Transfer of boundary conditions for DAEs of index 1*

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Abstract

In this paper, the concept of Abramov's method for transferring boundary conditions posed for regular ordinary differential equations is applied to index 1 DAEs. Having discussed the reduction of inhomogeneous problems to homogeneous ones and analyzed the underlying ideas of Abramov's method, we consider boundary value problems for index 1 linear DAEs both with constant and varying leading matrix. We describe the relations defining the subspaces of solutions satisfying the prescribed boundary conditions at one end of the interval. The index 1 DAEs that realize the transfer are given and their properties are studied. The results are reformulated for inhomogeneous index 1 DAEs, as well.

Key words: Differential-algebraic equations, boundary value problems, transfer method

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1 Preliminaries

1.1 Transforming to homogeneous systems

In the theory of regular linear ordinary differential equations, there exist simple tricks that allow to transform inhomogeneous systems into homogeneous ones (of higher dimension) and, at least at the theoretical level, the investigations may be carried out only for homogeneous systems. This approach simplifies the theory. Of course, the new homogeneous system is of a special form. Therefore, when handling inhomogeneous systems, especially when constructing efficient numerical algorithms, these specialities have to be taken into account. Let the boundary value problem be of the following form:

$$y' + B(x)y = f(x), \quad x_l \le x \le x_r, \tag{1.1.1}$$

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$$y = (y_1, \dots, y_n)^T, \quad f = (f_1, \dots, f_n)^T, \quad B = (b_{ij})_{i,j=1}^n,$$

$$CY = d, \tag{1.1.2}$$

$$C = (C_l | C_r), \quad C_l, C_r \in R^{m\times n}, \quad C_l = (c_{ij}^l),\ C_r = (c_{ij}^r),$$

$$Y = (y^T(x_l) | y^T(x_r))^T, \quad d = (d_1, \dots, d_m).$$

We remove the inhomogeneity as done by Abramov [1]. Let $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n, \hat{y}_{n+1})^T$, $\hat{B} = (\hat{b}_{ij})_{i,j=1}^{n+1}$, $\hat{Y} = (\hat{y}^T(x_l) | \hat{y}^T(x_r))^T$,

$$\hat{b}_{ij} = \begin{cases} b_{ij}, & i,j = 1, \dots, n, \\ -f_i, & j = n+1, i = 1, \dots, n \\ 0, & i = n+1, j = 1, \dots, n+1. \end{cases}$$

Next we turn to the boundary conditions. Let

$$\hat{C} = (\hat{C}_l | \hat{C}_r), \quad \hat{C}_l, \hat{C}_r \in R^{m\times (n+1)}, \quad \hat{C}_l = (\hat{c}_{ij}^l),\ \hat{C}_r = (\hat{c}_{ij}^r),$$

$$\hat{c}_{ij}^l = \begin{cases} c_{ij}^l, & i = 1, \dots, m, j = 1, \dots, n, \\ \beta_i, & j = n+1, i = 1, \dots, m, \end{cases} \qquad c_{ij}^r = \begin{cases} c_{ij}^r, & i = 1, \dots, m, j = 1, \dots, n, \\ -d_i - \beta_i, & j = n+1, i = 1, \dots, m. \end{cases}$$

Together with the problem (1.1.1), (1.1.2), consider the boundary value problem

$$\hat{y}' + \hat{B}(x)\hat{y} = 0, \tag{1.1.3}$$

$$\hat{C}\hat{Y} = 0. \tag{1.1.4}$$

One easily verifies the following statements:

S1: Let $y(x)$ be a solution vector of (1.1.1), (1.1.2). Then, for an arbitrarily chosen parameter vector $\beta = (\beta_1, \dots, \beta_m)^T$ and scalar $c \neq 0$, the vector function $\hat{y}(x) \stackrel{\text{def}}{=} (cy^T(x)|c)^T$ (i.e. the last component is a nonzero constant) satisfies the equations (1.1.3), (1.1.4).

S2: Let $\hat{y}(x)$ be a solution vector of the problem (1.1.3), (1.1.4) for some value of $\beta = (\beta_1, \dots, \beta_m)^T$, such that for some x^0, $\hat{y}_{n+1}(x^0) = c \neq 0$. Then the vector function composed of the first n components of $\hat{y}(x)$, each multiplied by $1/c$, satisfies (1.1.1) and (1.1.2).

PROOF *S1* can be obtained by simple substitution. When checking *S2*, it should be noticed that due to the last equation in (1.1.3), $\hat{y}_{n+1} \equiv c = const$. Then, due to the homogeneity, $\tilde{y} = \frac{1}{c}\hat{y}$ is also a solution of (1.1.3) and (1.1.4) with $\tilde{y}_{n+1} \equiv 1$ and so $y(x) = (\tilde{y}_1(x), \dots, \tilde{y}_n(x))$ satisfies the equations (1.1.1), (1.1.2).

In this context the problems (1.1.1), (1.1.2) and (1.1.3), (1.1.4) are related. Each solution of (1.1.1), (1.1.2) gives rise to a one-dimensional subspace of solutions of (1.1.3), (1.1.4) and any solution of (1.1.3), (1.1.4) with a non-trivial last component results in a solution of (1.1.1), (1.1.2). Provided that there exists no solution with the property $\hat{y}_{n+1} \neq 0$, the problem (1.1.1), (1.1.2) has no solution, too.

This relationship between the inhomogeneous systems and their new homogeneous counterparts may be useful when $B(x)$ and $f(x)$ have some common properties (say, smoothness). In this case, the statements can be formulated more simply for the homogeneous systems. However, the price of this problem reduction are in new problem formulations. When having a homogeneous system, we are usually interested in the nonexistence of nontrivial solutions and/or in the existence of nontrivial solutions whatever they are. Now we are interested in special nontrivial solutions (with nonzero, constant last component), too, provided the system arose by "homogenization".

It is worth mentioning that boardering $B(x)$ up to $\hat{B}(x)$ adds just one zero eigenvalue to the set of eigenvalues of $B(x)$.

