement} \cap U_1(p(0))^{\complement}$. It implies first that $p^*(0) > v(0)$, otherwise the profit would be 0. Besides, it implies that it is incentive compatible and optimal to set the price to $\min\{p(x_1, 0), p(0, x_2)\}$ in this region. The problem reduces then to:

$$\max_{p(x)\in\mathcal{C}} \int_{\cup_i U_i(p(0))} \Lambda_T(p(x), x)[w - p(x)]f(x)dx$$

$$\text{s.t. for all } i \leq N,\ x \in X,\ z \in X_i :$$

$$\mathbb{1}_{\{p(x)\geq v_i(x)\}}[p(x) - v_i(x)] \geq \mathbb{1}_{\{p(z,x_{-i})\geq v_i(x)\}}[p(z, x_{-i}) - v_i(x)]$$

Let $p(x)$ be an incentive compatible price belonging to $\mathcal{C}$. Let x be in the interior of $\cup_i U_i(p(0)) \setminus \cup_i D_i(p(0))$. There is a seller i such that $v_i(x) < p(0)$ and $v_j(x_j, 0) < p(0)$. Hence, there is a path from 0 to x verifying the conditions of lemma 1 and $p(x) = p(0)$. The border of $\cup_i U_i(p(0)) \setminus \cup_i D_i(p(0))$ is of mass 0, because $f(\cdot)$ has no mass point[22]. Therefore, the price charged in these points is irrelevant, as long as the resulting price

[22] If it were not the case, the property would not fundamentally change, but the border would have to be studied separately and new cases distinctions introduced.

function is incentive compatible. In particular, setting $p(x) = p(0)$ on the border is incentive compatible. The buyer can then reduce the set over which he maximizes to functions that are constant on $\cup_i U_i(p(0)) \setminus \cup_i D_i(p(0))$.

Let x be in $D_j(p(0))$. If the buyer buys from i, lemma 2 implies that $p(0) \geq p(x) = p(0, x_j) \geq v_i(x) \geq v_i(0, x_i)$ which contradicts $v_i(x) > p(0)$. Hence, seller i does not tender his shares in $D_j(p^*(0))$. Moreover, if j sells in x, lemma 2 implies that the buyer proposes a price $p(x) = p(x_i, 0) \leq p(0)$. Therefore, I can further reduce the set over which the buyer maximizes. Splitting the problem on these three regions and I can rewrite the problem as:

$$\max_{\substack{p(0)\in\mathbb{R}\\ p(x_1,0)\leq p(0)\\ p(0,x_2)\leq p(0)}} \left\{ \int_{x\in\cup_i U_i(p(0))\setminus\cup_i D_i(p(0))} \Lambda_T(p(0), x)[w - p(0)]f(x)dx + \lambda_1 \int_{x\in D_1(p(0))} \mathbb{1}_{\{p(0,x_2)\geq v_1(x)\}}[w - p(0, x_2)]f(x)dx \right.$$
$$\left. + \lambda_2 \int_{x\in D_2(p(0))} \mathbb{1}_{\{p(x_1,0)\geq v_2(x)\}}[w - p(x_1, 0)]f(x)dx \right\}$$

Maximizing first over $(p(x_1, 0), p(0, x_2))$ given $p(0)$ and taking the maximum over $p(x_i, 0)$ in the integral over x_i, I can rewrite the problem as:

$$\max_{p(0)\in\mathbb{R}} \left\{ \lambda_1 \int_{x\in U_1(p(0))\setminus D_1(p(0))} [w - p(0)]f(x)dx + \lambda_2 \int_{x\in U_2(p(0))\setminus D_2(p(0))} [w - p(0)]f(x)dx \right.$$
$$+ \lambda_1 \int_{x_2\in X_{D1}(p(0))} \max_{p(0,x_2)\leq p(0)} [w - p(0, x_2)]G_1(p(0, x_2) \mid x_2) f_2(x_2)dx_2 \tag{2}$$
$$\left. + \lambda_2 \int_{x_1\in X_{D2}(p(0))} \max_{p(x_1,0)\leq p_0} [w - p(x_1, 0)]G_2(p(x_1, 0) \mid x_1) f_1(x_1)dx_1 \right\}$$

Where $X_{Di}(p(0)) = \{x_j \in X_j : (0, x_j) \in D_i(p(0))\}$. The two subproblems inside the integrals are the same as the unregulated problem stated in proposition 3. Therefore, the regularity of the problem for every seller i given X_{-i} implies that for all $x \in D_j(p^*(0))$, $p^*(x) = p^*(x_i, 0) = \min\{p^*(0), p_j^0(w, x_i)\}$. □

The generalization of proposition 5 can be found in the appendix. The principle is strictly the same. In a point where a seller has a higher value than the price in 0, the buyer may use this seller's information to post a smaller price to the others. The definition of the regions is the direct generalization of the one presented above. The proof of proposition 5 used the symmetry on the axes' intersections only through the lemma 2, to ensure that the two sellers sell in 0. The next proposition uses this fact to enlarge the set of valuations for which property 5 holds when $n = 2$. Even if the idea generalizes to the case where $n > 2$, the proof is beyond the scope of this paper[23].

Proposition 6. *When $n = 2$, if the problem is regular but $v_2(0) > v_1(0)$:*

- *Proposition 5 holds if and only if the solution of problem 2 is strictly greater than $v_2(0)$. Otherwise, the buyer posts $p^*(x) = \min\{v_2(0), p_1^0(w, x_2)\}$ and only 1 may sell.*

- *In particular, proposition 5 holds if $p_1^0(w, 0) > v_2(0)$ or if w is big enough and $v_1(\bar{x}, 0) > v_2(0)$.*

Proof. Let $v_2(0) > v_1(0)$. I split the problem in two and first find $p^{(1)}(x)$, the incentive compatible function such that $v_2(0) \geq p(0)$ that maximizes the profit. The monotonicity of v_2 implies that $v_2(x) > v_2(0) \geq \max\{v_1(0), p(0)\}$ for all $x_2 > 0$. Proposition 4 implies that $\max\{v_1(0), p(0)\} \geq p(x_1, 0)$ and, therefore, $v_2(x) > p(x_1, 0)$. If seller 2 sells in $x_2 > 0$, proposition 4 implies that $p(x_1, 0) = p(x) \geq v_2(x)$ which contradicts the last inequality. Hence, seller 2 never sells. Besides, Figure 6b implies that $v_2(0) \geq p(0, x_2)$ and, therefore, for all x such that 1 sells $p(x) = p(0, x_2) \leq v_2(0)$. Finally, because the signals such that $x_2 = 0$ are of mass 0, $p^{(1)}(x)$ maximizes the unregulated program for seller 1 with the additional constraint that $p(x) \leq v_2(0)$. So $p^{(1)}(x) = p^{(1)}(0, x_2) = \min\{v_2(0), p_1^0(w, x_2)\}$ and I denote the resulting profit Π_1. I want now to find $p^{(2)}(x)$, the incentive compatible function such that $p(0) > v_2(0)$ that maximizes the profit. The two sellers sell in 0, therefore, corollary 6 holds and I can rewrite the proof of proposition 5. Finally, $p^{(2)}(x)$ is the solution of the problem 2 with the additional constraint that $p(0) > v_2(0)$. Besides, the objective of problem 2 is continuous in $p(0)$, equal Π_1 in $v_2(0)$ and increasing in $p(0)$ for $p(0) < v_2(0)$. Therefore, the solution of problem 2 is strictly greater than $v_2(0)$ if and only if proposition 5 holds.

I will now prove the second item. $p_1^0(w, 0) > v_2(0)$ together with the first item imply that $p^*(0) \geq v_2(0)$. Because $p_1^0(w, \cdot)$ and $v_2(0, \cdot)$ are continuous, there exists $x_l > 0$ such that for all $x_2 < x_l$, $p_1^0(w, x_2) > v_2(0, x_l)$. The regularity of the problem implies that for all $v < p_1^0(w, x_2)$, $(w - v)G_1(v \mid x_2)$ increases. Therefore, for all $x_2 < x_l$ and $p(x_2) \leq v_2(0)$, $p_1^0(w, x_2) > v_2(0, x_l) > p(x_2)$ and I have: $(w - v_2(0, x_l))\, G_1(v_2(0, x_l) \mid x_2) > [w - p(x_2)] G_1(p(x_2) \mid x_2)$. The profit of the buyer when $p(0) = v_2(0)$ can then be

[23] to prove the first item $p(0, 0, x_{-\{i,j\}})$ should be replaced in the proof in the appendix and in lemma 2 by $\min\{p(0, 0, x_{-\{i,j\}}), v_i(0, 0, x_{-\{i,j\}})\}$.

written as:

$$\begin{aligned}
\Pi_1 &= \lambda_1 \int \max_{p(x_2)\leq v_2(0)} [w - p(x_2)] G_1\left(p\left(x_2\right) \mid x_2\right) f_2(x_2) dx_2 \\
&< \lambda_1 \left(\int\limits_{x_2 \leq x_l} \left(w - v_2\left(0, x_l\right)\right) G_1\left(v_2\left(0, x_l\right) \mid x_2\right) f_2(x_2) dx_2 \right. \\
&\qquad \left. + \int\limits_{x_2 \geq x_l} \max_{p(x_2)\leq v_2(0)} [w - p(x_2)] G_1\left(p\left(x_2\right) \mid x_2\right) f_2(x_2) dx_2 \right) \\
&< \lambda_1 \left(\int\limits_{x \in U_1(v_2(x_l,0)) \setminus D_1(v_2(x_l,0))} \left(w - v_2\left(x_l, 0\right)\right) f(x) dx \right. \\
&\qquad \left. + \int\limits_{x_2 \in X_{D1}(v_2(x_l,0))} \max_{p(x_2,0)\leq v_2(x_l,0)} [w - p(x_2,0)] G_1\left(p\left(x_2, 0\right) \mid x_2\right) f_2(x_2) dx_2 \right)
\end{aligned}$$

Where the last line comes from the fact that $U_1(v_2\left(x_l, 0\right)) \setminus D_1(v_2\left(x_l, 0\right)) = \{x \in X : v_2(0, x_2) \leq v_2(0, x_l) \text{ and } v_1(x) \leq v_2(0, x_l)\}$, $X_{D1}(v_2\left(x_l, 0\right)) = \{x_2 \in X : v_2(0, x_2) \geq v_2(0, x_l) \text{ and } v_1(x) \leq v_2(0, x_l)\}$, the max is increasing in the upper bound and $v_2(0, x_l) > v_2(0)$. Finally, the last line is smaller than the objective of problem 2 in $p(0) = v_2(0, x_l)$ and, therefore, $p^*(0) > v_2(0)$. If w is high enough, $p_1^0(w, 0)$ approaches $v_1(\bar{x}, 0) > v_2(0)$ and the property is verified. □

The optimal price at the origin $p^*(0)$ is difficult to derive, even in the simple examples presented in this paper. This complicates a lot the general analysis of the mechanism. The next corollary provides a simple test to see if the optimal mechanism uses the sellers' reports. It also formally proves that if the buyer doesn't discriminate at the optimum, the optimal mechanism will be the same as the one with optimal constant price.

Corollary 7.

- *If the optimal mechanism doesn't use the reports, the buyer posts p^* on the entire signals' space as in section 4.1. In particular, the reports will be used if the buyer can discriminate when $p(0) = p^*$.*

- *When $n = 2$, the buyer never uses i's reports if for all $x_i \in X_i$, $p_j^0(w, x_i) \geq v_i(x_i, 0)$.*

- *When $n = 2$, $p_1^* \geq p_2^*$ and w is small enough, if seller 1's information is negative for 2 and seller 1 is strictly more optimistic than 2' when $x_2 = 0$, the buyer discriminates for some signals.*

Proof. To prove the first part, I denote $\Pi_{nc}(p)$ the profit function of the buyer without discrimination when he charges p. Similarly, I define $\Pi(p)$ the objective function inside the first maximum in (2) and $p^*(0)$ the price at the origin. If the optimum mechanism cannot discriminate, $\Pi\left(p^*\left(0\right)\right) = \Pi_{nc}\left(p^*\left(0\right)\right)$. Moreover, $\Pi(p^*) \geq \Pi_{nc}(p^*)$ because posting a constant price equal to p^* is also feasible when communication is allowed. Finally, $\Pi_{nc}(p^*) \geq \Pi_{nc}\left(p^*\left(0\right)\right)$ because p^* is the optimal price without communication. Therefore, $\Pi(p^*) \geq \Pi\left(p^*\left(0\right)\right)$ and setting $p^* = p^*(0)$ is optimal. If the buyer can discriminate when $p(0) = p^*$, $\Pi\left(p^*\left(0\right)\right) > \Pi_{nc}(p^*)$. This is impossible if there is no price discrimination.

For the second part, the posted price is a function of x_i only when when $v_i(x_i, 0) > p^*(0)$ and $p^*(0) \geq p^0_j(w, x_i)$. This is impossible if $p^0_j(w, x_i) > v_i(x_i, 0)$. Therefore, the buyer will never use i's report.

Let $n = 2$ and $p^*_1 \geq p^*_2$. Proposition 1 implies that $p^* \geq p^*_2$. If information of 1 is negative for 2, corollary 4 implies that $p^0_2(w, x_1)$ crosses p^*_2 only once and from above in the support of $x_1 \mid v_2 = p^*_2$. Therefore, because v_2 is increasing, $p^* > p^0_2(w, x_1)$ for high signals in the support of $x_1|v_2 = p^*$. Besides, if seller 1 is more optimistic than 2 when seller 2's signal is 0: for all x_1, $v_1(0, x_1) > v_2(0, x_1)$. Therefore, if w is small enough, p^* approaches $v(0)$ and $D_2(p^*)$ is non empty. Finally, $X_{D2}(p^*)$ contains some signals where $p^* > p^0_2(w, x_1)$ because it contains the highest signals on the support of $x_1 \mid v_2 = p^*$. Therefore, the buyer discriminates for some signals when $p(0) = p^*$. □

Finally, I use a well-known result of the discrimination literature to derive a simple test to check if communication reduces welfare.

Corollary 8. *If $p^*(0) \leq p^*$, allowing the players to communicate before the buyer posts the price reduces welfare.*

Proof. If $p^*(0) \leq p^*$, all sellers face a lower price for all signals when beforehand communication is allowed. The quantity sold in each market decreases and, following Varian (1985), welfare decreases. □

Example 9. *In the first example, the weak symmetry around* 0 *imposes that $\alpha^1_0 = \alpha^2_0$. If the signals are independent, proposition 6 extends the result to all cases where* $\min\{\alpha^j_j, w - \alpha^i_0\} \geq \alpha^i_0 - \alpha^j_0 \geq -\min\{\alpha^i_i, w - \alpha^j_0\}$. *As in the previous section, I focus on the symmetric cases. I already showed that the optimal mechanisms with a uniform and a constant price are the same if $\alpha_1 \leq 1/2$. Similarly, when the signals are independent and $w \geq 3\alpha_1 - 1$, $v_i(x_i, 0) < p^0_j(w, x_i)$ and corollary 7 implies that beforehand communication will be useless. Actually, with the help of mathematica, I can prove that reporting signals* *significantly raises the profit of the buyer, only if the parameters belong to the gridded area of figure 14, i.e.when* α_1 *is close enough to* 1 *and* w *close enough to* 0 *.*

Figure 8 represents the first example with independent signals, $\alpha_1 = 5/6$ *and* $w = 1/4$*. Allowing the buyer to collect reports rises the price at the origin:* $p^*(0) = 0.171 > p^* = 1/6$*. The buyer buys from* 2 *in the vertically hashed area and from* 1 *in the horizontaly hashed one. The buyer uses the information of a seller to post a lower price to the other in the shaded area. On the remaining areas, the price posted is equal to* $p^*(0)$*. Allowing beforehand communication increase the profit by* 1.1%*. The potential gain from removing all regulations as in subsection 4.2 amounts only to* 8.3%*. Figure 9 draws the profits for the constant price (black) and for the uniform price (red) as a function of the price posted in* 0*. The comparison with the previous mechanisms in terms of efficiency is not obvious. For the signals where* $p^*(x) < p^*$*, the surplus is lower than the surplus without communication. In the remaining signals' space, the price increases, so the quantity sold increases and the surplus increases. However, in this case, the price in* 0 *does not rise enough to compensate for the first loss. The welfare surplus is* 0.6% *smaller than with a constant posted price. The welfare with no regulation is in comparison* 2.5% *smaller than with a constant posted price. In this case, imposing a uniform price mitigates the consequences of deregulation. In particular, the surplus of the sellers decreases by* 3.2% *compared to the case with no communication. Without regulation, it decreases instead by* 18%*.*

If the sellers are not symmetric, the picture does not change fundamentally. If the signals are correlated, the picture changes as in the previous subsection. Depending on where the correlation takes place on the x_1 *axis, it may help or not the buyer to achieve a higher profit. For* $n > 2$*, the weak symmetry on the axes' intersections imposes additional restrictions on the possible* α^i_j*. The* α^i_i*, how much a seller weights his own signal, can still be freely chosen. But the* $(\alpha^j_i)_{j \neq i}$*, how the others value his signal, must be equal.*

In this section, I proved that forbidding price and quantity discrimination may still let enough flexibility for meaningful communication to take place before the biding price is announced. In particular, it may happen when the sellers' valuations are far away one from another. In such cases, allowing communication may enable the buyer to identify sellers with high valuations. He may then use their information to post a lower price to the remaining sellers. When the conditions for meaningful communication are not fulfilled, the buyer can do no better that use his prior about the sellers' valuations to post the optimal constant price of the classical monopolist. In the following part, I study

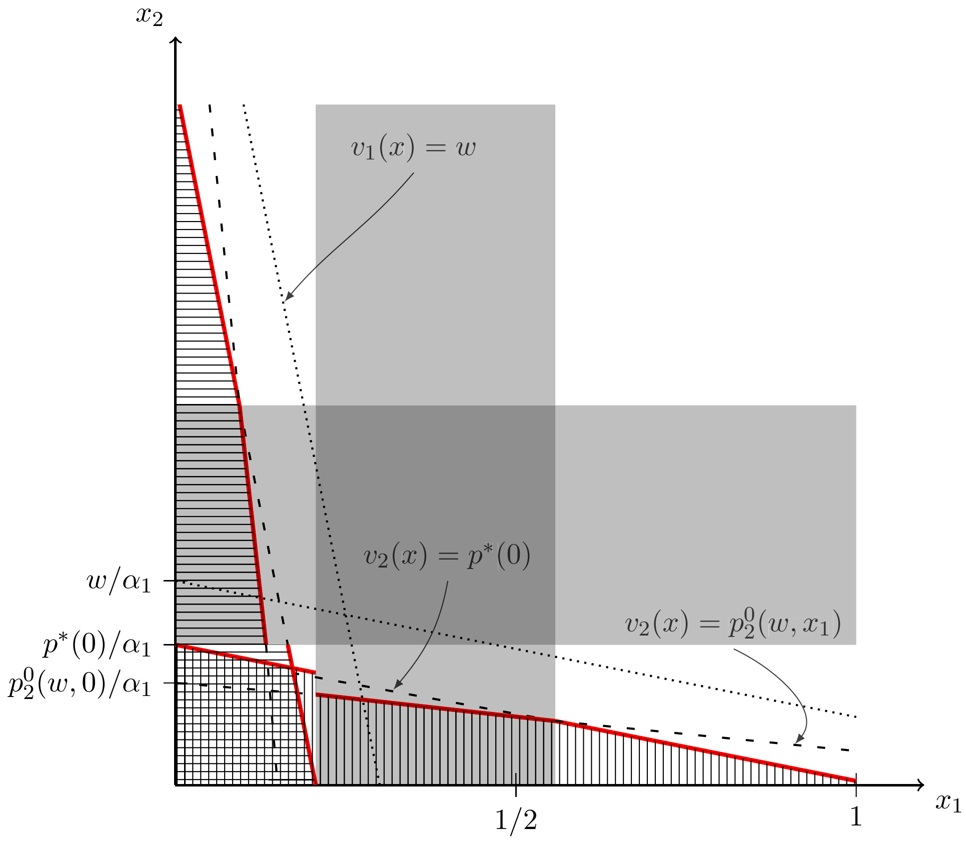


FIGURE 8: Linear utilities with a uniform price

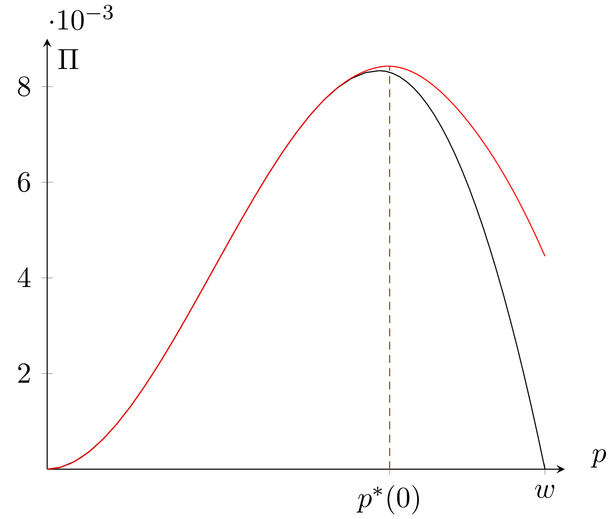


FIGURE 9: Profit with a uniform price

whether allowing the monopolist to set a lower bound on the quantity traded allows him to do better.

6 Optimal Uniform Posted Price and Demand Lower Bound

When the buyer cannot use price and quantity discrimination but can link his offer with a minimum on the quantity traded, Section 3 showed that the problem of the buyer can be described as finding two functions belonging to $\mathcal{C}$, $p(x)$ and $\Lambda(x)$, solving the following maximization program:

$$\max_{(p(x),\Lambda(x))\in\mathcal{C}^2} \int_X \mathbb{1}_{\{\Lambda_T(p(x),x)\geq\Lambda(x)\}}\Lambda_T\left(p\left(x\right),x\right)\left[w-p(x)\right]f(x)dx$$

$$\text{s.t. for all } i\leq n,\ x\in X,\ z\in X_i\text{:}$$

$$\mathbb{1}_{\{\Lambda_T(p(x),x)\geq\Lambda(x)\}}\mathbb{1}_{\{p(x)\geq v_i(x)\}}[p(x)-v_i(x)]\geq$$

$$\mathbb{1}_{\{\Lambda_T(p(z,x_{-i}),x)\geq\Lambda(z,x_{-i})\}}\mathbb{1}_{\{p(z,x_{-i})\geq v_i(x)\}}[p(z,x_{-i})-v_i(x)]$$

Where $\Lambda_T(p,x)=\sum_{i=1}^{n}\lambda_i\mathbb{1}_{\{p\geq v_i(x)\}}$ is the number of shares tendered in x when the price offered is p. In the following, I denote $IC_i(z,x)$, the incentive constraint of seller i in $x\in X$ when he reports $z\in X_i$. I also define $S(p,x)=\{i: p\geq v_i(x)\}$, the set of sellers who tender their shares in x when the price offered is p. $\Lambda_T(p,x)$ and $S(p,x)$ are increasing in p and decreasing in x. Therefore, $\Lambda_T(p(z,x_{-i}),x)$ is decreasing in x_i.

First, I show that the buyer can restrict his attention to incentive compatible mechanisms where $\Lambda(x) = \Lambda_T(p(x), x)$. In the spirit of the revelation principle, I show that for any incentive compatible mechanism $(p(x), \Lambda(x))$, there is an incentive compatible mechanism, $(p'(x), \Lambda'(x))$ with $\Lambda'(x) = \Lambda_T(p'(x), x)$, such that the buyer's profit is the same in any point of the signals' space.

Proposition 7. *The buyer can w.o.l.g. restrict to incentive compatible mechanisms where* $\Lambda(x) = \Lambda_T\left(p\left(x\right), x\right)$.

Proof. Let $\left(p\left(x\right), \Lambda\left(x\right)\right)$ be incentive compatible. I will show that a function $p'(x) \in \mathcal{C}$ exists such that $\left(p'\left(x\right), \Lambda_T\left(p'\left(x\right), x\right)\right)$ is incentive compatible and the buyer's profit in x doesn't change. I denote $U = \{x \in X : \Lambda_T\left(p\left(x\right), x\right) \geq \Lambda(x)\}$ the set of signals such that a sale takes place. I define $p'(x)$ as the price function such that $p'(x) = p(x)$ for all $x \in U$ and $p'(x) = 0$ otherwise. Hence, for all $x \in U$, the sale takes place in both mechanisms at the same price. Whereas for all $x \notin U$, no sale takes place in both mechanisms. Therefore, for all x, the buyer's profit is the same in the two mechanisms. It remains to show that the new mechanism is incentive compatible. For all $x \in X$, i, and $z_i \in X_i$, the left hand side of $IC_i(z_i, x)$ doesn't change. It stays to 0 if $x \notin U$ and the price doesn't change otherwise. If $(z_i, x_{-i}) \in U$, the new quantity lower bound rises because $\Lambda'(z_i, x_{-i}) = \Lambda_T\left(p\left(z_i, x_{-i}\right), z_i, x_{-i}\right) \geq \Lambda(z_i, x_{-i})$. Therefore, the right hand side of $IC_i(z_i, x)$ decreases. If $(z_i, x_{-i}) \notin U$, $\Lambda_T\left(p'\left(z_i, x_{-i}\right), z_i, x_{-i}\right) = 0$, and the right hand side of $IC_i(z_i, x)$ also decreases. In any case, the slack of $IC_i(z_i, x)$ increases with the new mechanism. □

In the rest of the article, I suppose that $\Lambda(x) = \Lambda_T\left(p\left(x\right), x\right)$. This proposition simplifies a lot the problem and allows me to focus on only one variable, the price function. Besides, it shows that the quantity lower bound is never violated in equilibrium. So this additional tool might only be used to control out of equilibrium behaviors. As in section 5, I consider the incentive constraint of one seller fixing x_{-i} and see how it restricts the price function $p(\cdot, x_{-i})$ that can be offered by the buyer. I first show that as in figure 6b, if seller i doesn't sell in x, he won't sell for any higher signal.

Proposition 8. *If* $p(x_i^0, x_{-i}) \leq v_i(x_i^0, x_{-i})$, *the incentive constraint of seller* i *implies that for all* $x_i > x_i^0$, $p(x) \leq v_i(x_i^0, x_{-i})$.

Proof. The proposition comes directly from $IC(x_i, x_i^0, x_{-i})$ when $x_i > x_i^0$. Indeed, if seller i makes no profit in (x_i^0, x_{-i}) and $p(x) > v_i(x_i^0, x_{-i})$, seller i would lie in (x_i^0, x_{-i}). He would report x_i and make positive profit because $\Lambda_T\left(p\left(x\right), \cdot, x_{-i}\right)$ decreases and, therefore, $\Lambda_T\left(p\left(x\right), x_i^0, x_{-i}\right) \geq \Lambda_T\left(p\left(x\right), x\right) = \Lambda(x)$. □

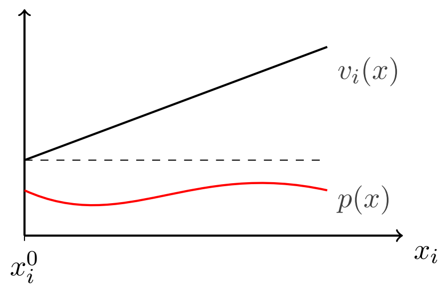


FIGURE 10: Price function with $p(x_i^0, x_{-i}) \leq v_i(x_i^0, x_{-i})$

The lower bound on the quantity of the transaction extends the possible price functions that the buyer can implement. $p(x)$ doesn't have to be flat for all signals where seller i sells anymore. Instead, the next proposition shows that the buyer is now able to implement a weakly decreasing step function in these points. Starting from $x_i = 0$ with $p_0 = p(0, x_{-i}) > v_i(0, x_{-i})$, the price must be constant for all signals where the number of shares tendered, $\Lambda_T(p_0, x)$, is constant. But as one seller drops out of $S(p_0, x)$, the buyer is able to set a new price p_1, weakly lower than the precedent. If seller i tenders his shares with this new price, the $p(x)$ must once again be constant until a new seller drops out of $S(p_1, x)$. This pattern carries on until seller i stops tendering his shares at the price proposed. Setting the right $\Lambda(x)$ is crucial to be able to set a p_1 strictly lower than p_0. Indeed, to prevent a seller i facing p_1 in x_i to report a lower z_i and get p_0, the buyer must set $\Lambda(z_i, x_{-i})$ higher than $\Lambda_T(p_0, x)$, the number of shares tendered by the sellers in x when they face p_0. This is only possible if the price change happens exactly when one seller drops out of $S(p_0, x)$.

Proposition 9. *If $p(0, x_{-i}) > v_i(0, x_{-i})$, the incentive constraint of seller i implies that there exist $K+1$ prices $p(0, x_{-i}) = p_0 \geq p_1 ... \geq p_K$ such that $p(x) = p_k \geq v_i(x)$ for all $x_i \in (x_i^k; x_i^{k+1}]$. Where $x_i^0 = 0$, and x_i^{k+1} is the first signal after x_i^k such that $S(p_k, x)$ strictly decreases. Moreover, $p(x) \leq v_i(x_i^{K+1}, x_{-i})$ for $x_i > x_i^{K+1}$*[24]*.*

Proof. Let $x \in X$ such that $p(x) > v_i(x)$. $IC(x_i', x_i, x_{-i})$ implies that $p(x) \geq p(x_i', x_{-i})$ for all $x_i' \geq x_i$. Indeed, if $p(x) < p(x_i', x_{-i})$, because $\Lambda_T\left(p\left(x_i', x_{-i}\right), \cdot, x_{-i}\right)$ decreases, $\Lambda_T\left(p\left(x_i', x_{-i}\right), x_i, x_{-i}\right) \geq \Lambda_T\left(p\left(x_i', x_{-i}\right), x_i', x_{-i}\right) = \Lambda(x_i', x_{-i})$, and seller i would lie in x, report x_i' and make a higher profit.

Together with $p(0, x_{-i}) > v_i(0, x_{-i})$ and the monotonicity of v_i, it implies that there is a $x_i^K > 0$ such that $p(x)$ is decreasing and strictly greater than $v_i(x)$ for all $x_i < x_i^K$.

