10-9-2012

How Often Are Two Permutations Comparable?

Adam J. Hammett
Cedarville University, ahammett@cedarville.edu

Boris Pittel
The Ohio State University

How often are two permutations comparable?

Adam Hammett[1]
(joint work with Boris Pittel[2])

[1]Department of Mathematical Sciences
Bethel College

[2]Department of Mathematics
The Ohio State University

October 9, 2012

Talk Outline

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.
- We say $\pi \leq \sigma$ in Bruhat order iff π can be obtained from σ by "undoing" some inversions of σ.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.
- We say $\pi \leq \sigma$ in Bruhat order iff π can be obtained from σ by "undoing" some inversions of σ.
- Intuitively, $\pi \leq \sigma$ means π is more orderly (closer to $12\cdots n$) than σ.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.
- We say $\pi \leq \sigma$ in Bruhat order iff π can be obtained from σ by "undoing" some inversions of σ.
- Intuitively, $\pi \leq \sigma$ means π is more orderly (closer to $12\cdots n$) than σ.
- $(\mathfrak{S}_n, \leq)$ is only a *partially*-ordered set (poset), i.e. it may happen that given π, σ are incomparable.

Definition of Bruhat Order

- $\mathfrak{S}_n$, the set of permutations of $[n] := \{1, 2, \ldots, n\}$.
- Write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ in one-line array notation.
- An inversion of π is an ordered pair (i, j) with $i > j$ and $\pi^{-1}(i) < \pi^{-1}(j)$.
  Example. $(3, 1)$ and $(5, 4)$ are inversions of 35412.
- We say $\pi \leq \sigma$ in Bruhat order iff π can be obtained from σ by "undoing" some inversions of σ.
- Intuitively, $\pi \leq \sigma$ means π is more orderly (closer to $12 \cdots n$) than σ.
- $(\mathfrak{S}_n, \leq)$ is only a *partially*-ordered set (poset), i.e. it may happen that given π, σ are incomparable.
- Bruhat ordering can be extended to general Coxeter groups, but we studied $\mathfrak{S}_n$ only.

Example of Bruhat Comparability

Example. $3412 > 1324$:

Example of Bruhat Comparability

Example. $3412 > 1324$: 3412

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 >$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 >$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 >$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

- There's no need to restrict to undoing adjacent inversions as we have here (such a restriction gives rise to another partial-ordering: weak ordering).

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

- There's no need to restrict to undoing adjacent inversions as we have here (such a restriction gives rise to another partial-ordering: weak ordering).
- Despite no restrictions on which inversions we destroy, the total number of inversions strictly decreases each time we destroy one.

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

- There's no need to restrict to undoing adjacent inversions as we have here (such a restriction gives rise to another partial-ordering: weak ordering).
- Despite no restrictions on which inversions we destroy, the total number of inversions strictly decreases each time we destroy one.
- **Problem:** From the definition alone, checking Bruhat comparability is far from algorithmic.

Example of Bruhat Comparability

Example. $3412 > 1324$: $3412 > 3142 > 3124 > 1324$

- There's no need to restrict to undoing adjacent inversions as we have here (such a restriction gives rise to another partial-ordering: weak ordering).
- Despite no restrictions on which inversions we destroy, the total number of inversions strictly decreases each time we destroy one.
- **Problem:** From the definition alone, checking Bruhat comparability is far from algorithmic.
- To get around this, we used two comparability criteria that are algorithmic in nature: the Ehresmann Tableaux and $\{0, 1\}$-matrix criteria.

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

2 4

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

1	2		4	5
2			4	

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

$$\begin{array}{ccc} 1 & 2 & 5 \\ 1 & 2 & \\ 2 & & \end{array} \qquad\qquad \begin{array}{ccc} 3 & 4 & 5 \\ 4 & 5 & \\ 4 & & \end{array}$$

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

1	2	3	5		1	3	4	5
1	2	5			3	4	5	
1	2				4	5		
2					4			

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.

1	2	3	4	5	1	2	3	4	5
1	2	3	5		1	3	4	5	
1	2	5			3	4	5		
1	2				4	5			
2					4				

The Ehresmann Criterion

- Discovered by C. Ehresmann in 1934.
- Let $\pi = 21534$, $\sigma = 45312$. We build a pair of staircase tableaux from these permutations.
- Then $\pi \leq \sigma$ iff the tableau for π is dominated entry-wise by that for σ.

1	2	3	4	5	1	2	3	4	5
1	2	3	5		1	3	4	5	
1	2	5			3	4	5		
1	2				4	5			
2					4				

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

X				

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

X				
	X			

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

		X		
X				
	X			

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

			X	
		X		
X				
	X			

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

			X	
				X
		X		
X				
	X			

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi,\sigma)$

			X	
O				X
		X		
X				
	X			

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		X		
X				
	X			

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				
	X			

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi,\sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		X O		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi,\sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		X O		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		X O		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0,1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$. Get that $\pi < \sigma$.

$M(\pi, \sigma)$

	O		X	
O				X
		XO		
X				O
	X		O	

The $\{0, 1\}$-criterion (our primary focus)

Example. Let $\pi = 21354$, $\sigma = 45312$. Get that $\pi < \sigma$.
Advantage: Algorithmic way to check comparability.

$M(\pi, \sigma)$

	O		X	
O				X
		X O		
X				O
	X		O	

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$.

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have 132 $<$ 231 in Bruhat order, but 132 $\not\preceq$ 231.
  Indeed, 132

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$.
  Indeed, $132 \prec$

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$.
  Indeed, $132 \prec 312$

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\preceq 231$.
  Indeed, $132 \prec 312$

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$.
  Indeed, $132 \prec 312 \prec$

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\preceq 231$.
  Indeed, $132 \prec 312 \prec 321$,

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
Example. We have $132 < 231$ in Bruhat order, but $132 \not\preceq 231$.
Indeed, $132 \prec 312 \prec 321$,

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132.

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \le \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, 132

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\preceq 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, $132 <$

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, $132 < 231$.

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\preceq 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, $132 < 231$.

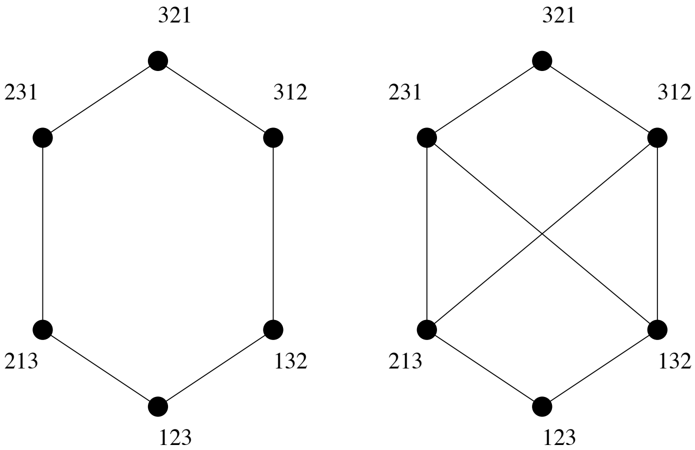
Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, $132 < 231$.
- The weak order poset $(\mathfrak{S}_n, \preceq)$ is a lattice (C. Berge, "Principles of Combinatorics"), i.e. infimums and supremums exist.

Definition of Weak Order

- Given $\pi, \sigma \in \mathfrak{S}_n$, we have $\pi \leq \sigma$ in weak order (written $\pi \preceq \sigma$) if π can be obtained from σ by undoing adjacent inversions.
  Example. We have $132 < 231$ in Bruhat order, but $132 \not\prec 231$. Indeed, $132 \prec 312 \prec 321$, and there are no other chains in weak order starting with the permutation 132. However, $132 < 231$.
- The weak order poset $(\mathfrak{S}_n, \preceq)$ is a lattice (C. Berge, “Principles of Combinatorics”), i.e. infimums and supremums exist. This is *not* so for Bruhat order.