The inhomogeneous systems of DAEs behave more complicated than their homogeneous pairs. It is interesting to know the results of the trick above when it is applied to DAE-s. The existence and the behaviour of solutions of DAE problems is closely connected with the index of the system. Thus, it is worth looking at the result of the above "homogenization" in DAEs. In this paper we consider only DAE systems of index 1:

$$A(x)y' + B(x)y = f(x), \quad x_l \leq x \leq x_r. \tag{1.1.5}$$

A necessary and sufficient condition for this is that *rank* A = const $< n$ and the matrix $G = A + B(I - A^+A)$ is invertible. Here A^+ is the generalized inverse of A, $I - A^+A$ is an orthoprojector onto *Ker* A. The enlarged system reads

$$\hat{A}\hat{y}' + \hat{B}\hat{y} = 0, \tag{1.1.6}$$

where

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} B & -f \\ 0 & 0 \end{pmatrix}.$$

Since

$$\hat{A}^+\hat{A} = \begin{pmatrix} A^+A & 0 \\ 0 & 1 \end{pmatrix},$$

we find

$$\hat{G} = \hat{A} + \hat{B}(I - \hat{A}^+\hat{A}) = \begin{pmatrix} A + B(I - A^+A) & 0 \\ 0 & 1 \end{pmatrix}$$

and the new system remains of index 1.

Together with (1.1.5), let us consider the boundary condition

$$CY = d, \tag{1.1.7}$$

where C, Y, d are as before. Proceeding as before, we set

$$\hat{C}\hat{Y} = 0. \tag{1.1.8}$$

It turns out that both statements *S1* and *S2* remain valid analogously.

1.2 Transfer of boundary conditions in regular ODEs

Let us return to the problems (1.1.1), (1.1.2) and (1.1.3), (1.1.4). Let us assume that in (1.1.2) the boundary conditions are separated, i.e.

(1.2.1)
$$C_l = \begin{pmatrix} C_{l1} \\ 0 \end{pmatrix}, C_r = \begin{pmatrix} 0 \\ C_{r2} \end{pmatrix},$$

$$C_{l1} \in R^{m_l \times n}, \quad C_{r2} \in R^{m_r \times n}, \quad m_l \leq n, \quad m_r \leq n, \quad m_l + m_r = m,$$
$$rank\ C_{l1} = m_l \quad rank\ C_{r2} = m_r.$$

Provided the conditions are non-separated, there are several tricks to transform the system into another one with separated boundary conditions, we refer to [2, 3]. From now on we assume the original problems to have separated boundary conditions. Set

$$\beta_i = \begin{cases} -d_i, & i = 1, \ldots, m_l, \\ 0, & i = m_l + 1, \ldots, m, \end{cases}$$

then the conditions (1.1.4) are separated too.

Here and in following sections we will consider homogeneous boundary value problems, i.e. systems of differential equations

(1.2.2)
$$y' + B(x)y = 0, \quad x_l \leq x \leq x_r,$$

and systems of differential algebraic equations of index 1

(1.2.3)
$$A(x)y' + B(x)y = 0, \quad x_l \leq x \leq x_r,$$

with boundary conditions

(1.2.4)
$$CY = 0,$$

not taking into account for the moment whether they arose originally or they originated from an inhomogeneous problem (consequently, the mark " ~" will not be used until needed). The matrix C is assumed to be of the form given by (1.2.1). Throughout the whole paper we assume $B(x), f(x) \in C[x_l, x_r]$ and $A(x) \in C^1[x_l, x_r]$.

Let M denote an $(n-m)$-dimensional linear subspace of vectors $y \in R^n$. This subspace can be described by a relation

(1.2.5)
$$\phi^T y = 0, \quad (y \in Ker\ \phi^T),$$

where $\phi \in R^{n \times m}$, $rank\ \phi = m$, and so the columns of the matrix ϕ span $M^\perp$, in other words $M^\perp = Im\ \phi$. The columns of ϕ form a basis in $M^\perp$.

The linear subspace $M(x)$ of solutions of (1.2.2) for which $M(\bar{x})$ (or a equivalently, $M^\perp(\bar{x})$) is prescribed for some $\bar{x}$ may be obtained using the (unique) solution of the initial value problem for the adjoint equation

(1.2.6)
$$\phi' - B^T(x)\phi = 0,$$

taking any basis in $M^\perp(\bar{x})$ as $\phi(\bar{x})$ at $x = \bar{x}$ [4]. Since, obviously, the only requirement for the (full rank) matrix $\phi(x)$ is to span the subspace $M^\perp(x)$, there are several ways different from (1.2.6) to define it. This is precisely the central question in the so-called "initial value methods" [3] for solving two-point boundary value problems: Find the $(n - m_i)$-dimensional linear subspaces $M_i(x)$ of solutions of (1.2.2) satisfying boundary conditions $C_{i'} y(x_i) = 0$ prescribed at points x_i, i.e. define an appropriate matrix function $\phi_i(x)$,

$$\phi_i^T(x) y(x) = 0. \tag{1.2.7}$$

(Here the index i stands for either l or r and i' stands for either $l1$ or $r2$, respectively.) Each method has its own advantages and disadvantages. In [5], the following choice is proposed. Take the (matrix) solution of the problem below:

$$\phi_i' - (I - \phi_i(\phi_i^T \phi_i)^{-1} \phi_i^T) B^T \phi_i = 0, \tag{1.2.8}$$

$$\phi_i(x_i) \ \text{ is any basis in } \ M_i^\perp(x_i). \tag{1.2.9}$$

Since we will return to this equation and its variants repeatedly, let us recall several results from [5, 6].

R1: The matrix function $\phi_i(x)$ realizes a special choice of the basis in $M_i^\perp(x)$:

$$\phi_i^T(x) \phi_i'(x) = 0. \tag{1.2.10}$$

R2: The initial value problem (1.2.8), (1.2.9) has a unique solution on the whole interval $[x_l, x_r]$.

R3: Let $\phi_i(x)$ be the solution of (1.2.8), (1.2.9). Then the relation (1.2.7) holds iff $y(x) \in M_i(x)$

Remark 1. The main point in the proof is the representation of $M_i^\perp(x)$ in a basis with the property (1.2.10), that is, the representation of the solution of (1.2.6) using the unknown basis transformation matrix within $M_i^\perp(x)$ and showing that $|\phi_i^T(x)\phi_i(x)|$ and $|(\phi_i^T(x)\phi_i(x))^{-1}|$ are uniformly bounded on the whole interval.

Remark 2. In contrast to *R2*, if one applies other basis transformations yielding for example a Riccati equation, it may happen that the solution of the corresponding initial value problem exists on a shorter interval only.