[24] if $v_i(x_i^{K+1}, x_{-i}) = p_K$ any $p(x_i^{K+1}, x_{-i}) \leq v_i(x_i^{K+1}, x_{-i})$ is IC and the last interval is open.

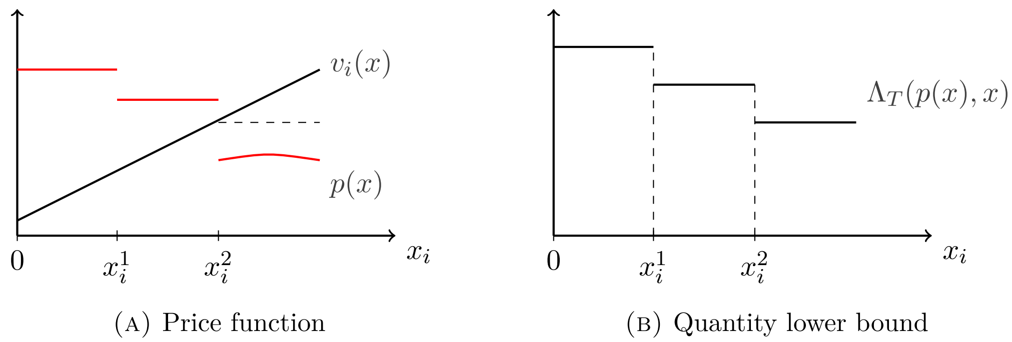


FIGURE 11: $p(0, x_{-i}) > v_i(0, x_{-i})$

This implies in turn that $\Lambda_T\left(p\left(\cdot, x_{-i}\right), \cdot, x_{-i}\right)$ decreases also on the same set. Moreover, it must be a decreasing step function because it takes its value in a discrete set. Therefore, there exist K different signals, $x_i^1, ..., x_i^K$, where it jumps.

$IC(x_i, x_i', x_{-i})$ implies that for all $x_i, x_i' \in (x_i^k, x_i^{k+1})$, $p(x) = p(x_i', x_{-i})$. Let $x_i' \geq x_i$. The first paragraph implies that $p(x) \geq p(x_i', x_{-i})$. $x_i, x_i' \in (x_i^k, x_i^{k+1})$ implies that $\Lambda_T\left(p\left(x_i', x_{-i}\right), x_i', x_{-i}\right) = \Lambda_T\left(p\left(x\right), x\right) = \Lambda(x)$. Moreover, if $p(x) > p(x_i', x_{-i})$, $\Lambda_T\left(p\left(x\right), x_i', x_{-i}\right) \geq \Lambda_T\left(p\left(x_i', x_{-i}\right), x_i', x_{-i}\right)$ and seller i would lie in (x_i', x_{-i}), report x_i and make a higher profit.

Let p_k be the price on (x_i^k, x_i^{k+1}). x_i^{k+1} is the last signal after x_i^k such that $\Lambda_T(p_k, \cdot, x_{-i})$ is constant. Otherwise, if there is $x_i < x_i^{k+1} < x_i'$ such that $\Lambda_T(p_k, x_i', x_{-i}) = \Lambda_T(p_k, x) = \Lambda(x)$, $p(x) = p_k > p(x_i', x_{-i})$ because $\Lambda_T\left(p\left(\cdot, x_{-i}\right), \cdot, x_{-i}\right)$ jumps in x_i^{k+1}. Therefore, seller i would lie in (x_i', x_{-i}), report x_i and make a higher profit.

Finally, if $p(x_i^{k+1}, x_{-i}) > v_i(x_i^{k+1}, x_{-i})$, $p(x_i^{k+1}, x_{-i}) = p_k$. Otherwise, $x_i \in (x_i^k, x_i^{k+1})$ implies that $\Lambda_T(p_k, x_i^{k+1}, x_{-i}) = \Lambda(x)$. If $p_k > p(x_i^{k+1}, x_{-i})$, seller i would lie in (x_i', x_{-i}), report x_i and make a higher profit. □

As in the previous section, I show that the conditions stated above on $p(\cdot, x_{-i})$ and $\Lambda(\cdot, x_{-i})$ imply that for all $x \in X$ and $z_i \in X_i$, $IC_i(z_i, x)$ is verified.

Proposition 10. *The conditions stated in propositions 7, 8, and 9 are sufficient conditions for the mechanism to be incentive compatible.*

Proof. If seller i reports $z_i > x_i$, he will get a lower price and his profit will decrease. If seller i reports $z_i < x_i$ such that $p(z_i, x_{-i}) > p(x)$, $\Lambda_T\left(p\left(z_i, x_{-i}\right), \cdot, x_{-i}\right)$ will strictly decrease between z_i and x_i. Therefore, $\Lambda(z_i, x_{-i}) = \Lambda_T\left(p\left(z_i, x_{-i}\right), z_i, x_{-i}\right) > \Lambda_T\left(p\left(z_i, x_{-i}\right), x\right)$ and the number of shares tendered will be below the threshold set by the buyer. The profit will be 0 for everybody. □

The mechanism spots the lie of the seller through the other sellers' behaviors. The lower bound on the quantity tendered works like a threat on the seller. It cancels the sale if the reports contradict the number of shares tendered. The threat loosens the restrictions imposed on the price by the incentive constraints compared with the previous section. But these constraints are still linked through the price function. Applying proposition 9 in x for all sellers in $S(p(x),x)$, I can narrow down the possible price functions on entire areas of the signals' space. In general, the price must be constant on every subspace where the set of sellers and the signals of non-sellers are constant.

Lemma 3. *For all $x \in X$, $J \subset S\left(p\left(x\right),x\right)$ and $x'_J \in X_J$: $p(x) = p(x_{N-J}, x'_J)$, if there is $k(\cdot)$, an ordering of the sellers in J, such that for all $1 \leq j \leq |J|$:*

$$S\left(p\left(x\right),x\right) = S\left(p\left(x\right), x_{N\backslash J}, x_{k(1)}, .., x_{k(j-1)}, x'_{k(j)}, ..., x'_{k(|J|)}\right)$$

$$v_{k(j)}(x_{N\backslash J}, x_{k(1)}, .., x_{k(j-1)}, x'_{k(j)}, ..., x'_{k(|J|)}) < p(x)$$

Proof. The proof consists of iteratively using figure 11a for seller $k(|J|)$ until $k(1)$ along the path $(x), (x_{N\backslash k(|J|)}, x'_{k(|J|)}), ..., (x_{N\backslash\{J\backslash k(1)\}}, x'_{k(2)}, ..., x'_{k(|J|)}), (x_{N\backslash J}, x'_J)$. Figure 11a and the following equations imply that $p(x) = p(x_{N\backslash k(|J|)}, x'_{k(|J|)})$

$$k(|J|) \in S\left(p\left(x\right),x\right) = S\left(p\left(x\right), x_{N\backslash k(|J|)}, x'_{k(|J|)}\right)$$

$$v_{k(|J|)}(x_{N\backslash k(|J|)}, x'_{k(|J|)}) < p(x)$$

Similarly, figure 11a together with the following equations

$$v_{k(|J|-1)}(x_{N\backslash k(|J|)\backslash k(|J|-1)}, x'_{k(|J-1|)}, x'_{k(|J|)}) < p(x)$$

$$\begin{aligned} k(|J|-1) \in S\left(p\left(x\right),x\right) &= S\left(p\left(x\right), x_{N\backslash k(|J|)}, x'_{k(|J|)}\right) \\ &= S\left(p\left(x\right), x_{N\backslash k(|J|)\backslash k(|J|-1)}, x'_{k(|J-1|)}, x'_{k(|J|)}\right) \end{aligned}$$

imply that $p(x) = p(x_{N\backslash k(|J|)\backslash k(|J|-1)}, x'_{k(|J-1|)}, x'_{k(|J|)})$. Proceeding like this until $k(1)$ shows that $p(x) = p(x_{N\backslash J}, x'_J)$. □

This lemma is the equivalent of lemma 1 in the previous section. The first item of corollary 5 still holds with the additional freedom given to the buyer.

Corollary 9. *For all $x \in X$ such that* $\max_i\{v_i(x)\} < p(0)$, $p(x) = p(0)$.

Contrary to the corollary 5 of the previous section, if a seller is weakly more pessimistic than any other seller with a signal equal to 0, the information of i may be used to set the price of some transactions. Therefore, allowing players to communicate before the buyer posts the price may be useful in the first example when $\alpha_i^i \leq \alpha_j^i$ and in the second example. As in the previous section, I suppose that the valuations are symmetric on the axes' intersections to show that the price decreases as the number of non-zero signals increases.

Lemma 4. *If the valuations are symmetric on the axes' intersections and seller i sells in $x \in X$, then every seller j sells in $(0, x_{-\{i,j\}})$ and, therefore, $p(0, x_{-\{i,j\}}) \geq p(0, x_{-i}) \geq p(x)$.*

Proof. The proof is almost identical to the proof of lemma 2. If seller i sells in x, applying proposition 9 gives that $p(0, x_{-i}) \geq p(x) \geq v_i(x) \geq v_i(0, x_{-\{i,j\}}) = v_j(0, x_{-\{i,j\}})$, where the last equality comes from the symmetry on the axes' intersections. But if $p(0, x_j, x_{-\{i,j\}}) = p(0, x_{-i}) \geq v_j(0, x_{-\{i,j\}})$, proposition 8 implies that seller j must sell in $(0, x_{-\{i,j\}})$. Applying once again proposition 9 along the j axis, gives $p(0, x_{-\{i,j\}}) \geq p(0, x_{-i}) \geq p(x)$. □

The properties stated in the corollary 6 are still valid in this problem.

Corollary 10.

- *For all $x \in X$ such that a sale takes place, $p(x) \leq p(0)$.*
- *For all $x \in X$ such that $\min_i v_i(x) > p(0)$, no sale takes place.*

Compared to the previous section, the main change is the definition of the region where the price is constant and where the buyer can use the sellers' information. The lower bound on the total quantity of the transaction increases the areas where the buyer can price discriminate. As in the previous part, I restrict to the case where $n = 2$ in the core of this article. The generalization is not presented in the appendix but follows the same lines as the one of the previous section. The buyer can now price discriminate on $U_i(p^*(0)) \setminus U_j(p^*(0))$. On this space, he proposes a lower price to i using seller j's signal. Figures 15 and 17 present the regions of the signals' space where the mechanism imposes different restrictions on the price structure for the two examples. The gridded areas represent $U_1(p^*(0)) \cap U_2(p^*(0))$, the regions where the price must be equal to the price in the origin. The vertically hashed areas represent $U_2(p^*(0)) \setminus U_1(p^*(0))$, the regions where the buyer can use seller 1's information to post a lower price to seller 2. The horizontally hashed areas represent $U_1(p^*(0)) \setminus U_2(p^*(0))$, the regions where the buyer can use seller 2's information to post a lower price to seller 1. Finally, in the remaining areas, the sale never happens.

Proposition 11. *If the problem is regular for every seller j given X_i and the valuations are symmetric on the axes' intersections, the optimal mechanism is such that $p^*(0) > v(0)$ and:*

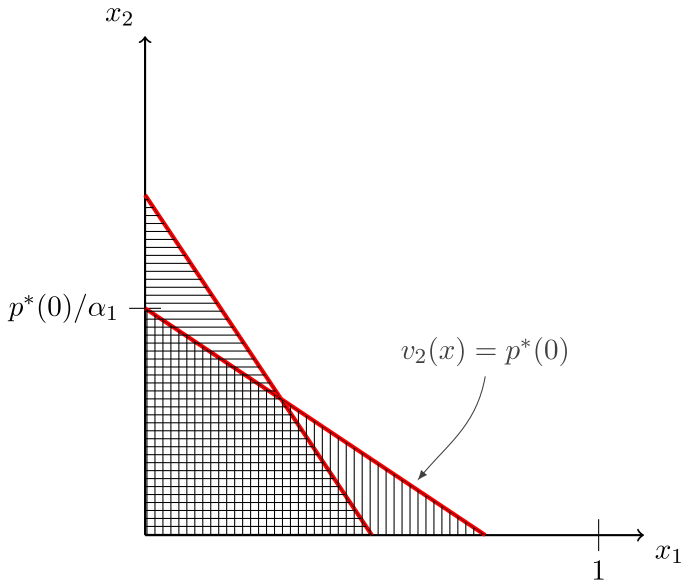


FIGURE 12: Linear utilities with a uniform price and a demand lower bound

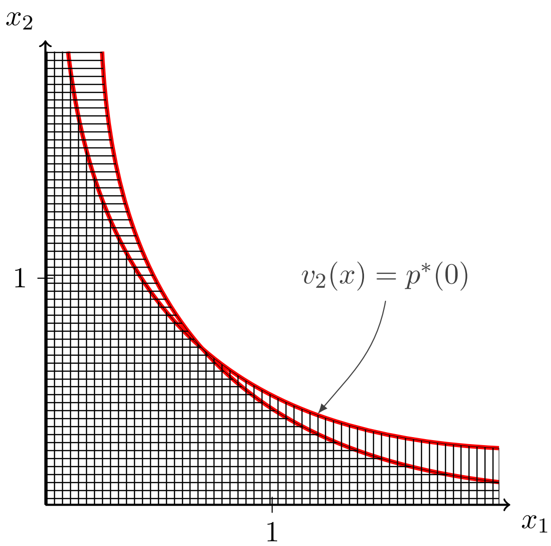


FIGURE 13: Log-linear utilities with a uniform price and a demand lower bound

- *In* $x \in U_1(p^*(0)) \cap U_2(p^*(0))$, *the buyer posts* $p^*(x) = p^*(0)$, $\Lambda^*(x) = \lambda_1 + \lambda_2$ *and both sellers sell.*
- *In* $x \in U_j(p^*(0)) \setminus U_i(p^*(0))$ *the buyer posts* $p^*(x) = \min\{p^*(0), p_j(w, p^*(0), x_i)\}$, $\Lambda^*(x) = \lambda_j$ *and only seller* j *may sell.* $p_j(w, p, x_i)$ *is the optimal price for* j *knowing* x_i *and* $v_i(x_i, x_j) \geq p$. *In particular,* $p_j(w, p, x_i) = p_j^0(w, x_i)$ *if* $x \in D_j(p)$ *and* $p_j(w, p, x_i) > p_j^0(w, x_i)$ *otherwise.*
- *In* $x \notin U_1(p^*(0)) \cup U_2(p^*(0))$, *nobody sells. In particular, it is optimal for the buyer to post* $p^*(x) = 0$ *and* $\Lambda^*(x) = 0$.

Proof. The proof of this proposition follows the same lines as the proof of proposition 5. Let $p(x)$ be an incentive compatible price function. Proposition 7 implies that the buyer can restrict to mechanisms where $\Lambda(x) = \Lambda_T(p(x), x)$. Moreover, the second item of corollary 10 implies that nobody sells in $x \notin U_2(p(0)) \cup U_1(p(0))$. Therefore, the set of possible price functions can be restricted to $\mathcal{C}'$, the set of functions from X to $\mathbb{R}$ such that for all $x \notin \cup_i U_i(p(0))$, $p(x) = 0$. And the problem reduces to:

$$\max_{p(x) \in \mathcal{C}'} \int_{\cup_i U_i(p(0))} \Lambda_T(p(x), x)[w - p(x)] f(x) dx$$

$$\text{s.t. for all } i \leq N,\ x \in X,\ z \in X_i :$$

$$\mathbb{1}_{\{p(x) \geq v_i(x)\}}[p(x) - v_i(x)] \geq \mathbb{1}_{\{p(z, x_{-i}) \geq v_i(x)\}}[p(z, x_{-i}) - v_i(x)]$$

Let $p(x)$ be an incentive compatible price in $\mathcal{C}'$. Let x be in the interior of $U_1(p(0)) \cap U_2(p(0))$. Corollary 9 implies that $p(x) = p(0)$. The border of $U_1(p(0)) \cap U_2(p(0))$ is of mass 0 because $f(\cdot)$ has no mass point[25]. Therefore, the price charged in these points is irrelevant as long as the resulting price function is incentive compatible. In particular, setting $p(x) = p(0)$ in these points is incentive compatible. The buyer can then reduce the set over which he maximizes to functions that are constant on $U_1(p(0)) \cap U_2(p(0))$.

Let $x \in U_j(p(0)) \setminus U_i(p(0))$. If the buyer buys from i, lemma 4 implies that $p(0) \geq p(0, x_j) \geq p(x) \geq v_i(x)$ which contradicts $x \notin U_i(p(0))$. Hence, seller i does not tender his shares in $U_j(p(0)) \setminus U_i(p(0))$. Moreover, if j sells in x and $(x_i, x_j') \in U_j(p(0)) \setminus U_i(p(0))$, lemma 3 implies that $p(x) = p(x_i, x_j')$ and lemma 4 implies that $p(x) = p(x') \leq p(x_i, 0) \leq p(0)$. Therefore, I can further reduce the set over which the buyer maximizes.

[25] If it were not the case, the property would not fundamentally change, but the border would have to be studied separately and new cases distinctions introduced.

Splitting the problem on these three regions, I can rewrite the problem as:

$$\max_{\substack{p(0)\in\mathbb{R}\\ p_2(x_1)\leq p(0)\\ p_1(x_2)\leq p(0)}} \left\{ (\lambda_1+\lambda_2)\int\limits_{x\in U_1(p(0))\cap U_2(p(0))} [w-p(0)]f(x)dx + \lambda_1 \int\limits_{x\in U_1(p(0))\setminus U_2(p(0))} \mathbb{1}_{\{p_1(x_2)\geq v_1(x)\}}[w-p_1(x_2)]f(x)dx \right.$$

$$\left. +\lambda_2 \int\limits_{x\in U_2(p(0))\setminus U_1(p(0))} \mathbb{1}_{\{p_2(x_1)\geq v_2(x)\}}[w-p_2(x_1)]f(x)dx \right\}$$

Maximizing first over $(p_2(x_1), p_1(x_2))$ given $p(0)$ and taking the maximum over $p_j(x_i)$ in the integral over x_i, I can rewrite the problem as:

$$\max_{p(0)\in\mathbb{R}} \left\{ (\lambda_1+\lambda_2)\int\limits_{x\in U_1(p(0))\cap U_2(p(0))} [w-p(0)]f(x)dx \right.$$

$$+\lambda_1 \int\limits_{x_2\in X_{U1\setminus U2}(p(0))} \max_{p_1(x_2)\leq p(0)} [w-p_1(x_2)][F_1(x_1(p_1(x_2),x_2),x_2) - F_1\left(x_1^2(p(0),x_2),x_2\right)]^+ dx_2 \tag{3}$$

$$\left. +\lambda_2 \int\limits_{x_1\in X_{U2\setminus U1}(p(0))} \max_{p_2(x_1)\leq p(0)} [w-p_2(x_1)][F_2(x_2(p_2(x_1),x_1),x_1) - F_2\left(x_2^1(p(0),x_1),x_1\right)]^+ dx_1 \right\}$$

Where:

$$X_{Uj\setminus Ui}(p(0)) = \{x_i\in X_i : \exists x_j\in X_j : x\in U_j(p(0))\setminus U_i(p(0))\}$$
$$x_j^i(p(0),x_i) = \sup\{x_j\in X_j : v_i(x)\leq p(0)\}$$

Dividing the objective of the two subproblems by $F_j\left(x_j^i(p(0),x_i),x_i\right) = G_i(p(0)\mid x_i)$, they become:

$$\max_{p_j(x_i)\leq p(0)} [w-p_j(x_i)]G_j(p_j(x_i)\mid x_i, v_i\geq p(0)) \tag{4}$$

It is the same problem as the unregulated one stated in proposition 3 where the distribution of $v_j \mid x_i$ is replaced by the distribution of $v_j \mid x_i, v_i \geq p(0)$. Therefore, defining $p_j(w,p,x_i)$ as the optimal price for j knowing x_i and $v_i(x_i,x_j)\geq p$ implies that $p^*(x) = \min\{p^*(0), p_j(w,p^*(0),x_i)\}$. The regularity of the problem for every seller i given X_j implies that for all $x\in D_j(p)$, $p_j(w,p,x_i) = p_j^0(w,x_i)$. Otherwise, because $G_i(p(0)\mid x_i) > 0$ and because the FOC of the original problem crosses 0 only once from above in $p_j^0(w,x_i)$, the FOC of the maximization problem is strictly negative for all $p\leq p_j^0(w,x_i)$. Therefore, $p_j(w,p,x_i) > p_j^0(w,x_i)$. □

The generalization of proposition 11 to more than two sellers follows the same principle as the one presented in the appendix for proposition 5. In the regions where some sellers don't sell, the buyer uses their information to post a lower price to the rest. The definition of the subspace is the direct generalization of the one presented here. However, to properly write the upper bound on the price in the generalization of problem 4, I would have to introduce cumbersome notations to generalize the function $x_j^i\left(p\left(0\right), x_i\right)$. This is the main difference with the proof of the previous section and the reason why I am not presenting it in the appendix. For the same reason as in the previous section, I can enlarge the set of valuations for which property 11 holds when $n = 2$. As before, the generalization of proposition 12 to the case where $n > 2$ is beyond the scope of this paper.

Proposition 12. *When $n = 2$, if the problem is regular but $v_2(0) > v_1(0)$:*

- *Proposition 11 holds if and only if the solution of problem 3 is strictly greater than $v_2(0)$. Otherwise the buyer posts $p^*(x) = \min\{v_2(0), p_1^0(w, x_2)\}$ and only 1 may sell.*
- *In particular, proposition 11 holds if $p_1(w, x_2) > v_2(0)$ or if w is big enough and $v_1(\bar{x}, 0) > v_2(0)$.*

Proof. The proof of the first item is exactly the same as the proof of the first item of proposition 6. As before, if $v_2(0) \geq p(0)$, proposition 8 and 9 imply that seller 2 cannot sell in x for all $x_2 > 0$. But if only one seller sells, there is no difference between the two sections for $v_2(0) \geq p(0)$. The rest of the proof holds because problem 3 is also continuous in $p(0)$ equal Π_1 in $v_2(0)$ and is increasing before.

The proof of the second item is also almost the same as in the previous section. x_l is defined in the same manner. If $p(0) = v_2(0, x_l)$, for $x_2 < x_l$, and $x_1 \geq x_1^2\left(v_2(0, x_l), x_2\right)$, $p(x) = \min\{v_2(0, x_l), p_1(w, v_2(0, x_l), x_2)\} = v_2(0, x_l)$ because

$$p_1(w, v_2(0, x_l), x_2) \geq p_1^0(w, x_2) > v_2(0, x_l)$$

Therefore, the prices posted are the same as in the previous section for $p(0) = v_2(0)$ and for $p(0) = v_2(0, x_l)$. So the profit are the same and the proof of the previous section holds. □

The optimal $p^*(0)$ is still complicated to derive even for the examples presented in this paper. The corollary 7 is still verified. Besides, the set of valuations for which beforehand communication makes a difference strictly increases when the buyer can set a lower bound on the quantity of the transaction.

Corollary 11. *If the optimal mechanism doesn't use the reports, the buyer posts p^* on the entire signals' space as is section 4.1 with constant price. In particular, the reports are used if the buyer can discriminate when $p(0) = p^*$ or if it is already the case without lower bound in section 5.*

Proof. If the buyer can price discriminate when $\Lambda(x) = 0$, $\Pi\left(p^*\left(0\right)\right) > \Pi_{nc}(p^*)$ which is impossible if there is no price discrimination. □

Corollary 8 still holds. The next proposition gives also some sufficient conditions for the price at the origin to be greater that the optimal constant price. These conditions open, therefore, some room for the welfare to increase. It generalizes some special cases of the examples presented in this paper. I denote $W_i(p)$ seller i's signals such that discrimination takes place when $p(0) = p$. It corresponds to the subset of $X_{Uj\setminus Ui}(p)$ where $p_j(w, p, x_i) < p$.

Proposition 13.
- *If $p^*(0) \leq p^*$, allowing beforehand communication reduces welfare.*
- *When $n = 2$, $p^*(0) > p^*$ if for all $p \leq p^*$ and $x_i \in W_i(p)$: $G_j(\cdot \mid x_i, v_i \geq p)$ is log-concave and if there is $\bar{v} \in (p_j(w, p, x_i); p]$ such that $g_j(\bar{v} \mid x_i) \geq \max\{g_j(p \mid x_i), g_i(p \mid x_i)\}$.*

Proof. The proof of the first item is the same as in the previous section. The proof of the second item can be found in the appendix. I provide only an intuition here. The first condition ensures that the problem stated in equation (4) has a unique and interior solution. Raising the price at the origin $p(0)$ has two contradictory effects on the size of the region where the buyer discriminates using i's information. It increases it through the increase of $U_j(p(0))$ and decreases it through the increase of $U_i\left(p\left(0\right)\right)$. The second condition ensures that the positive effect wins and that the overall size of the region increases. □

The conditions of the second item are verified in the two examples when signals are not too negatively correlated and each seller trusts the other's signal more than his own. It corresponds to the cases where α_1 or β_1 are below 1/2 and w is low enough [26].

[26] In these cases, the distribution of $v_i \mid x_i$ is a spread (not necessarily mean preserving) of the distribution of $v_j \mid x_i$. $g_j(v \mid x_i)$ is above $g_i(v \mid x_i)$ if and only if $v \in [v_l(x_i); v_h(x_i)]$ and both are decreasing after $v_h(x_i)$. Moreover, $x_i \in W_i(p)$ if and only if $G_j(p \mid x_i) \geq G_i(p \mid x_i)$ and, therefore, only if $p \geq v_l(x_i)$. Finally, if w is low enough, $p_j(w, p^*, x_i) \leq v_h(x_i)$ for all x_i because $G_j(\cdot \mid x_i, v_i \geq p)$ is log-concave

Example 10. *As in the previous section, I focus on the symmetric case. It can be extended to asymmetric cases as long as* $\min\{\alpha_j^j, w - \alpha_0^i\} \geq \alpha_0^i - \alpha_0^j \geq -\min\{\alpha_i^i, w - \alpha_0^j\}$*. Figure 14 presents the set of parameters* (α_1, w) *such that the buyer can use the information of one seller to post a price to the other. In the gridded region, the buyer can use price discrimination without lower bound. Corollary 11 ensures that I can do it also with a lower bound on a bigger set of signals. The vertically hashed region represents the parameters for which the buyer can use price discrimination only if he can set a lower bound on demand. For parameters in the shaded area, the buyer posts* p^* *for any signals because* $p_j(w, x_i) > p^*(0)$ *on the set of signals where the buyer could price discriminate.*

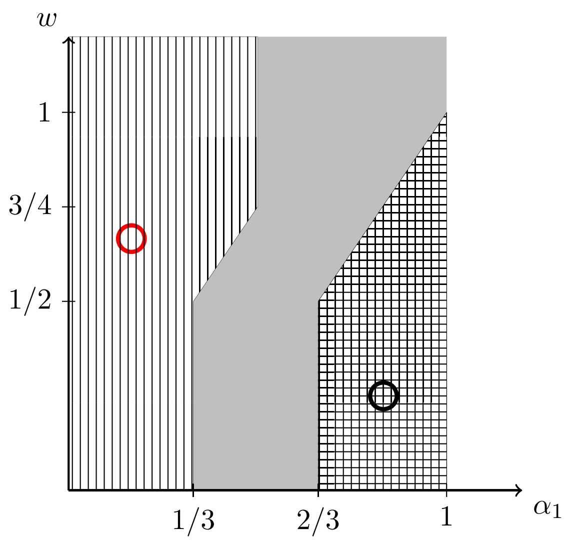


FIGURE 14: Parameters and discrimination with linear utilities

In the gridded area, the consequences of allowing a strictly positive lower bound are very limited. The black circle in figure 14 shows the parameters of figure 8. In this last figure, the area where the lower bound allows additional price discrimination is the small triangle between the shade and the gridded area. The increases of the price in the origin and of the profit are negligible.