The Posets $(\mathfrak{S}_3, \preceq)$ and $(\mathfrak{S}_3, \leq)$



Equivalent Definition of Weak Order

- Again, checking $\pi \preceq \sigma$ is clumsy from the definition alone. Is there an *algorithmic* way?

Equivalent Definition of Weak Order

- Again, checking $\pi \preceq \sigma$ is clumsy from the definition alone. Is there an *algorithmic* way?
- Yes! For each $i \in [n]$, introduce

$$E_i(\pi) = \{j < i \,:\, \pi^{-1}(j) < \pi^{-1}(i)\}.$$

Equivalent Definition of Weak Order

- Again, checking $\pi \preceq \sigma$ is clumsy from the definition alone. Is there an *algorithmic* way?
- Yes! For each $i \in [n]$, introduce

$$E_i(\pi) = \{j < i \,:\, \pi^{-1}(j) < \pi^{-1}(i)\}.$$

Then $\cup_{i \in [n]} \{(j, i) \,:\, j \in E_i(\pi)\}$ is the set of non-inversions of π. We have $\pi \preceq \sigma$ iff $E_i(\pi) \supseteq E_i(\sigma)$ for each $i \in [n]$.

Problems Studied

Problems Studied

(1) $(\mathfrak{S}_n, \leq)$ is only *partially*-ordered. So how likely is it that for independent, uniformly random $\pi, \sigma \in \mathfrak{S}_n$ we have $\pi \leq \sigma$? That is, what are bounds for $P(\pi \leq \sigma)$? (Skandera, MIT, 2004.)

Problems Studied

(1) $(\mathfrak{S}_n, \leq)$ is only *partially*-ordered. So how likely is it that for independent, uniformly random $\pi, \sigma \in \mathfrak{S}_n$ we have $\pi \leq \sigma$? That is, what are bounds for $P(\pi \leq \sigma)$? (Skandera, MIT, 2004.)

(2) More generally, for independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$, what are bounds for $P(\pi_1 \leq \cdots \leq \pi_r)$?

Problems Studied

(1) $(\mathfrak{S}_n, \leq)$ is only *partially*-ordered. So how likely is it that for independent, uniformly random $\pi, \sigma \in \mathfrak{S}_n$ we have $\pi \leq \sigma$? That is, what are bounds for $P(\pi \leq \sigma)$? (Skandera, MIT, 2004.)

(2) More generally, for independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$, what are bounds for $P(\pi_1 \leq \cdots \leq \pi_r)$?

(3) Same questions for the weak order.

Problems Studied

(1) $(\mathfrak{S}_n, \leq)$ is only *partially*-ordered. So how likely is it that for independent, uniformly random $\pi, \sigma \in \mathfrak{S}_n$ we have $\pi \leq \sigma$? That is, what are bounds for $P(\pi \leq \sigma)$? (Skandera, MIT, 2004.)

(2) More generally, for independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$, what are bounds for $P(\pi_1 \leq \cdots \leq \pi_r)$?

(3) Same questions for the weak order.

(4) As $(\mathfrak{S}_n, \preceq)$ is a lattice, how likely is it that independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$ have minimal infimum, $12 \cdots n$?

Problems Studied

(1) $(\mathfrak{S}_n, \leq)$ is only *partially*-ordered. So how likely is it that for independent, uniformly random $\pi, \sigma \in \mathfrak{S}_n$ we have $\pi \leq \sigma$? That is, what are bounds for $P(\pi \leq \sigma)$? (Skandera, MIT, 2004.)

(2) More generally, for independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$, what are bounds for $P(\pi_1 \leq \cdots \leq \pi_r)$?

(3) Same questions for the weak order.

(4) As $(\mathfrak{S}_n, \preceq)$ is a lattice, how likely is it that independent, uniformly random $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$ have minimal infimum, $12 \cdots n$?

- Pittel studied the analogous problems for the poset of integer partitions under dominance order, and for the poset of set partitions ordered by refinement.

Bruhat Order Results

Theorem

Let $\pi_1, \dots, \pi_r \in \mathfrak{S}_n$ be independent and uniformly random. Then there are uniform constants $c_1 = c_1(\epsilon), c_2 > 0$ such that

$$c_1 \left(\frac{1}{r!} - \epsilon \right)^n \le P(\pi_1 \le \cdots \le \pi_r) \le c_2 n^{-r(r-1)}, \quad \forall \epsilon > 0.$$

Equivalently, there are at least $(n!)^r c_1 (1/r! - \epsilon)^n$ and at most $(n!)^r c_2 n^{-r(r-1)}$ length r chains in Bruhat order. In the case $r = 2$, there is a uniform constant $c > 0$ such that

$$P(\pi_1 \le \pi_2) \ge c(0.708)^n.$$

Bruhat Order Results

Theorem

Let $\pi_1, \ldots, \pi_r \in \mathfrak{S}_n$ be independent and uniformly random. Then there are uniform constants $c_1 = c_1(\epsilon), c_2 > 0$ such that

$$c_1 \left(\frac{1}{r!} - \epsilon \right)^n \leq P(\pi_1 \leq \cdots \leq \pi_r) \leq c_2 n^{-r(r-1)}, \quad \forall \epsilon > 0.$$

Equivalently, there are at least $(n!)^r c_1 (1/r! - \epsilon)^n$ and at most $(n!)^r c_2 n^{-r(r-1)}$ length r chains in Bruhat order. In the case $r = 2$, there is a uniform constant $c > 0$ such that

$$P(\pi_1 \leq \pi_2) \geq c(0.708)^n.$$

- We will focus on the proof of the $r = 2$ upper bound.

Weak Order Comparability Results

Theorem

Let $\pi, \sigma \in \mathfrak{S}_n$ be independent, uniformly random, and write $P_n^* := P(\pi \preceq \sigma)$. Then, as a function of n, P_n^* is submultiplicative, i.e. $P_{n_1+n_2}^* \le P_{n_1}^* P_{n_2}^*$. So (Fekete lemma) there exists $\rho = \lim_n (P_n^*)^{1/n} = \inf_k (P_k^*)^{1/k}$. Furthermore, there exists an absolute constant $c > 0$ such that

$$\prod_{i=1}^{n} H(i)/i \le P_n^* \le c(0.362)^n;$$

here $H(i) = \sum_{j=1}^{i} 1/j$. Consequently $\rho \le 0.362$.

Weak Order Comparability Results

Theorem

Let $\pi, \sigma \in \mathfrak{S}_n$ *be independent, uniformly random, and write* $P_n^* := P(\pi \preceq \sigma)$. *Then, as a function of n,* P_n^* *is submultiplicative, i.e.* $P_{n_1+n_2}^* \le P_{n_1}^* P_{n_2}^*$. *So (Fekete lemma) there exists* $\rho = \lim_n (P_n^*)^{1/n} = \inf_k (P_k^*)^{1/k}$. *Furthermore, there exists an absolute constant* $c > 0$ *such that*

$$\prod_{i=1}^{n} H(i)/i \le P_n^* \le c(0.362)^n;$$

here $H(i) = \sum_{j=1}^{i} 1/j$. *Consequently* $\rho \le 0.362$.

- Note that in any case (Bruhat or weak ordering) we have $P(\bullet) \to 0$, $n \to \infty$.

Weak Order Lattice-property Results

Theorem

Write $P_{n,r} := P(\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n)$*. Then, as a function of* n*,* $P_{n,r}$ *is submultiplicative, and*

$$\lim_{n\to\infty} (P_{n,r})^{1/n} = 1/z^*;$$

here, $z^* = z^*(r) \in (1,2)$ *is the unique (positive) root of the equation* $\sum_{j\geq 0}(-1)^j z^j/(j!)^r = 0$ *within the disk* $|z| \leq 2$*.*

Weak Order Lattice-property Results

Theorem

Write $P_{n,r} := P(\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n)$*. Then, as a function of* n*,* $P_{n,r}$ *is submultiplicative, and*

$$\lim_{n \to \infty} (P_{n,r})^{1/n} = 1/z^*;$$

here, $z^* = z^*(r) \in (1, 2)$ *is the unique (positive) root of the equation* $\sum_{j \geq 0} (-1)^j z^j / (j!)^r = 0$ *within the disk* $|z| \leq 2$*.*

- In the case $r = 2$, we have $1/z^* \approx 0.69$. Note that, for r fixed, $P_{n,r} \to 0$ exponentially fast as $n \to \infty$.