Remark 3. In fact, *R1* expresses that $\phi_i(x)$ changes as smoothly as the subspace $M_i^\perp(x)$ does. This may not be the case for the solution of (1.2.6), where the basis vectors may change very fast or become nearly linearly dependent even in smoothly varying $M_i^\perp(x)$. This property allows larger steps in the numerical integration of (1.2.8), which in turn gives savings in time despite of the complicated form of the equation. As a consequence of (1.2.10), one obtains

$$\phi_i^T(x)\phi_i(x) = const = \Phi,$$

i.e. an extra possibility for checking the accuracy of numerical integration.

Remark 4. This special choice of the basis in $M_i^{\perp}(x)$ results in the most remarkable property of this method. It preserves the well-conditioning of the boundary value problem, which may not be the case for some initial value methods like simple shooting or can only be achieved at least by extra efforts.

It is interesting to notice that, due to assumption (1.2.9), equation (1.2.8) can be rewritten as

$$\phi_i' - (I - \phi_i\phi_i^+)B^T\phi_i = 0. \tag{1.2.11}$$

There are two "natural" choices for $\phi_i(x_i)$:

$$\phi_i(x_i) = C_{i'}^T \quad \text{and} \quad \phi_i(x_i) = C_{i'}^T L_{i'},$$

where $L_{i'}$ is chosen such that $L_{i'}L_{i'}^T = (C_{i'}C_{i'}^T)^{-1}$ holds. The latter one provides $\Phi = I$, where I is the $m_i \times m_i$ identity matrix.

Now we recall that (1.2.7) forms a system of linear algebraic equations to determine $y(x)$. Thus, one obtain at the following existence theorem:

Theorem 1.1 *Let $\phi_l(x)$ and $\phi_r(x)$ be the solutions of the problem* (1.2.8),(1.2.9). *Then the space $M(x) \stackrel{\text{def}}{=} M_l(x) \cap M_r(x)$ of solutions of the boundary value problem* (1.2.2), (1.2.4) *is of dimension $n - k$, where*

$$k = rank\ (\phi_l(x)|\phi_r(x)) \leq m,$$

and k is independent of $x \in [x_l, x_r]$. It holds that $M^{\perp}(x) = Im\ (\phi_l(x)|\phi_r(x))$.

Proof: Take into account that *R3* holds for both $i = l$ and $i = r$. The resulting linear system is simply

$$(\phi_l(x)|\phi_r(x))^T y(x) = 0.$$

Notice that no restriction was made on m in the formulation; we only assumed $m_l \leq n$ and $m_r \leq n$.

As a particular case of the above, we obtain the following

Corollary 1.1 *The trivial solution $y(x) \equiv 0$ is the unique solution iff $k = n$.(This implies $m \geq n$.)*

Assume that the problem (1.2.2), (1.2.4) has originated from an inhomogeneous problem (now n may be considered to be an $n+1$ of Section 1.1, B for $\hat{B}$ there, etc.).

Theorem 1.2 *The inhomogeneous problem (of dimension $n - 1$) is solvable iff for the "homogenized" boundary value problem* (1.2.2), (1.2.4)

$$rank\ (\phi_l(x)|\phi_r(x)|e_n) = rank \begin{pmatrix} \phi_l(x) & \phi_r(x) & e_n \\ & e_{m+1}^T & \end{pmatrix}, \tag{1.2.12}$$

where e_k denotes a k-component column with the first $k-1$ components equal to 0 and the last one equal to 1.

The (original) inhomogeneous problem has a unique solution iff the matrix on the left-hand side of the equation (1.2.12) *is of rank n.(This implies $m \geq n$.)*

2 Transfer of boundary conditions in index 1 DAEs

2.1 Auxiliary statements for index 1 DAEs

Now we return to the homogeneous DAE (1.2.3) and recall that it is said to be of index 1 iff $rank\ A(x)$ is $const < n$ and $G(x) \stackrel{\text{def}}{=} A(x) + B(x)(I - A(x)^+A(x))$ is nonsingular for any x. In this case the decomposition

$$R^n = Ker\ A(x) \oplus S(x) \tag{2.1.1}$$

holds where

$$S(x) \stackrel{\text{def}}{=} \{z \in R^n : \quad B(x)z \in Im\ A(x)\} = Ker\ [(I - A(x)A(x)^+)B(x)].$$

$S(x)$ is precisely the solution space for (1.2.3) and $\oplus$ denotes the direct sum. Exactly one solution passes through each (x_0, y_0), $x_0 \in [x_l, x_r]$, $y_0 \in S(x_0)$. Therefore the initial value problem for (1.2.3) with the initial value $y(x_0) = y_0$ is solvable iff $y_0 \in S(x_0)$. Such an initial value is called consistent. We are interested in formulating initial value problems with consistent initial values.

Let $\tilde{y}_0 \in R^n$ be an arbitrary vector. Choose $y(x_0) \in S(x_0)$ such that

$$A(x_0)(y(x_0) - \tilde{y}_0) = 0,$$

or equivalently, $A(x_0)^+A(x_0)(y(x_0) - \tilde{y}_0) = 0$, then

$$y(x_0) = P_s(x_0)y(x_0) = P_s(x_0)\tilde{y}_0, \quad y_0 := P_s(x_0)\tilde{y}_0 \in S(x_0). \tag{2.1.2}$$

Here $P_s(x)$ denotes the projector onto $S(x)$ along $Ker\ A(x)$.

A fundamental solution matrix $Y(x)$ may be defined by

$$AY' + BY = 0, \quad A(x_0)(Y(x_0) - I) = 0.$$

One easily verifies the relations

$$Y(x) = P_s(x)Z(x)A(x_0)^+A(x_0), \quad Im\ Y(x) = S(x),$$

where Z(x) denotes the classical nonsingular fundamental matrix of the inherent regular ODE:

$$Z' + RZ = 0, \quad Z(x_0) = I, \quad R \stackrel{\text{def}}{=} A^+AG^{-1}B - (A^+A)'P_s.$$

Now consider separated boundary conditions (1.2.4), (1.2.1), keeping the assumptions of section 1.2. The special formulation of initial conditions leads to the natural conditions

$$C_{i'}A(x_i)^+A(x_i) = C_{i'}, \quad l' \stackrel{\text{def}}{=} l1, \quad r' \stackrel{\text{def}}{=} r2, \tag{2.1.3}$$

which we assume to be valid. The formula (2.1.3) means that the boundary conditions are related to the inherent regular ODE. It immediately results in $m_i \leq rank\ A$.