On the contrary, the effects of allowing a strictly positive lower bound can be quite large in the vertically hashed area of figure 14. Figure 15 represents the optimal mechanism when the parameter are $\alpha_1 = 1/6$ *and* $w = 2/3$*, i.e. in the red circle of figure 14. In the gridded area, the buyer buys from both sellers at price* $p^*(0) = 0.422 > 0.375 = p^*$ *and* $\Lambda(x) = 2$*. In the shaded and vertically hashed area, the buyer buys from seller 2* *at price* $p^*(x) = \alpha_1 + (1-\alpha_1)x_1$ *and* $\Lambda(x) = 1$. *In the clear vertically hashed area,* $p_2(w, p^*(0), x_1) > p^*(0)$ *and the buyer buys from seller* 2 *at price* $p^*(0)$. *Symmetrically, the buyer buys from seller* 1 *at price* $p^*(x) = \alpha_1 + (1-\alpha_1)x_2$ *in the shaded horizontally hashed area and at price* $p^*(0)$ *in the remaining horizontally hashed area.*

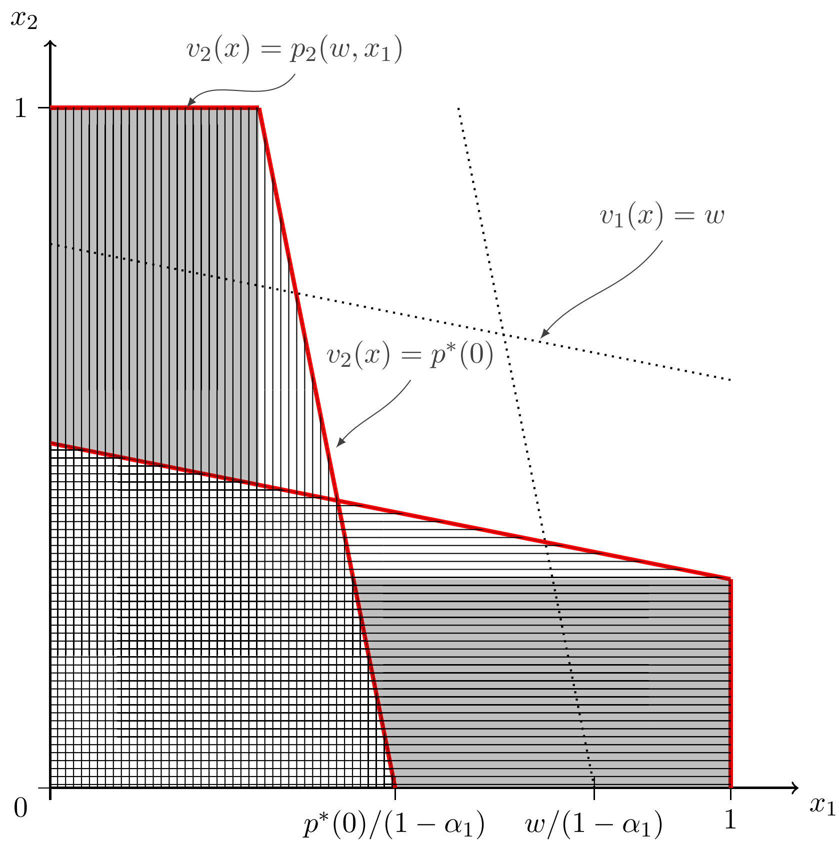


FIGURE 15: Linear utilities with a uniform price and a demand lower bound

Figure 16 represents the profit of the buyer as a function of the price at the origin. Compared to the profit with a constant price, the expected profit for the buyer in this new mechanism is 17% *higher whereas its potential gain from removing all regulation amounts to* 52%. *Contrary to the previous section, in this case, the price in* 0 *rises enough to compensate the sellers' surplus losses on the region where the buyer can price discriminate. Compared to the constant price mechanism, the surplus of the sellers is* 77% *higher whereas when there is no regulation it falls by* 15%. *Finally, summing both effects, the total welfare is* 37% *higher than with a constant price whereas when there is*

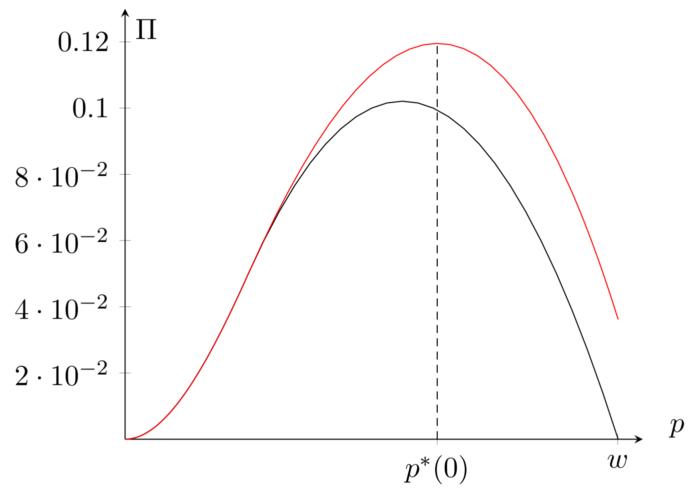


FIGURE 16: Profit with a uniform price and a demand lower bound

no regulation it rises only by 29%. *Asymmetries and correlation have the same effects as in the previous section.*

Example 11. *As in the first example, the new mechanism differs from the one with a constant posted price only when the valuation functions are asymmetric enough. I cannot explicitly derive the price function with no regulation and, therefore, neither do it in the case studied in this section. However, I can calculate numerically the price without communication* p^*. *If for* $p(0) = p^*$, *the contour plot of the first order condition for* $p_j(w, p^*, x_i)$ *passes through* $U_j(p^*) \setminus U_i(p^*)$, *corollary 11 implies that the buyer uses the information of* i *to set the price for* j. *When* $\mu_1 = 0$ *and* $\sigma_1 = 1$, *this is the case if* $\beta_1 < 1/3$ *and* $\rho > 0$, *or if* $\beta > 2/3$ *and* $w < 1/2 - 3/2(1-\beta_1)$ *or if* $\rho \in (0, 2\beta - 1)$. *When the parameters verify one of these conditions, the graph describing the mechanism is similar to figure 15 or figure 8. On the contrary, when* $\beta_1 > 1/2$ *and* $\rho < -(1-\beta)/\beta$, *the information of one seller is negative for the others. Moreover,* $x_j^i(p, x_i)$ *goes to* 0 *when* x_i *goes to infinity. I can rewrite the proof of corollary 7 using limits and show that the buyer uses price discrimination for high values of* x_i. *Figure 17 shows that discrimination is possible when* $p^*(0) = p^*$, $\beta_1 = 5/6$, $w = 2$ *and* $\rho = -3/4$. *In this case, the optimal price without communication is* $p^* = 1.31$.

These two examples illustrate how the buyer can use the quantity lower bound as a threat to punish the inconsistent behaviors triggered by a lying seller. It shows also how it allows more flexible prices in comparison to the previous cases. Besides, it highlights that two phenomena drive the optimal mechanism away from the optimal constant posted

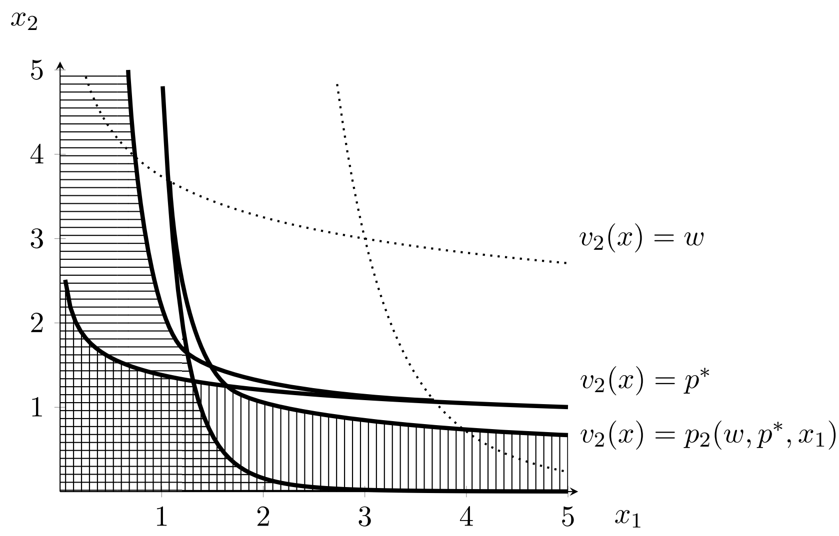


FIGURE 17: Log-linear utilities with a uniform price and a demand lower bound

price. First, it is necessary that the sellers' valuations be not too correlated. Otherwise, the sellers have almost a common value and the buyer in a sense faces only one market. He either buys from all sellers or from nobody. Therefore, price discrimination requires valuations that are asymmetric or private enough with not too correlated signals. But being able to distinguish the sellers is not enough. The price posted must be a function of the sellers or of their signals. If the valuations are symmetrically and independently distributed, the information gathered is worthless. The buyer always posts the same price. As in the classical price discrimination, the buyer posts different prices for different sellers if the valuations are independent but asymmetrically distributed. The price is a function of the signals if the valuations are correlated with the signals of the others. In the next section, I first show the optimal mechanism can be implemented and then come back to my motivating example. I also explain how the results can be extended to weaker restrictions on quantity discrimination and address the law in force in the US

7 Implementation and Extension

In this section, I present a simple way to implement the optimal mechanism with a uniform posted price and quantity lower bound when there are only two sellers. This implementation allows me to discuss the consequences of additional restrictions imposed by the member states of the European Union. In a last part, I will try to see how the result can be used to study the consequences of additional discrimination tools allowed in the United States.

Proposition 14. *If n=2, the optimal buying mechanism with a uniform price and a demand lower bound can be implemented sequentially as follows:*

1. *The buyer posts a price* p_0*. If all sellers or no seller tender their shares, the process ends and the buyer buys the tendered shares.*
2. *If exactly one seller doesn't tender his shares, the buyer collects his signal and uses it to post a new price,* p_1*, lower than the first one. The process ends and the buyer buys the tendered shares.*

Proof. I will show that, in equilibrium, the buyer posts the same price as in the previous section and the sellers tender their shares exactly when the price proposed is above their valuations. In the second round, every seller tenders his shares if and only if his valuation is lower than the last posted price. I will show that this is also the case in the first round. If $v_i > p_0$, seller i never makes a positive payoff because $p_0 \geq p_1$. He won't, therefore, tender his shares directly. If $v_i \leq p_0$ and $v_j \leq p_0$, seller i will tender his shares directly because $p_0 \geq p_1$. If $v_i \leq p_0$ and $v_j > p_0$, he may end up with a positive payoff if he tenders his shares whereas the sale will end directly without transaction if he doesn't. Hence, if a seller doesn't tender his shares in the first round, he won't do it in the second one. He will, therefore, report his information truthfully at the beginning of the second round. I will now study the equilibrium strategy of the buyer. In the second round, given that he posted p_0 and that j tendered his shares in the first round, he posts a price which solves the following problem:

$$\max_{p_j(x_i) \leq p_0} [w - p_j(x_i)] G_j\left(p_j\left(x_i\right) \mid x_i, v_i \geq p_0, v_j \leq p_0\right)$$

Multiplying the objective by $G_j(p_0 \mid x_i, v_i \geq p_0)$ gives the same as the one of problem 4. Therefore, $p_1 = p_j(w, x_1, p_0)$. Finally, given the equilibrium strategies in the second round and the equilibrium strategies of the sellers in the first round, the buyer solves problem 3 in the first round. Therefore, $p_0 = p^*(0)$. □

Generalizing this proposition to more than two sellers requires additional work. The prices that can be posted in the second round may have an upper bound strictly lower than p_0. In particular, if the number of sellers not tendering their shares belongs to $\{2, ..., n-1\}$, this upper bound will depend on these sellers' identities and information.

The price posted must be below all prices that the buyer would post if an additional seller had tendered his shares. Therefore, the implementation may be facilitated if the buyer can exclude only one seller in each round.

This implementation process has the big advantage to require little commitment power from the buyer. Section 3 requires that the information exchange between the players be contractible and that the buyer can commit to use predefined price and lower bound functions. In comparison, in proposition 14, the information doesn't need to be contractible. Like in a usual procurement open auction, the buyer must only commit to post descending prices because the regulation requires already the buyer to post uniform prices. The report written by Marccus Partners and the Center for European Policy Studies (2012) explicitly mentions that the directive allows upward and downward revisions of the price and the acceptance level after an offer lapses. But national supervisory authorities can impose restrictions on this matter. Posting sequentially decreasing levels of acceptance is generally allowed by the European member states' regulators. Actually, according to Bonellierede et al. (2016), in France, Italy, Germany, the Netherlands and the UK, the buyer must lower his level of acceptance if he wants to make a new offer when the previous one lapsed. However, in all member states of the European Union studied by these law firms, regulations explicitly forbid the buyer to post sequentially decreasing prices. The implementation presented above is, therefore, ruled out. The consequences of these additional regulations at the national level crucially depend on the commitment power that the buyer may have. If the information exchange between the players is contractible and the buyer can commit to use predefined price and lower bound functions, the regulation has no bite. If the buyer decides to make an offer, the level of acceptance will be reached at the price posted. If he cannot commit to the price and level of acceptance functions, he has to give away potential profit and post the constant price presented in subsection 4.1. This feature also provides a justification for the importance of intermediaries in such transaction. Indeed, a bank specialized in such transaction may have more commitment power due to reputation effects.

It is also interesting to see what would happen if the regulator allowed further quantity discrimination. In particular, what would happen if the buyer could make a partial offer as in the United States. In a partial offer, in addition to his price and level of acceptance, the buyer submits a maximum amount of shares that he is ready to buy. Contrary to the level of acceptance, the sale is not cancelled when the upper bound is reached. The buyer must purchase the shares tendered on a "pro rata" basis. If Λ_T is greater than the announced upper bound $L(x)$, each seller tendering his shares will be able to sell $\lambda_i * \Lambda_T(x)/L(x)$. Such a tie-breaking rule doesn't change the IR constraints. The sellers tender all their shares as soon as the price proposed is above their valuations. On the contrary, it may weaken the IC constraints by reducing the attractiveness of higher prices. If a seller lies to trigger a higher price, more sellers than expected will tender their shares. If the upper bound is set to the expected amount of tendered shares, the sellers won't be able to sell all their shares. Therefore, they will receive a lower payoff than without upper bound. This fact may allow the buyer to enlarge the set of possible prices presented in figure 11. A price increasing in a seller's signal may be incentive compatible. The buyer could for instance choose a mechanism resembling the one presented in figure 8 with an additional smaller fixed price around 0. The mechanisms resulting from allowing such an additional tool are too complicated to be treated in this article. But its results shed some light on what they could look like.

8 Conclusion

This article addresses the problem of a potential buyer facing many sellers with private signals and interdependent valuations. Supposing that the buyer considers only mechanisms respecting ex-post constraints, it studies the effect of imposing a uniform pricing rule with different restrictions on quantity discrimination. It first derives the optimal mechanisms in the two extreme cases where there is no regulation and where the buyer must post a constant price for all signals' realizations. These two benchmarks situate the problem in the literature of mechanism design and price discrimination in a monopoly setting. They offer also interpretation and derivation tools used throughout the paper. The first contribution of this paper is to derive, in proposition 5, the optimal mechanism when the buyer must post a uniform price to all sellers. In comparison to the benchmark with the constant price, the price may depend on the sellers' private information in some regions of the signals' space. I provide in corollaries 5 and 7 sufficient and necessary conditions so that the buyer can profitably extract information from the sellers. All the results rely on a regularity assumption and a weak symmetry assumption. The first one is standard and its relaxation has been studied extensively in the mechanism design literature. In proposition 6, I show how the latter can be relaxed in the case of two sellers.

Using similar techniques, I am able to extend the study to the case where the buyer can set a lower bound on the total quantity he is ready to buy. The optimal mechanism, presented in proposition 11, simply uses the information of the agents, who are not selling, to post a price to the rest. The lower bound on quantity is never violated in equilibrium. It works as a threat that cancels the sale if the reports indicate a higher number of shares that the one tendered. This threat enables the buyer to post a price that increases when the number of tendered shares increases. In corollary 11, I show that this additional tool enlarges the set of valuation functions and signals' distributions where the buyer can collect meaningful information from the sellers. Even if discrimination is possible without a lower bound on the demand, it can help to extract information on a bigger set of signals. In the case of two sellers, the mechanism can be implemented by a sequential game where the buyer collects the information of the sellers who opted out in the previous round and post a new price lower than the previous one. This implementation doesn't require contractible information or strong commitment power.

The paper addresses also the comparison of these mechanisms in terms of efficiency. As it is often the case in price discrimination settings, efficiency can increase or decrease with price discrimination. I present a simple test to compare the efficiency of the uniform posting price mechanisms. Two examples illustrate the mechanisms and show that the most efficient mechanism can be in between the constant price and the unregulated mechanism.

Finally, the results are used to get insight on the consequences of the takeover regulation in the European Union. First, the efficiency analysis shows that forbidding price discrimination is not " takeover deterrent ", contrary to the claim of the economic report to the commission. Analyzing the consequences in terms of quantity traded would require to calibrate the distribution of valuations and signals on real world data. Moreover, the paper provides a justification for the apparent simplicity of the real world takeover processes. As in the optimal mechanism, it seems that the offered price rarely rises after an offer lapses. On the other hand, regulations on the national level prevent price decrease. Together with weak commitment power, and contracting limitation, it provides a justification for the importance of intermediaries and the fact that sellers rarely communicate their valuations to the buyer. The paper provides also insight on what might be the consequences of a different regulation in forced in the United States.

9 Appendix

Proof. **Corollary 2**. Let v_i belong to the support of $v_i|x_{-i}$.

$$\frac{dx_i}{dv}(v,x_{-i}) = \frac{1}{\frac{dv_i}{dx_i}\left(x_i\left(v,x_{-i}\right),x_{-i}\right)} > 0$$

$$G(v_i \mid x_{-i}) = P\left(v_i\left(x\right) \leq v_i \mid x_{-i}\right) = F\left(x_i\left(v_i,x_{-i}\right) \mid x_{-i}\right)$$

$$\frac{G(v_i \mid x_{-i})}{g(v_i \mid x_{-i})} = \frac{F\left(x_i\left(v_i,x_{-i}\right) \mid x_{-i}\right)}{f\left(x_i\left(v_i,x_{-i}\right) \mid x_{-i}\right)}\frac{dv_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) = \gamma_i\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) \tag{5}$$

Therefore, $V_i(v,x_{-i}) = m_i\left(x_i\left(\cdot,x_{-i}\right),x_{-i}\right)$ crosses w at most once and from below if and only if $m_i(\cdot,x_{-i})$ crosses w at most once and from below. □

Corollary 12. *If the problem is regular for i given $x_{-i} \in X_{-i}$, the signal of j is positive for i given $x_{-i} \in X_{-i}$ if and only if the indifference curve $v_i(x) = w$ in the plane (e_j, e_i) crosses from above the indifference curve of $m_i(\cdot)$.*

It means that if x_j increases and x_i moves along the indifference curve of $m_i(x)$, $v_i(x)$ increases. In other terms, $MRS_{j,i}v_i(x) \geq MRS_{j,i}m_i(x)$ and switching from true to virtual valuations increases the relative importance of x_j over x_i. The property that the signal of $j \in J$ is negative for i given $x_J \in X_J$ is defined similarly using generalized Leibniz integral rule. The previous corollaries can be then adapted.

Proof. Let v_i belong to the support of $v_i \mid x_{-i}$. The information of j is positive for i in v_i given x_{-i} if and only if $\frac{d}{dx_j}\left(\frac{G(v_i|x_{-i})}{g(v_i|x_{-i})}\right) \leq 0$. Differentiating equation 5 gives:

$$\frac{dx_i}{dx_j}(v_i,x_{-i})\frac{d\gamma_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) + \frac{d\gamma_i}{dx_j}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) \leq 0$$

and because $\frac{dx_i}{dx_j}(v,x_{-i}) = -\frac{\frac{dv_i}{dx_j}(x_i(v,x_{-i}),x_{-i})}{\frac{dv_i}{dx_i}(x_i(v,x_{-i}),x_{-i})} \leq 0$ and $\frac{dv_i}{dx_i}(x) > 0$, it implies that:

$$\frac{dv_i}{dx_j}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)\frac{d\gamma_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) \geq \frac{d\gamma_i}{dx_j}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)\frac{dv_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)$$

and because $\frac{d\gamma_i}{dx_j}(x) = \frac{dm_i}{dx_j}(x) - \frac{dv_i}{dx_j}(x)$:

$$\frac{dv_i}{dx_j}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)\frac{dm_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) \geq \frac{dm_i}{dx_j}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)\frac{dv_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)$$

Finally, $\frac{dm_i}{dx_i}\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) > 0$ because the problem is regular for i in v_i knowing x_{-i}. Therefore, $MRS_{j,i}v_i\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right) \geq MRS_{j,i}m_i\left(x_i\left(v_i,x_{-i}\right),x_{-i}\right)$ □

Proof. **Proposition 1**. If the problem is regular for all sellers, for each i, $G_i(p)$ crosses $[w-p]g_i(p)$ at most once and from below. Therefore, if $p^* \le \min_i\{p_i^*\}$ the $-G_i(p^*) + [w-p^*]g_i(p^*)$ are all positive and so is their sum and rising p^* is profitable. Similarly, if $p^* \ge \max_i\{p_i^*\}$, lowering p^* is profitable.

Now if the buyer changes his beliefs about a seller i such that $p_i^* \ge p^*$, and switches from $G_i(v)$ to $G_i^+(v)$ that dominates in term of reverse hazard rate the original one. $p_i^* \ge p^*$ and the regularity of the problem imply that:

$$\frac{G_i(p^*)}{g_i(p^*)} \le (w - p^*)$$

The reverse hazard rate domination implies that that for all v:

$$\frac{G_i^+(v) - G_i(v)}{g_i^+(v) - g_i(v)} \le \frac{G_i(v)}{g_i(v)}$$

Therefore, combining the two equations with $v = p^*$ and adding on each side the derivative of the profit on the other sellers in p^*:

$$-G_i^+(p^*) + [w-p^*]g_i^+(p^*) \ge -G_i(p^*) + [w-p^*]g_i(p^*)$$
$$-Q^+(p^*) + [w-p^*](Q^+)'(p^*) \ge 0$$

So rising p^* increases profit with the new belief. Moreover, jumping to any local maximum p^- lower than p^* lower the profit. Indeed, denoting $\pi_j(p)$ the profit made with seller j at price p, $\pi_{-i}(p^-) - \pi_{-i}(p^*) \le \pi_i(p^*) - \pi_i(p^-)$ because p^* is optimal with the original distribution. In the previous equations p^* can be replaced by any $p \le p_i^*$. Therefore, for all $p \le p_i^*$, $-G_i^+(p) + [w-p]g_i^+(p) \ge -G_i(p) + [w-p]g_i(p)$ and integrating between p^- and p^* gives that $\pi_i^+(p^*) - \pi_i^+(p^-) \ge \pi_i(p^*) - \pi_i(p^-)$. Finally, the profit with the new distribution is smaller in p^- than in p^*. □

Proof. **Proposition 2**. This is a adaption the proof of revenue equivalence theorem in Perry and Reny (1999). If I define $u_i(x) = c_i(x) - q_i(x)v_i(x)$, the incentive constraint is equivalent to the following equation:

$$u_i(x) = \max_{z \in X_i} c_i(z, x_{-i}) - q_i(z, x_{-i})v_i(x)$$

The function in the max is Lipschitz in x_i because $v_i(\cdot, x_{-i})$ is continuously differentiable in x_i. Because the sup of Lipschitz function is Lipschitz, $u(x)$ must be Lipschitz in x_i and, therefore, it must be differentiable almost everywhere in x_i. Taking $z \ge x_i \in X_i$, $x_{-i} \in X_{-i}$ and expressing the IC constraint in x and in (z, x_{-i}), I can prove that $q_i(x)$

is decreasing in x_i. Finally, letting z goes toward x_i implies that:

$$\frac{du_i}{dx_i}(x) = -q_i(x)\frac{dv_i}{dx_i}(x)$$

These two conditions also imply the original IC condition. Integrating with respect to x_i gives an expression for $u_i(x)$ that I can use to rewrite the objective function:

$$q_i(x)w - c_i(x) = q_i(x)(w - v_i(x)) + \int_0^{x_i} q_i(z, x_{-i})\frac{dv_i}{dx_i}(z, x_{-i})dz - u_i(0, x_{-i})$$

Because the IC constraints for two different sellers are independent, by taking the sum out of the integral, I can split the problem in as many problems as there are sellers. Each of these problems can be written as the following:

$$\max_{(q_i, u_i(0,x_{-i}))\in \mathcal{Q}_i\times\mathcal{C}_{-i}} \int_X \left(q_i(x)[w - v_i(x)] + \int_0^{x_i} q_i(z, x_{-i})\frac{dv_i}{dx_i}(z, x_{-i})dz - u_i(0, x_{-i})\right) f(x)dx$$

$$\text{s.t. for all } x \in X\text{: } u_i(0, x_{-i}) \geq \int_0^{x_i} q_i(z, x_{-i})\frac{dv_i}{dx_i}(z, x_{-i})dz$$

Therefore, at the optimum it must be that $u_i(0, x_{-i}) = \int_0^{\bar{x}_i} q_i(z, x_{-i})\frac{dv_i}{dx_i}(z, x_{-i})dz$. Replacing it in the objective, and changing the order of integration gives [27]:

$$\max_{q_i\in \mathcal{Q}_i\times\mathcal{C}_{-i}} \int_X q_i(x)\left(w - v_i(x) - \frac{dv_i}{dx_i}(x)\frac{F(x_i \mid x_{-i})}{f(x_i \mid x_{-i})}\right) f(x)dx$$

Finally, substituting with $m_i(x)$ and denoting for all i and $x_{-i} \in X_{-i}$ such that $m_i(0, x_{-i}) \leq w$ $s_i(w, x_{-i}) = \max\{x_i \in X_i \mid m_i(x) \leq w\}$, the regularity of the problem implies that:

$$q_i^*(x) = \begin{cases} \lambda_i & \text{if } m_i(x) \leq w \\ 0 & \text{otherwise} \end{cases} \quad c_i^*(x) = \begin{cases} \lambda_i v_i\left(s_i\left(w, x_{-i}\right), x_{-i}\right) & \text{if } m_i(x) \leq w \\ 0 & \text{otherwise} \end{cases}$$

$q_i^*(x)$ is decreasing in x_i because the problem is regular. □

Proof. **Proposition 3**. If the problem is regular for seller i knowing x_{-i} and w belongs to the support of $m_i \mid x_{-i}$, $q_i(x) = \lambda_i$ if and only if $x_i \leq s_i(w, x_{-i})$. Then monotonicity of $v(\cdot, x_{-i})$ implies:

$$q_i^*(x) = \begin{cases} \lambda_i & \text{if } v_i(x) \leq p_i^0(w, x_{-i}) \\ 0 & \text{otherwise} \end{cases} \quad c_i^*(x) = \begin{cases} \lambda_i p_i^0(w, x_{-i}) & \text{if } v_i(x) \leq p_i^0(w, x_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

[27] Because the set of signals where $f(x)$ is of measure 0, I can choose $q_i(x)$ as I want on this set as long as it is decreasing.

Because the problem is regular for i given X_{-i}, if $w \in [v(0,x_{-i}), m(\bar{x},x_{-i})]$ and the optimal price posted by the monopsonist p must verify:

$$-G_i(p \mid x_{-i}) + (w-p)g_i(p \mid x_{-i}) = 0$$
$$w - m_i\left(x_i\left(p, x_{-i}\right), x_{-i}\right) = 0$$

Because $p_i^0(w,x_{-i}) = v_i\left(s_i\left(w,x_{-i}\right), x_{-i}\right)$, $x_i\left(p_i^0\left(w,x_{-i}\right), x_{-i}\right) = s_i(w,x_{-i})$, and $p_i^0(w,x_{-i})$ is the unique solution of the problem.

If $w \leq v(0,x_{-i})$, $p^* = v(0,x_{-i})$ and if $w \geq m(\bar{x},x_{-i})$, $p^* = v(\bar{x},x_{-i})$ are solutions. Deriving $p_i^0(w,x_{-i}) = v_i\left(s_i\left(w,x_{-i}\right), x_{-i}\right)$ with respect to w, I can show that the price is increasing in w and that if the cumulative distribution of $v_i \mid x_{-i}$ is log-concave in $p_i^0(w,x_{-i})$, the derivative is smaller than 1. □

Proof. **Corollary 4**. Suppose that $p_i^* > p_i^0(w,x_{-i})$ for all x_{-i} on the support of $x_{-i} \mid v_i$. The regularity of the problem for i given X_{-i} and the definition of $p_i^0(w,x_{-i})$ implies that:

$$\begin{aligned} G_i(p_i^* \mid x_{-i}) + (w-p_i^*)g_i(p_i^* \mid x_{-i}) &< G_i\left(p_i^0\left(w,x_{-i}\right) \mid x_{-i}\right) \\ &\quad + [w - p_i^0(w,x_{-i})]g_i\left(p_i^0\left(w,x_{-i}\right) \mid x_{-i}\right) \\ &< 0 \end{aligned}$$

Multiplying by $f(x_{-i})$ and integrating over X_i gives:

$$G_i(p_i^*) + (w-p_i^*)g_i(p_i^*) < 0$$

which contradicts the definition of p_i^*. $p_i^* < p_i^0(w,x_{-i})$ for all x_{-i} on the support of $x_{-i}|v_i$ leads in the same way to a contradiction. □

Proof. **Corollary 4**. If w belongs to the support of $m_i \mid x_{-i}$, $m_i\left(s_i\left(w,x_{-i}\right), x_{-i}\right) = w$. Moreover, the regularity of the problem for i given x_{-i} implies that:

$$\frac{ds_i}{dx_j}(w,x_{-i}) = -\frac{\frac{dm_i}{dx_j}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)}{\frac{dm_i}{dx_i}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)}$$

$$\begin{aligned} \frac{dp_i^0}{dx_j}(w,x_{-i}) &= \frac{dv_i}{dx_i}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)\frac{ds_i}{dx_j}(w,x_{-i}) + \frac{dv_i}{dx_j}\left(s_i\left(w,x_{-i}\right), x_{-i}\right) \\ &= -\frac{dv_i}{dx_i}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)\frac{\frac{dm_i}{dx_j}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)}{\frac{dm_i}{dx_i}\left(s_i\left(w,x_{-i}\right), x_{-i}\right)} + \frac{dv_i}{dx_j}\left(s_i\left(w,x_{-i}\right), x_{-i}\right) \end{aligned}$$

Moreover, $x_i(v_i, x_{-i}) = s_i(w, x_{-i})$ if the information of j is positive for i in $v_i = v_i\left(s_i\left(w, x_{-i}\right), x_{-i}\right)$ and corollary 12 implies that :

$$\frac{\frac{dm_i}{dx_j}\left(s_i\left(w, x_{-i}\right), x_{-i}\right)}{\frac{dm_i}{dx_i}\left(s_i\left(w, x_{-i}\right), x_{-i}\right)} \leq \frac{\frac{dv_i}{dx_j}\left(s_i\left(w, x_{-i}\right), x_{-i}\right)}{\frac{dv_i}{dx_i}\left(s_i\left(w, x_{-i}\right), x_{-i}\right)}$$

If w does not belong to the support of $m_i \mid x_{-i}$ the derivative is equal to 0. □

Proof. **Proposition 5**. Let $p(x)$ be an incentive compatible price function, for $J \subset N$ I denote:

$$D_J\left(\left(p\left(x_{N\setminus J\setminus\{i\}}, 0\right)\right)_{i\in N\setminus J}\right) = \Big\{x_{N\setminus J} \in X_{N\setminus J} \text{ s.t. for all } i \in N \setminus J, \quad v_i\left(x_{N\setminus J}, 0\right) > p\left(x_{N\setminus J\setminus\{i\}}, 0, 0\right)\Big\}$$

the set of signals in $X_{N\setminus J}$ such that no i in $N \setminus J$ sells at the price proposed in the hyperplane $(x_{N\setminus J\setminus\{i\}}, 0)$.