Toward the Proof of the Bruhat Order Upper Bound (r=2)

Toward the Proof of the Bruhat Order Upper Bound (r=2)

- Exact enumeration of pairs (π, σ) with $\pi \leq \sigma$ seems hopeless.

Toward the Proof of the Bruhat Order Upper Bound (r=2)

- Exact enumeration of pairs (π, σ) with $\pi \leq \sigma$ seems hopeless.
- We need to select a subset of conditions necessary for $\pi \leq \sigma$ that are sufficiently simple, so that we can compute (estimate) the number of these pairs.

Toward the Proof of the Bruhat Order Upper Bound (r=2)

- Exact enumeration of pairs (π, σ) with $\pi \le \sigma$ seems hopeless.
- We need to select a subset of conditions necessary for $\pi \le \sigma$ that are sufficiently simple, so that we can compute (estimate) the number of these pairs.
- On the other hand, these conditions need to stringent enough so that they collectively have probability $o(1)$.

Toward the Proof of the Bruhat Order Upper Bound

Toward the Proof of the Bruhat Order Upper Bound

- First Advance: the Ehresmann Criterion implies that for each $k \leq n$,

$$\{\pi \leq \sigma\} \subseteq \left\{ \sum_{i=1}^{j} \pi(i) \leq \sum_{i=1}^{j} \sigma(i),\ \forall j \leq k \right\}. \tag{1}$$

Toward the Proof of the Bruhat Order Upper Bound

- First Advance: the Ehresmann Criterion implies that for each $k \leq n$,

$$\{\pi \leq \sigma\} \subseteq \left\{ \sum_{i=1}^{j} \pi(i) \leq \sum_{i=1}^{j} \sigma(i),\ \forall j \leq k \right\}. \tag{1}$$

- So long as $k = o\left(n^{1/2}\right)$, the first k entries of a random permutation are asymptotically independent and uniform on $[n]$.

Toward the Proof of the Bruhat Order Upper Bound

- First Advance: the Ehresmann Criterion implies that for each $k \le n$,

$$\{\pi \le \sigma\} \subseteq \left\{ \sum_{i=1}^{j} \pi(i) \le \sum_{i=1}^{j} \sigma(i),\ \forall j \le k \right\}. \tag{1}$$

- So long as $k = o\left(n^{1/2}\right)$, the first k entries of a random permutation are asymptotically independent and uniform on $[n]$.
- So letting $k \to \infty$ "slowly" with n, we obtain from (1)

$$P(\pi \le \sigma) = O\left(n^{-1/2}\right)$$

by using a certain connection with a random walk on the real line (Feller, "Intro. to Prob. Theory, Vol. II").

Toward the Proof of the Bruhat Order Upper Bound

Toward the Proof of the Bruhat Order Upper Bound

- Second Advance: how are the Ehresmann and $\{0, 1\}$ -criteria equivalent?

Toward the Proof of the Bruhat Order Upper Bound

- Second Advance: how are the Ehresmann and $\{0, 1\}$ -criteria equivalent?
- Look at the sets of entries $\{\pi(1), \pi(2), \dots, \pi(k)\}$, $\{\sigma(1), \sigma(2), \dots, \sigma(k)\}$ and sort them $\pi(i_1) < \cdots < \pi(i_k)$, $\sigma(j_1) < \cdots < \sigma(j_k)$. The k-th Ehresmann condition says

$$\pi(i_1) \leq \sigma(j_1), \dots, \pi(i_k) \leq \sigma(j_k).$$

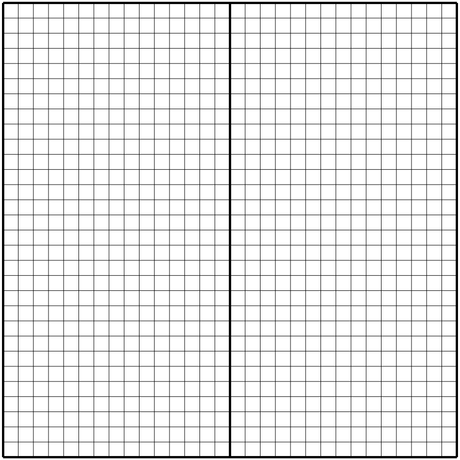
Toward the Proof of the Bruhat Order Upper Bound

- Second Advance: how are the Ehresmann and $\{0,1\}$ -criteria equivalent?
- Look at the sets of entries $\{\pi(1),\pi(2),\dots,\pi(k)\}$, $\{\sigma(1),\sigma(2),\dots,\sigma(k)\}$ and sort them $\pi(i_1)<\cdots<\pi(i_k)$, $\sigma(j_1)<\cdots<\sigma(j_k)$. The k-th Ehresmann condition says

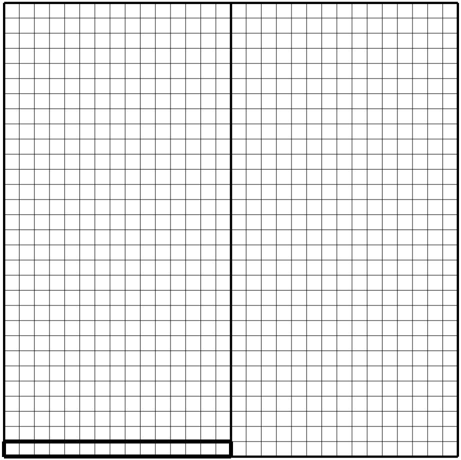
$$\pi(i_1)\le\sigma(j_1),\dots,\pi(i_k)\le\sigma(j_k).$$

- This is the same as "reading-off" rows of the first k columns of $M(\pi,\sigma)$, bottom to top, with the # X's (for π) always more than the # O's (for σ) at any intermediate point.

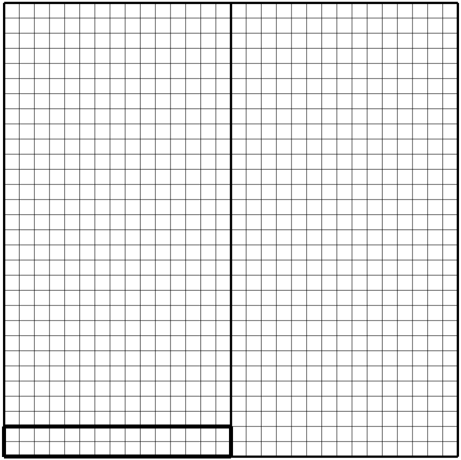
Equivalence of Ehresmann and $\{0, 1\}$-criteria



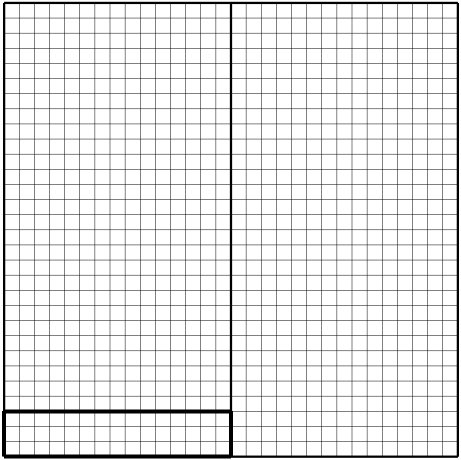
Equivalence of Ehresmann and $\{0, 1\}$-criteria