Introducing the orthoprojector $P(x) \overset{\text{def}}{=} A(x)^+A(x)$ we may reformulate equation (1.2.3) as

$$A(x)P(x)y'(x) + B(x)y(x) = 0,$$

hence, as

$$A(x)\{[P(x)y(x)]' - P'(x)y(x)\} + B(x)y(x) = 0.$$

This indicates the function space

$$C_A^1 \overset{\text{def}}{=} \{y(.) : y(.) \in C[x_\ell, x_r], (Py)(.) \in C^1[x_l, x_r]\}$$

to be the appropriate one for the solutions of (1.2.3).
$P(x)$ is a C^1 matrix function since $A(x)$ is so. However, $P_s(x)$ is not necessarily C^1 because the second coefficient matrix $B(x)$ is supposed to be continuous only. Consequently, the solutions may have components that are continuous only.

For a detailed discussion of the material above we refer to [7, 8].

2.2 Transfer of boundary conditions. The constant A case

In this section we assume $A(x) \equiv A$. The adjoint equation

$$A^T\phi'(x) - B(x)^T\phi(x) = 0 \tag{2.2.1}$$

is also an index 1 DAE [9]. The consistent initial values belong to

$$S_*(x) \overset{\text{def}}{=} \{\xi \in R^n : \quad B(x)^T\xi \in Im\ A^T\} = Ker\ [(I - A^+A)B(x)^T].$$

In analogy with (2.1.1), we have

$$R^n = Ker\ A^T \oplus S_*(x).$$

Let $P_{*s}(x)$ denote the projector onto $S_*(x)$ along $Ker\ A^T$.

For each pair $y(x)$, $\phi(x)$ solving (1.2.3) and (2.2.1), respectively, we have

$$(\phi^T(x)Ay(x))' = \phi^{T\prime}(x)Ay(x) + \phi^T(x)Ay'(x) = \phi^T(x)By(x) - \phi^T(x)By(x) = 0.$$

If

$$\phi(x_i) = P_{*s}(x_i)A^{T+}C_{i'}^T \tag{2.2.2}$$

is taken, i.e. $A^T\phi_i(x_i) = C_{i'}^T = A^+AC_{i'}^T$ is chosen and $\phi_i(x)$ denotes the solution of (2.2.1) with the initial value (2.2.2), then

$$\begin{aligned} \phi_i^T(x)Ay(x) &\equiv \phi_i^T(x_i)Ay(x_i) = \\ &= C_{i'}A^+P_{*s}^T(x_i)Ay(x_i) = C_{i'}A^+Ay(x_i) = C_{i'}y(x_i), \end{aligned} \tag{2.2.3}$$

thus the solution subspace $M_i(x)$ (determined by the condition $C_{i'}y(x_i) = 0$) is given by

$$\phi_i^T(x)Ay(x) = 0, \quad (I - AA^+)By(x) = 0, \tag{2.2.4}$$

i.e.

$$M_i(x) = Ker\ \phi_i^T(x)A \cap S(x) = Ker\ (A^T\phi_i(x))^T \cap S(x) = [Im\ A^T\phi_i(x)]^\perp \cap S(x).$$

According to section 1.2, there arises another natural choice for the value of $\phi_i(x_i)$ if one replaces $C_{i'}$ by $L_{i'}^T C_{i'}$ in (2.2.2).

Unfortunately, solutions of (2.2.1) may behave unstably, even if the subspace defined by the boundary condition changes slowly. As done for the regular differential problems in [4, 5], an orthogonalization process helps to avoid these problems. So, instead of finding the solution $\phi(x)$ of the initial value problem for the adjoint equation we will look for a matrix $\psi(x)$ that varies more smoothly. Since the subspace $M_i(x)$ is fixed, there should be a nonsingular matrix $T(x)$ such that $A^T\psi(x)T(x) = A^T\phi(x)$ holds.

Consider the equation

$$A^T\psi' - [I - A^T\psi(\psi^TAA^T\psi)^{-1}\psi^TA]B^T\psi = 0. \tag{2.2.5}$$

Lemma 2.1 *If $\psi(x)$ solves the equation* (2.2.5), *we have*

$$(A^T\psi)^T(A^T\psi)' \equiv \psi^TAA^T\psi' \equiv 0. \tag{2.2.6}$$

Proof: Premultiplying (2.2.5) by ψ^TA gives

$$\psi^TAA^T\psi' = \psi^TA[I - A^T\psi(\psi^TAA^T\psi)^{-1}\psi^TA]B^T\psi = 0.$$

Since $I - A^T\psi(\psi^TAA^T\psi)^{-1}\psi^TA$ projects onto $[Im\ A^T\psi]^\perp = Ker\ \psi^TA$, the result simply follows.

Corollary 2.1

$$\psi^T(x)AA^T\psi(x) \equiv const = \psi^T(x_i)AA^T\psi(x_i) =: \Pi$$

Proof:

$$(\psi^TAA^T\psi)' = \psi^{T\prime}AA^T\psi + \psi^TAA^T\psi' \equiv 0,$$

since the transpose of (2.2.6) is also valid.

Now choose the initial value for ψ_i equal to that for ϕ, say, as in (2.2.2), i.e.

$$\begin{array}{rcl} A^T\psi_i(x_i) &=& C_{i'}^T, \\ \psi_i(x_i) &=& P_{*s}(x_i)A^{T+}C_{i'}^T \quad \text{respectively.} \end{array} \tag{2.2.7}$$

We do not show the existence of the nonsingular matrix $T(x)$ explicitly, we rather verify the validity of the following

Lemma 2.2 *Let ψ and y solve the equations* (1.2.3) *and* (2.2.5), *respectively. Then*

$$\psi^T(x)Ay(x) \equiv 0 \quad \Longleftrightarrow \quad \psi(x')^TAy(x') = 0 \quad \text{at some } x' \in [x_l, x_r]$$

PROOF:

$$\begin{aligned}(\psi^T Ay)' &= \psi^{T\prime} Ay + \psi^T Ay' \\ &= \psi^T B[I - A^T\psi(\psi^T AA^T\psi)^{-1}\psi^T A]y - \psi^T By \\ &= -\psi^T BA^T\psi(\psi^T AA^T\psi)^{-1}\psi^T Ay.\end{aligned}$$

This differential equation is linear with respect to $\psi^T Ay$. The zero initial value prescribed at $x = x'$ proves the assertion from the right to the left.