I will first show that for all $j \in N$ and given the $N-1$ price functions $p\left(x_{-\{j,i\}}, 0, 0\right)$, only seller j can sell in all $x \in D_j \times X_j$ and that if the sale takes place the buyer must propose any price $p(0, x_{-j}) \leq \min_{i\neq j} p(x_{N\setminus\{j,i\}}, 0, 0)$. Lemma 2 implies that if $i \neq j$ sells in x, $p(x) = p(0, x_{-i}) \leq p(0, 0, x_{-\{i,j\}})$ which by definition of D_j is strictly smaller that $v_i(0, x_{-j})$ which by monotonicity is smaller than $v_i(x)$. Therefore, $p(x) < v_i(x)$ and for all $i \neq j$, i cannot sell in x. If j sells in x, then lemma 2 implies that $p(x) = p(0, x_{-j}) \leq \min_{i\neq j} p(0, 0, x_{-\{i,j\}})$.

If $x \notin B_1 = \cup_j\{D_j \times X_j\}$ and the buyer buys in x, there must exist two sellers (i, j) such that $p(x) = p(0, 0, x_{-\{i,j\}})$. Let j be the seller who sells, it must hold that $p(x) = p(0, x_{-j})$. Because $x \notin A_j \times X_j$ there must exist $i \neq j$ such that $p(0, 0, x_{-\{i,j\}}) \geq v_i(0, x_{-j}) \geq v_i(0, 0, x_{-\{i,j\}})$. Figure 6a of proposition 4 implies that the buyer buys from i in $(0, x_{-j})$ and $(0, 0, x_{-\{i,j\}})$ and that $p(0, 0, x_{-\{i,j\}}) = p(0, x_{-j}) = p(x)$.

As in the case where $n = 2$, I will rewrite the maximization problem in a recursive way. I will split the integral between the $D_j \times X_j$ and the rest. On each of this space I will maximize over $p(0, x_{-j})$ given all prices on the subspaces of dimension lower than $n-2$. Taking this maximum in the intergral give n maximizations of the form:

$$\lambda_j \int_{x_{-j}\in D_j\left(\left(p\left(x_{-\{i,j\}},0,0\right)\right)_{i\neq j}\right)} \max_{p(x_{-j},0)\leq \min_{i\neq j}\{p(x_{-\{i,j\}},0,0)\}} [w - p(x_{-j}, 0)] F_j\left(x_j\left(p\left(x_{-j}, 0\right), x_{-j}\right), x_{-j}\right) dx_{-j}$$

The objective in the maximum is the same as the objective the unregulated problem stated in proposition 3 and, therefore, the regularity of the problem for every seller j given

x_{-J}, implies that for all $j \in N$, $x \in D_j\left(\left(p\left(x_{-\{i,j\}},0\right)\right)_{i\neq j}\right) \times X_j$, $p^*(x) = p^*(x_{-j},0) = \min\left\{\min_{i\neq j}\left\{p^*\left(x_{-\{i,j\}},0\right)\right\}, p_i^0\left(w, x_{-j}\right)\right\}$.

Defining by induction $B_{j+1} = B_j \cup_{J\subset N:|J|=j+1} D_J \times X_J$, I will prove by induction on j, that for all $J \subset N$ if $x \in D_J \times X_J \setminus B_j$ the buyer can sell at most to sellers in J at price $p(0, x_{N-J})$ and that if $x \notin B_{j+1}$ and the buyer buys in x, there must exist an ordering of $j+1$ sellers verifying the condition of lemma 1.

Let $J \subset N$ such that $|J| = j$ and $x \in D_J \times X_J \setminus B_j$.

Because $x \in D_J \times X_J$ and lemma 2 implies that for all $k \in N \setminus J$ and $J' \subset J$, k cannot sell in $(x_{N\setminus J}, x_{J'}, 0)$. Otherwise repetitively using lemma 2 would imply that $v_k(x_{N\setminus J}, x_{J'}, 0) \leq p(x_{N\setminus J}, x_{J'}, 0) = p(x_{N\setminus J\setminus\{k\}}, 0, x_{J'}, 0) \leq p(x_{N\setminus J\setminus\{k\}}, 0)$. Moreover, v_k increasing would imply $v_k(x_{N\setminus J}, 0, 0) \leq v_k(x_{N\setminus J}, x_{J'}, 0)$. But these two inequalities together contradict the fact that $x \in D_J \times X_J$. In particular, the buyer can sell at most to sellers in J in x.

Because $x \notin B_j$ and because of the induction hypothesis, if somebody is selling in x, there exists an ordering of j sellers verifying the condition of lemma 1. Beside the previous paragraph ensures that this ordering is a reordering of J. The price offered will be, therefore, $p(x) = p(x_{N\setminus J}, 0)$.

Let $x \notin B_{j+1} = B_j \cup_{J\subset N:|J|=j+1} D_J \times X_J$ such that the buyer buys in x. Because $x \notin B_j$ there must exist $J \subset N$ such that a reordering of these sellers verifies the condition of lemma 1. But because $x \notin D_J \times X_J$ there must exist $k \in N \setminus J$ such that $p(x_{N\setminus J\setminus\{k\}}, 0) \geq v_k(x_{N\setminus J}, 0)$, and figure 6a of proposition 4 implies that the buyer buys from k in $(x_{N\setminus J}, 0)$ and $(x_{N\setminus J\setminus\{k\}}, 0)$ and that $p(x_{N\setminus J\setminus\{k\}}, 0) = p(x_{N\setminus J}, 0) = p(x)$. Adding this seller in front of the pervious ordering gives the new one.

Finally, I rewrite once again the problem recursively. Having determined the profit on B_j as a function of the price $p(x_{N\setminus J}, 0)$, I can pin down the price $p(x_{N\setminus J}, 0)$ and the profit in $D_J \times X_J$ as a function of the price $p(x_{N\setminus J\setminus\{k\}}, 0))_{k\in N\setminus J})$ by solving the following problem:

$$\sum_{j\in J} \lambda_j \int_{x_{N\setminus J}\in D_J\left(\left(p\left(x_{N\setminus J\setminus\{k\}},0,0\right)\right)_{k\in N\setminus J}\right)} \max_{p(x_{N\setminus J},0)\leq \min_{k\in N\setminus J}\{p(x_{N\setminus J\setminus\{k\}},0,0)\}} [w - p(x_{N-J},0)] G_j\left(p\left(x_{N\setminus J},0\right) \mid x_J\right) dx_{N\setminus J} + \Pi_{B_j}\left(p\left(x_{N\setminus J},0\right)\right)$$

The objective in the maximum is here the objective of a monopsonist who have to post a fixed price to buy from the sellers in J knowing the signals of the sellers in $N \setminus J$.

The second term is in general also a function of $p(x_{N\setminus J}, 0, 0)$, because the constraints in the maximization problem for the previous set of sellers are in general be binding.

$\square$

Proof. **Proposition 13**. To prove the second bullet point let $p \leq p^*$. Let us denote $W_i(p)$ the subset of $X_{Uj\setminus Ui}(p)$ such that $p_j(w, p, x_i) \leq p$, $\Pi_j^d(p, x_i) = (w-p)F_j\left(x_j^i(p, x_i), x_i\right) + [w - p_j(w, p, x_i)][F_j(x_j(p_j(w, p, x_i), x_i), x_i) - F_j\left(x_j^i(p, x_i), x_i\right)]$ the profit earned from buyer j when $x_i \in W_i(p)$ and $\Pi_j^{nc}(p, x_i) = (w-p)F_j(x_j(p, x_i), x_i)$ the profit from buying from j when $x_i \in X_i \setminus W_i(p)$. Evaluating the derivative of problem (2) in p gives:

$$\frac{d\Pi}{dp}(p) = \sum_i \int\limits_{X_i\setminus W_i(p)} \frac{d\Pi_j^{nc}}{dp}(p, x_i)dx_i + \int\limits_{\frac{d}{dp}W_i(p)} \Pi_j^{nc}(p, x_i) - \Pi_j^d(p, x_i)dx_i + \int\limits_{W_i(p)} \frac{d\Pi_j^d}{dp}(p, x_i)dx_i$$

Besides, $p_j(w, p, x_i) = p$ on $\frac{d}{dp}W_i(p)$. Indeed, if x_i is on the border of $X_{Uj\setminus Ui}(p)$ either $v_i(0, x_i) > p = v_j(0, x_i)$ or $v_i\left(x_j^i(p, x_i), x_i\right) = p = v_j\left(x_j^i(p, x_i), x_i\right)$ and for all $x_j < x_j^i(p, x_i)$, $v_j(x) < p$ and $v_i(x) \leq p$. In either case, $(w-p)G_j(p \mid x_i, v_i \geq p) = 0$ and the maximum must be greater than p. Therefore, $\Pi_j^{nc}(p, x_i) = \Pi_j^d(p, x_i)$. Finally, because $p_j(w, p, x_i)$ is the price maximizing $(w-s)[F_j(x_j(s, x_i), x_i) - F_j\left(x_j^i(p, x_i), x_i\right)]$, I can rewrite:

$$\begin{aligned}\frac{d\Pi}{dp}(p) = \sum_i \int\limits_{X_i\setminus W_i(p)} \frac{d\Pi_j^{nc}}{dp}(p, x_i)dx_i \\ + \int\limits_{W_i(p)} [p_j(w, p, x_i) - p] f\left(x_j^i(p, x_i), x_i\right) \frac{dx_j^i}{dp}(p, x_i) - F\left(x_j^i(p, x_i), x_i\right) dx_i\end{aligned}$$

On the other hand:

$$\begin{aligned}\frac{d\Pi^{nc}}{dp}(p) &= \sum_i \int\limits_{X_i\setminus W_i(p)} \frac{d\Pi_j^{nc}}{dp}(p, x_i)dx_i + \int\limits_{W_i(p)} \frac{d\Pi_j^{nc}}{dp}(p, x_i)dx_i \\ &= \sum_i \int\limits_{X_i\setminus W_i(p)} \frac{d\Pi_j^{nc}}{dp}(p, x_i)dx_i + \int\limits_{W_i(p)} (w-p) f(x_j(p, x_i), x_i) \frac{dx_j}{dp}(p, x_i) \\ &\quad - F_j(x_j(p, x_i), x_i) dx_i\end{aligned}$$

Therefore, because p^* is the maximum of Π^{nc}, $p^*(0) \geq p^*$ if for all $x_i \in W_i(p)$:

$$(w-p) f\left(x_j\left(p, x_i\right), x_i\right) \frac{d x_j}{d p}\left(p, x_i\right)-F_j\left(x_j\left(p, x_i\right), x_i\right) \leq \\ \left[p_j(w, p, x_i)-p\right] f\left(x_j^i\left(p, x_i\right), x_i\right) \frac{d x_j^i}{d p}\left(p, x_i\right)-F_j\left(x_j^i\left(p, x_i\right), x_i\right) \tag{6}$$

Let $x_i \in W_i(p)$ and $\bar{v}$ such that $p \geq \bar{v} > p_j(w, p, x_i)$ and $g_j(\bar{v} \mid x_i) \geq \max\{g_j(p \mid x_i), g_i(p \mid x_i)\}$. Because $\bar{v} > p_j(w, p, x_i)$ and $g_i(p \mid x_i)/g_j(\bar{v} \mid x_i) \leq 1$, I can find a $\bar{p} \leq p$ such that:

$$\bar{v} \geq p' = \bar{p} - \frac{g_i(p \mid x_i)}{g_j(\bar{v} \mid x_i)} p + \frac{g_i(p \mid x_i)}{g_j(\bar{v} \mid x_i)} p_j(w, p, x_i) \geq p_j(w, p, x_i)$$

These two inequalities together with the fact that $G_j(v \mid x_i, v_i \geq p)$ is log-concave and the problem described in equation (3) imply that

$$w - p' \leq \frac{G_j(p' \mid x_i, v_i \geq p)}{g_j(p' \mid x_i)} \leq \frac{G_j(\bar{v} \mid x_i, v_i \geq p)}{g_j(\bar{v} \mid x_i)}$$

Replacing p' and $G_j(\bar{v} \mid x_i, v_i \geq p) = [F_j\left(x_j\left(\bar{v}, x_i\right), x_i\right) - F_j\left(x_j^i\left(p, x_i\right), x_i\right)]/f_i(x_i)$ gives:

$$\left(w - \bar{p} + p\frac{g_i(p \mid x_i)}{g_j(\bar{v} \mid x_i)} - p_j(w, p, x_i)\frac{g_i(p \mid x_i)}{g_j(\bar{v} \mid x_i)}\right) g_j(\bar{v} \mid x_i) \leq G_j(\bar{v} \mid x_i, v_i \geq p)$$

$$(w - \bar{p}) g_j(\bar{v} \mid x_i) - G_j(\bar{v} \mid x_i, v_i \geq p) \leq [p_j(w, p, x_i) - p] g_i(p \mid x_i)$$

$$(w-\bar{p}) f\left(x_j\left(\bar{v}, x_i\right), x_i\right) \frac{d x_j}{d p}\left(\bar{v}, x_i\right)-F_j\left(x_j\left(\bar{v}, x_i\right), x_i\right) \leq \\ \left[p_j(w, p, x_i)-p\right] f\left(x_j^i\left(p, x_i\right), x_i\right) \frac{d x_j^i}{d p}\left(p, x_i\right)-F_j\left(x_j^i\left(p, x_i\right), x_i\right)$$

Because $g_j(\bar{v} \mid x_i) \geq g_j(p \mid x_i)$, and $p \geq \bar{p}$, the last inequality implies (6). $\square$

Chapter II

House Allocation with Limited Externalities

Based on Frys (2018a).

1 Introduction

This paper studies the allocation of houses to students, when students have preferences not only over the houses they receive but also over where their friends live. I consider the case in which the set of students can be partitioned into groups of friends who care about each other assignment. So if A and B are friends, both care about where they both end up. While I impose the assumption that these preferences are common, so that friends agree on how to rank different allocations, I don't require friends to prefer to be assigned to the same house.

For example, there might be one house on the campus that has access to a pool, while another has access to a tennis court. Two friends, A and B, might prefer to be assigned separately, one in each of the houses, so that both can enjoy the pool as well as the tennis court. On the contrary, A and B might prioritize convenience and prefer to live together in the house with the pool. If there is not enough slots left in this house, they might prefer to be assigned separately in the two houses rather than to be assigned together in a third house with no facilities. What I do rule out are situations in which A wants to live close to B, while B wants to live far away from A. Otherwise, if preferences of friends would not coincide, aspects of the roommate problem, which I abstract from here, would reappear. In practice, a student's utility could be affected by the presence in a house of other students who are already assigned and remain there with certainty. The presence of such existing tenants can be treated as a characteristic of the house and abstracting from it is harmless. Fundamentally, I call friends people who are able to construct a common preference ordering over their allocation and can commit to stick to it during the duration of the game. We can even imagine that students who particularly dislike each other form a group to be sure to be allocated far from each other.

This set-up can be transposed to other one sided matching markets. A particularly relevant market is the allocation of refugees' families to countries or geographical regions. Such matching mechanism would be a good complement to the quotas system that some members of the European union would like to implement. It could also be used to allocate time slots of sport facilities or to allocate tasks to working groups.

A common mechanism used to assign students' accommodation is known in the literature as the random serial dictatorship (RSD). In this mechanism, students are ordered randomly and choose in that order their most preferred room among the ones still available. In my setting, I show that the sub-game perfect equilibrium of the game induced by this mechanism fails to be efficient. Because students can only express preferences over houses, this procedure is not a direct mechanism. To implement it as a direct mechanism, the students would instead have to report their ranking of the assignments of their friends to all houses.

Therefore, I present a modification of RSD, which I call the random serial group dictatorship (GD). In this mechanism, friends report their friendship. They are then treated as a single agent in the RSD and asked for their preferences over joint assignments. This procedure was used, for example, at King's College, at Cambridge University and at Harvard College. Conditional on groups of friends having been truthfully reported, submitting truthful preferences over allocations is a dominant strategy for each group. However, I show that there are cases in which students may strictly benefit from lying about their friendship and that these strategies form a new Nash equilibrium. Because a group of friends is treated as a single agent, splitting the group increases the number of draws a group have in the lottery. Some groups may then use this strategically and let the first subgroup block an allocation for the rest of the group. Similarly, two groups of students may merge and report themselves as only one group to avoid the ordering of the groups that lead to bad outcome for them. However, to exploit these incentives, students need a lot of information. Indeed, if enough singles are telling the truth and if there is enough demand for the most preferred houses, I show that reporting a group truthfully is never stochastically dominated.

The weak incentives properties of the GD mechanism stem partly from the dependence of the lottery on the submitted information. I propose a new direct mechanism, which I call the random serial bossy (RSB) mechanism. This mechanism asks each agent to report her preferences over assignments and ranks them individually. The agents' preferences are then considered in that order. Whenever it is an agent's turn, the best feasible allocation according to that agent's reported preferences is implemented. I show that RSB is both efficient and strategy-proof. However, as for the GD mechanism, a group of students may collectively deviate from truth-telling to an other Nash equilibrium that is stochastically strictly better. This happens for all market's sizes and for almost all relevant preferences' profile of the students. In the light of their respective properties, I discuss the pros and cons of implementing each mechanism in practice.

The next subsection discusses related literature. In Section 3, I present the basic model and definitions as well as the restrictions on the set of admissible preferences. Section 4 analyzes the performance of the standard random serial dictatorship in my setting. Section 5 introduces and discusses the random serial group dictatorship mechanism. Section 6 presents the random serial bossy mechanism and its properties. Section 7 concludes.

2 Literature Overview

There are a number of papers analyzing matching problems with externalities between agents. In particular, a part of the literature on two-sided matchings studies the problem of matchings with couples. In the mechanism matching new doctors to hospitals in the U.S., Roth and Peranson (1999) noted that the increasing share of couples of doctors led to a decrease in the use of the centralized algorithm, because couples were unable to identify themselves as such. Even after the market has been redesigned to allow couples to express their joint preferences over pairs of jobs, the existence of a stable matching cannot generally be guaranteed, Roth and Sotomayor (1992). Klaus and Klijn (2005) showed that a restriction of the domain of preferences can ensure the existence of a stable matching even when couples are present. This restriction essentially states that couples cannot have preferences such that they prefer to be matched to similar geographic areas. This preference for closeness, however, appears to be the main feature of couples' preferences empirically. In addition, Klaus et al. (2007) noted that the Roth-Perranson algorithm developed in Roth and Peranson (1999) may fail to find a stable matching even under responsive preferences. They also showed that couples might have an incentive to misreprort their preferences. Under the assumption of weakly responsive preferences, Klaus and Klijn (2007) constructed an algorithm to obtain a stable matching from an arbitrary matching even in the presence of couples. Finally, Kojima et al. (2013) showed that when there are relatively few couples and the size of the market tends toward infinity, the probability that a stable matching exists tends to one.

Sasaki and Toda (1996) considered a general model of two-sided matching markets with externalities. In this general case, agents have preferences over the entire set of matchings. Their definition of stability supposed that two matched agents considering a deviation are pessimistic and compare the actual matching to the worst in their expectation sets. They showed that if the expectation sets don't depend on agents' preferences, the existence of a stable matching is guaranteed only if a deviating pair considers that all matchings where they are together can arise after the deviation. They defined rational expectations as the case where a pair considers that a matching is possible only if it is stable in the subgame where they are matched together. They showed that this reduced game consistency guarantees the existence of a stable matching only when agents have weak externalities. Hafalir (2007) emphasized the lack of rationality of the players in Sasaki and Toda's solution. He defined an intermediate notion of rational set of expectation that ensures the existence of a stable matching. However, this definition seems too complicated to find real world application. Ultimately, he showed that the presence of externalities in matching problems makes them unsolvable in general. Externalities create cycles of couples that break matchings one after the other. The only way to circumvent the problem is hence to focus on narrowed set-up such as mine.

House allocation problems in which groups of students are to be allocated have been mentioned in some previous papers. Collins and Krishna (1997) analyzed the house allocation procedure in Harvard College. A feature of that house allocation procedure is that groups of students are allocated together. However, they ignored this aspect of the procedure, which hints at the presence of externalities, but focused instead on an empirical examination of the extent to which students make rational choices. Che and Kojima (2010) noted that the mechanism used at Columbia University to allocate rooms explicitly allows groups of agents to report joint preferences. They noted that one advantage of random serial dictatorship over the Probabilistic Serial (PS) mechanism of Bogomolnaia and Moulin (2001) is that it easily allows groups to apply jointly. Under the PS mechanism agents reserve shares of houses at a constant speed. The share of a house reserved by an agent represents the probability of obtaining this house. It is difficult to extend the PS mechanism to cases where agents have externalities. It is indeed not clear how to construct lotteries over matchings in a way that is consistent with the expected allocation of all participating agents.

Finally, a part of the literature on allocation mechanism considers problems where multiple goods can be allocated to the same agent. This problem is the same as the one presented in this article if the compositions of the groups of friends are known. In this case, a group of k friends could be seen as an individual agent choosing k rooms.

If the goods are heterogeneous and the preferences over sets of goods are strict, Pápai (2001) showed that an allocation rule is strategy-proof, nonbossy and satisfy citizen sovereignty if and only if it is a sequential dictatorship. However, in the model presented in the next section, there are multiple units of goods, preferences displays indifferences and the k friends can at most receive k goods. In a more recent article, Budish et al. (2013) extended the pseudo market mechanism of Hylland and Zeckhauser (1979) to the case where students may receive multiple goods and where goods may have multiple exemplars. However, they suppose that a good can fulfill different "roles", that students have cardinal utilities for each role of each good and that these utilities are additive. In the model presented in the next section, I impose less structure on the set of possible preferences.

3 The Model

I consider a house allocation problem, which consists of i) S, a finite set of students; ii) H, a finite set of houses; and iii) $(q_h)_{h\in H}$, a vector listing the number of rooms available in each house. A matching μ is defined as mapping from the set of students to the set of houses such that for every $s \in S$ and $h \in H$:

1 $\mu(s) \in H \cup \emptyset$

2 $\mu^{-1}(h) \subseteq S$

3 $|\mu^{-1}(h)| \leq q_h$

A matching μ associates to each student either a house h in the set of houses, or no house, which I denote by $\emptyset$. Furthermore, for μ to be a matching, each house must be assigned to less students than its capacity. I denote the set of all matchings by $\mathcal{M}$. I denote $\mathcal{Q}$ the set of transitive and complete preference relations over $\mathcal{M}$. Each students has an ordinal preference relation over the set of matchings, $\succcurlyeq_s \in \mathcal{Q}$. I denote by $\succ_s$ and $\sim_s$ the strict and indifferent preference relations of student s and $\succcurlyeq \in \mathcal{Q}^{|S|}$ the preference profile of all students.

To compare probability distributions over matchings, I will use the partial order induced by first order stochastic dominance. A probability distribution Δ over the set of matchings is weakly preferred to another probability distribution Δ' by student s, $\Delta \succcurlyeq_s^{sd} \Delta'$, if renaming the matching according to the preferences $\succcurlyeq_s$, we have that for all $n \leq |\mathcal{M}|$ $\sum_{i=1}^{n} \delta(\mu_i) \geq \sum_{i=1}^{n} \delta'(\mu_i)$. It is equivalent to suppose that for all cardinal utilities compatible with $\succcurlyeq_s$, if s is expected utility maximizer, s prefers Δ to Δ'. I use $\succ_s^{sd}$ when there is a n such that the inequality is strict and $\mu_n \succ_s \mu_{n+1}$.

I will say that two students are friends if they have the same preferences. In usual housing markets, two agents with the same preferences want to be allocated to the same houses and are therefore in competition for theses houses. Because the preferences here are on the set of matchings, agents with the same preferences are not competing against each other but are promoting the same outcome. So even if they are not friends in real life, they are in practice allied when they face the allocation problem. It could also be interpreted in a different way. A group of friends would be fundamentally a set of students who are able to express common preferences. Friends have more power on each other. They are, therefore, able to better negotiate with each other to define common preferences before the beginning of the game and can commit to stick to it during its entire duration.

Definition 4. *The* ***group of friends*** *of student s, denoted $L(s)$, is the set of students such that: for all $s' \in L(s)$, $\succcurlyeq_s = \succcurlyeq'_s$.*

This definition has straightforward consequences. There is no disagreement about friendship. There is no case where s_1 is a friend of s_2 but the inverse doesn't hold. Moreover, each group of friends forms a clique. Each of its members is a friend of all other members and no other student. In particular, this rules out continuous networks of friendship.

Remark 1. *The sets of groups of friends forms a partition of S:*

- *Friendship is reciprocal: for all $s, s' \in S$, if $s' \in L(s)$ then $s \in L(s')$*
- *Friendship is transitive: for all $s, s', \hat{s} \in S$, if $\hat{s} \in L(s')$ and $s' \in L(s)$ then $\hat{s} \in L(s)$.*

N is the number of (mutually exclusive) groups of friends. A typical group of students is denoted by g_i. I denote by G the set of groups, so we have $N = |G|$ and $\sum_i |g_i| = |S|$. Similarly, for all $S' \subset S$, I denote $L(S')$, the set of groups of friends that have a member in S'.

I haven't yet restricted the class of preference relations that I want to consider. I want to study a market with a specific type of limited externalities, where students do not care about the allocation of every other participant but only about the allocation of a defined closed set of students. I will suppose that students only care about the allocation of their friends. This hypothesis is the major restriction of the model.

Assumption 1. *For all $i \leq N$, $\mu \sim_{g_i} \nu$ if for all $s \in g_i$, $\mu(s) = \nu(s)$.*

This first assumption restricts a lot the possible externalities. A student is allowed to have preferences over another student's allocation only if these preferences coincide with her own preferences. It is the smallest step away from a selfish agent model. Letting people express such preferences seems a priori harmless.

A second assumption on the preferences ensures that the model is a generalization of the classical case of selfish agents with strict preferences. I will suppose that groups of friends have strict preferences over the set of houses they are assigned to.

Assumption 2. *For all $i \leq N$, $\mu \sim_{g_i} \nu$ implies that for all $h \in H$, $|\mu^{-1}(h) \cap g_i| = |\nu^{-1}(h) \cap g_i|$.*

It implies in particular that if a student has no friend, she must have strict preferences over the set of houses. This assumption allows a group of friends to be indifferent between two different matchings of its members. Because friends agree on preferences, they might be able to reshuffle their allocation after the mechanism. They might, therefore, be indifferent between all allocations that give them the same set of houses.

These assumptions allow for two special cases. One in which $L(s) = \{s\}$ for all s, we are then back in the standard case without externalities. The other when $L(s) = S$, meaning that all students are friends of each other and agree on the ranking of the matchings. It also implies that students cannot be friends accidentally. Students know their group of friends because they know their preferences. They would in particular be able to report their group of friends if they have to. I denote the set of preferences for which assumptions 1 and 2 hold by $\mathcal{P} \subset \mathcal{Q}^{|S|}$. I assume throughout the paper that the preference profile belongs to $\mathcal{P}$. While I have made a number of restrictions concerning the shape that externalities can take, it still leaves a fairly large space of possible preferences to consider. I don't request that friends prefer to be allocated to the same house, as it has been the case for problems of allocation of couples. Instead, the restrictions fix the set of people over whose assignment an agent cares. Hence, an agent may care about the allocation of another agent even when this agent is not allocated to the same house. I will use this fact to recover the group of friends associated with a preference order over matchings.

Definition 5. *The* ***group of friends associated with*** *$\succcurlyeq \in \mathcal{Q}$, denoted $L(\succcurlyeq)$, is the set of students $s \in S$ such that: for all $\mu, \nu \in \mathcal{M}$ with $\mu(s') = \nu(s')$ for all $s' \neq s$ and $\mu(s) \neq \emptyset = \nu(s)$, either $\mu \succ \nu$ or $\nu \succ \mu$.*

If a preference profile $\succcurlyeq$ belongs to $\mathcal{P}$, the group of friends associated with a student's preferences coincides with her group of friends defined in definition 4. To simplify the analysis, I will first consider direct mechanisms.

Definition 6. *A (Random)* ***Direct Mechanism*** Ψ *is a mapping from the set of preferences to the set of probability distributions over the set of matchings, i.e.* $\Psi : \mathcal{Q}^{|S|} \to \Delta(\mathcal{M})$.

I allow students to report preferences outside of $\mathcal{P}$. Otherwise, the set of admissible preferences for each student would depend on the preferences reported by her friends. I next define desirable properties that direct mechanisms should satisfy.

Definition 7. *A matching* μ *is* ***Pareto efficient*** *if there does not exist* $\mu' \in \mathcal{M} \setminus \{\mu\}$ *such that for all* $s \in S$, $\mu' \succcurlyeq_s \mu$ *and for at least one* $s \in S$ $\mu' \succ_s \mu$.

The definition of Pareto efficiency then extends easily to direct mechanisms:

Definition 8. *A direct mechanism* Ψ *is (ex-post)* ***Pareto efficient*** *with respect to the reported profile* $\succcurlyeq \in \mathcal{Q}^{|S|}$ *if all matchings in the support of* $\Psi(\succcurlyeq)$ *are Pareto efficient.*

I next introduce two notions of truth-telling incentives for students. A mechanism is strategy-proof if for each student, reporting her true preferences is a dominant strategy.

Definition 9. *A random direct mechanism* Ψ *is* ***strategy-proof*** *if for all* $s \in S$, $\succcurlyeq \in \mathcal{P}$, $\succcurlyeq' \in \mathcal{Q}^{|S|}$, $\Psi(\succcurlyeq_s, \succcurlyeq'_{-s}) \succcurlyeq_s^{sd} \Psi(\succcurlyeq')$.

A weaker notion than strategy-proofness is that truth-telling forms a Nash equilibrium. It means that no student has an incentive to lie about her preferences given that all other students report their preferences truthfully.

Definition 10. ***Truth-telling is a Nash equilibrium*** *of a random direct mechanism* Ψ *if for all* $\succcurlyeq \in \mathcal{P}$, *and for all* $s \in S$ *and* $\succcurlyeq'_s \in \mathcal{Q}$, $\Psi(\succcurlyeq) \succcurlyeq_s^{sd} \Psi(\succcurlyeq'_s, \succcurlyeq_{-s})$.

These definitions are using the first order stochastic partial order. Therefore, I can define a weak version of each notion requiring that there be no report strictly preferred than telling the truth. One interpretation of definition 4 is that friends are able to negotiate to submit common preferences. Therefore, we should expect some students to be able to coordinate their actions. I thus introduce a notion of truth-telling that allows students to consider joint deviations.