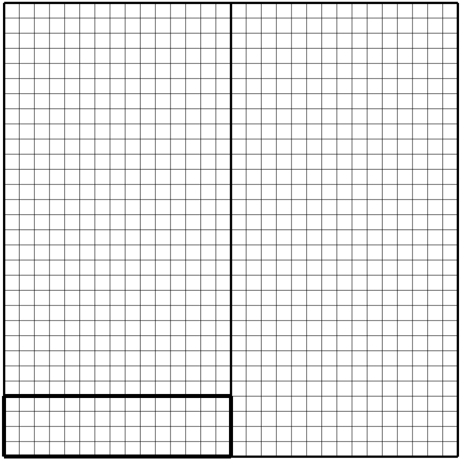
Equivalence of Ehresmann and $\{0, 1\}$-criteria



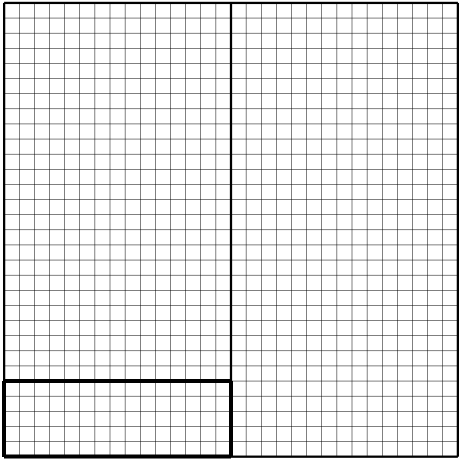
Equivalence of Ehresmann and $\{0, 1\}$-criteria



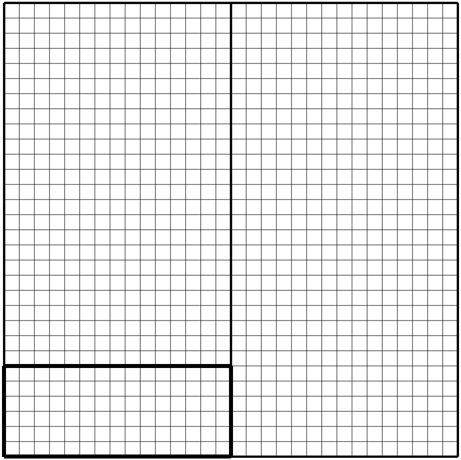
Equivalence of Ehresmann and $\{0, 1\}$-criteria



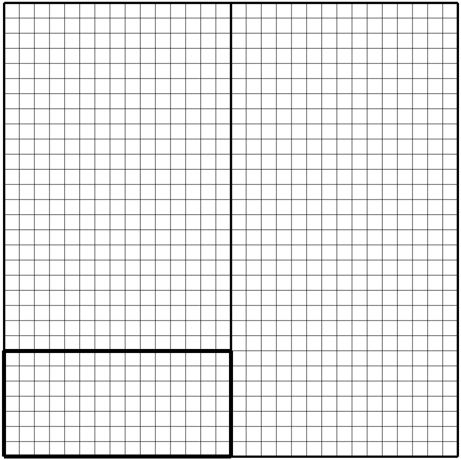
Equivalence of Ehresmann and $\{0, 1\}$-criteria



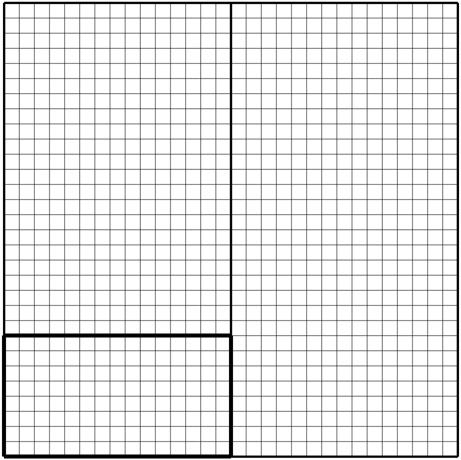
Equivalence of Ehresmann and $\{0,1\}$-criteria



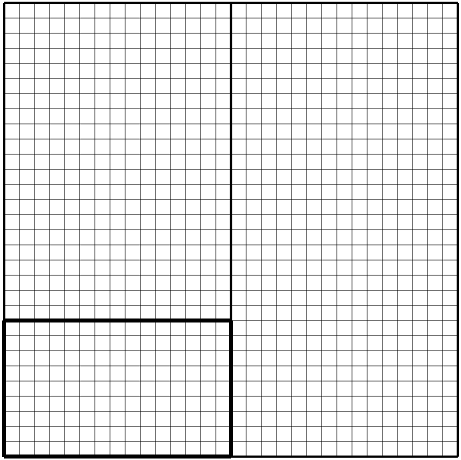
Equivalence of Ehresmann and $\{0, 1\}$-criteria



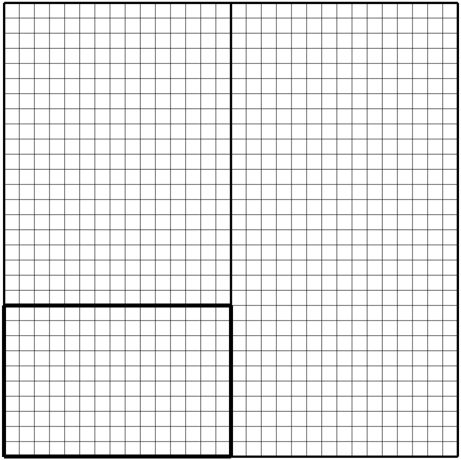
Equivalence of Ehresmann and $\{0, 1\}$-criteria



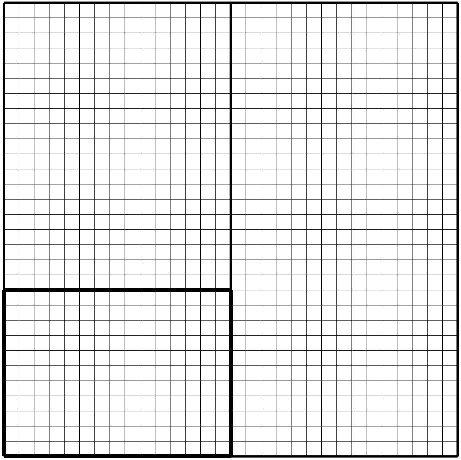
Equivalence of Ehresmann and $\{0, 1\}$-criteria



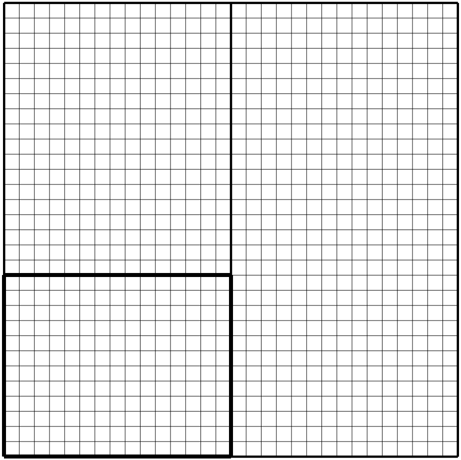
Equivalence of Ehresmann and $\{0, 1\}$-criteria



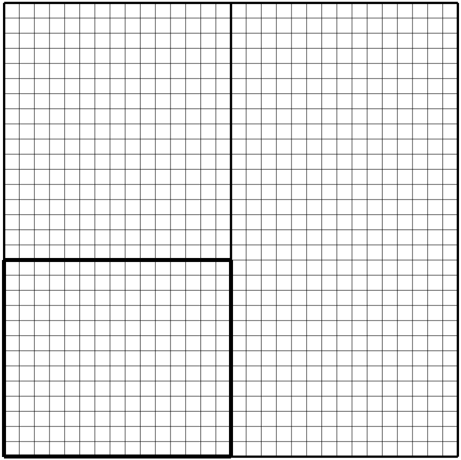
Equivalence of Ehresmann and $\{0,1\}$-criteria