Thus we have proved that the solution ψ of (2.2.5), (2.2.7) determines precisely the solution subspace $M_i(x)$ we are interested in, provided $\psi(x)$ exists on the whole interval $[x_l, x_r]$. The validity of the last assumption is the main point in the following

Theorem 2.1 *The initial value problem* (2.2.5), (2.2.7) *has a unique solution* ψ_i *defined on* $[x_l, x_r]$*;* $\psi_i(x)$ *determines the solution subspace* $M_i(x)$ *for* (1.2.3) *related to the condition* $C_{i'}y(x_i) = 0$ *by*

$$M_i(x) = Ker\ \psi_i^T(x)A \cap S(x) = Ker\ (A^T\psi_i(x))^T \cap S(x) = [Im\ A^T\psi_i(x)]^\perp \cap S(x).$$

PROOF: It remains to show the solvability of (2.2.5), (2.2.7) on the whole interval $[x_l, x_r]$. Consider the initial value problem

$$\begin{aligned}\eta' - [I - \eta(\eta^T\eta)^{-1}\eta^T]B^T P_{*s} A^{T+}\eta &= 0, \\ \eta(x_i) &= C_{i'}^T.\end{aligned} \tag{2.2.8}$$

Below, the background of (2.2.8) becomes transparent: This equation may be obtained from (2.2.5) by means of the transformation $\eta = A^T\psi$, $\psi = P_{*s}A^{T+}\eta$.

This problem is of the type (1.2.8), (1.2.9); therefore it is solvable on $[x_l, x_r]$ [5, 6]. Since $\eta(x)^T\eta(x) = const = C_{i'}C_{i'}^T$, the function $\eta(x)$ solves also the initial value problem

$$\eta' - [I - \eta(C_{i'}C_{i'}^T)^{-1}\eta^T]B^T P_{*s}A^{T+}\eta = 0,$$

$$\eta(x_i) = C_{i'}^T.$$

Multiplying by A^+A and taking into account that $(I - A^+A)B^T P_{*s} = 0$, $A^{T+} = A^{T+}A^T A^{T+} = A^{T+}A^+A$, we obtain

$$(A^+A\eta)' - [I - A^+A\eta(C_{i'}C_{i'}^T)^{-1}(A^+A\eta)^T]B^T P_{*s}A^{T+}A^+A\eta = 0,$$

$$\eta(x_i) = A^+AC_{i'}^T = C_{i'}^T.$$

Thus, both η and $A^+A\eta$ solve the same initial value problem, which is, at least locally, uniquely solvable. Hence, $\eta(x) \equiv A^+A\eta(x)$. Now, if $\psi := P_{*s}A^{T+}\eta$, then it becomes a solution of (2.2.5), (2.2.7) on the same interval, as the following equations will show. $A^T\psi(x) = A^T A^{T+}\eta(x) = A^+A\eta(x)$, $\psi(x)^T AA^T\psi(x) = \eta(x)^T A^+ P_{*s}^T AA^T P_{*s} A^{T+}\eta(x)$, $\eta(x)^T A^+A\eta(x) = \eta(x)^T\eta(x) = C_{i'}C_{i'}^T$.

Multiplying (2.2.8) by $A^TA^{T+} = A^+A$ and using the relation $(I - A^+A)B^TP_{*s} = 0$ again, we conclude

$$A^T\psi' - [I - A^T\psi(\psi^TAA^T\psi)^{-1}\eta^TA^+A]B^TP_{*s}\psi = 0,$$

but $\eta^TA^+A = \eta^TA^+P_{*s}^TA = \psi^TA$.

The equality $P_{*s}\psi = \psi$ is obvious and the uniqueness of ψ comes from the theory of initial value problems for nonlinear index 1 DAEs, see [10]. In fact, as for linear problems, it depends on the correct formulation of initial values.

Let us briefly return to the inherent regular ODE of (1.2.3), i.e. to

$$z' + Rz = 0, \quad R := A^+AG^{-1}B. \tag{2.2.9}$$

Naturally, the question arises whether (2.2.8) represents the original Abramov's method for this ODE. The next theorem shows that this applies, indeed.

Theorem 2.2 $R^T = B^TP_{*s}A^{T+}$.

PROOF: Use the notations $Q := I-P$, $P := A^+A$, $Q_* := I-P_*$, $P_* := AA^+$, $G := A+BQ$, $G_* := A^T+B^TQ_*$, $P_{*s} := I-Q_*G_*^{-1}B^T$ and $\mathcal{G} := A+Q_*BQ$. Since A, Q_*B form a regular index 1 matrix pencil, $\mathcal{G}$ is nonsingular and we have

$$\mathcal{G}^{-1}A = P, \quad \mathcal{G}^{-1}Q_*BQ = Q, \quad \mathcal{G}^{-1}P_* = \mathcal{G}^{-1}AA^+ = PA^+,$$

and further $Q\mathcal{G}^{-1} = Q\mathcal{G}^{-1}(P_* + Q_*) = Q\mathcal{G}^{-1}Q_*$. Next we compute

$$G^{-1} = (I - \mathcal{G}^{-1}P_*BQ)\mathcal{G}^{-1}, \quad G_*^{-1} = (I - \mathcal{G}^{-1T}PB^TQ_*)\mathcal{G}^{-1T}.$$

This leads to

$$\begin{aligned} B^TP_{*s}A^{T+} &= B^TA^{T+} - B^TQ_*\mathcal{G}^{-1T}QB^TA^{T+} = (I - B^TQ_*\mathcal{G}^{-1T}Q)B^TA^{T+} \\ &= (I - B^T\mathcal{G}^{-1T}Q)B^TA^{T+} \end{aligned}$$

and, on the other hand,

$$\begin{aligned} R^T &= B^TG^{-1T}A^TA^{T+} = B^T(AG^{-1})^TA^{T+} \\ &= B^T[(A + BQ)G^{-1} - BQG^{-1}]^TA^{T+} \\ &= B^T[I - BQ(I - \mathcal{G}^{-1}P_*BQ)\mathcal{G}^{-1}]^TA^{T+} \\ &= B^T(I - BQ\mathcal{G}^{-1})^TA^{T+} = (I - B^T\mathcal{G}^{-1T}Q)B^TA^{T+}. \end{aligned}$$

Remark. Numerical integration methods applied to inital value problems for (1.2.3) are known to work well. There is no reason for finding the inherent ODE (2.2.9) in practice. By the same argument, instead of (2.2.8), the problem (2.2.5) should be solved numerically. However, for a proper understanding of the situation, it is nice to have the property stated in the Theorem 2.2.