Definition 11. *Truth-telling is a Nash equilibrium of a random direct mechanism* Ψ ***robust to coalitions of*** l ***groups of friends*** *if truth-telling is a Nash equilibrium and for all coalition* $S' \subset S$ *such that* $|L(S')| \leq l$, *for all* $\succcurlyeq \in \mathcal{P}$, *there is no* $\succcurlyeq'_{S'} \in \mathcal{Q}^{|S'|}$ *such that:*

- $\Psi(\succcurlyeq'_{S'}, \succcurlyeq_{S \setminus S'}) \succcurlyeq_g^{sd} \Psi(\succcurlyeq)$ *for all* $g \in L(S')$, *with strict inequality for one* $g \in L(S')$

- $\Psi(\succsim'_{S'}, \succsim_{S\setminus S'}) \succsim_g^{sd} \Psi(\succsim'_{S'\setminus g}, \succsim_g, \succsim_{S\setminus S'})$ *for all* $g \in L(S')$,

A Nash equilibrium is robust to coalitions of l groups of friends if no coalition of less than l groups of friends can trigger a better outcome for them by jointly reporting a stable deviation from truth-telling. This definition departs from classical definitions of coalition-proofness through three weakening restrictions. First, the size of the coalition is limited. If $l = 1$, we consider only joint deviations of members of the same group of friends and truth-telling must be a weak Nash Equilibrium for groups of friends. Second, the first bullet point requires that lying do not stochastically dominate truth-telling, rather than requiring that truth-telling stochastically dominate lying. Finally, the second bullet point requires the deviation strategy to be credible or stable in the sense that a group of friends taking part in the lying coalition never wants to jointly switch back to the truth.

If truth-telling is a strictly dominant strategy, truth-telling is a Nash equilibrium robust to all sizes of coalitions. There is no direct relation with simple strategy-proofness. However, in strategy-proof mechanisms, truth-telling is not robust to coalitions of size l only if at the report dominating truth-telling, every student is indifferent between telling the truth and not. Therefore, the mechanism must be bossy, i.e. a student changing his report can change another student's allocation without changing its own. In the case without externalities and strict preferences, truth-telling in RSD is a strict dominant strategy and therefore weakly coalition-proof. A random mechanism that would not verify this definition for a small l would be very sensitive to joint deviations. A small subset of students would be able to deviate to get a strictly better outcome only through cheap talk, with no commitment power.

However, given the size of the preferences set, direct mechanisms are unlikely to be used in practice. The size of the preferences that students would have to report might be too large. If all houses have more than k slots, a group containing k members would have to report a ranking of $|H + 1|^k$ alternatives. Therefore, I will also study indirect implementations of the mechanisms studied. In most of the cases developed in the following sections, the strategy space in each information set will consist of reporting information contained in the preferences of the student. This would in particular be the case when students have to report their groups of friends or to choose the best allocation among a set. I could also say that students face a partition of $\mathcal{P}$ and have to report the set to which their preferences belong. In these cases, I can extend the incentive definitions by saying that a report is truthful if a student reports the true set. If the game is sequential, I will use perfect Nash equilibrium instead of simple Nash equilibrium. I will also extend the efficiency property to these types of indirect mechanisms. I will say that it is Pareto efficient if the outcome of truthful strategies is a distribution over Pareto efficient matchings.

4 Random Serial Dictatorship Revisited

In the standard case, where students have preferences over the set of houses, random serial dictatorship (RSD) is a direct mechanism. It works as follows: each student reports a ranking of houses, a lottery is drawn to order the students and they are allocated sequentially according to their lottery number to their most preferred house in the set of the remaining houses. In the standard case, it can also be implemented by ordering the students directly and letting them choose sequentially their match. I will call this game the sequential RSD. In this game, the students know the complete ranking of the students and the match of the preceding students at the time of their decision. Without externalities, each game has a unique equilibrium in dominant strategy. The two equilibrium strategies are linked by a bijective function and lead to the same outcome.

This equivalence breaks down in the presence of externalities, even if they respect assumption 1 and 2. In different information sets of the sequential RSD, the relative ordering of two houses may vary as the preferences over these houses depends on which students are likely to be allocated to them. There is no bijection between the optimal ranking of houses in RSD and the optimal strategy in the sequential RSD. Moreover, I will show that the two mechanisms can implement different matchings. In the first mechanism, no information is revealed to the students before they submit their preferences. In the second one, students are aware of the choices of all students who were ahead in the lottery. Hence, they will find it optimal to condition their choice on the set of houses available, on the actions of their friends already allocated and on the ranking of the students who will move after her.

In the present setting with externalities, the two implementations of RSD are no longer direct mechanisms. They do not take students' preferences as an input. They restrict students to report rankings of houses instead of rankings of matchings. It rules out any incentive properties defined in section 3. I cannot interpret the strategy spaces as a collection of subsets of $\mathcal{P}$ forming a partition to extend these notions. Actually, there may be no Nash equilibrium of the classical RSD. However, I can still say something about efficiency. For the two versions of RSD, if the resulting game has a Nash equilibrium, I will show that the equilibrium outcome may not be a Pareto efficient matching. Therefore, the direct mechanisms implementing the Nash equilibrium of RSD mechanisms are not Pareto efficient.

Proposition 1. *RSD may have no Nash equilibrium. If a Nash equilibrium exists, it may implement Pareto inefficient matchings.*

Proof. For the first part, it is sufficient to find an house allocation problem and a preference profile in $\mathcal{P}$ such that RSD has no Nash equilibrium. Let us consider a problem with three students 1, 2, 3 and two houses, h_1 and h_2, with capacities $q_1 = 2$ and $q_2 = 1$. $\{1, 2\}$ forms a group of friends and the agents' preferences are given by:

$$\begin{aligned} (1,2): &\quad (h_1, h_1) \succ (h_1, h_2) \succ (h_2, h_1) \\ 3: &\quad h_1 \succ h_2 \end{aligned}$$

Reporting her true preferences is a dominant strategy for 3. If the couple reports that they both prefer h_1 over h_2, they will end up with probability 1/3 in each of their possible match. If 2 reports instead that she prefers h_2 over h_1, they will end up with probability one in (h_1, h_2). Both strategies are weak Nash equilibrium but none of them is a Nash equilibrium.

In the appendix, I present also another example where couples are indifferent when we reshuffle their allocation. This game has a Nash equilibrium and the equilibrium distribution over matchings puts positive mass on Pareto inefficient matchings. □

In the sequential RSD, on the contrary, there is always a subgame perfect Nash equilibrium because there is no uncertainty at the points where the players move. But as before, the resulting matching may not be Pareto efficient.

Proposition 2. *Sequential RSD does not implement Pareto efficient outcome in perfect Nash equilibrium.*

Proof. I keep the same problem as in the example of the appendix but suppose the preferences are the following:

$$\begin{aligned} (1,2): &\quad (h_2, h_2) \succ (h_1, h_1) \succ (h_2, h_1) \succ (h_2, h_3) \\ (3,4): &\quad (h_1, h_1) \succ (h_2, h_2) \succ (h_2, h_3) \succ (h_1, h_3) \\ 5: &\quad h_1 \succ h_3 \succ h_2 \end{aligned}$$

If we consider the order $(1, 3, 2, 5, 4)$, the outcome of the prefect Nash equilibrium is the matching $\mu((1, 2, 3, 4, 5)) = (h_1, h_1, h_2, h_2, h_3)$. It is strictly dominated by $\mu'((1, 2, 3, 4, 5)) = (h_2, h_2, h_1, h_1, h_3)$. The exact calculations are detailed in the appendix. □

This section highlighted the weakness of these classical implementations of RSD. In the next section, I will propose another mechanism inspired from RSD that will have improved incentive and efficiency properties.

5 The Random Serial Group Dictatorship Mechanism

In this section, I present a modified version of RSD, which allows students to submit preferences over the matchings. The lottery used in the mechanism does not produce a priority order over students, but instead a priority order over sets of students. It allows students of the same group of friends to move together to choose their most preferred allocation among the available ones after the groups with higher priorities made their choices. The following mechanism implements this procedure as a direct mechanism. I will call this mechanism the random serial group dictatorship mechanism and denote it ψ^{GD}.

Definition 12. *The **random serial group dictatorship mechanism**, Ψ^{GD}, associates for each profile of reported preferences, $\succsim \in \mathcal{Q}^{|S|}$, the outcome of the following algorithm:*

- ***Step 0:*** *For all $s \in S$ and $s' \in L(\succsim_s)$, if $s \notin L(\succsim_{s'})$ assign s to $\emptyset$. Define the groups as the set of students who are connected through a chain of friendship.*
- ***Step 1:*** *Randomly draw a priority ordering of the groups.*
- ***Step 2:*** *Allocate the students in g_1 according to one the most preferred matchings of one of its members. Reduce the set of feasible matchings to those respecting these allocations.*
- ***Step k:*** *Allocate the students in g_{k-1} according to one of the most preferred matchings of one of its members among the set of feasible matchings. Reduce further the set of feasible matchings to those respecting these allocations.*

The procedure terminates when the group with the lowest priority has chosen its assignment.

Step 0 ensures that the friendships are reciprocal. No student can report herself as a member of a group without the consent of a member of this group. However, the reported preferences may not belong to $\mathcal{P}$. Later on, I will show how modifying this step changes the properties of the mechanism.

As for RSD, I will define a similar sequential GD mechanism. In step 0 of this indirect mechanism, students report only their group of friends. As in the direct mechanism, it allocates students reporting a non-reciprocal friendship to the empty set, constructs and orders the groups in the same manner. After learning the outcome of the lottery and the allocation of the preceding groups, a member of the next group will choose a slot among the available ones for each member of this group. This indirect mechanism is a lot easier to implement in practice. Students don't have to rank all the possible matchings, which could quickly amount to an impractical number. Reporting a group of friends and picking an allocation for a subset of students amount to choose a set in a partition of $\mathcal{P}$. Therefore, I can extend the properties of direct mechanisms as explained in section 3. I will show that the results of this section are also valid for the sequential GD mechanism.

Proposition 3. *The GD mechanism is Pareto-efficient with respect to any reported preference profile in $\mathcal{P}$.*

Proof. The GD mechanism is efficient in this set-up for the same reasons as the RSD mechanism when there are strict preferences and no externality. Suppose by contradiction that there exist $\succcurlyeq \in \mathcal{P}$, $\nu \in \mathcal{M}$ and μ in the support of $\Psi^{GD}(\succcurlyeq)$, such that ν Pareto dominates μ. Let's order the groups of friends $g_1, ..., g_N$ according to one of the draws leading to μ. Definition 4 implies that there is a first group g_i such that all students in g_i strictly prefer ν over μ and such that all students of the preceding groups are indifferent between the two matchings. Therefore, assumption 2 implies that all groups before g_i are matched with the same set of houses under μ and ν. Hence, ν is feasible when the turn of g_i arrives. Assumption 1 implies that all groups weakly prefer μ over all matchings feasible when their turn arrives. Therefore, $\mu \succcurlyeq_{g_i} \nu$ which is a contradiction. □

The next proposition addresses the question of the incentives of the students. First, I will show that a student has an incentive to report truthfully her preferences over matchings if all other students report truthfully.

Proposition 4. *Truth-telling is a Nash equilibrium of the GD mechanism.*

Proof. Consider $\succcurlyeq \in \mathcal{P}$ and a student $s \in S$. If s lies such that the group of friends associated with her preferences doesn't change, her reported preferences will be used with positive probability and the outcome will be dominated. If she reports as a friend a student outside her true group of friends, the friendship won't be reciprocal, she will be allocated to the empty set and the outcome will be dominated. Finally, if she excludes a true friend from her reported group of friends, this student will be allocated to the empty set and because of assumption 2, the outcome will be dominated for s. □

This proof shows why the step 0 is important in the procedure described above. It ensures that a group of friends cannot be hijacked by another one. A student cannot pretend to be part of another group of friends without their consent. Note that if a member of a couple lies and reports herself as single, the other member of that couple has an incentive to lie also to avoid being allocated to the empty set. Hence, truthful reporting of the preferences cannot be a dominant strategy and therefore the mechanism is not strategy-proof.

Remark 2. *In the sequential game, if the group of friends of a student has been reported truthfully, it is a dominant strategy for her to report her preferences over matchings truthfully.*

This result seems appealing, especially when the mechanism designer already knows the groups of friends. It might be true if these groups are used also for other purposes, forcing students to report them truthfully. In this case, I could delete step 0 of the GD mechanism and ask students to report their preferences before or after the ordering of groups and the mechanism would be ex-post efficient and truth-telling would be a strictly dominant strategy. This is usually the mechanism the literature refers to as the extension of RSD to groups of agents.

However, in the general case, there is no reason why the mechanism should exclude the reporting of groups of friends from students' strategy space. As we will see in the next proposition the students can indeed use the reports of groups strategically to improve their distribution over the possible matchings. The example below shows that the members of a group may have an incentive to split in smaller groups. Because lotteries are taken over groups, splitting a group increases the number of draws of the groups. Moreover, the subgroup choosing first may be able to secure slots for the subgroups coming afterward. The combination of these two features may help the group to stochastically improve its allocation.

Proposition 5. *The GD mechanism is not strategy-proof and truth-telling is not a Nash equilibrium robust to coalitions of size* 1.

Proof. It is sufficient to show that there exist a house allocation problem, $\succcurlyeq \in \mathcal{P}$, $g_i \in G$ and $\succcurlyeq'_{g_i} \in \mathcal{Q}^{|g_i|}$ such that: $\Psi^{GD}(\succcurlyeq'_{g_i}, \succcurlyeq_{G\setminus g_i})$ stochastically dominates $\Psi^{GD}(\succcurlyeq)$ and the new strategies form a Nash equilibrium.

Consider a house allocation problem with two houses with two slots each, h_1 and h_2, and four students. The students are composed of two couples, which have the following preferences:

g_1: $(h_1, h_1) \succ (h_2, h_2) \succ (h_1, h_2)$
g_2: $(h_1, h_1) \succ (h_2, h_2) \succ (h_1, h_2)$

Suppose that the groups report their preferences truthfully. Each group is allocated to (h_1, h_1) with probability one half and to (h_2, h_2) with probability one half. However, if g_1 does not report itself as a group but as two singles who prefer h_1 over h_2, they will be allocated to (h_1, h_1) with probability two third and to (h_2, h_2) with probability one third. This last probability distribution stochastically dominates the first one. Besides, the new reports form a Nash equilibrium. $\square$

This proposition highlights also the importance of the punishment procedure in step 0.

Corollary 1. *Truth-telling is not a weak Nash-equilibrium of the mechanisms where there is no punishment procedure in step* 0 *and where groups are defined as either:*

- *The sets of students who are connected through a chain of reciprocal friendships*
- *The sets of students who have the same preferences over matchings.*

Proof. In the mechanism without the punishment procedure, the previous example still works if one student of g_1 reports $h_1 \succ h_2$ and the other one tells the truth. $\square$

On the contrary, making the punishment procedure stricter and forcing the students to report preferences in $\mathcal{P}$ doesn't solves the problem. Finally, this counter example works also for the sequential *GD* mechanisms.

Corollary 2. *The sequential GD mechanism is not strategy-proof and truth-telling is not a Nash equilibrium robust to coalitions of size* 1*.*

Splitting groups is not the only incentive to lie coordinately for groups of people. Students may also have an incentive to merge their groups of friends to get a better outcome as the following example shows.

Example 12. *Consider a house allocation problem with three houses with one slot each, h_1, h_2 and h_3 and three students. Students are singles and have the following preferences:*

$1 : h_1 \succ h_2 \succ h_3 \quad 2 : h_1 \succ h_3 \succ h_2 \quad 3 : h_3 \succ h_1 \succ h_2$

Suppose that the students report their preferences truthfully. Students 1 *is allocated to h_1 with probability one half and to h_2 with probability one half. Student* 3 *is allocated to h_3 with probability* $5/6$ *and to h_2 with probability* $1/6$. *However, if* 1 *and* 3 *collude and report themselves as a couple preferring (h_1, h_3) over (h_2, h_3) over the other matchings, student* 1 *won't change her distribution over houses and student* 3 *will be allocated to h_3 with probability one. These strategies form a Nash equilibrium of the one shot game as of the sequential one. Indeed,* 3 *cannot do better, and* 2 *gets the same outcome if she breaks the coalition and kicks* 3 *out of the game, whereas if she doesn't break the group, she cannot achieve a better outcome by changing her reports.*

In this example, student 3 doesn't get her first choice only in the draw where student 2 already chose it. Therefore, the preferences of students 1 and 3 are never conflicting and they can form a stable coalition. Such incentive can also stem from the ordinally inefficiency of RSD. Indeed, if two groups can improve their ex-ante allocation by exchanging some probabilities of allocation in some slots, they may be able to do so by merging before the start of the mechanism and submit common preferences. However, these coalitions may not verify the stability condition. This is for instance the case in the example presented in the introduction of Bogomolnaia and Moulin (2001).

In general, it seems that the students need a lot of information to use profitably the weak incentive properties of the GD mechanisms. In the next subsection, I will study this point in greater details. I will derive sufficient conditions on the structure of the house allocation problem and on students' preferences so that these bad incentives disappear when the market is big enough.

5.1 Truth-telling in Large Markets

In several real matching markets, even if the mechanism is not strategy-proof, it can perform well in some markets because the incentives to lie disappear as the market increases in size. There are many different manners to define a large market. In my case, it could either have very large capacities q_{h_i} and a small number of houses or a large number of houses but small capacities. On the students' side, I face the same problem. I could have a small number of very large groups or a very large number of small groups. In the matching market literature most of the problems fit well to the first solution with large capacities. It's also simpler because students' preferences stay the same as the size of the market increases. Whereas if I add houses, markets with different sizes have different preference profiles. Moreover, if I keep small capacities, I can built a counterexample for any market size by repeatedly duplicating the counterexample of proposition 5. It creates an artificial global market made of infinity many small markets. In such a market, each group has still an incentive to lie on its composition to get a better probability distribution over matchings.

Nevertheless, in our case, it makes less sense to let the houses' capacities tend to infinity. First of all, on the students' side of the market, it seems natural to let the number of groups tend to infinity and not the number of students per group. Apart from the fact that reporting a close to infinite number of friends seems unrealistic, I supposed that friends negotiate to agree on a common preference order. This hypothesis is less realistic when there is a very large number of students in each group of friends. But if groups of friends are finite, it also makes more sense to have small capacities for houses. If the sizes of the groups of friends were negligible compared to the sizes of the houses, there may be few benefits for friends to be in the same house. If they end up in the same house but in rooms far from each other, the positive externality may disappear. The solution would be then to divide the big houses in smaller areas so that friends could live in the same area of the house. But it would be equivalent to the case of an large number of houses.

The counterexample constructed by duplicating the market presented in the proof of proposition 5 shows that I need additional hypotheses to ensure that truth-telling is a Nash equilibrium robust to coalitions of size one. In particular, the demand for slots in a house desired by a group of friends must be high enough, even when this house is almost full. Therefore, I will suppose that there are enough singles also desiring a slot in this house.

Proposition 6. *For N large enough, truth-telling is a Nash equilibrium robust to coalitions of size one if for each group g_i with more than two students, there are k_i sets of houses $H_{i,1}, ..., H_{i,k_i}$ with a total of q_i slots such that:*

1. *g_i prefers being allocated together in one of the $H_{i,j}$ over all other matchings.*
2. *The number of singles preferring a slot in $H_{i,1}, ..., H_{i,k_i}$ over the rest, $\alpha_i(N)$, is such that $N = o(\alpha_i(N)^2)$ and $q_i = o(\frac{\alpha_i(N)^2}{N \ln(\alpha_i(N))})$.*

Proof. It is sufficient to show that for N large enough the students of a same group of friends have a higher probability to be allocated together in one the $H_{i,j}$ when they say the truth than when they split. The proof is rather technical and can be found in the appendix. The intuition why the strategies used in the proof of proposition 5 don't work is rather simple. If there are enough singles, when a group splits in two parts, the highest ranked subgroup cannot block with high probability slots in one of the best houses for the lowest ranked subgroup. The singles will fill these slots before the second subgroup can make its choice. □

Condition 1 ensures that for each group I am be able to separate H into a small set of good houses and a big one of bad houses. The requirement on the group' preferences allows for two special cases. One in which $k_i = 1$ and $|H_{i,1}| = 1$, the group prefers then to be allocated together in one house. The other in which $k_i = 1$ and $H_{i,1} = H$, the group prefers then to have all its members allocated rather than having one member matched to $\emptyset$. In each case, the number of singles with similar preferences must be large enough in comparison to the number of groups and size of the set of good houses. $H_{i,j}$ can be interpreted as a meta-house to which the members of g_i want to be allocated together. In practice, $H_{i,j}$ might be a campus with different dormitories or a dormitory separated in different areas. In the case of refugees allocation, $H_{i,j}$ might be a country with different cities or a city with different accommodation centers. If μ allocate all members of g_i together in $H_{i,j}$ and ν allocates them together in $H_{i,j'}$, the order of preference between them is not restricted. It is also true if none of these matchings allocate them together in one of the $H_{i,j}$. The conditions stated in the previous proposition rules out markets where the popular houses for couples and singles are different. Indeed, if a dormitory has for example only double beds rooms, couples have a clear incentive to split. If the first partner choose their most preferred double bed room, and nobody until the other partner have an interest to choose the remaining slot in this room.

Corollary 3. *For N large enough, truth-telling is a Nash equilibrium of the GD mechanism robust to coalitions of size one if the fraction of singles is bounded away from* 0 *and either:*

- *The number of rooms is smaller than $N/\ln(N)^2$ and all rooms are desirable.*
- *There is a set of houses with less than $N/\ln(N)^2$ rooms where all groups prefer being allocated together rather than having one member outside.*

The conditions presented in proposition 6 are not sufficient to ensure that truth-telling is a Nash equilibrium robust to coalitions of size 2. Indeed, the counterexample constructed by duplicating the markets presented in example 12 verify these conditions. To ensure that the market is not a simple aggregation of small markets, I need some sort of correlation in the preferences of students. However, there is no clear definition of correlations as the groups are composed of different numbers of students. To do so, I will use the restriction of the previous proposition and suppose that the collection of meta-houses defined before must be the same for all groups of students.

Lemma 1. *In the GD mechanism, no coalition formed of l merged groups of students is stable if there are k sets of houses $H_1, ..., H_k$ with a total of q slots, such that:*

1. *All groups prefer being allocated together in one of the H_j over all other matchings.*
2. *For all $k \neq k'$, the number of singles preferring the houses in H_k over the houses in H'_k, $\alpha(k, k')$, is bigger than q.*

Proof. It is sufficient to show that for N large enough, if l groups merge, one of them has a higher probability to see all its members allocated together in one of the H_j when she switches back to the truth. The intuition why the strategies used in the example 12 aren't stable is the following. Under the given hypothesis, when two groups merge, for some order, they won't agree on how to allocate the slots that are still available. Therefore, each group has an incentive to deviate from the first lie and not to report its coalition partner as a friend. By doing so, it will allocate its partner to $\emptyset$ and will be able to improve its allocation for the draw where there is a disagreement without changing the allocation for the other draws. $\square$

These two conditions includes the cases described in corollary 3 with weaker restrictions on the number of singles and slots. If $k = 1$, there is a set of houses such that all groups prefer to have all their members allocated in one of its slots rather than having one member allocated outside. In this case, the condition 2 reduces to the requirement that the number of singles is bigger than the number of slots in H_1. If $k > 1$, the restrictions impose that the preferences of the singles over the houses in $H_1, ..., H_k$ are diverse enough.

Proposition 7. *Let $r < 1$. For N large enough, truth-telling is a Nash equilibrium of the GD mechanism robust to coalitions of size $r * \alpha(N)$ if there are k sets of houses $H_1, ..., H_k$ with a total of q slots such that:*

1. *All groups prefer being allocated together in one of the H_j over all other matchings.*
2. *The number of singles, $\alpha(N)$, is such that $N = o(\alpha(N)^2)$ and $q = o(\frac{\alpha(N)^2}{N \ln(\alpha(N))})$.*
3. *For all $k \neq k'$, the number of singles preferring the houses of H_k over those of H'_k, $\alpha(k, k')$, is bigger than $q + l$.*

Proof. Let $l \leq r * \alpha(N)$ with $r < 1$. For N large enough, the number of singles not colluding is such that $N = o(\alpha(N)^2)$ and $q = o(\frac{\alpha(N)^2}{N \ln(\alpha(N))})$. Therefore, proposition 6 ensures that all students of the same group report themselves as part of the same group. In particular, all groups reported in the coalition must be the result of the merging of true groups. Lemma 1 guarantees that all merged groups have an incentive to switch back to the truth because among the $\alpha(k, k')$ singles, the number not colluding is bigger that q. Therefore, in large markets, telling the truth is a Nash equilibrium robust to size l coalitions. $\square$

Corollary 4. *Under the same hypotheses as proposition 7, truth-telling is a Nash equilibrium of the sequential GD mechanism robust to coalitions of size $r * \alpha(N)$.*

Proof. In the proof of proposition 6 and lemma 1, the groups' strategies can actually depend on the draw they received and on the choices of the preceding groups. □

6 The Random Serial Bossy Mechanism

To address the shortcomings of the random serial group dictatorship mechanism identified above, I propose a new mechanism that is efficient and strategy-proof. This mechanism can be seen as another natural extension of the random serial dictatorship to the domain of preferences studied. As in the RSD, the lottery used by the mechanism designer produces a priority order over students. When the turn of a student arrives, she is made a full dictator again and can choose one of her preferred outcomes among the feasible ones. The following mechanism implements this procedure as a direct mechanism. I will call it the random serial bossy mechanism and denote it ψ^{RSB}.

Definition 13. *The* ***random serial bossy mechanism****,* Ψ^{RSB}*, associates for each profile of reported preferences,* $\succcurlyeq \in \mathcal{Q}^{|S|}$*, the outcome of the following algorithm:*

- ***Step 0:*** *Randomly draw a priority ordering of the students.*
- ***Step 1:*** *Consider the highest ranked student* s_1*. Allocate the students of* $L(\succcurlyeq_{s_1})$ *according to one of its most preferred matchings. Reduce the set of feasible matchings to those respecting these allocations.*
- ***Step k:*** *Consider the* k^{th} *highest ranked student* s_k*. If* s_k *has been allocated in an earlier step, move to step* $k+1$*. Otherwise, allocate the students of* $L(\succcurlyeq_{s_k})$ *according to one of its most preferred matchings among the feasible ones. Reduce further the set of feasible matchings to those respecting these allocations.*

The procedure terminates when all students have been assigned.

As before, I will define a sequential RSB mechanism that will be easier to implement in practice. In this mechanism, students don't report their preferences but choose sequentially the allocation of all the students they care about. I will show that the results of this section are also true for this sequential mechanism. For the same reason as for the GD mechanism, the RSB mechanism is efficient in my set-up.

Proposition 8. *The RSB mechanism is Pareto-efficient with respect to any reported preference profile in* $\mathcal{P}$

Proof. The proof is almost the same as the proof of proposition 3. Suppose by contradiction that there exist $\succcurlyeq \in \mathcal{P}$, $\nu \in \mathcal{M}$ and μ in the support of $\Psi^{SRB}(\succcurlyeq)$, such that ν Pareto dominates μ. Let's order the groups of friends $g_1, ..., g_N$ according the draw of their first member in one of the orderings that lead to μ. Definition 4 implies that there is a group g_i whose members strictly prefer ν over μ and such that all students of the preceding groups are indifferent between the two matchings. Therefore, assumption 2 implies that all groups before g_i are matched with the same set of houses under μ and ν. Hence, ν is feasible when the turn of g_i arrives. Assumption 1 implies that all groups weakly prefer μ over all matchings feasible when their turn arrives. Therefore, $\mu \succcurlyeq_{g_i} \nu$ which is a contradiction. □

Proposition 9. *The RSB mechanism is strategy-proof.*

Proof. Consider $\succcurlyeq \in \mathcal{P}$, student $s \in S$ and $\succcurlyeq' \in \mathcal{Q}^{|S|}$. If s is ranked lower than any s' such that $s \in L(\succcurlyeq'_{s'})$, then s can only be weakly worse off by lying. Otherwise, the report that s makes is irrelevant to the resulting allocation and s is indifferent between truth-telling and lying. Thus, telling the truth sd-dominates any other reports. □

As the GD mechanism, the RSB mechanism is a bossy mechanism: a student can change the allocation of another student without changing its own. But contrary to the GD mechanism, the RSB mechanism doesn't ask for the consent of these students. This is innocuous when students tell the truth because assumption 1 ensures that a students wants only to allocate another student if they share the same preferences. But if this assumption is slightly violated, it may create inefficiencies as the first student may choose a Pareto dominated matching among the set of her most preferred matchings or a matching that is not individually rational for all students. If I want to ensure the consent of the people allocated, it would come at the expense of strategy-proofness, since student s may decide to acquiesce to the lie of one of her friend to be allocated with her.

Contrary to the RSB mechanism, the GD mechanism uses a two-steps procedure, where the priority ordering depends on information contained in the preferences. It allows the students to affect the lottery with their reports, which creates incentives to deviate from truth-telling. These incentives are not present in the RSB mechanism.

Proposition 10. *Truth-telling is a Nash equilibrium of the RSB mechanism robust to coalitions of size one.*

Proof. Consider $\succcurlyeq \in \mathcal{P}$, a group $g_i \in G$ and $\succcurlyeq' \in \mathcal{Q}^{|g_i|}$ such that all members of the group are strictly better off. Fix the order of the students and denote s_1 the first student of

g_i. Because all other groups tell the truth, no student of g_i has been allocated before s_1. Because all members of the coalition have the same preferences as s_1, she can only make them weakly worse off by reporting $\succcurlyeq'_{s_1}$ independently of their reports. Thus, for any order, misreporting can only make all members weakly worse off. Therefore, the resulting distribution cannot strictly first order stochastically dominate the distribution resulting from truth-telling. □

But again, students have an incentive to jointly misreport their preferences. This incentive comes from the bossiness of the mechanism and the fact that forming a bigger group allows students to achieve a better position in the lottery. Since each member of a group of friends has a draw in the lottery, larger groups are likely to choose earlier their allocation than smaller groups.