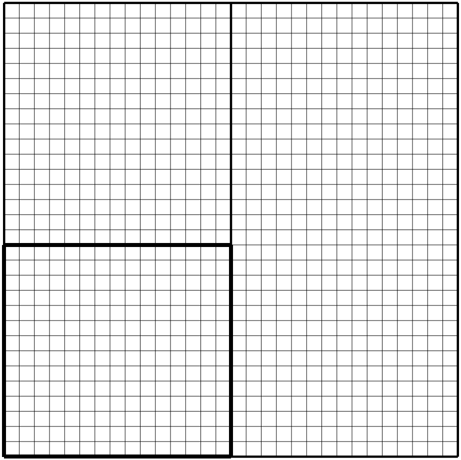
Equivalence of Ehresmann and $\{0, 1\}$-criteria



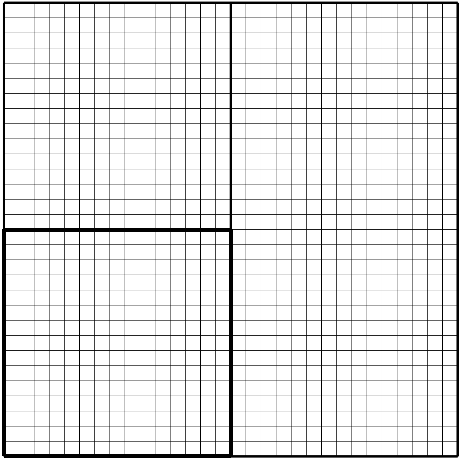
Equivalence of Ehresmann and $\{0, 1\}$-criteria



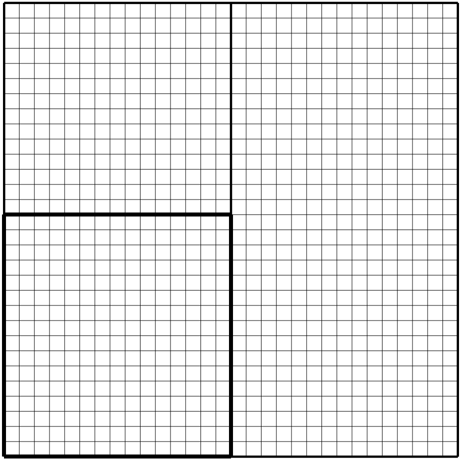
Equivalence of Ehresmann and $\{0, 1\}$-criteria



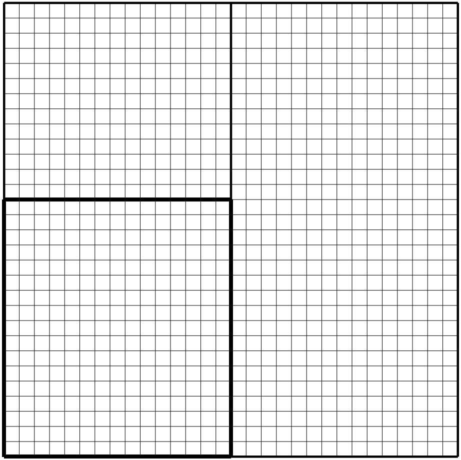
Equivalence of Ehresmann and $\{0, 1\}$-criteria



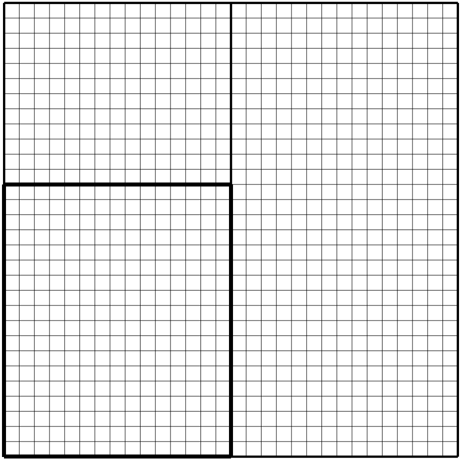
Equivalence of Ehresmann and $\{0, 1\}$-criteria



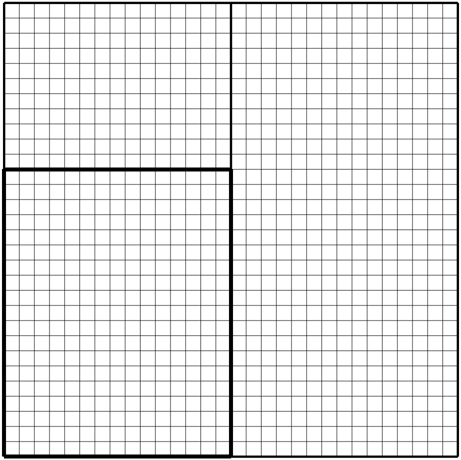
Equivalence of Ehresmann and $\{0, 1\}$-criteria



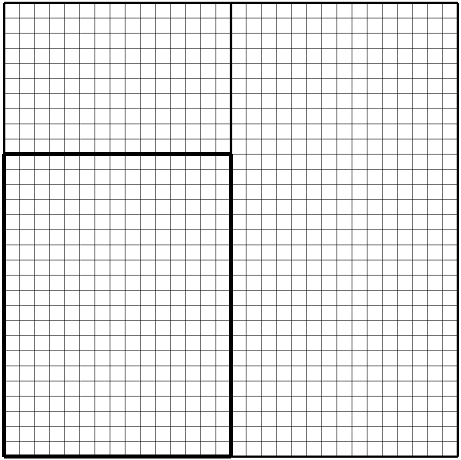
Equivalence of Ehresmann and $\{0, 1\}$-criteria



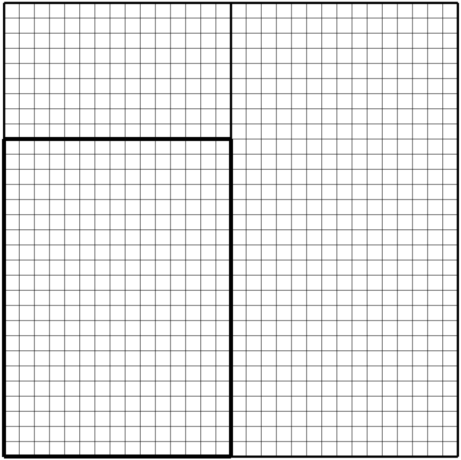
Equivalence of Ehresmann and $\{0, 1\}$-criteria



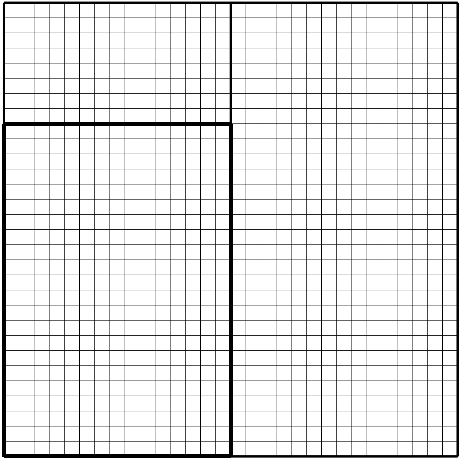
Equivalence of Ehresmann and $\{0,1\}$-criteria



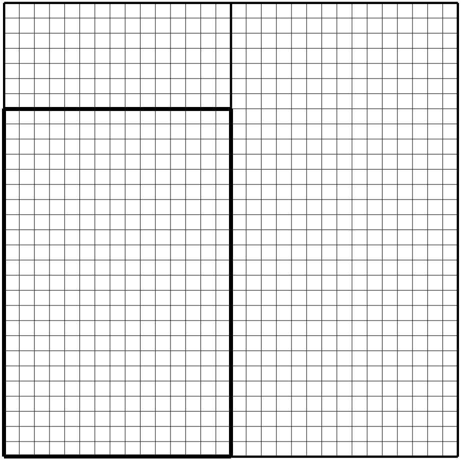
Equivalence of Ehresmann and $\{0, 1\}$-criteria



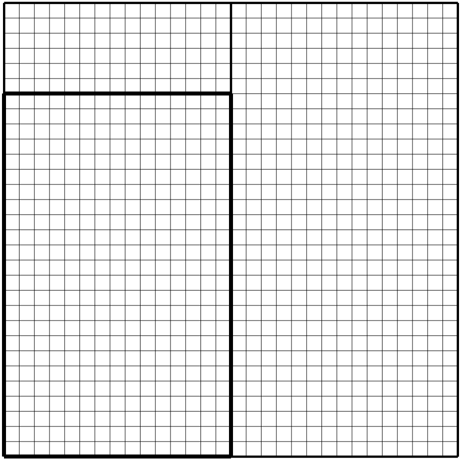
Equivalence of Ehresmann and $\{0, 1\}$-criteria