From the results above we can now formulate the existence theorem:

Theorem 2.3 *Let $\psi_l(x)$, $\psi_r(x)$ be the solutions of* (2.2.5) *with $A^T\psi_l(x_l) = C_{l1}^T$ and $A^T\psi_r(x_r) = C_{r2}^T$, respectively. Then the solution subspace $M(x) = M_l(x) \cap M_r(x)$ of the boundary value problem* (1.2.3), (1.2.4) *is of dimension $n-k$, where*

$$k = rank\ (A^T\psi_l(x)|A^T\psi_r(x)|B^T(I-AA^+)).$$

Moreover,

$$M(x) = Ker \begin{pmatrix} \psi_l(x)^T A \\ \psi_r(x)^T A \\ (I-AA^+)B \end{pmatrix} \quad and \quad M^{\perp}(x) = Im\ (A^T\psi_l(x)|A^T\psi_r(x)|B^T(I-AA^+)).$$

2.3 Transfer of boundary conditions. The varying A case

Consider the index 1 DAE

$$A(x)y'(x) + B(x)y(x) = 0, \quad x_l \le x \le x_r \tag{2.3.1}$$

with the assumptions on $A(x)$ and $B(x)$ posed at the beginning of section 2.1. Denote

$$P(x) := A(x)^+A(x), \quad P_*(x) := A(x)A(x)^+.$$

Obviously, $A(x) = P_*(x)A(x) = A(x)P(x)$, and P, P_* are C^1 matrix functions.

Rewrite (2.3.1) as

$$\begin{aligned} A(x)\{[P(x)y(x)]' - P'(x)y(x)\} + B(x)y(x) &= 0, \quad \text{or} \\ A(x)[P(x)y(x)]' + [B(x) - A(x)P'(x)]y(x) &= 0. \end{aligned} \tag{2.3.2}$$

Again, we are looking for equations which provide a function $\phi(x)$ describing the solution manifolds under consideration by

$$\phi(x)^T A(x)y(x) = 0 \tag{2.3.3}$$

as before. For this purpose, we first turn to

$$A^T(P_*\phi)' - (B^T - A^{T\prime}P_*)\phi = 0, \tag{2.3.4}$$

or, in slightly reformulated version, to

$$A^T(P_*\phi)' - (B^T + A^TP_*' - A^{T\prime})\phi = 0 \tag{2.3.5}$$

$$(A^T\phi)' - B^T\phi = 0. \tag{2.3.6}$$

Lemma 2.3 *The identity $\phi(x)^T A(x)y(x) \equiv const$ is true for all pairs of solutions $y \in C_A^1$, $\phi \in C_{A^T}^1$ of* (2.3.1) *and* (2.3.4) *(or* (2.3.5) *or* (2.3.6)*), respectively.*

PROOF: Let $Q := I - P$.

$$
\begin{aligned}
(\phi^T Ay)' &= (\phi^T P_* APy)' = (\phi^T P_*)'Ay + \phi^T P_* A'Py + \phi^T A(Py)' \\
&= \phi^T(B - P_*A')y + \phi^T P_* A'Py - \phi^T(B - AP')y \\
&= -\phi^T P_* A'y + \phi^T P_* A'Py + \phi^T AP'y = -\phi^T P_* A'Qy + \phi^T AP'y \\
&= \phi^T P_* AQ'y + \phi^T AP'y = \phi^T A(Q' + P')y = 0.
\end{aligned}
$$

Remark that the DAE (2.3.4) ((2.3.5), (2.3.6)) is of index 1 simultaneously with (2.3.1).

Concerning initial and boundary conditions, all that was explained in section 2.1 remains valid. Hence, the solution $\phi_i(x)$ of the initial value problem for (2.3.4) with $A(x_i)^T\phi(x_i) = C_{i'}^T$ provides a tool for describing the solution manifold $M_i(x)$ related to the (initial value) problem for (2.3.1) with $C_{i'}y(x_i) = 0$. Namely,

$$M_i(x) = Ker\ \phi_i(x)^T A(x) \cap S(x)$$

is true.

Unfortunately, this method cannot be expected to behave well, in general. The same stability problems as in the simpler case of regular ODEs or DAEs with constant A may occur. This is why we are looking for a more stable representation of $M_i(x)$ again.

Following the concept of [5, 6] as done in section 2.2, we transform $\phi(x)$ by a nonsingular matrix $T(x)$, $\phi(x) = \psi(x)T(x)$. The transformation matrix is chosen to achieve

$$\psi(x)^T A(x)(A(x)^T\psi(x))' = 0.$$

This leads to the nonlinear DAE

$$
\begin{aligned}
&(2.3.7) & (A^T\psi)' - [I - A^T\psi(\psi^T AA^T\psi)^{-1}\psi^T A]B^T\psi &= 0 \quad \text{or} \\
&(2.3.8) & A^T(P_*\psi)' + A^{T\prime}P_*\psi - [I - A^T\psi(\psi^T AA^T\psi)^{-1}\psi^T A]B^T\psi &= 0.
\end{aligned}
$$

Lemma 2.4 *All solutions $\psi \in C^1_{A^T}$ of* (2.3.7) *have the property*

$$(2.3.9) \qquad \psi(x)^T A(x)(A(x)^T\psi(x))' \equiv 0.$$

PROOF: The proof is exactly the same as for Lemma 2.1.

Corollary 2.2

$$\psi(x)^T A(x)A(x)^T\psi(x) \equiv const$$

PROOF: see Corollary 2.1.