Proposition 11. *In any house allocation problem:*

- *Its is a dominant strategy for all students to report a larger group of friends*
- *Truth-telling is a Nash equilibrium robust to coalitions of size two if and only if for all $g \in G$, once g has chosen its allocation, the remaining students agree on the most preferred feasible matching.*

Proof. Let N, H, S be a house allocation problem and $\succcurlyeq \in \mathcal{P}$ the preference profile of the students. For all $g_i \in G$, g_i is weakly better off by submitting preferences $\succcurlyeq'_{g_i}$ where the indifferences in $\succcurlyeq_{g_i}$ are resolved according to the preferences of $s' \notin g_i$.

To prove the sufficient part of the second item, suppose that g_1 and g_2 contemplate to form a coalition. Fix the order of the students. If g_1 and g_2 don't choose first, both get their most preferred matching given the choice of the first group. Therefore, a coalition can only make them weakly worse off. If g_i chooses first, to strictly improve the allocation of g_j, g_i must change the number of seats chosen in one house. But assumption 2 implies that it will be strictly worse off. To make up for this loss, g_j must change the number of seats chosen in one house in an order where it chooses first. Finally, the coalition can at best exchange some of the orders in which they get their first choice and none of them can be strictly better off.

To prove the necessity part of the second item, suppose that there is $g_1 \in G$ such that when it chooses first, the remaining students don't agree on the most preferred feasible matching. There is a group g_2 whose allocation strictly worsens when the order changes from g_1 first and g_2 second to g_1 first and g_2 last. Therefore, if the two groups report the deviation explained in the first paragraph, the allocation of the deviating coalition will be strictly improved. □

Corollary 5. *The previous propositions are also verified by the sequential RSB mechanism.*

Corollary 6. *Under the assumptions described in lemma 1 or proposition 6 or 7, truth-telling is not robust to coalitions of size two.*

Proof. Because the number of singles preferring one of the most popular houses is bigger than the capacity of this set of houses, the singles don't agree on the most preferred matching once a single has made her choice. □

The deviation strategies are very robust. They form a new equilibrium in dominant strategy. Even more problematic, any two groups can form a coalition and weakly improve the allocation of their members for any realizations of the lottery. Therefore, the information needed to implement this strategy is quasi null. In the sequential mechanism, the groups don't even need to know each other's preferences. The first group chooses its best matching and names any group as its direct follower. These incentives to merge groups of friends appear in practice in virtually all house allocation problems of interest.

Some modification of this mechanism might mitigate the incentive to build coalitions. A first solution would be to check that the submitted preference profile belongs to $\mathcal{P}$. However, such a test would be not implementable in practice due to the size of the set. Another solution would be to add a punishment procedure, similar to the step 0 of the GD mechanism. A third solution would be to let random friend of s_k choose the allocation when s_k's turn arrives. It would incentivize groups of a coalition with partially conflicting preferences to break it. The mechanism would lose its strategy-proofness, but truth-telling would be still a Nash equilibrium robust to coalitions of size one. However, example 13 shows that two groups might sill improve their allocation by building a stable coalition, and that the incentive might still be stronger than in the GD mechanism.

Example 13. *Consider once again the house allocation problem presented in example 12. There are three houses with one slot each, h_1, h_2 and h_3 and three students. Students are singles and have the following preferences:*

$1 : h_1 \succ h_2 \succ h_3 \quad 2 : h_1 \succ h_3 \succ h_2 \quad 3 : h_3 \succ h_1 \succ h_2$

In the GD mechanism only $(1,3)$ had an incentive to lie. In the RSB mechanism without a punishment procedure, $(1,2)$ and $(2,3)$ also have an incentive to report themselves as a couple. The random allocation when they tell the truth is the same as before:

	h_1	h_2	h_3
1:	1/2	1/2	0
2:	1/2	1/3	1/6
3:	0	1/6	5/6

If 1 *and* 3 *report themselves as a couple with the following preferences* $(1,3) : (h_1, h_3) \succ (h_2, h_3)$*, the resulting random allocation is the following:*

	h_1	h_2	h_3
1:	2/3	1/3	0
2:	1/3	2/3	0
3:	0	0	1

The new random assignment is strictly better for both of them. Contrary to the coalitions $(1,2)$ *and* $(2,3)$*, the coalition* $(1,3)$ *is stable even if we add a punishment procedure or if the friends of a student choose her allocation. Besides, the incentive to lie is even greater than in the GD mechanism.*

7 Conclusion

This paper extends the classical house allocation problem to a restrictive case of non-selfish agents. An agent can express preferences over another student's allocation only if they share the same preferences over matchings. The focus is set on the random serial dictatorship and on how it can accommodate this set-up.

Modifying this mechanism to allow agents to form a group and to participate as such in a random serial dictatorship is a natural extension. It has already been mentioned in the literature, but never analyzed in detail. In particular, the fact that groups have to be reported to the mechanism has been overlooked. I show that this procedure, which I call the random serial group dictatorship, is efficient and that truth-telling is a Nash equilibrium of the induced game. However, this last property relies on a punishment procedure whose implementation might be questioned. Moreover, the mechanism incentivizes students to collectively lie to improve their allocation. But I show that these incentive problems disappear in large markets if there is enough competition for the most preferred houses. Therefore, they may be irrelevant in practice.

I also present a new mechanism, which pushes the power of the dictator to the limit and allows her to select the set of matchings she likes the most. This mechanism has stronger incentive properties when only agents with the same preferences can collectively deviate. However, increasing the number of agents he cares about is a weakly dominant strategy for all students and is a profitable deviation for two agents in almost all markets of interest. This could offset its first advantage over the random serial group dictatorship and could explain the large prevalence of the latter in real life.

This kind of externalities occurs in several matching markets. The market for students' accommodation used throughout the paper is a good illustrative example. Another relevant case is the allocation of asylum-seeker to different regions of a larger geographic area. The European Union is trying to reform its asylum policy and to create quotas of refugees for each of its members. This would require an allocation mechanism. But a country offering asylum to someone has to offer also asylum to his family. The two problems are then equivalent if we replace the groups of friends by asylum-seekers' families and houses by the member states or smaller geographic areas. Because the family ties are in this case often difficult to prove, the reporting of the groups would be a relevant step of this mechanism. Besides, the number of participants and the popularity of a few geographic regions likely guaranty that the conditions for the large market results hold. This would support the application of the random serial group dictatorship.

A lot of questions remain open and would require further investigations. The paper highlights the importance of the procedure punishing non consistant reports and the lottery ordering the students. These two aspects open the door to new mechanisms that could outperform those presented here. Finally, it would be interesting to study how the definition of large market of this paper relates to the more classic ones and how envyfreeness can be extended beyond the equal treatment of equal, which is obviously verified here.

8 Appendix

Proof. **Proposition 1**. Let us consider an environment with five students 1, 2, 3, 4, 5 and three houses, h_1, h_2 and h_3 with capacities $q_1 = 2$, $q_2 = 2$ and $q_3 = 1$. There are two couples $\{1,2\}$ and $\{3,4\}$ and a single 5. The agents' preferences are given by:

$(1,2):\quad (h_2,h_2) \succ (h_1,h_3) \sim (h_3,h_1) \succ (h_2,h_3) \sim (h_3,h_2)$

$(3,4):\quad (h_1,h_1) \succ (h_2,h_2) \succ (h_1,h_2) \sim (h_1,h_2) \succ (h_1,h_3) \sim (h_1,h_3)$

$5:\quad h_1 \succ h_2 \succ h_3$

I will show that the following reports form a Nash equilibrium:

1, 2: $h_2 \succ h_1 \succ h_3$ 3, 4, 5: $h_1 \succ h_2 \succ h_3$

Reporting $h_1 \succ h_2 \succ h_3$ is a best reply for 5 because she has no externality and truthful report is then a dominant strategy.

Reporting $h_1 \succ h_2 \succ h_3$ is a best reply for 3 and 4. It leads to the following distribution over matchings: $P(h_1,h_1) = 1/3$, $P(h_1,h_2) = P(h_2,h_1) = 2/15$ and $P(h_1,h_3) = P(h_3,h_1) = 1/5$. If one of them reports h_3 before h_1 or h_2, for some lottery outcomes, she will receive h_3 instead of h_1 or h_2. So the probability to end up in (h_1,h_1), (h_2,h_2) or (h_1,h_2) will decrease. The resulting distribution will be stochastically dominated. If 3 exchanges h_1 and h_2, the outcome is equivalent to the previous one where I permute h_1 and h_2 as well as the players $(1,2)$ and $(5,4)$. The probability that 3 or 4 ends up in h_3 stays the same, the probability to end up in (h_1,h_1) is $1/15$, and $P(h_1,h_2) = P(h_2,h_1) = 2/15$. Therefore, the resulting distribution is stochastically dominated by the original one.

Reporting $h_2 \succ h_1 \succ h_3$ is a best reply for 1 and 2. It leads to the following distribution over matchings: $P(h_2,h_2) = 3/5$ and $P(h_2,h_3) = P(h_3,h_2) = 1/5$. Because they never get h_1, exchanging it with h_3 doesn't change the distribution. If one of them reports h_3 first, she will with probability one end up in h_3 and the outcome will be stochastically dominated. If 1 reports $h_1 \succ h_2 \succ h_3$, the probability to end up in (h_2,h_2) is $1/5$, $P(h_1,h_3) = 1/10$ and $P(h_3,h_1) = 0$. Therefore, the resulting distribution is stochastically dominated by the original one.

But if I consider the order $(5,3,4,1,2)$, the resulting matching $\mu((1,2,3,4,5)) = (h_2,h_3,h_1,h_2,h_1)$ is strictly dominated by $\mu'((1,2,3,4,5)) = (h_1,h_3,h_2,h_2,h_1)$. □

Proof. **Proposition 2**. Here, the strategies consist of the choice of a house for each history of the game, i.e. for each lottery outcome and choice of the preceding students. A backward induction shows that for each lottery outcome there is a subgame perfect Nash equilibrium. I will describe the one following the lottery outcome $1,3,2,5,4$. The first thing to notice is that if 1 chooses h_2, 3 will also choose h_2 because her partner 4 is so low ranked that she wont succeed to secure a place in their preferred house h_1. Indeed, when 1 chooses h_2, if 3 chooses h_1, 2 will choose h_2, 5 will choose h_1 and 4 will end up in h_3. Hence, $(3,4)$ will be assigned to (h_1, h_3). Whereas if 3 chooses h_2, 2 will choose h_1, 5 will choose h_1, and 4 will end up in h_3. Hence, $(3,4)$ will be assigned to (h_2, h_3) and $(1,2)$ to (h_2, h_1). On the other hand, if 1 chooses h_1, 3 will choose h_2, 2 will choose h_1. Hence, $(1,2)$ will end up with (h_1, h_1) and $(3,4)$ with (h_2, h_2). This last matching is better for 1 so it will be the equilibrium matching. □

I will first show proposition 6 with stronger assumptions and generalize the result afterwards. I will first suppose that:

Assumption 3. *For N big enough:*

1. *There is a house h_1 with $\max_{i \in G_N} |g_i| \leq q_{h_1}$ such that all groups prefer being together in this house over all other alternatives.*

2. *There is $\alpha > 0$ such that the fraction of singles $\alpha(N)/N \geq \alpha$ and $q_{h_1} = o(\frac{\alpha(N)}{\ln(\alpha(N))})$*

Proof. **Lemma 1**. I want to prove that for N large enough, the students of a same group of friends have a higher probability to be allocated to their top choice when they say the truth than when they split and report then selves in different groups.

Consider a group g who splits in $l \in \{2, ..|g|\}$ groups $\{g_1, ..., g_l\}$ which report themselves as part of l groups $\{g_{11} \cup g_1, ..., g_{ll} \cup g_l\}$. Fix the order of all other reported groups O_{-1}. I will denote $k_{O_{-1}}(|g|)$ the biggest draw such that there are $|g|$ free slots in h_1. If g switches back to the truth, $\{g_{11}, ..., g_{ll}\}$ will be allocated to $\emptyset$ because of step 0 and the probability for all students of g to end up in h_1 is:

$$\begin{aligned} P_T(h_1) &= \sum_{O_{-1}} P_T(h_1 \mid O_{-1}) * P(O_{-1}) \\ &= \sum_{O_{-1}} \frac{k_{O_{-1}}(|g|)}{N - l + 1} * P(O_{-1}) \end{aligned}$$

If g sticks to her lie, the strategy that maximizes the chance of all students of g to be allocated together in h_1 is for $\{g_{11} \cup g_1, ..., g_{ll} \cup g_l\}$ to rank allocations where the students of g_i end all in h_1 in the first position. The probability for g to end up in h_1 is then:

$$P_L(h_1) = \sum_{O_{-1}} P_L(h_1 \mid O_{-1}) * P(O_{-1})$$

$$P_L(h_1 \mid O_{-1}) = P(\{g_1 \in h_1\} \cap ... \cap \{g_l \in h_1\} | O_{-1})$$

The allocation of rank in the algorithm is equivalent to the following allocation of rank: first randomly pick O_{-1}, then randomly pick $\{r_1, ..., r_l\} \in \{1, ..., N - l + 1\}$ with replacement and assign all $g_{ii} \cup g_i$ and $g_{jj} \cup g_j$ such that $r_i = r_j$, in a random order in the r_i^{th} position in O_{-1}. If all g_i end up in h_1, $\min\{r_i\} \leq k_{O_{-1}}(|g|)$. Otherwise, there are at most $|g|-1$ slots left in h_1 when the turn of the first group arrives. Therefore, there will be not enough slots left for the other ones when they have to choose later on. It must also be that $\max\{r_i\} \leq S_{O_{-1}}(|g|)$, the rank of the $(q_h - |g|)^{th}$ single in O_{-1}. Otherwise, there are not enough slots left when the turn of the last group arrives. Therefore, I can write:

$$\begin{aligned} P_L(h_1 \mid O_{-1}) &\leq \left(\frac{S_{O_{-1}}(|g|)}{N-l+1}\right)^l \left(1 - \left(1 - \frac{k_{O_{-1}}(|g|)}{S_{O_{-1}}(|g|)}\right)^l\right) \\ &\leq k_{O_{-1}}(|g|) * l * \frac{S_{O_{-1}}(|g|)^{l-1}}{(N-l+1)^l} \end{aligned}$$

The last inequality comes from the factorization of $a^n - b^n$ and $k_{O_{-1}}(|g|) \leq S_{O_{-1}}(|g|)$. If splitting the groups stochastically dominates truth-telling:

$$\begin{aligned} P_T(h_1) - P_L(h_1) &\leq 0 \\ \sum_{O_{-1}} \frac{k_{O_{-1}}(|g|)}{N-l+1} \left(1 - l\left(\frac{S_{O_{-1}}(|g|)}{N-l+1}\right)^{l-1}\right) * P(O_{-1}) &\leq 0 \\ \sum_{O_{-1}|1-l\left(\frac{S_{O_{-1}}(|g|)}{N-l+1}\right)^{l-1} \geq L(N)} L(N) * P(O_{-1}) - \sum_{O_{-1}|1-l\left(\frac{S_{O_{-1}}(|g|)}{N-l+1}\right)^{l-1} < L(N)} (l-1) * P(O_{-1}) &\leq 0 \\ P\left(O_{-1} \mid S_{O_{-1}}(|g|) \leq \left(\frac{1-L(N)}{l}\right)^{\frac{1}{l-1}} (N-l+1)\right) - & \\ \frac{l-1}{L(N)} P\left(O_{-1} \mid S_{O_{-1}}(|g|) > \left(\frac{1-L(N)}{l}\right)^{\frac{1}{l-1}} (N-l+1)\right) &\leq 0 \end{aligned} \tag{1}$$

Where $L(N) \in [0;1)$. Moreover:

$$P\left(O_{-1} \mid S_{O_{-1}}(|g|) > f(N)\right) \leq P\left(O_{-1} \mid \exists \text{ less than } q_h - |g| \text{ singles before rank } f(N)\right)$$

Define the set of events A_k as the set of all O_{-1} such that there are k singles before rank $f(N)$ when there are N groups and $\alpha(N)$ singles. If $f(N) - k > N - \alpha(N)$ or $\alpha(N) < k$, I have $\#(A_k) = 0$. If $k \leq 1$:

$$\#(A_k) = \binom{\alpha(N)}{k} * \binom{N - \alpha(N)}{f(N) - k} * f(N)! * (N - f(N))!$$

$$\begin{aligned}
P(A_k) =& \frac{1}{N!} * \frac{(\alpha(N))!}{k!(\alpha(N)-k)!} * \frac{(N-\alpha(N))!}{(f(N)-k)!(N-\alpha(N)-f(N)+k)!} * f(N)! * (N-f(N))! \\
=& \frac{\alpha(N)\cdots(\alpha(N)-k+1)}{k!} * \frac{(N-\alpha(N))\cdots(N-\alpha(N)-f(N)+k+1)}{N\cdots(N-f(N)+1)} \\
& * f(N)\cdots(f(N)-k+1) \\
\leq & \frac{1}{k}\left(\frac{\alpha(N) * f(N)}{N-\bar{m}(N)}\right)^k \left(\frac{N-\bar{m}(N)}{N+1-\underline{m}(N)}\right)^{\underline{m}(N)}
\end{aligned}$$

Where $\bar{m}(N) = \max\{\alpha(N), f(N)\}$ and $\underline{m}(N) = \min\{\alpha(N), f(N)\}$. Besides, $\bar{m}(N) + \underline{m}(N) = \alpha(N) + f(N)$. The formula extends to $k = 0$ if the first term is replaced by 1.

In the case considered here, the number of groups is $N - l$, the number of remaining singles $\alpha(N-l)$ and the limit rank $f(N) = (\frac{1-L(N)}{l})^{\frac{1}{l-1}}(N-l+1)$. Because $l \geq 2$, I can pick $L(N) \geq \epsilon > 0$ such that $(\frac{1-L(N)}{l})^{\frac{1}{l-1}}(N-l+1) = \frac{\alpha(N-l)}{2}$. It implies that $\underline{m}(N) = \alpha(N-l)/2$, $\bar{m}(N) = \alpha(N-l)$ and therefore:

$$\frac{l-1}{L(N)} * q_h * P(A_{q_h}) = \frac{l-1}{L(N)} e^{q_h \ln\left(\frac{\alpha(N-l)^2}{2(N-l-\alpha(N-l))}\right) - \frac{\alpha(N-l)}{2}\ln\left(1+\frac{\alpha(N-l)+2}{2(N-l-\alpha(N-l))}\right)}$$

Which goes to 0 as N goes to $+\infty$ because $l \leq q_h = o(N)$, $q_h = o(\frac{\alpha(N)}{\ln(\alpha(N))})$ and $\alpha(N)/N \geq \alpha > 0$. Moreover, for N big enough, it is increasing in q_h and finally:

$$P(O_{-1} \mid \exists \text{ less than } q_h \text{ singles before rank } f(N)) = \sum_{k=0}^{q_h-1} P(A_k) < q_h * P(A_{q_h})$$

$$\frac{l-1}{L(N)} P\left(O_{-1} \mid S_{O_{-1}}(|g|) > \left(\frac{1-L(N)}{l}\right)^{\frac{1}{l-1}}(N-l+1)\right) \xrightarrow[N\to\infty]{} 0$$

$$P\left(O_{-1} \mid S_{O_{-1}}^{|g|} \leq \left(\frac{1-L(N)}{l}\right)^{\frac{1}{l-1}}(N-l+1)\right) \xrightarrow[N\to\infty]{} 1$$

Which contradicts (1).

I can easily generalize this proof to the conditions presented in proposition 6 by replacing $k_{O_{-1}}(|g|)$ by the last draw such that there are $|g|$ slots left in one of the H_k and replace q_{h_1} by the sum of the capacities in $H_1, ..., H_K$. The proof is also still valid in the cases where the proportion of singles goes to 0 as long as $\alpha(N)^2/N$ goes to infinity and $q_h = o(\frac{\alpha(N)^2}{N\ln(\alpha(N))})$. □

Proof. **Lemma 1**. I keep the same notations as in the proof of proposition 6. The only change is that l groups $\{g_1, ...g_l\}$, with $|g_1| \leq ... \leq |g_l|$, merged to form a group g of size $\sum_i |g_i|$. If g_i switches back to the truth all the other groups will be allocated to $\emptyset$ because of the punishment procedure. The probability for g_i to be allocated to h_1 is therefore:

$$P_T(g_i \in h_1 \mid O_{-1}) = \frac{k_{O_{-1}}(|g_i|)}{N-l+1}$$

If they all stick to their lie and merge, in the best case, if they have a draw below $k_{O_{-1}}(\sum_i |g_i|)$ they all end up in h_1. For each draw between $k_{O_{-1}}(\sum_i |g_i|)$ and $k_{O_{-1}}(\sum_{i=1}^{l-1} |g_i|)$, at most $l-1$ of them end up in h_1. If their draw is between $k_{O_{-1}}(|g_2|+|g_1|)$ and $k_{O_{-1}}(|g_1|)$, g_1 ends up alone in h_1. Therefore:

$$\begin{aligned}\sum_i P_L(g_i \in h_1 \mid O_{-1}) \leq & l * \frac{k_{O_{-1}}(\sum_i |g_i|)}{N-l+1} + \sum_{i=1}^{l-1} i * \frac{k_{O_{-1}}(\sum_{j=1}^{i} |g_j|) - k_{O_{-1}}(\sum_{j=1}^{i+1} |g_j|)}{N-l+1} \\ \leq & \sum_{i=1}^{l} \frac{k_{O_{-1}}(\sum_{j=1}^{i} |g_j|)}{N-l+1}\end{aligned}$$

If no group wants to switch back to the truth:

$$\begin{aligned}\sum_{O_{-1}} \sum_i \left(\frac{k_{O_{-1}}(|g_i|)}{N-l+1} - \frac{k_{O_{-1}}(\sum_{j=1}^{i} |g_j|)}{N-l+1} \right) P(O_{-1}) \leq & 0 \\ \sum_i \sum_{O_{-1}} \left(k_{O_{-1}}(|g_i|) - k_{O_{-1}}(\sum_{j=1}^{i} |g_j|) \right) P(O_{-1}) \leq & 0 \\ \forall i \geq 2 : P \left(O_{-1} \mid k_{O_{-1}}(|g_i|) > k_{O_{-1}}(\sum_{j=1}^{i} |g_j|) \right) = & 0 \end{aligned} \tag{2}$$

Because $|g_i| \leq \sum_{j=1}^{i} |g_j|$ and $k_{O_{-1}}(i+1) \leq k_{O_{-1}}(i)$.

In particular, if there are $s \geq 2$ singles among the l groups:

$$P\left(O_{-1} \mid k_{O_{-1}}(1) > k_{O_{-1}}(s)\right) = 0$$

If $s > q_{h_1}$, it is impossible because $k_{O_{-1}}(1) \geq 1 > 0 = k_{O_{-1}}(s)$. Otherwise,

$$P\left(O_{-1} \mid k_{O_{-1}}(1) > k_{O_{-1}}(s)\right) \geq P(\exists \text{ a single in rank } k_{O_{-1}}(s))$$

Besides, there must be at least $\alpha(N-l) - q_h + s = \alpha(N) - q_h$ singles after $k_{O_{-1}}(s)$. Therefore:

$$P(\exists \text{ a single in rank in } k_{O_{-1}}(s)) \geq \sum_k \frac{\alpha(N) - q_h}{N-l-k} P(k = k_{O_{-1}}(s)) \geq \frac{\alpha(N) - q_h}{N-l}$$

which is strictly positive. If there is less than one single in the group, I can rewrite the same lines with $|g_1|$ instead of 1 and $|g_1| + |g_2|$ instead of s. This contradicts (2). The proof is valid as long as $\alpha(N) > q_h$.

I now extend the proof to the assumptions presented in lemma 1. To do so, I first replace $k_{O_{-1}}(\sum_{j=1}^{i} |g_j|))$ by $k_{O_{-1}}(|g_1|, ..., |g_i|)$, the last draw such that each $g_1, ...g_i$ can end up in of the H_k, and q_h by $\sum_k q_{H_k}$. A group has an interest to switch back to the truth as soon as there is an order such that for one $i \leq l$, $k_{O_{-1}}(|g_1|, ..., |g_i|) < k_{O_{-1}}(|g_i|)$.

If there are $s \geq 2$ singles among the l groups, $P\left(O_{-1} \mid k_{O_{-1}}(1) > k_{O_{-1}}(1, ..., 1)\right) = 0$. But as before, because $\alpha(N) > q$, with positive probability, a single is in position $k_{O_{-1}}(1, ..., 1)$ and $k_{O_{-1}}(1) > k_{O_{-1}}(1, ..., 1)$. So there must be less than one single among the l groups.

If there is an order such that $k_{O_{-1}}(|g_1|, |g_2|) \leq k_{O_{-1}}(|g_2|+1)$. Because $\alpha(N) > q$, there is an order keeping the same $k_{O_{-1}}(|g_1|, |g_2|) - 1$ first groups with a single in the following rank. For this order, there is a H_k with $|g_2|$ free slots after this single. Therefore, for all orders, all H_k have less than $|g_2|$ slots free in $k_{O_{-1}}(|g_1|, |g_2|)$.

Moreover, if for some order $k_{O_{-1}}(|g_1|, |g_2|) = k_{O_{-1}}(|g_2|, |g_2|)$. Because $\alpha(N) > q$, there is an order keeping the same $k_{O_{-1}}(|g_1|, |g_2|) - 1$ first groups with a single in the following rank. For this order, there is a H_k with $|g_2|$ free slots after this single. Therefore, for all orders, there exists a unique H_k with more than $|g_2|$ free slots in $k_{O_{-1}}(|g_1|, |g_2|))$. In particular, $|g_1| < |g_2|$.

For all order there is H_1 is the only set of houses with $|g_2|$ free slots in $k_{O_{-1}}(|g_1|, |g_2|)$. All other H_k have less than $|g_2| - 1$ slots and at least one have more than $|g_1|$ slots. Because $\alpha(N) > q$, there is an order keeping the same $k_{O_{-1}}(|g_1|, |g_2|) - 1$ first groups with a single in the following rank. All the remaining singles must prefer one slot in H_1 over all slots remaining in the H_k. This contradicts item 2. $\square$

Chapter III

University Entrance Test and High Schools Segregation

Based on Foucart and Frys (2018).

1 Introduction

The goal of this paper is to study how the design of entrance university exams can be used to influence the composition of high schools and universities, and ultimately a country's ability to educate students and select them in higher education. We identify situations in which students face a trade-off between benefiting from peer effects in high school and increasing their chance of getting into a prestigious institution of higher education. In particular, if a segregated high school market is inefficient, forbidding comparison of performance across high schools can restore efficiency. This policy may provide an incentive based substitute to quotas policies in school choice problems. For desegregation to happen, tests must be sufficiently precise for the high types to be willing to match with the low types and lose some peer effects. It must also be sufficiently noisy for the low types to accept a match with a high type, and lower her chance to be selected. In addition, we show that forbidding comparison of performance across high schools doesn't always hurt the precision of the selection of students in higher education. If the available testing technology is noisy or if the peer effects for high type students are low, selecting the best students of each school maximizes the peer effects in high schools and picks more often the best students than when comparison is also made across schools.

Across countries there are different ways to select the high school students who attend the best universities. For the purpose of this introduction, we cluster them in three groups. In centralized tests, all the students who compete for the same university's slots write the same exam at the same time and are accepted to university solely according to their score in this test. Such entrance university exams are used for example in China, Turkey, Korea and France. On the other side of the spectrum are the purely decentralized tests, where the students competing for the same university's slots write exams in their high schools and are accepted to university not solely according to their test score but also according to their high school's reputation, the one of the professor who wrote their recommendation letter, or their results in external certifications. Such entrance university exams are widely used in Anglo-Saxon countries but also in private universities in other parts of the world. Our main focus in this paper is on intermediate systems, in which students pass tests in their respective high school but are accepted according only to their rankings within schools. Such a selection is in place in Texas and was also recently introduced in some universities in France. The programs were openly designed in order to desegregate universities, but the effect on the composition of high schools only became a subject of study after the laws were voted (Cullen et al., 2013; Estevan et al., 2017).

We build a model to understand the impact of such mechanisms on school choice, the ability of a country to select the best students, but also to make the most of the peer effects in secondary education. We make two major assumptions through the paper. First, we assume that peer effects in high school exist, are important and cannot be internalized by direct utility transfers. Peer effects imply that the educative achievement of a student depends positively on the level of her high school peers (Hoxby, 2000; Pop-Eleches and Urquiola, 2013). In particular, we assume that the benefits from peer effects display *decreasing differences* (Summers and Wolfe, 1977; Kang, 2007). This implies that low ability students benefit the most from being surrounded by high ability students. We know at least since Becker (1973) that, if utility is not perfectly transferable within groups, decreasing differences may lead to inefficient matching. In our framework, we can derive from Legros and Newman (2007) that the driving force behind such inefficiency is the impossibility for low-ability students to compensate the high ability ones - for instance using monetary transfers - for joining their group.[1]

Second, we assume that there exists an intrinsic benefit from joining a prestigious institution. In the spirit of Spence (1973), being accepted in a good university can be interpreted as a signal of high ability. In the US context, Zimmerman (2014) shows that the marginal admission yields earning gains for 22% between 8 and 14 years after high school completion.

[1]This restriction can be related to the borrowing constraint studied by Fernandez and Gali (1999), in particular if we assume ability to be positively correlated with socio-economic background.

In centralized tests, what matters the most is one's absolute individual level of education. Hence, students of high ability have no incentive to mix with the other ones and one can expect the high school market to be segregated. In fully decentralized tests, if what matters is a combination of individual achievements and high school reputation, the effect is further exacerbated. In intermediate tests, however, students may face the following trade-off: joining a school populated by privileged students allows to benefit from peer effects, but if university admission depends on ranking *within* a high school, it may be strategic to join one populated by students with non-privileged background. Such a strategic effect has been measured in the state of Texas by Cullen et al. (2013) and Estevan et al. (2017). Since 1997, this state has guaranteed school admission to any in-state public higher education institution, including the flagships, to all students in the top 10% of their high school.[2] Students with high ability may hence be willing to implicitly trade their peer effects against increased probability of being accepted in a privileged institution.