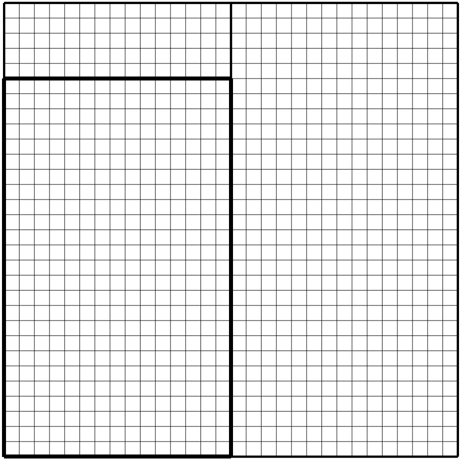
Equivalence of Ehresmann and $\{0, 1\}$-criteria



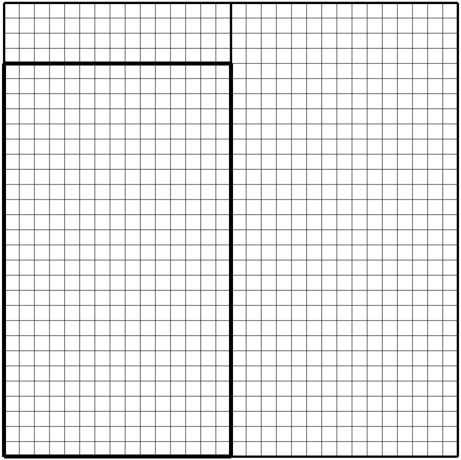
Equivalence of Ehresmann and $\{0, 1\}$-criteria



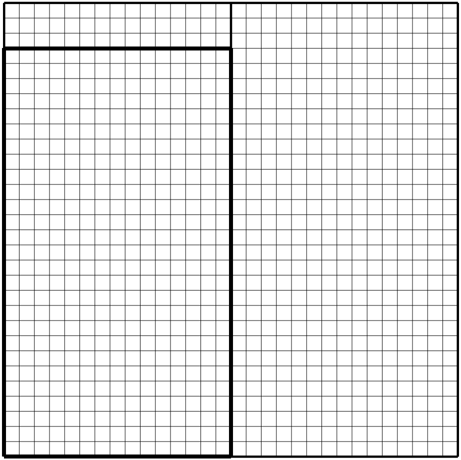
Equivalence of Ehresmann and $\{0, 1\}$-criteria



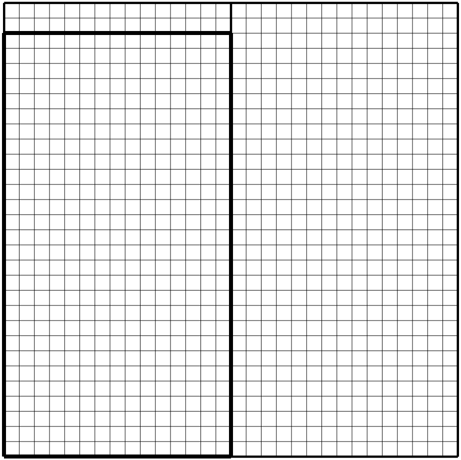
Equivalence of Ehresmann and $\{0, 1\}$-criteria



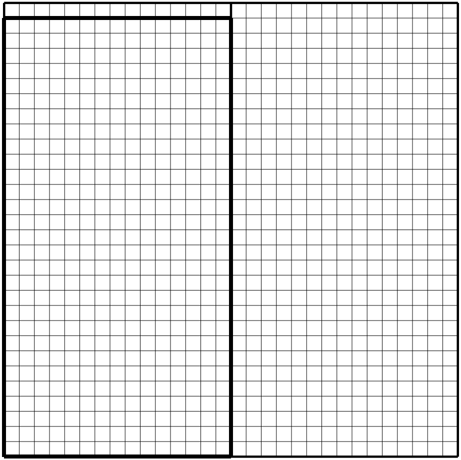
Equivalence of Ehresmann and $\{0,1\}$-criteria



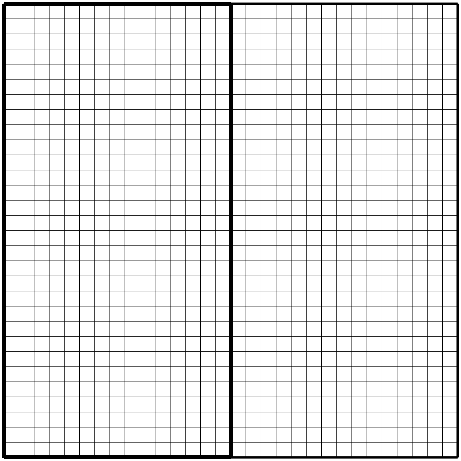
Equivalence of Ehresmann and $\{0, 1\}$-criteria



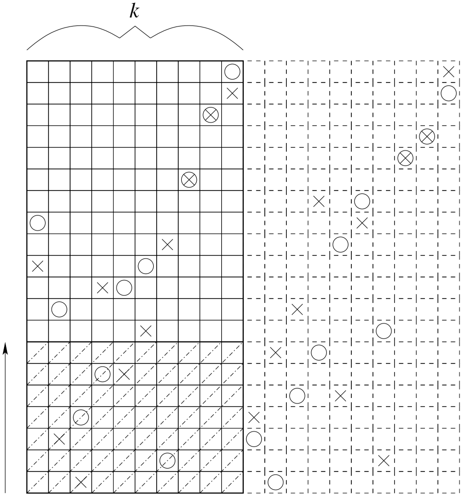
Equivalence of Ehresmann and $\{0, 1\}$-criteria



Equivalence of Ehresmann and $\{0, 1\}$-criteria



Equivalence of Ehresmann and $\{0, 1\}$-criteria



Upper Bound Proof (2nd advance, cont.)

Upper Bound Proof (2nd advance, cont.)

- We have

$$\{\pi \leq \sigma\} \subseteq A_k,$$

where $A_k = \{$"the k-th Ehresmann cond. is satisfied"$\}$.

Upper Bound Proof (2nd advance, cont.)

- We have
$$\{\pi \le \sigma\} \subseteq A_k,$$
where $A_k = \{$"the k-th Ehresmann cond. is satisfied"$\}$.
- We can compute $|A_k|$. There are exactly k X's and k O's to distribute "properly" in these first k columns, i.e. in accordance with the "ballot condition".

Upper Bound Proof (2nd advance, cont.)

- We have
$$\{\pi \le \sigma\} \subseteq A_k,$$
where $A_k = \{$"the k-th Ehresmann cond. is satisfied"$\}$.
- We can compute $|A_k|$. There are exactly k X's and k O's to distribute "properly" in these first k columns, i.e. in accordance with the "ballot condition".
- $\ell :=$ # of rows with both an X and O.

Upper Bound Proof (2nd advance, cont.)

- We have
$$\{\pi \leq \sigma\} \subseteq A_k,$$
where $A_k = \{$"the k-th Ehresmann cond. is satisfied"$\}$.
- We can compute $|A_k|$. There are exactly k X's and k O's to distribute "properly" in these first k columns, i.e. in accordance with the "ballot condition".
- $\ell :=$ # of rows with both an X and O.
- Other $2(k-\ell)$ rows need to be split between π (for X's) and σ (for O's) according to the ballot condition.

Upper Bound Proof (2nd advance, cont.)