Lemma 2.5 *For solutions $y \in C^1_A$ and $\psi \in C^1_{A^T}$ of* (2.3.1) *and* (2.3.7)*, respectively, the following holds*

$$\psi^T(x)A(x)y(x) \equiv 0 \quad \Longleftrightarrow \quad \psi(x')^T A(x')y(x') = 0 \quad \text{at some } x' \in [x_l, x_r].$$

Proof:

$$\begin{aligned}(\psi^T Ay)' &= (\psi^T P_* APy)' = (\psi^T P_*)' Ay + \psi^T P_* A' Py + \psi^T A(Py)' \\ &= -\psi^T P_* A' y + \psi^T B[I - A^T\psi(\psi^T AA^T\psi)^{-1}\psi^T A]y + \psi^T P_* A' Py - \psi^T (B - AP')y \\ &= -\psi^T BA^T\psi(\psi^T AA^T\psi)^{-1}\psi^T Ay - \psi^T P_* A' Qy + \psi^T AP' y \\ &= -\psi^T BA^T\psi(\psi^T AA^T\psi)^{-1}\psi^T Ay + \psi^T A(Q' + P')y \\ &= -\psi^T BA^T\psi(\psi^T AA^T\psi)^{-1}\psi^T Ay =: D\psi^T Ay.\end{aligned}$$

Again, the equation turns out to be linear with respect to $\psi^T Ay$.

As in section 2.2, the transformation $\eta = A^T\psi$, $\psi = P_{*s}A^{+T}\eta$ connects (2.3.7) with the regular ODE (2.2.8) (now with varying A). Obviously, Theorem 2.3 about the solution subspace remains valid (again after replacing equation (2.2.1) by (2.3.7)).

However, now the regular ODE (2.2.8) does no more represent the original Abramov's method applied to the inherent ODE

$$z' + Rz = 0, \quad R := A^+AG^{-1}B - (A^+A)'P_s$$

in general. That means, Theorem 2.2 does not hold in case of a variable coefficient $A(x)$. This fact is shown by Abramov [11] by means of the following example.

Example. Let $A(x) = \begin{pmatrix} \cos x & \sin x \\ 0 & 0 \end{pmatrix}$, $B(x) = \begin{pmatrix} 0 & 0 \\ -\sin x & \cos x \end{pmatrix}$.

Compute $A^+AG^{-1}B = 0$, $B^T P_{*s} A^{T+} = 0$, but $R(x) = -\begin{pmatrix} \cos x \sin x & -\sin^2 x \\ \cos^2 x & \sin x \cos x \end{pmatrix} \neq 0$.

3 Assesment of the results for the inhomogeneous systems

3.1 Boundary value problems for regular inhomogeneous systems

Theorem 1.2 about the solution subspace may be not sufficiently transparent. It seems appropriate to rewrite it in terms of the original inhomogeneous problem. Therefore, we work out the intermediate results in the inhomogeneous context, too.

Let us return to the problem (1.1.1), (1.1.2), assuming that $B(x)$ and $f(x)$ are continuous and the boundary conditions are separated, i.e. (1.2.1) holds.

We are interested in determining the manifolds of solutions $M_i(x) \ni y(x)$ such that $y(x)$ solves the system and satisfies the boundary condition $C_{i'}y(x_i) = d_i, \quad d = \begin{pmatrix} d_l \\ d_r \end{pmatrix}$.
The results given below are based on the natural splitting of the "boardered" matrices arising by the "homogenization" and the specialization of the previous results for these

systems. The matrix $\phi(x)$ used in section 1.2 is also split, namely, the last row of $\phi(x)$ is considered to be a transposed column taken with negative sign and it can be handled separately:

$$y(x) \in M_i(x) \quad \text{involves} \quad \chi_i^T(x)y(x) = \delta_i(x).$$

The adjoint equation leads to the representation

$$\begin{aligned} \chi_i' - B^T\chi_i &= 0, \qquad \chi_i(x_i) = C_{i'}^T, \\ \delta_i' &= \chi_i^T f, \quad \delta_i(x_i) = d_i. \end{aligned} \tag{3.1.1}$$

Instead of solving the adjoint problem (3.1.1), the smooth transfer of boundary conditions can be realized by

$$\chi' - (I - \chi W^{-1}\chi^T)B^T\chi + \chi W^{-1}\delta f^T\chi = 0, \tag{3.1.2}$$

$$\delta' + \chi^T B\chi W^{-1}\delta - (1 - \delta^T W^{-1}\delta)\chi^T f = 0. \tag{3.1.3}$$

Here $W := \chi^T\chi + \delta\delta^T$. (The indices i and i' are omitted for brevity.)

The smoothness is achieved by setting

$$\chi^T\chi' + \delta\delta^{T\prime} \equiv 0, \tag{3.1.4}$$

which in turn yields $\chi^T\chi + \delta\delta^T \equiv const.$

Finally, Theorem 1.1 implies immediately the following one.

Theorem 3.1 *The inhomogeneous problem* (1.1.1), (1.1.2) *is solvable iff*

$$\begin{pmatrix} \delta_l \\ \delta_r \end{pmatrix} \in Im\ (\chi_l^T | \chi_r^T),$$

or, equivalently, the solution(s) is (are) available from the system

$$\begin{aligned} \chi_l^T(x)y(x) &= \delta_l(x), \\ \chi_r^T(x)y(x) &= \delta_r(x), \end{aligned}$$

for any $x \in [x_l, x_r]$. The rank of the matrix $(\chi_l|\chi_r)$ is independent of x.

Remark. In [5], a less severe constraint is considered. Namely, the construction there is built upon the requirement

$$\chi^T\chi' \equiv 0, \quad \chi^T\chi \equiv const, \tag{3.1.5}$$

while in [6] both (3.1.4) and (3.1.5) are mentioned.

3.2 Boundary value problems for index 1 inhomogeneous DAEs

Now we determine the manifolds of solutions $M_i(x) \ni y(x)$ such that $y(x)$ solves the system (1.1.5) and satisfies the separated boundary conditions

$$C_{i'}y(x_i) = d_i, \quad d = \begin{pmatrix} d_l \\ d_r \end{pmatrix}.$$

Again, (1.2.1) is assumed. We look for representations of the form

$$\chi_i^T(x)A(x)y(x) = \delta_i(x), \quad B(x)y(x) - f(x) \in Im\ A(x). \tag{3.2.1}$$

Corresponding to the consistent initial value problem formulation, we have to assume that $C_{i'} = C_{i'}A^+A$. Provided $A(x) \equiv const$, the adjoint system results in

$$\begin{array}{rcll} A^T\chi_i' - B^T\chi & = & 0, & \chi_i(x_i) = P_{*s}(x_i)A^{T+}C_{i'}^T, \\ \delta_i' & = & \chi^T f, & \delta_i(x_i) = d_i, \end{array} \tag{3.2.2}$$

see (2.2.1). Here P_{*s} denotes the projector onto the solution space of the first equation in (3.2.2) along $Ker\ A^T$. The transfer of boundary conditions suggested in 2.2. is carried out by the equations

$$A^T\chi' - (I - A^T\chi W^{-1}\chi^T A)B^T\chi + A^T\chi W^{-1}\delta f^T\chi = 0, \tag{3.2.3}$$

$$\delta' + \chi^T BA^T\chi W^{-1}\delta - (1 - \delta^T W^{-1}\delta)\chi^T f = 0 \tag{3.2.4}$$

with initial values as above. Here $W := \chi^T AA^T\chi + \delta\delta^T$.