Besides the case of Texas, features of this intermediate system are present in Germany and France. In Germany, for the most demanded fields of study, a central clearinghouse reserves 20% of the university slots for students who received the best *Abitur* grade, the university entrance exam (Braun et al., 2010). This exam is different across the different German states and until 2005 was even different across high schools in most of the states. From 2005 on, some of the states introduced centralized tests, sometimes one subject at a time each year. However, in most of the states, the exam is only a part of the final grade (1/3 in Berlin, Bayern, Baden-wurtenberg, Hessen, Hamburg and Bremen)[3] which consists for the other part of the grades of the last two years in high school. Moreover, even the centralized *abitur* are graded in a decentralized way. It is well documented that the overall *abitur* grade is largely influenced by a frame-of-reference: a student with the same ability would get a better grade in a school of lower average ability (see for instance Neumann et al., 2011) and this effect is more important when the share of the decentralized grade is more important (Schwerdt and Woessmann, 2017).

In France, students pass a centralized exam at the end of high school called the *baccalaureat*. The exam and grading are fully centralized and therefore not subject to frame-of-reference effects. Access to higher education is granted for students who passed the exam. In general, the final grade doesn't enter the selection process of the most prestigious institutions of higher education. However, each student who is in the best 10% of her high school has priority over the others to enter some reserved slots at selective

[2]Texas House Bill 588

[3]*Qualificationphase vs Abiturpruefung*

universities. The stated objective of this law[4] was to increase diversity in university, and promote "republican elitism", the idea that anyone could succeed by thriving in high schools, regardless of one's background. Critics of the system, however, argue that it actually favors socially privileged students, as it is not combined with any social background criterion.[5]

The remainder of the paper is organized as follows. We briefly relate our work to the literature in Section 2. The basic setup is presented in Section 3. Section 4 derives the equilibriums matchings in high schools. Section 5 studies the optimal decision of a central planner. We conclude in Section 6.

2 Literature Overview

A model showing that a system similar to Texas' "top 10 percent" can lead to high school desegregation is provided in Section 2 of Estevan et al. (2017). In their model, students' abilities are fixed and determine their valuations of the university whose admission process is deterministic. High schools are segregated at the beginning of the game and good students move at a cost to a bad high school to enter university. Our approach differs in many dimensions: our high school market is smaller, modeled as a one sided matching market with fixed capacities for high schools and we focus on the design of the tests, not only on their (de)centralized nature. Besides, students' abilities display peer effects and while the university valuation is common to all students, the university entrance test is random.

Our paper also relates to a literature on the role of decentralized matching and education, going back at least to Benabou (1996) and Durlauf (1996), and showing that a free market may lead to too much segregation. Pre-matching investment has been studied among others by Hopkins (2012) and Hoppe et al. (2009). Our contribution is also related to Gall et al. (2015), who study investment pre-college with affirmative action and a local re-match policy. To the best of our knowledge, however, none of these paper specifically study the effect of test designs. The role of institutions on decentralized matching has been studied in a very different context by Booth and Coles (2010), on the role of romance in marriage markets. Finally, the strategic effects we suggest have also been measured in the presence of different "tracks" in high schools, with students strategically choosing a school in which they could be on the good track, at the cost of the other students of the school being of "lower" background (De Bartolome, 1990).

[4] *Dispositif meilleurs bacheliers*, article L.612.3.1 of the code of education.

[5] "Comment les meilleurs bacheliers sont poussés en prépa", Challenges, March 1, 2013.

3 The Model

The economy is composed of four students $I = \{1, 2, 3, 4\}$ and two institutes of higher education $U = \{u, o\}$, the university u and the outside option o which have two slots each. Two students are of high ability θ_h whereas the two others are of low ability θ_l with $\theta_h > \theta_l$. We call $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ the vector of abilities of the students and $q_o = q_u = 2$ the capacities of the university and the outside option. Finally there is a test designer maximizing its objective function by choosing the university entrance exam. The type θ is observable to the students but no to the test designer who has a uniform prior over the possible θ.

3.1 Matching

Before passing the test, students form pairs to acquire a level of education. We call these pairs high schools and denote μ the one sided matching function representing these pairs. Hence, $\mu(i)$ denotes the student matched with student i. Education displays peer effects, so that the level of education of a student i, e_i, is a function of her type, θ_i, and of the type of her match, $\theta_{\mu(i)}$. We denote by e the vector of education. For each student a test implemented by the test designer results in a two sided matching η and a probability of entering university, $P(\eta(i) = u) = p_i(e, \mu)$, function of the vector of education and of the repartition in high schools. Students have additively separable utilities and value the level of education acquired in high school as well as the possibility to secure a place at the university. The utility function of student i can then be written as

$$u_i = p_i(e, \mu).\pi_u + v(e_i)$$

π_u is the monetary value of entering university and $v(\cdot)$ is the valuation function of the student's education. We suppose that $v(\cdot)$ and π_u are constant across individuals, and that $v(\cdot)$ is increasing and concave. We normalize the value of the outside option to 0. $v(e)$ can be interpreted either as the intrinsic value of education or as the increase in productivity of a student who acquired a level of education e in high school in comparison to a student with no education and hence can also have a monetary interpretation. The separability assumption is relatively restrictive. In particular, it means that the additional utility of being accepted in university for a student is independent of her level of education in high school. This will for instance not be verified if firms can screen students who attended university through a variable correlated to their levels of education in high school, or if the tuition fees are a function of similar variables.

The timing is the following. First, the test designer publicly announces which test she commits herself to using later on. Then, students form two stable pairs, education levels realize and the test designer learns who is matched with whom (but not their ability or education). Finally, the test takes place, students are selected and utilities are realized.

3.2 Testing technology

We restrict the set of possible tests to two elements. The first test is centralized: all students pass the same exam and the two best performing students are selected. It models the tests used in China, Turkey, Korea and Japan, but also tests based on external certifications. The probability of student i entering university, $p_i^c(e) = p^c(e_i, e_{-i})$, is derivable, strictly increasing in e_i, strictly decreasing and symmetric in e_{-i} and such that $\sum_i p_i^c(e) = 2$. It implies that this test is fair, i.e. that if $e = \lambda.(1,1,1,1)$, $p_i^c(e) = 1/2$ for all i. The second possible test is partially decentralized: students from the same high schools write the same test and the best of each high school is selected by the university. The probability of student i entering university, $p_i^d(e) = p^d(e_i, e_{\mu(i)})$, is derivable, strictly increasing in e_i, strictly decreasing in $e_{\mu(i)}$ and such that $p^d(e_i, e_{\mu(i)}) + p^d(e_{\mu(i)}, e_i) = 1$. It also implies that this test is fair, i.e. that if $e = \lambda.(1,1,1,1)$, $p_i^d(e) = 1/2$. Moreover to be able to disentangle the effects of the design of the test from the effects of its precision, we suppose that the testing technology is the same for the two tests. The only difference between the two tests are their perimeter (who takes the same exam) and how the test scores are used to select students. The marginal probability that a student with education e is better ranked than a student of education e' is the same in both test. Therefore, in partially decentralized tests, it makes no difference if students from two different high schools take the same test or not. Hence, this test can model the selection system in place in Texas as well as the French desegregation program. We provide in Appendix an example of such test with all the properties described above. We also discuss in appendix how to extend the model to the test in place in Germany where the students pass different tests and the two best performers overall are selected, as well as to the test where the university is completely free to choose its students.[6]

[6]If we want to model the tests in place in Germany the probability of being taken would not be symmetric in e_{-i}. If we define the test as the sum of the vector of education levels and a positively correlated white noise for those who take the same exam, the probability of being taken depends on μ. It will be "more decreasing" in $e_{\mu i}$ than in the others' education. see Appendix

3.3 Peer effects

The partner with whom a student is matched in a high school pins down her level of education though the function $e(\theta_i, \theta_{\mu(i)})$, where:

$$
\begin{aligned}
e(\theta_h, \theta_l) &= \theta_h \\
e(\theta_h, \theta_h) &= \theta_h + \alpha \\
e(\theta_l, \theta_l) &= \theta_l \\
e(\theta_l, \theta_h) &= \theta_l + \beta
\end{aligned}
$$

Education displays therefore peer effects. We assume that $\beta > \alpha > 0$, so that the outcome of the match displays *decreasing differences*. The peer effects are such that the impact of being matched with a high type is higher for low type students than for high type students. This is equivalent to saying that students' types are strategic complements or that the education function is submodular. We also assume that $\theta_l + \beta < \theta_h$, so that the types are persistent or that a low ability student matched with a high ability one remains of lower ability. Knowing how the test uses their level of education and the composition of high schools to select who will attend university, students form stable pairs. A student is free to pair with anyone she likes, as long as the other agrees to be matched with her and cannot convince another student to form another school instead. Crucially, we assume utilities to be non-transferable, so that a low ability student cannot compensate a high ability one for joining the same school.

4 Equilibrium matchings

We focus on the case where the test designer can choose only between the two types of test described in Section 3. We proceed backward and study first the different equilibria arising in the upstream matching market as a function of the test implemented. We use these results in Section 5 to study which of the two tests should the designer choose as a function of her objective.

Because the four players have only two types, the game can induce only two feasible matchings, a negative assortative matching and a positive assortative matching. The negative assortative matching, μ_N, is the matching where every high type is matched with a low type. The positive assortative matching, μ_P, is the matching where the students of the same types are matched together. Formally we have:

$$\begin{cases}\mu_N(\theta_l) = \theta_h \\ \mu_N(\theta_h) = \theta_l\end{cases} \quad \begin{cases}\mu_P(\theta_l) = \theta_l \\ \mu_P(\theta_h) = \theta_h\end{cases}$$

They respectively give rise to the levels of education:

$$\begin{cases}e(\theta_l, \mu_N) = \theta_l + \beta \\ e(\theta_h, \mu_N) = \theta_h\end{cases} \quad \begin{cases}e(\theta_l, \mu_P) = \theta_l \\ e(\theta_h, \mu_P) = \theta_h + \alpha\end{cases}$$

The *decreasing differences* hypothesis implies that μ_N leads to a higher average level of education and is better in terms of education for the low type whereas μ_P is better in terms of education for the high type.

Lemma 1. *If the exam is centralized, the unique equilibrium is a positive assortative matching, where high schools are segregated.*

The formal proof is in Appendix. When the exam is centralized, the high types refuse to be matched with a low type, because it both lowers their education level and their chance of going to university. Indeed, in the positive assortative matching, a high type student has the same education as the other high type student, and much higher education than the low type ones. In the negative assortative matching, she still has the same education as the other high type student – albeit lower than under positive assortative matching – but the gap with the low type students is smaller. As the exam is centralized, the probability of a high type being selected is thus lower. The result is independent of the decreasing differences assumption: it is enough that some peer effects exist for positive assortative matching to be the unique equilibrium. However, with decreasing differences, the average level of education is lower in the stable equilibrium matching than in the other possible matching.

Lemma 2. *If the exam is partially decentralized, there exists $\bar{\pi}$ and $\underline{\pi}$, with $\bar{\pi} > \underline{\pi} > 0$, such that negative assortative matching is the unique equilibrium if $\pi_u \in [\underline{\pi}; \bar{\pi}]$ and positive assortative matching is the unique equilibrium otherwise.*

The formal proof and the definition of the threshold values are in Appendix. When the university selects the best student of each school and the value of going to university is very high, i.e. $\pi_u > \bar{\pi}$, the only equilibrium is positive assortative matching as for the centralized test. Because the value of education is comparatively small, the low types do not want to reduce their chance to go to university by competing with a high type in the same high school in exchange for a higher education. When the value of going

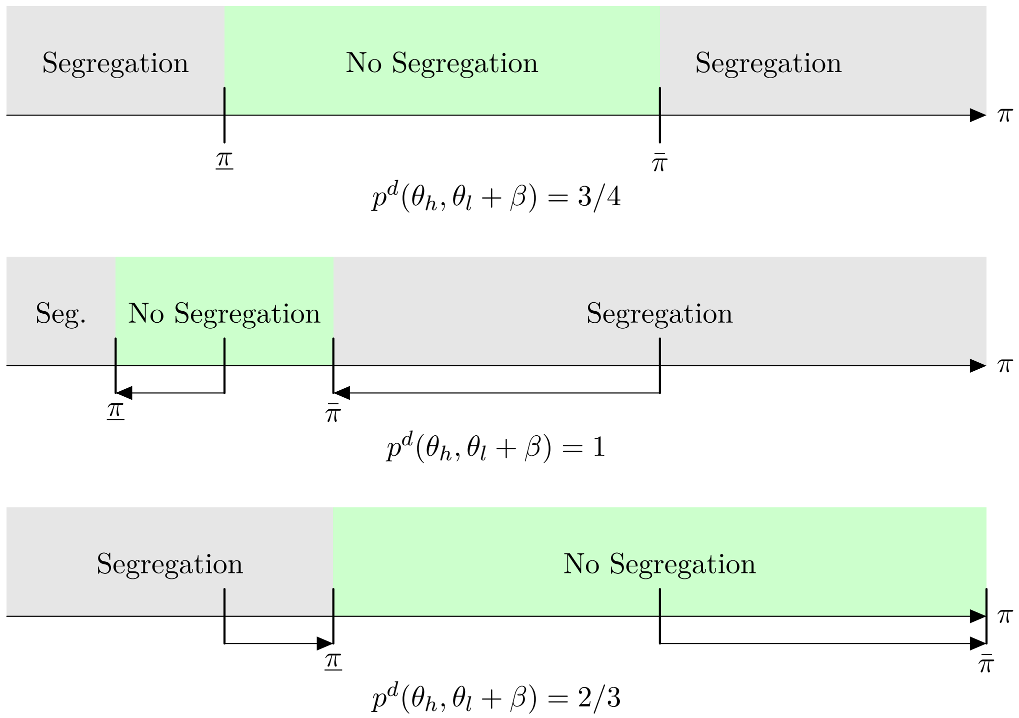


FIGURE 1: Equilibria of the decentralized test, $v(e) = e$; $\alpha = 1$, $\beta = 3$.

to university is very low, i.e. $\underline{\pi} > \pi_u$, the only equilibrium is also positive assortative matching. Here contrary to the first case, because the value of education is comparatively high, the high types do not want to lower their level of education in exchange for a higher probability of going to university. For the intermediate case, when the value of going to university is between $\bar{\pi}$ and $\underline{\pi}$, the good types are ready to exchange with the bad types probabilities of getting into university against education, and the negative assortative matching is stable. To do comparative statics, we normalize $\theta_l = 0$ and $v(0) = 0$. When θ_h rises, i.e. when the difference between the types increases, or when $p^d(\theta_h, \theta_l + \beta)$ increases, i.e. when the precision of the test increases, the interval where negative assortative matching is the equilibrium shifts to the left. When α rises, $\underline{\pi}$ rises whereas $\bar{\pi}$ stays constant, and when β rises, the interval shifts to the right. In the limit when the decreasing differences hypothesis is maximal, i.e. when $\alpha \approx 0$ and $\beta \approx \theta_h - \theta_l$, the interval is the whole positive real line. On the contrary, if the decreasing differences hypothesis is very small, i.e. $\alpha \approx \beta$, the length of the interval is bounded below by a positive amount, as long as the peer effects are strictly positive ($\alpha > 0$) and the function $v(\cdot)$ is strictly concave.

We illustrate the role of the precision of the test in Figure 1. In a decentralized test, when matching is positive assortative, the opponent of each student is of identical ability. Therefore, the assumption on the testing technology implies that the probability of being selected in university in equal to 1/2, independently of the precision of the test. On the graph above, we present a case in which $p^d(\theta_h, \theta_l + \beta) = 3/4$. This means that the probability of a high type being selected increases by 1/4 whenever she accepts to be matched with a low type. This also implies that the probability of a low type going to university under negative assortative matching lower to 1/4. As we assume the decreasing differences to be fairly high ($\beta = 3, \alpha = 1$), there is a fairly large area in the center corresponding to possible values of the university degree such that both types of students prefer negative assortative matching.

The second figure starting from the top represents a dramatic increase in the precision of the testing technology, so that in the case of negative assortative matching the high type is absolutely certain to be selected, $p^d(\theta_h, \theta_l + \beta) = 1$. In that case, the zone where negative assortative matching is the equilibrium moves to the left. Only if the value of going to university is low enough, the low type is willing to abandon all hope to be selected. It must at the same time still be sufficiently high for the high type to accept giving away education in exchange. Therefore, the area where negative assortative matching is an equilibrium shrinks.

Finally, the third figure represents a much less precise technology, with $p^d(\theta_h, \theta_l + \beta) = 2/3$. The area now translates to the right, as the benefits of university need to be sufficiently high for the high type to accept the loss of peer effects in exchange for a slightly higher probability of being selected.

Proposition 1. *(i) For any given precision of the test, $p^d(\theta_h, \theta + \beta)$, there exists a value of the university degree, π_u, such that a partially decentralized exam leads to a negative assortative matching.*

(ii) If the value of the university degree is sufficiently high, $\pi_u > 2\,(v(\theta_h + \alpha) - v(\theta_h))$, there exists a level of precision of the test, $p^d(\theta_h, \theta + \beta)$, such that a partially decentralized exam leads to the negative assortative matching. Otherwise, the positive assortative matching is the unique equilibrium, regardless of the exam type.

The formal proof is in appendix. We illustrate the logic of the proposition in Figure 2. The first part is just a corollary of Lemma 2: because $\bar{\pi} > \underline{\pi}$, it is always possible to find a value of π_u between those two curves. On the contrary, it is not always possible to reach the negative assortative matching by adjusting the precision of the test. For the lowest values of π_u, even promising a university slot to a high type student is not enough to convince her to match with a low ability one. For all the other values, a test designer

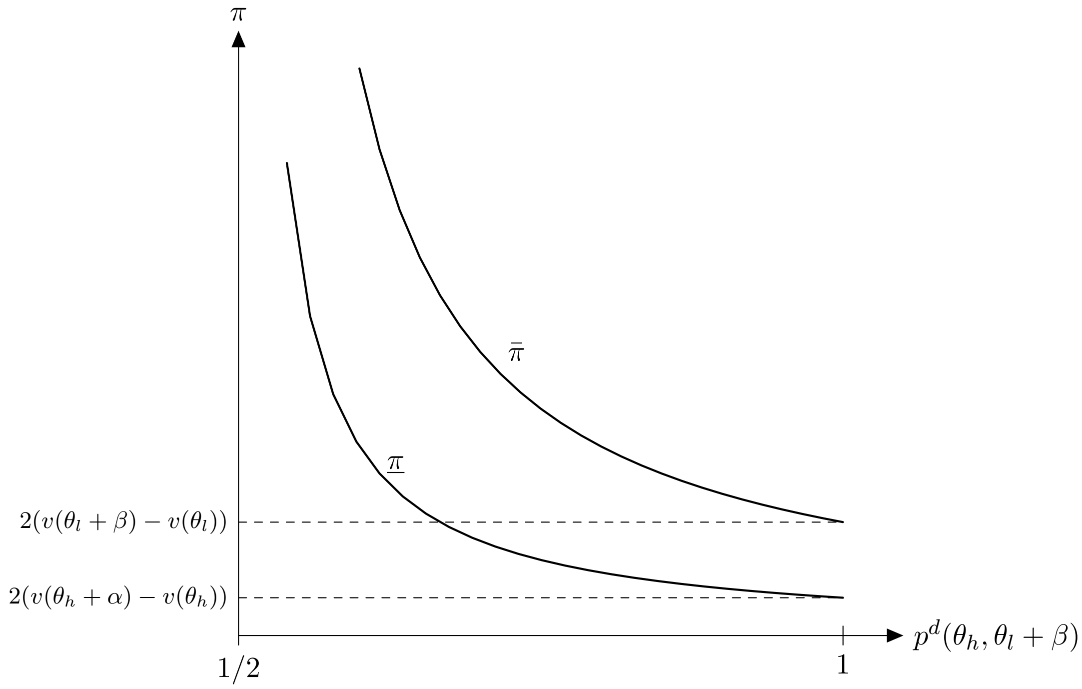


FIGURE 2: threshold values of $\underline{\pi}, \bar{\pi}$, $v(e)=e$; $\alpha=1$, $\beta=3$.

able to choose the precision of the test can induce the desired matching by choosing a partially decentralized one. In practice, the test designer biggest tools to influence the net monetary value of attending university are probably university's fees. Perhaps surprisingly, increasing university fees can induce desegregation in the high school market. While negative assortative matching is often attainable by a social planner with control of the test precision and/or of the value of entering university, whether it is desirable is a different question. In the next section, we take as given the precision of the test and the value of a university degree, and study how the test designer's preference for centralization changes given her objective function.

5 Choice of the test

From the above section, we know that the choice of the test changes the nature of the match and, therefore, the segregation in high school when the monetary value of university π_u is intermediate, i.e. belongs to $[\underline{\pi};\bar{\pi}]$.

Using our results on the influence of the two tests on the composition of high schools, we can determine which of the two tests should a designer choose as a function of her objective. A priori, defining the objective of the test designer is not a straightforward question here. In reality, the entity defining the test varies and, therefore its objective, across countries. In some cases, the state is designing the test and directly assigns students to universities. In others, it defines the test but lets universities use it as they want. It can also let each high school design a test but regulate how universities use them to screen students. Finally, in other countries, universities and high schools are totally unconstrained. We consider in this subsection three different objectives that epitomize classical beliefs about the role of education. First of all, we determine which test is the best when the designer social choice function is purely utilitarian, which here happens to be the same as maximizing the average level of education of all students. Then, we study the case where a “selective” test designer wants to maximize the average ability of the students accepted to university. Finally, we study an “elitist” objective function in which the test designer cares about the average level of education of the students selected by the university.

5.1 Utilitarian social planner

Perhaps the most natural objective of a social planner is to maximize the sum of the utilities of the students. The hypotheses imposed on utilities imply that maximizing this objective is the same as maximizing the average level of education, which is a natural goal in itself. The previous section tells us that the choice of one or the other test is relevant to achieve this goal. Indeed, different tests imply different upstream equilibrium matchings and therefore different levels of education through the peer effects. Our first result shows that choosing the partially decentralized test maximizes the average level of education for all values of the parameters.

Proposition 2. *A test designer who wants to maximize the the sum of the utilities of the students prefers the partially decentralized test rather than the centralized test for all* π_u *and strictly prefers it for* $\pi_u \in [\underline{\pi}; \bar{\pi}]$.

The formal proof is in Appendix B. The fact that the assignment of university's slots to students is welfare irrelevant is rather extreme. It relies heavily on the hypothesis that the additional utility of attending university for a student is independent of her level of education and of her type. But Proposition 2 is robust to small variations in our model and would still hold if the utility of attending university were slightly increasing with the type or with the level of education of the students. Even if we go further away from our stylized model, this proposition highlights an effect that would remain and probably enter in a trade off with the efficiency of the allocation of university's slots to students.

From Proposition 1, we know that the condition on π_u can easily be reached if the test designer has some influence on the net monetary value of university or on the precision of the test.

5.2 Selective social planner

We now turn to the case where the test designer is not an utilitarian social planner but rather the university. Her objective focuses on the characteristics of the two students she selects and disregards the students that do not attend university. The objective depends on the economic meanings of the education level and of the type of the students as well as on economic role of the university. We first consider the case where the university wants to maximize the average type of its students.

This objective can be justified by a two stages screening model where the productivity of a worker is determined solely by her type. Firms compete to attract the future students of each university by proposing a wage, then universities compete to attract students by choosing a test and advertising the highest wage proposed by the firms, and finally a student goes to the university that advertised the highest wage and accepted her.[7]

To be able to compare the two tests, we have to impose further regularity on the test. More precisely, we suppose that education improves the probability of being better ranked than another student weakly uniformly, i.e. independently of the fact that these two students are the best or the worst ranked. This assumption rules out for instance cases where the bad type performs either very badly or very well whereas the good type performs always well. The particular condition we need is that, regardless of whether they are among the best or the worse performers of a test, a high type is on expectation ranked better than a low type. The two tests presented in the appendix verify this assumption. In the next lemma we present an important implication of this restriction on the comparison of the centralized a partially decentralized tests. For any vector of education e, we denote $r_i(e)$ the random variable describing the rank given by the test to student with education e_i.

Lemma 3. *Let $e_1 = e_3 = e_h > e_2 = e_4 = e_l$. We have $p^d(e_h, e_l) > p^c(e_h, e_h, e_l, e_l)$, if in the centralized test:*

$$P\left(r_1(e) \geq r_2(e) | r_1, r_2 \in \{3, 4\}\right) > P\left(r_2(e) \geq r_1(e) | r_1, r_2 \in \{1, 2\}\right)$$

[7] It supposes that the academic degree is the only screening device that firms can use (no test scores), that students are price taker on π_u and that the number of students accepted by the university is exogenously fixed. In this set up if the university does not choose the test that separate the most the workers' ability, another university would use it and offer an higher π_u, throwing the first university out of the market.

Therefore, when the test verifies a weak uniformity assumption, a highly educated student prefers to compete for one slot against a bad student rather than to compete for two slots against two bad students and one good student. The chance of being better ranked than a low ability student is the same in the two tests and the chance of beating the other high type is one half. However, the chance of being selected in the market where we duplicated the number of students and the number of slots are not the same. Indeed, when the test changes from partially decentralized to centralized, the high type is worse off in the case where she won against its low type high school partner but lose against the two other. On the contrary, she is better off in the case where she arrived second behind her high school partner. But the first case happens with higher probability than the second one, making her globally worse off. In other words, the partially decentralized test performs better in selecting the high types than the centralized test when students are negatively assorted because the high school match has already partially sorted the good and the bad students.

Our second result shows that the test maximizing the average type of the students selected by the university changes as the parameters of the model changes.

Proposition 3. *A test designer maximizing the average type of the students attending university,*

(i) prefers the centralized test rather than the partially decentralized test if $\pi_u \in [0; \underline{\pi}] \cup [\bar{\pi}; +\infty]$.

(ii) Otherwise, she prefers the centralized test when the peer effects for the low type are important and the partially decentralized test when the peer effects for the two types are small.

The formal proof is in Appendix. In the case where π_u is intermediary, the designer faces a trade off. By implementing the centralized test, she increases the difference between the level of education of the good and bad type and, therefore, helps the test distinguish them more often. On the other hand, by implementing a partially decentralized test, she partially delegates the sorting task to the students. Indeed, the students facilitate the distinction of the types by sorting themselves into two pairs of different types.

We can see directly that the optimal test changes if we enlarge the set of feasible tests and allow the university use the rankings to select its students. Indeed, if she implements a centralized test, the university knows that the high schools' matching is positive assortative in equilibrium, i.e. that the two high ability students are in the same high school. Using this particular prior on types distribution, if the best and the third best ranked students are from the same school, the student with the third best score have a higher probability of being of high ability than the student with the second best score. The university will then deviate and select the best and third best students. Students rationally anticipate this behavior and the incentive to have segregative equilibrium in the high school market is reinforced. There is no such effect when the test designer chooses a partially decentralized test. However, it would now be dominated by the improved centralized test for all parameters.

5.3 Elitist social planner

The assumption that the productivity of a student is determined by her type and is independent of her education level is extreme. Here, we suppose to the contrary that the type reflects only the socio-economic status of the parents of a student. In this case, education rather than ability determines the future productivity. The university, for the same reason as in the previous proposition, is willing to maximize the level of education of the average students it selects. In this model, $v(e)$ may either represent the intrinsic value for education, or the fact that in the long run a firm will be able to screen a student's productivity whether or not she went to university and pays her the according wage. Our third result shows that the test maximizing the average education of the selected students changes as the parameters of the model change.

Proposition 4. *A test designer maximizing the average education of the selected students,*

(i) prefers the centralized test rather than the partially decentralized test if $\pi_u \in [0; \underline{\pi}] \cup [\bar{\pi}; +\infty]$.

(ii) Otherwise, she prefers the centralized test when the test is very accurate or when the peer effects are all important; whereas she prefer the partially decentralized test when the test is very inaccurate or when the peer effects are low or very decreasing.

The formal proof is in the Appendix. When π_u is intermediary and the peer effects are small, the education levels are close in both matching equilibria and lemma 3 implies that the decentralized test is more accurate and therefore performs better. On the contrary when the peer effects are important, the centralized test makes fewer mistake and the test designer faces once again a trade-off. When the two tests select a student with the highest education level, the one selected by the centralized test has a higher education level because $\alpha > 0$. However, when the centralized test makes a mistake, the mistake is worse than when the partially decentralized test make a mistake, because $\beta > 0$. Hence, when α increases, $p^c(\theta_h, \mu_P)$ increases and the centralized test performs better.

Increasing the precision of the test can change the nature of the optimal test, but always increases the average education of the student selected. It helps the test make fewer mistakes and incentivizes positive assortative matchings. Assume now the university is forced to use a partially decentralized test. Following the logic of Proposition 1, it is possible to induce positive or negative assortative matching by changing the net monetary value of university and the precision of the test. If the university induces positive assortative matching, she receives a high ability and well-trained student and a low ability and low trained student, so that the expected ability is equal to $\frac{1}{2}(\theta_h + \alpha + \theta_l)$ for any test precision and value of being selected to university π_u. If she induces negative assortative matching, if the test is precise enough, she can get two high ability but low trained students, with expected ability equal to θ_h. This may be worse than the centralized exam but is better than a decentralized exam with positive assortative matching. Thus, a social planner who wants to implement a negative assortative matching only has to impose the partially decentralized test and can let the university choose the precision of the test and the university fee (seen as a way to choose π_u).

As for proposition 3, the test chosen is not optimal anymore if we enlarge the set of possible tests. Particularly, the centralized test is dominated by another test where the university updates its prior about the vector of the education level of the students taking into account the equilibrium matching in the high school market. As in the discussion of the proposition 3, it would rather sometimes take two students from the same school even if they are not the two best ranked students. This would not happen for the partially decentralized test because when the equilibrium matching is negative assortative, it is optimal to select one student from each school even when we update the prior of the university. However, as before, this new test makes positive assortative matching more attractive to the university and shrinks the parameters ranges in which the partially decentralized test is optimal. When the test designer can only choose between our two tests, the commitment in the first period to use a test in the second period is not important because the test designer never wants to change the test ex-post. However, if we enlarge the test space it would probably become an important question.