- We have
$$\{\pi \leq \sigma\} \subseteq A_k,$$
where $A_k = \{$"the k-th Ehresmann cond. is satisfied"$\}$.
- We can compute $|A_k|$. There are exactly k X's and k O's to distribute "properly" in these first k columns, i.e. in accordance with the "ballot condition".
- $\ell :=$ # of rows with both an X and O.
- Other $2(k-\ell)$ rows need to be split between π (for X's) and σ (for O's) according to the ballot condition.
- The number of ways to do this, for given ℓ, is
$$\binom{n}{\ell}\binom{n-\ell}{2(k-\ell)}\frac{1}{k-\ell+1}\binom{2(k-\ell)}{k-\ell},$$
the last two factors coming from the classic Ballot Theorem.

Statement of the Ballot Theorem

Theorem

Candidate A receives a votes, B gets b votes, $a > b$. Then the number of ballot tallies (counted 1 vote at a time) such that A is always strictly ahead of B equals

$$\frac{a-b}{a+b}\binom{a+b}{a}.$$

Statement of the Ballot Theorem

Theorem

Candidate A receives a votes, B gets b votes, $a > b$. Then the number of ballot tallies (counted 1 vote at a time) such that A is always strictly ahead of B equals

$$\frac{a-b}{a+b}\binom{a+b}{a}.$$

- Equ., starting at $(0,0)$, we can make a rightward unit move each time A gets a vote, and an upward unit move each time B gets a vote. Then this theorem counts the number of lattice paths with these moves, joining the points $(0,0) - (a,b)$, that never *touch* the diagonal $y = x$.

Ballot Theorem cont.

- In our case, we are *allowed* to touch the diagonal, as “ties” in the cumulative counts are permitted.

Ballot Theorem cont.

- In our case, we are *allowed* to touch the diagonal, as “ties” in the cumulative counts are permitted.
- To compensate for this, we “shift” the diagonal left 1 unit, and the Ballot Theorem count changes to, for $a \geq b$,

$$\frac{a+1-b}{a+1+b}\binom{a+1+b}{a+1} = \frac{a-b+1}{a+1}\binom{a+b}{a}.$$

Ballot Theorem cont.

- In our case, we are *allowed* to touch the diagonal, as “ties” in the cumulative counts are permitted.
- To compensate for this, we “shift” the diagonal left 1 unit, and the Ballot Theorem count changes to, for $a \geq b$,

$$\frac{a+1-b}{a+1+b}\binom{a+1+b}{a+1} = \frac{a-b+1}{a+1}\binom{a+b}{a}.$$

- For us, $a = b = k - \ell$, and this delivers the count

$$\binom{n}{\ell}\binom{n-\ell}{2(k-\ell)}\frac{1}{k-\ell+1}\binom{2(k-\ell)}{k-\ell}$$

we claimed for the total number of admissible row selections for π (to contain X’s) and for σ (to contain O’s) with overlap size ℓ.

Upper Bound Proof (2nd advance cont.)

- To complete the construction of pairs $(\pi, \sigma) \in A_k$, we need to decide where to *put* the X's and O's in these chosen rows, and also place the remaining $n-k$ X's and $n-k$ O's somewhere in the remaining rows/columns. The total number of ways to do this is

$$(k!)^2 (n-k)!^2.$$

Upper Bound Proof (2nd advance cont.)

- To complete the construction of pairs $(\pi, \sigma) \in A_k$, we need to decide where to *put* the X's and O's in these chosen rows, and also place the remaining $n-k$ X's and $n-k$ O's somewhere in the remaining rows/columns. The total number of ways to do this is

$$(k!)^2(n-k)!^2.$$

- Putting these pieces together, and summing over all $\ell \leq k$, we get

Upper Bound Proof (2nd advance cont.)

$$P(\pi \le \sigma) \le P(A_k) = \sum_{\ell \le k} \frac{\binom{n}{\ell}\binom{n-\ell}{2(k-\ell)}\binom{2(k-\ell)}{k-\ell}(k!)^2(n-k)!^2}{(n!)^2(k-\ell+1)}$$

$$= \frac{n+1}{(n-k+1)(k+1)} \sum_{\ell \le k} \frac{\binom{k}{\ell}\binom{n+1-k}{k+1-\ell}}{\binom{n+1}{k+1}}$$

$$= \frac{n+1}{(n-k+1)(k+1)} = O\left(n^{-1}\right).$$

Upper Bound Proof (2nd advance cont.)

$$
\begin{aligned}
P(\pi \le \sigma) \le P(A_k) &= \sum_{\ell \le k} \frac{\binom{n}{\ell}\binom{n-\ell}{2(k-\ell)}\binom{2(k-\ell)}{k-\ell}(k!)^2(n-k)!^2}{(n!)^2(k-\ell+1)} \\
&= \frac{n+1}{(n-k+1)(k+1)} \sum_{\ell \le k} \frac{\binom{k}{\ell}\binom{n+1-k}{k+1-\ell}}{\binom{n+1}{k+1}} \\
&= \frac{n+1}{(n-k+1)(k+1)} = O\left(n^{-1}\right).
\end{aligned}
$$

- For fixed k and $n \to \infty$, $P(A_k) \sim (k+1)^{-1}$, which is in accordance with our intuition.

Upper Bound Proof (final form)

Upper Bound Proof (final form)

- To obtain the final result, $P(\pi \leq \sigma) = O(1/n^2)$, how do we take into account an even larger subset of conditions?

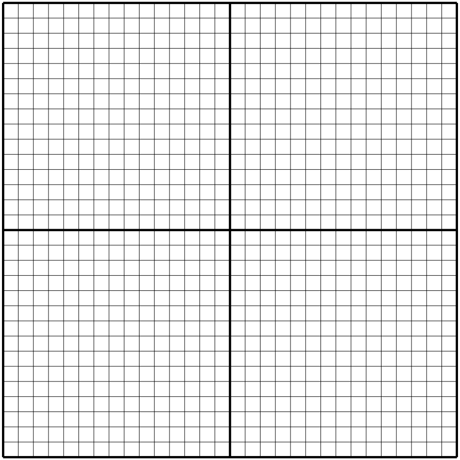
Upper Bound Proof (final form)

- To obtain the final result, $P(\pi \leq \sigma) = O(1/n^2)$, how do we take into account an even larger subset of conditions?
- Notice that we did not pay attention to the conditions in the last $n - k$ columns. So we need to incorporate them somehow, while still preserving our ability to enumerate the resulting pairs of permutations.

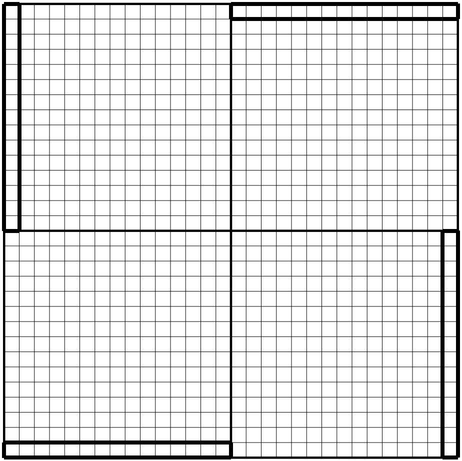
Upper Bound Proof (final form)

- To obtain the final result, $P(\pi \leq \sigma) = O(1/n^2)$, how do we take into account an even larger subset of conditions?
- Notice that we did not pay attention to the conditions in the last $n-k$ columns. So we need to incorporate them somehow, while still preserving our ability to enumerate the resulting pairs of permutations.
- With the ballot-like conditions we just encountered driving our intuition, we arrive at the following picture:

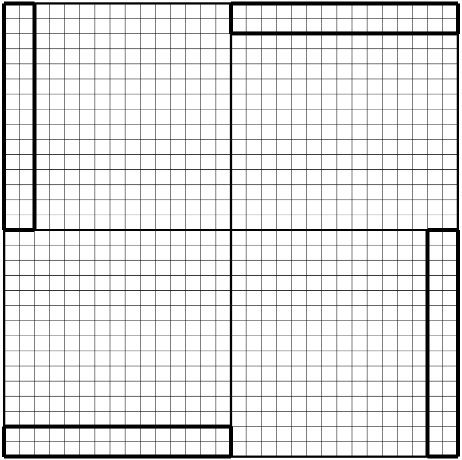
Finding a Necessary Condition for $\pi \leq \sigma$