These equations ensure $\chi^T AA^T\chi' + \delta\delta^{T\prime} \equiv 0$. The consequence of this identity is $\chi^T AA^T\chi + \delta\delta^T \equiv const = C_{i'}C_{i'}^T + d_{i'}d_{i'}^T$, which can be used for checking the accuracy of the numerical integration.

On this background, Theorem 2.1 obviously implies the next statement.

Theorem 3.2 *Let $\chi_l, \chi_r, \delta_l, \delta_r$ be the solutions of the initial value problems posed for the equations* (3.2.3) *and* (3.2.4) *with initial values given above. A function $y(x)$ solves the boundary value problem* (1.1.5) *with separated boundary conditions* (1.1.7) *iff it solves the system*

$$\begin{pmatrix} \chi_l^T(x) \\ \chi_r^T(x) \end{pmatrix} Ay(x) = \begin{pmatrix} \delta_l(x) \\ \delta_r(x) \end{pmatrix}, \quad (I - AA^+)(By(x) - f(x)) = 0, \quad x \in [x_l, x_r]. \tag{3.2.5}$$

If the leading matrix A depends on x, then the only change is that in the equations (3.2.2) and (3.2.3) the expression $A^T\chi'$ is replaced by $(A^T\chi)'$, or, equivalently, by $A^T(P_*\chi)' + A^{T\prime}P_*\chi$, where P_* is the orthogonal projector onto $Im\ A$. Anything else is the same as for the case of a constant leading matrix A.

4 Some general comments

1. As far as we know, this paper represents the first attempt to apply a transfer method that is known to be especially well-conditioned for regular linear ODE's to the case of DAE's. We hope for analogous advantages, e.g. with respect to shooting methods, as they are well-known for regular ODE's. However, we should like to emphasize that we consider here a special procedure for linear boundary value problems only.

2. The formally simple form of the regular ODE (2.2.8) might entice into performing the transfer on the basis of (2.2.8) and into forming $\Psi := P_{*s}A^{T+}\eta$ subsequently. Thus, one would have the possibility to work with simple explicit integration methods. However, one should take into account that $P_{*s} = I - Q_*(A^T - B^T Q_*)^{-1}B^T$ as well as A^{T+} would have to be computed and that (2.2.8) would lose in structure relative to the DAE (2.2.5). It seems to us that the direct integration of (2.2.5) is simpler and more favourable even though one has to go back to implicit methods.

3. We have not yet gained any appreciable experience concerning the selection of special integration methods. Of course, as for the regular DAE's, we cannot expect to find a general answer for the DAE's. We will report about corresponding experiments later.

4. This paper proposes a method of solution of boundary value problems for linear differential-algebraic equations of index 1. Due to its stability properties this method may play the same role among the methods of solution of this problem as the original Abramov's transfer plays among the so-called initial value methods for the solution of boundary value problems posed for regular ordinary differential equations. One might wish to know whether there are any reasons and hopes to extend the results to higher index problems.
One can easily check that the homogenization as it is proposed in §1 may be carried out in the same way and the enlarged homogeneous system preserves the index of the original problem. Some of basic assertions in §2.3, namely, Lemma 2.3 and Lemma 2.4, Corollary 2.1 and Lemma 2.5, are independent of the index.
We emphasize that the consistency condition

$$y(x) \in S(x) = \operatorname{im} P_s(x) = \ker Q_s(x), \quad \text{i.e. } Q_s(x)y(x) = 0,$$

which appears in the linear system to be solved has to be replaced by the corresponding higher index consistency conditions. For this, an accurate explicit description of the related lower dimensional subspaces of $S(x)$ is needed. For index 2 DAEs this subspace is given, e.g. in [12], to be the image space $\operatorname{im} \prod(x) \subset S(x)$ of a certain projector function, which is constructed on its turn by further special projections as well as their derivatives. The details take quite a bit of space (cf. [12]).
Next, recall that even for regular ODEs there is no advantage of using the linear adjoint equation (1.2.6) itself to realize the transfer. The point of Abramov's method is the nonlinear transfer by equation (1.2.8) and (2.3.7), respectively, with its nicer stability

behaviour (cf. §1.2, Remark 3).
In the higher index case, the transfer equation (2.3.7) becomes a higher index, nonlinear, non-Hessenberg matrix differential algebraic equation. Theorem 2.1 does not apply since the projector P_{*s} in (2.2.8) does not exist any more. Hence, the needed unique solvability on the whole given interval remains under question by now.
Furthermore, with the transfer method we aim at a more realiable possibility to carry out necessary numerical integrations. Thus, if we were not able to integrate (2.3.7) numerically fairly well, the whole game of the transfer would not be of any use at all. By now, there is no reliable integration method for that higher index nonlinear matrix DAE. These two questions are hoped to be answered positively for the index 2 case, but for that, considerable further effort is needed.

5.Concerning the concrete form of a possibly proper transfer for higher index DAEs, this is not expected to be related to the transferring equation for the inherent ODE of the DAE under consideration. As the counterexample at the end of §2 shows, even for index 1 problems with variable coefficients the relationship stated in Theorem 2.2 for constant coefficients is no more valid.

6. Another possibility for treating index 2 problems is to regularize them to index 1 systems. These regularization methods are analyzed e.g. in [13]. Let us mention e.g. the system

$$(A + \varepsilon BP)y' + (B + \varepsilon BPP')y = f,$$

which has index 1 for sufficiently small parameters $\varepsilon \neq 0$ provided that $Ay' + By = f$ is an index 2 tractable system.

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