6 Conclusion

The programs in France and in Texas were introduced to offer disadvantaged kid higher chance of entering university. But in a a model with peer effects with decreasing differences, they may serve efficiently also another objective namely to desegregate high schools. The intuition that the test used by university to select students allows them to imperfectly transfer the utility among each other is verified. The low ability students and the high ability students are able to trade education against probability of entering university to stabilize the desegregated equilibrium for some intermediate range of parameters. This range can always be reached by a social planner controlling the university's fees or, if university is attractive enough, by controlling the precision of the test. Even in the case where the parameters fall outside of this range, the partially decentralized test is optimal for an utilitarian social planner. Perhaps surprisingly, from a policy perspective, it means that a government wanting to desegregate high school might consider rising university fees or forbidding external certification tests. On the contrary, if the university chooses the test and peer effects are high for both type, partially decentralized tests and desegregated matchings never appear in equilibrium. In this case there is a trade-off between the average level education in the economy (higher in the decentralized system) and the ability of university to select the high type students (higher in the centralized system). More interestingly, if the technology of the test is relatively bad in assessing the education of the students or if the peer effects are weak or very decreasing with type, and universities want to screen highly educated students, the trade-off disappears. The decentralized system performs better on both criteria (aggregate education and education of the elite) and it can arise in equilibrium.

7 Appendix

A. Examples of testing technologies

Categorical random ranking

Denoting $j = min\{I/\{1, \mu(1)\}\}$, we define the vector of education as:

$$e(\theta, \mu) = (e(\theta_1, \theta_{\mu(1)}), e(\theta_{\mu(1)}, \theta_1), e(\theta_j \theta_{\mu(j)}) e(\theta_{\mu(j)}, \theta_j))$$

In the decentralized test, the probability of a student being selected is a Bernoulli distribution where the parameter is the relative level of education between the student and her opponent. We have then:

$$p^d(e_i, e_{\mu(i)}) = \frac{e_i}{e_i + e_{\mu(i)}}$$

In the centralized test, the probability of a student being first is a categorical distribution where the parameter is the relative level of education between the student and her opponents and the probability of a student being second is the same distribution where the student arrived first has been withdrawn from the pool. We have then:

$$\begin{aligned}
P(1^{st} = i) &= \frac{e_i}{\sum_j e_j} \\
P(2^{nd} = i | 1^{st} = k) &= \frac{e_i}{\sum_{j \neq k} e_j} \\
p^c(e_i, e_{-i}) &= P(1^{st} = i) + P(2^{nd} = i) \\
P(e_i \text{ before } e_{\mu(i)}) &= p^d(e_i, e_{\mu(i)})
\end{aligned}$$

Therefore, the test verifies all the properties assumed in section 3. Moreover, it verifies the property required in lemma 3. Indeed, if $e_1 = e_2 = e_h > e_{\mu(1)} = e_{\mu(2)} = e_l > 0$:

$$p^d(e_1, e_{\mu(1)}) - p^c(e_1, e_{-1}) = \frac{e_h.e_l(e_h - e_l)}{(e_h + e_l)(e_h + 2e_l)(2e_h + e_l)} > 0$$

Normal random score

We define the test score t as the sum of the vector of education $e(\theta, \mu)$ and a error term ϵ, where ϵ is a four dimensional random variable following a normal distribution of mean 0. We denote Σ_c the covariance matrix if all students pass the same test and Σ_d if students from different high schools pass different tests. The covariance matrices are the following:

$$\Sigma_c = \sigma^2. \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix} \qquad \Sigma_d = \sigma^2. \begin{pmatrix} 1 & \rho & 0 & 0 \\ \rho & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho \\ 0 & 0 & \rho & 1 \end{pmatrix}$$

Two students who passed the same test have positively correlated scores. It reflects the intuition that if a student received a low grade in comparison to her true level of education, the other students who passed the same test probably received also a low grade in comparison to their true level of education.[8] This is consistent with the fact

[8] A negative correlation would mean that if test gives to a student a too high grade in comparison to her true level of education, it would "compensate" and gives a too low grade in comparison to her true level of education to the other students.

that a test can be either to easy or too hard, i.e. biased in one or the other direction.[9]

As explained in section 3, it is irrelevant whether in the partially decentralized test students of different high school pass the same test or not. We can pin down the probability of being selected in a partially decentralized test $p^d(e_1, e_{\mu(1)})$:

$$\begin{aligned} p^d(e_1, e_{\mu(1)}) &= P(t_1 > t_{\mu(1)}|e_1, e_{\mu(1)}) = P(e_1 + \epsilon_1 > e_{\mu(1)} + \epsilon_{\mu(1)}|e_1, e_{\mu(1)}) \\ &= P(\epsilon_1 - \epsilon_{\mu(1)} > -(e_1 - e_{\mu(1)})|e_1, e_{\mu(1)}) \end{aligned}$$

Formula about linear combination of jointly normal distribution gives us that $\epsilon_1 - \epsilon_{\mu(1)}$ follow a normal distribution of mean 0 and variance $\sigma^2(2 - 2\rho)$. Therefore :

$$p^d(e_1, e_{\mu(1)}) \begin{cases} = 1/2 \text{ if } e_1 = e_{\mu(1)} \text{ and constant for all } \rho, \sigma \\ > 1/2 \text{ if } e_1 > e_{\mu(1)} \text{ and increasing (decreasing) with } \rho \ (\sigma) \\ < 1/2 \text{ if } e_1 < e_{\mu(1)} \text{ and decreasing (increasing) with } \rho \ (\sigma) \end{cases}$$

Here two parameters determine the precision of the test, ρ and σ. When ρ increases, the noise of students passing the same test is more correlated and therefore there is less often a difference between the sign of $e_i - e_{\mu(i)}$ and the sign of $t_i - t_{\mu(i)}$. When σ decreases, the noise of the test decreases and therefore the difference between e_i and t_i decreases. Notice that it is harder to compare students who took different tests than students who wrote the same test because the correlation is lower in the first case. This fact is in line with the heuristic about testing.

In the centralized test all students pass the same test, the covariance matrix is therefore Σ_c. The probability of being selected in a centralized test $p^c(e_1, e_{-1})$ is the following:

$$p^c(e_1, e_{-1}) = P(t_1 > t_{-1}^{(2)}|e)$$

However, the distribution of $t_{-1}^{(2)}$ is only approximately normally distributed and the derivation of its moments are cumbersome (Clark, 1961). But the test verifies all the properties assumed in section 3. We can moreover show that the test verifies the property required in lemma 3. Indeed:

$$P(t_{\mu(1)} \geq t_1 \geq t_{\mu(2)} \geq t_2|e) = P((-Y_1, Y_2, -Y_3) \geq 0|e)$$

Where $Y_1 = t_1 - t_{\mu(1)}$, $Y_2 = t_1 - t_{\mu(2)}$ and $Y_3 = t_2 - t_{\mu(2)}$. $(-Y_1, Y_2, -Y_3)$ follows

[9] It does not imply that test are biased toward the mean, i.e. that a high school with high educated students would grade its student in a more conservative way than a high school with low educated students. If we want to include such a bias, we can write the mean of the test of a student as the difference between her level of education and the average level of education of the high school. The probability of being taken depends then on μ. It is more decreasing in $e_{\mu i}$ than in the others' education and therefore symmetric in e_{-i}.

therefore a multivariate normal distribution with mean $(e_{\mu(1)} - e_1, e_1 - e_{\mu(2)}, e_{\mu(2)} - e_2)$ and covariance matrix:

$$\sigma^2(2-2\rho)\begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$$

When all students pass the same exam. Moreover:

$$P(t_1 \geq t_{\mu(1)} \geq t_2 \geq t_{\mu(2)} \geq t_2|e) = P((Y_1, -Y_4, Y_3) \geq 0|e)$$

Where $Y_4 = t_2 - t_{\mu(1)}$. $(Y_1, -Y_4, Y_3)$ follows also a multivariate normal distribution with mean $(e_1 - e_{\mu(1)}, e_{\mu(1)} - e_2, e_2 - e_{\mu(2)})$ and the same covariance matrix. We now suppose that $e_1 = e_2 = e_h > e_{\mu(1)} = e_{\mu(2)} = e_l$. Therefore, any point such tat $(-Y_1, Y_2, -Y_3) \geq 0$ as a symmetric such that $(Y_1, -Y_4, Y_3) \leq 0$. The diagonalization of the covariance matrix gives us that the axis going through the mean in the direction $(-1/\sqrt{2}, 0, 1/\sqrt{2})$ is a symmetric axis of the distribution of $(Y_1, -Y_4, Y_3)$. Therefore, any point such that $(Y_1, -Y_4, Y_3) \leq 0$ has a symmetric such that $(Y_1, Y_2, Y_3) \geq 0$. We have finally that:

$$\begin{aligned} p^d(e_1, e_{\mu(1)}) - p^c(e_1, e_{-1}) =& P(t_1 \geq t_{\mu(1)} \geq t_2 \geq t_{\mu(2)} \geq t_2|e) - P(t_{\mu(1)} \geq t_1 \geq t_{\mu(2)} \geq t_2|e) \\ &> 0 \end{aligned}$$

Using this form of test, we could enlarge the space of feasible selection process. In the German case for example, where the tests are different in different high school but where the university selects the two best performing student overall, the probability of being selected would simply be $p^g(e_1, e_{-1}) = P(t_1 \geq t^{(2)}|e)$. If the university is totally free to choose how to use the test to select its students. The university would build its priority ordering following the order of the $E[e(\theta, \mu)|t]$ using Bayes rule and a prior for the distribution of $e(\theta, \mu)$ reflecting the equilibrium of the roommate game.[10] The prior would be uniform on $\Theta_N = \{(\theta_h, \beta, \theta_h, \beta), (\theta_h, \beta, \beta, \theta_h), (\beta, \theta_h, \theta_h, \beta), (\beta, \theta_h, \beta, \theta_h)\}$ if in equilibrium μ is a negative assortative matching and uniform on $\Theta_P = \{(\theta_h + \alpha, \theta_h + \alpha, 0, 0), (0, 0, \theta_h + \alpha, \theta_h + \alpha)\}$ if in equilibrium μ is a positive assortative matching. This model would therefore closely resemble to a system where university can assign "weight" to high schools to correct for the difference of level between high schools.

B. Formal Proofs

Proof. **Lemma 1**

[10]Communication between the university and the student does not help because because the value of a slot in the university is the same for all students

Because for all e, $\sum_i p^c(e_i, e_{-i}) = 2$, we have that $p^c(e_h, e_h, e_l, e_l) + p^c(e_l, e_l, e_h, e_h) = 1$ for all (e_l, e_h). Therefore, deriving with respect to e_h, for all (e_l, e_h) we have that $\frac{dp^c}{de_1}(e_h, e_h, e_l, e_l) + \frac{dp^c}{de_2}(e_h, e_h, e_l, e_l) = -\frac{dp^c}{de_3}(e_l, e_l, e_h, e_h) - \frac{dp^c}{de_4}(e_l, e_l, e_h, e_h) > 0$ because p^c is strictly decreasing in e_{-i}.Therefore, $p^c(e_h, e_h, e_l, e_l)$ is strictly increasing in e_h for all (e_l, e_h). Because $\alpha > 0$ and $\beta > 0$, we must have that $p^c(\theta_h, \mu_P) > p^c(\theta_h, \mu_N)$, where $p^c(\theta_i, \mu_K) = p^c(e(\theta_i, \mu_K), e(\theta_{-i}, \mu_K))$. Finally, because $v(\cdot)$ is increasing, we have that the utility of the high type is higher in μ_P than in μ_N, so the only stable match is where the high type is together with a high type in high school. □

Proof. **Lemma 2**

Because $p^d(e_i, e_{\mu(i)}) + p^d(e_{\mu(i)}, e_i) = 1$, we have that for all e, $p^d(e, e) = 1/2$. Moreover, because $\theta_h > \theta_l + \beta$, we have that $p^d(\theta_h, \theta_l + \beta) = 1 - p^d(\theta_l + \beta, \theta_h) > 1/2$. Therefore denoting $u^d(\theta_h, \mu)$ the utility of the high type when the exam is partially decentralized and the high school matching is μ, we have that :

$$u^d(\theta_h, \mu_N) \geq u^d(\theta_h, \mu_P) \iff \pi_u \geq \frac{v(\theta_h + \alpha) - v(\theta_h)}{p^d(\theta_h, \theta_l + \beta) - 1/2} = \underline{\pi}$$
$$u^d(\theta_l, \mu_N) \geq u^d(\theta_l, \mu_P) \iff \pi_u \leq \frac{v(\theta_l + \beta) - v(\theta_l)}{p^d(\theta_h, \theta_l + \beta) - 1/2} = \bar{\pi}$$

Because $\beta > \alpha$, $\theta_h > \theta_l$ and $v(\cdot)$ is concave, we have that $\bar{\pi} > \underline{\pi}$. □

Proof. **Proposition 1**

Point (i) is a direct corollary of Lemma 2. As $\bar{\pi} > \underline{\pi}$, it is always possible to find a value of $\pi_u \in (\underline{\pi}, \bar{\pi})$ such that negative assortative matching is the equilibrium. Point (ii) follows from the definitions of $\underline{\pi}$. For $p^d(\theta_h, \theta + \beta) \to 1/2$, $\underline{\pi} \to \infty$ and both $\underline{\pi}$ and $\bar{\pi}$ are continuous and decreasing in $p^d(\theta_h, \theta + \beta)$. Thus, for the highest values of π_u, it is always possible to induce negative assortative matching. However, for $p^d(\theta_h, \theta + \beta) \to 1$, $\underline{\pi} \to \frac{v(\theta_h+\alpha)-v(\theta_h)}{2}$, so that it is never possible to convince a high type to match with a low type when π_u is below this threshold. □

Proof. **Proposition 2**

The sum of the education levels when the equilibrium matching is μ_N is $2\theta_h + 2(\theta_l + \beta)$ whereas it is $2(\theta_h + \alpha) + 2\theta_l$ when the equilibrium matching is μ_P. Because $\beta > \alpha$, the average education is higher with negative assortative matching. Because π_u is the same for every student, the sum of the utilities is independent of the probabilistic allocation of university's slots. Moreover, because $v(\cdot)$ is concave, $\theta_h > \theta_l$ and $\beta > \alpha$, the sum of the utilities is also higher with negative assortative matching. Using lemmas 1 and 2, we can conclude that the partially decentralized exam is strictly better than the centralized exam when $\pi_u \in [\underline{\pi}; \bar{\pi}]$ and weakly better otherwise. □

Proof. **Lemma 3**

Let $e_1 = e_3 = e_h > e_2 = e_4 = e_l$ and suppose that $P(r_1(e) \geq r_2(e) | r_1, r_2 \in \{3, 4\}) > P(r_2(e) \geq r_1(e) | r_1, r_2 \in \{1, 2\})$ Because the marginal of the two tests are the same, we have:

$$\begin{aligned} p^d(e_h, e_l) =& P(r_1(e) \geq r_2(e)) \\ =& P(r_1, r_3 \in \{1, 2\}) + P(r_1, r_4 \in \{1, 2\}) + P(r_1 \geq r_2 \cap r_1, r_2 \in \{1, 2\}) \\ &+ P(r_1 \geq r_2 \cap r_1, r_2 \in \{3, 4\}) \\ p^c(e_h, e_h, e_l, e_l) =& P(r_1, r_3 \in \{1, 2\}) + P(r_1, r_4 \in \{1, 2\}) + P(r_1, r_2 \in \{1, 2\}) \end{aligned}$$

Therefore, because the test is symmetric, we have:

$$\begin{aligned} p^d(e_h, e_l) - p^c(e_h, e_h, e_l, e_l) =& \left(P\left(r_1 \geq r_2 | r_1, r_2 \in \{3, 4\}\right) - P\left(r_2 \geq r_1 | r_1, r_2 \in \{1, 2\}\right)\right) \\ &* P(r_1, r_2 \in \{1, 2\}) \\ >& 0 \end{aligned}$$

□

Proof. **Proposition 3**

Denoting μ_t the equilibrium matching when test t is implemented, maximizing the average type of the selected students is the same as maximizing $p^t(\theta_h, \mu_t)$, the probability of a high type being selected. Indeed:

$$\begin{aligned} E[\sum_{i \in \eta^{-1}(u)} \theta_i / 2] &= \frac{1}{2} E[\sum \theta_i \mathbb{1}_{\eta(i)=u}] = \frac{1}{2} \sum \theta_i P(\eta(i) = u) = \theta_h p^t(\theta_h, \mu_t) + \theta_l p^t(\theta_l, \mu_t) \\ &= (\theta_h - \theta_l) p^t(\theta_h, \mu_t) + \theta_l \end{aligned}$$

If $\pi_u \in [0; \underline{\pi}] \cup [\bar{\pi}; +\infty]$, lemmas 1 and 2 tell us that the centralized and the decentralized tests lead to the positive assortative matching μ_P. Moreover, we saw in the proof of these lemmas that $p^c(e_h, e_h, e_l, e_l)$ is strictly increasing in e_h and that $p^c(e, e, e, e) = 1/2 = p^d(e, e)$. Therefore, we must have that $p^c(\theta_h, \mu_P) > 1/2 = p^d(\theta_h, \mu_P)$ and the centralized test is the optimal choice. If $\pi_u \in [\underline{\pi}; \bar{\pi}]$, lemma 1 tells us that the centralized tests leads to μ_P whereas lemma 2 tells us that the decentralized test leads to μ_N. Because of lemma 3, we have for all $\alpha \geq 0$:

$$\begin{aligned} p^c(\theta_h, \mu_P) &= p^c(\theta_h + \alpha, \theta_h + \alpha, \theta_l, \theta_l) < p^d(\theta_h + \alpha, \theta_l) \\ p^d(\theta_h, \mu_N) &= p^d(\theta_h, \theta_l + \beta) \end{aligned}$$

If the peer effects for low type student are big, i.e. $\beta \approx \theta_h - \theta_l$, $p^d(\theta_h, \mu_N) \approx 1/2$, whereas $p^c(\theta_h, \mu_P) > 1/2$. Therefore for all precision of test and α, the test designer prefers the centralized test over the partially decentralized test. If decreasing difference and peer effects are small, i.e. $\alpha \approx \beta \approx 0$, $p^c(\theta_h, \mu_P) < p^d(\theta_h + \alpha, \theta_l) \approx p^d(\theta_h, \theta_l + \beta)$ and for all precision of the test the designer prefers the the partially decentralized test over the centralized test. □

Proof. **Proposition 4**

As in proposition 3, if $\pi_u \in [0; \underline{\pi}] \cup [\overline{\pi}; +\infty]$, lemmas 1 and 2 tell us that the centralized and the decentralized tests lead to the positive assortative matching μ_P. Therefore, for test t, the expected average education of the students selected by the university is the following:

$$E^t[\sum_{i \in \eta^{-1}(u)} e_i/2] = (\theta_h + \alpha - \theta_l)p^t(\theta_h, \mu_P) + \theta_l$$

Finally, because $p^c(\theta_h, \mu_P) > 1/2 = p^d(\theta_h, \mu_P)$, the centralized test is the optimal choice.

If $\pi_u \in [\underline{\pi}; \overline{\pi}]$, the centralized test leads to the same matching and the same average education. On the contrary, the partially decentralized test leads to μ_N and the average education of the students selected by the university,

$$E^d[\sum_{i \in \eta^{-1}(u)} e_i/2] = (\theta_h - \theta_l - \beta)p^d(\theta_h, \mu_N) + \theta_l + \beta$$

So if the test technology is very accurate, i.e $p^d(\theta_h, \mu_N) \approx 1 \approx p^c(\theta_h, \mu_P)$, the centralized test will be better for all values of the parameters (because $\alpha \geq 0$). On the contrary, if the test is very inaccurate, i.e $p^d(\theta_h, \mu_N) \approx 1/2 \approx p^c(\theta_h, \mu_P)$, the decentralized test will be better (because $\beta \geq \alpha$). If the decreasing differences are sufficiently decreasing, i.e if $\alpha \approx 0$ and $\beta \approx \theta_h - \theta_l$, the partially decentralized test always selects students with a education close to θ_h whereas the centralized test sometimes selects a student with an education θ_l, so the partially decentralized test performs better. If all peer effects are small, i.e. if $\alpha \approx \beta \approx 0$, lemma 3 implies that $p^c(\theta_h, \mu_P) < p^d(\theta_h + \alpha, \theta_l) \approx p^d(\theta_h, \theta_l + \beta)$. Therefore, the partially decentralized test select more often a high educated student and performs better. On the contrary, if the decreasing difference are almost constant and peer effects important, i.e. if $\alpha \approx \beta \approx \theta_h - \theta_l$, the partially decentralized test always selects students with a education close to θ_h whereas the expected education of the student selected by the centralized test is bigger than θ_h because $p^c(\theta_h, \mu_P) > 1/2$, so the centralized test performs better. □

Bibliography

Aguirre, I., Cowan, S., and Vickers, J. (2010). Monopoly price discrimination and demand curvature. *The American Economic Review*, 100(4):1601–1615.

Ali, M. M., Mikhail, N., and Haq, M. S. (1978). A class of bivariate distributions including the bivariate logistic. *Journal of multivariate analysis*, 8(3):405–412.

Ausubel, L. M. and Cramton, P. (2002). Demand reduction and inefficiency in multi-unit auctions.

Bagnoli, M. and Bergstrom, T. (2005). Log-concave probability and its applications. *Economic theory*, 26(2):445–469.

Becker, G. S. (1973). A theory of marriage: Part i. *Journal of Political economy*, 81(4):813–846.

Benabou, R. (1996). Equity and efficiency in human capital investment: the local connection. *The Review of Economic Studies*, 63(2):237–264.

Bergemann, D. and Morris, S. (2005). Robust mechanism design. *Econometrica*, 73(6):1771–1813.

Bogomolnaia, A. and Moulin, H. (2001). A new solution to the random assignment problem. *Journal of Economic Theory*, 100(2):295–328.

Bonellierede, Prat, B., Brauw, D., Mueller, H., Slaughter, May, and Menendez, U. (2016). Guide to public takeovers in europe. `https://www.debrauw.com/wp-content/uploads/2016/06/Guide-to-Public-Takeovers-in-Europe-2016.pdf`.

Booth, A. and Coles, M. (2010). Education, matching, and the allocative value of romance. *Journal of the European Economic Association*, 8(4):744–775.

Braun, S., Dwenger, N., and Kübler, D. (2010). Telling the truth may not pay off: An empirical study of centralized university admissions in germany. *The BE Journal of Economic Analysis & Policy*, 10(1).

Budish, E., Che, Y.-K., Kojima, F., and Milgrom, P. (2013). Designing random allocation mechanisms: Theory and applications. *The American Economic Review*, 103(2):585–623.

Che, Y.-K. and Kojima, F. (2010). Asymptotic equivalence of probabilistic serial and random priority mechanisms. *Econometrica*, 78(5).

Chung, K.-S. and Ely, J. C. (2002). Ex-post incentive compatible mechanism design. *URL http://www. kellogg. northwestern. edu/research/math/dps/1339. pdf. Working Paper.*

Chung, K.-S. and Ely, J. C. (2007). Foundations of dominant-strategy mechanisms. *The Review of Economic Studies*, 74(2):447–476.

Clark, C. E. (1961). The greatest of a finite set of random variables. *Operations Research*, 9(2):145–162.

Collins, S. M. and Krishna, K. (1997). The harvard housing lottery: Rationality and reform. *Unpublished Working Paper.*

Cowan, S. (2016). Welfare-increasing third-degree price discrimination. *The RAND Journal of Economics*, 47(2):326–340.

Cullen, J. B., Long, M. C., and Reback, R. (2013). Jockeying for position: Strategic high school choice under texas' top ten percent plan. *Journal of Public Economics*, 97:32–48.

De Bartolome, C. A. (1990). Equilibrium and inefficiency in a community model with peer group effects. *Journal of Political Economy*, 98(1):110–133.

Durlauf, S. N. (1996). A theory of persistent income inequality. *Journal of Economic growth*, 1(1):75–93.

Estevan, F., Gall, T., Legros, P., and Newman, A. F. (2017). College admission and high school integration.

European Parliament and Council (2004). Directive 2004/25/ec on takeover bids. *Official Journal of the European Union*, L 142:12–23.

Fernandez, R. and Gali, J. (1999). To each according to...? markets, tournaments, and the matching problem with borrowing constraints. *The Review of Economic Studies*, 66(4):799–824.

Foucart, R. and Frys, L. (2018). University entrance test and high schools segregation. Working paper.

Frys, L. (2018a). House allocation with limited externalities. Working paper.

Frys, L. (2018b). Optimal uniform price mechanisms with ex-post constraints. Working paper.

Gall, T., Legros, P., and Newman, A. F. (2015). College diversity and investment incentives.

Green, J. R. and Laffont, J.-J. (1987). Posterior implementability in a two-person decision problem. *Econometrica: Journal of the Econometric Society*, pages 69–94.

Hafalir, I. E. (2007). Efficiency in coalition games with externalities. *Games and Economic Behavior*, 61(2):242–258.

Hopkins, E. (2012). Job market signaling of relative position, or becker married to spence. *Journal of the European Economic Association*, 10(2):290–322.

Hoppe, H. C., Moldovanu, B., and Sela, A. (2009). The theory of assortative matching based on costly signals. *The Review of Economic Studies*, 76(1):253–281.

Hoxby, C. (2000). Peer effects in the classroom: Learning from gender and race variation. Technical report, National Bureau of Economic Research.

Hylland, A. and Zeckhauser, R. (1979). The efficient allocation of individuals to positions. *Journal of Political economy*, 87(2):293–314.

Jehiel, P., Meyer-ter Vehn, M., Moldovanu, B., and Zame, W. R. (2006). The limits of ex post implementation. *Econometrica*, 74(3):585–610.

Kang, C. (2007). Classroom peer effects and academic achievement: Quasi-randomization evidence from south korea. *Journal of Urban Economics*, 61(3):458–495.

Karlin, S. and Rinott, Y. (1980). Classes of orderings of measures and related correlation inequalities. i. multivariate totally positive distributions. *Journal of Multivariate Analysis*, 10(4):467–498.

Klaus, B. and Klijn, F. (2005). Stable matchings and preferences of couples. *Journal of Economic Theory*, 121(1):75–106.

Klaus, B. and Klijn, F. (2007). Paths to stability for matching markets with couples. *Games and Economic Behavior*, 58(1):154–171.

Klaus, B., Klijn, F., and Massó, J. (2007). Some things couples always wanted to know about stable matchings (but were afraid to ask). *Review of Economic Design*, 11(3):175–184.

Kojima, F., Pathak, P. A., and Roth, A. E. (2013). Matching with couples: Stability and incentives in large markets. *The Quarterly Journal of Economics*, 128(4):1585–1632.

Legros, P. and Newman, A. F. (2007). Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities. *Econometrica*, 75(4):1073–1102.

Marccus Partners and the Center for European Policy Studies (2012). The takeover bids assessment report. `http://ec.europa.eu/internal_market/company/docs/takeoverbids/study/study_en.pdf`.

Maskin, E., Riley, J., and Hahn, F. (1989). *Optimal Multi-Unit Auctions*, pages 312–335. Oxford University Press. Reprinted in P. Klemperer, The Economic Theory of Auctions, London: Edward Elgar, 2000.

McAfee, R. P. and Reny, P. J. (1992). Correlated information and mecanism design. *Econometrica: Journal of the Econometric Society*, pages 395–421.

Milgrom, P. R. and Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, pages 1089–1122.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research*, 6(1):58–73.

Nahata, B., Ostaszewski, K., and Sahoo, P. K. (1990). Direction of price changes in third-degree price discrimination. *The American Economic Review*, 80(5):1254–1258.

Neumann, M., Trautwein, U., and Nagy, G. (2011). Do central examinations lead to greater grading comparability? a study of frame-of-reference effects on the university entrance qualification in germany. *Studies in Educational Evaluation*, 37(4):206–217.

Pápai, S. (2001). Strategyproof and nonbossy multiple assignments. *Journal of Public Economic Theory*, 3(3):257–271.

Perry, M. and Reny, P. J. (1999). *An ex-post efficient auction.* Maurice Falk Institute for Economic Research in Israel.

Perry, M. and Reny, P. J. (2002). An efficient auction. *Econometrica*, 70(3):1199–1212.

Pop-Eleches, C. and Urquiola, M. (2013). Going to a better school: Effects and behavioral responses. *The American Economic Review*, 103(4):1289–1324.

Roth, A. and Sotomayor, M. (1992). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Number 18 in Econometric Society Monographs. Cambridge University Press.

Roth, A. E. and Peranson, E. (1999). The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American Economic Review*, 89(4):748–782.

Sasaki, H. and Toda, M. (1996). Two-sided matching problems with externalities. *Journal of Economic Theory*, 70(1):93–108.

Schmalensee, R. (1981). Output and welfare implications of monopolistic third-degree price discrimination. *The American Economic Review*, 71(1):242–247.

Schwerdt, G. and Woessmann, L. (2017). The information value of central school exams. *Economics of Education Review*, 56:65–79.

Segal, I. (2003). Optimal pricing mechanisms with unknown demand. *The American economic review*, 93(3):509–529.

Spence, M. (1973). Job market signaling. *The Quarterly Journal of Economics*, 87(3):355–374.

Summers, A. A. and Wolfe, B. L. (1977). Do schools make a difference? *The American Economic Review*, 67(4):639–652.

Varian, H. R. (1985). Price discrimination and social welfare. *The American Economic Review*, 75(4):870–875.

Wilson, R. (1985). Game-theoretic analysis of trading processes. Technical report, DTIC Document.

Zimmerman, S. D. (2014). The returns to college admission for academically marginal students. *Journal of Labor Economics*, 32(4):711–754.

Selbständigkeitserklärung

Ich bezeuge durch meine Unterschrift, dass meine Angaben über die bei der Abfassung meiner Dissertation benutzten Hilfsmittel, über die mir zuteil gewordene Hilfe sowie über frühere Begutachtungen meiner Dissertation in jeder Hinsicht der Wahrheit entsprechen.

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