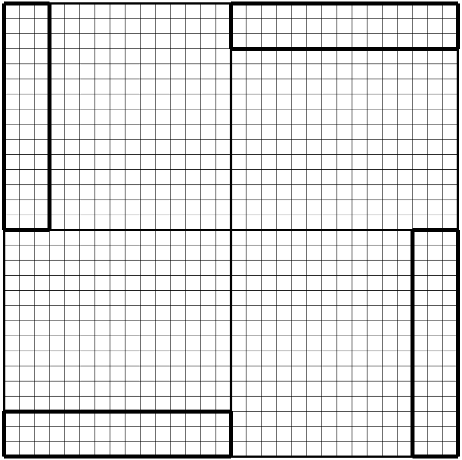
Finding a Necessary Condition for $\pi \leq \sigma$



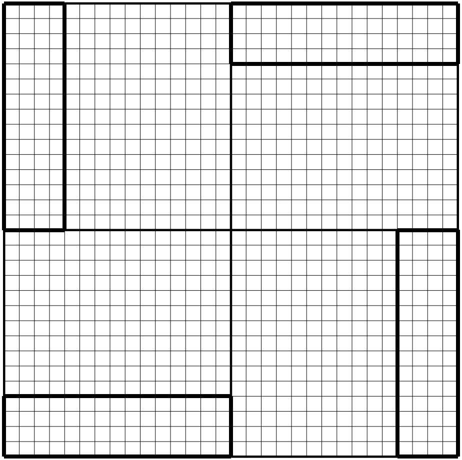
Finding a Necessary Condition for $\pi \leq \sigma$



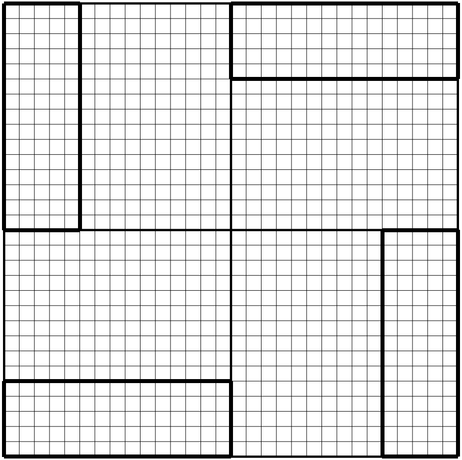
Finding a Necessary Condition for $\pi \leq \sigma$



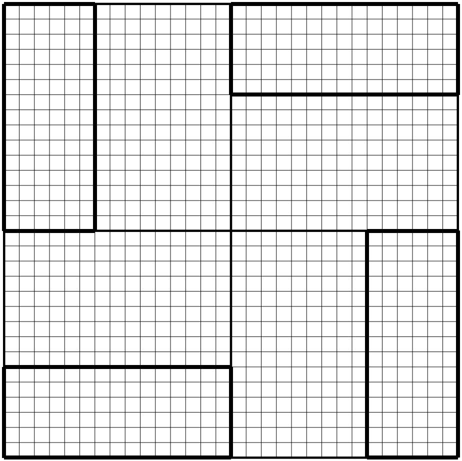
Finding a Necessary Condition for $\pi \leq \sigma$



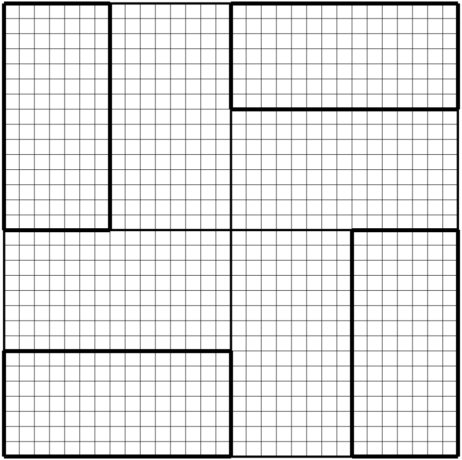
Finding a Necessary Condition for $\pi \leq \sigma$



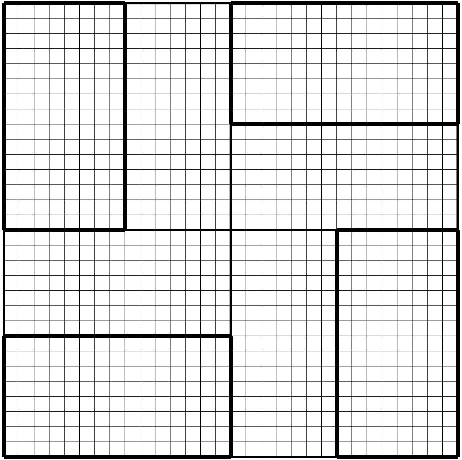
Finding a Necessary Condition for $\pi \leq \sigma$



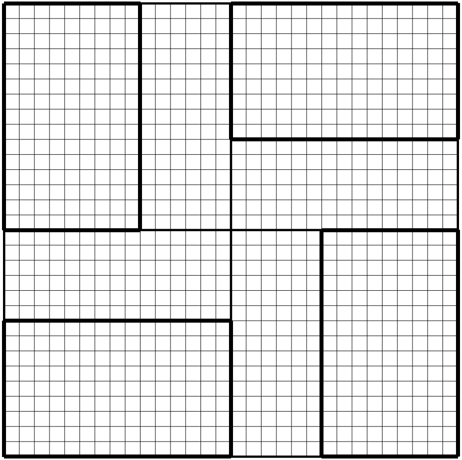
Finding a Necessary Condition for $\pi \leq \sigma$



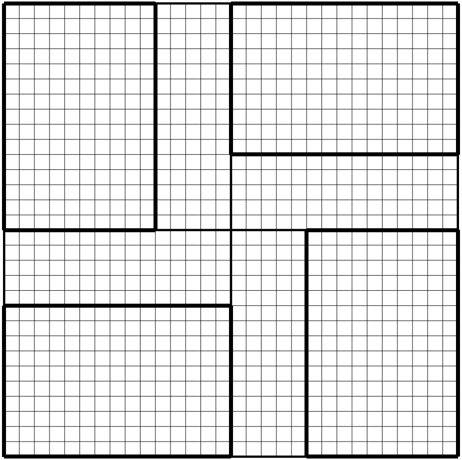
Finding a Necessary Condition for $\pi \leq \sigma$



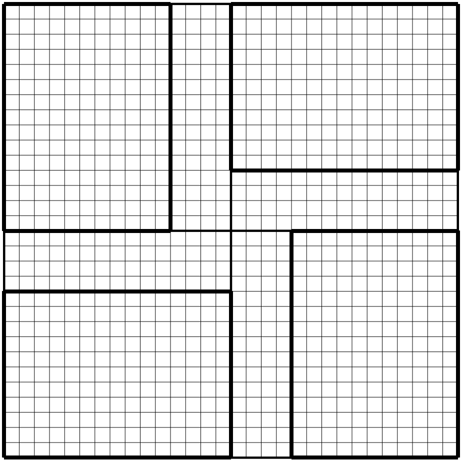
Finding a Necessary Condition for $\pi \leq \sigma$



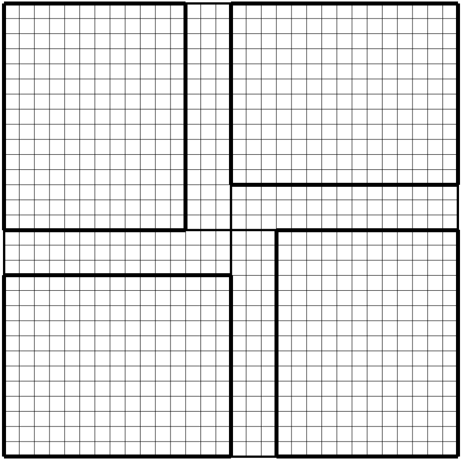
Finding a Necessary Condition for $\pi \leq \sigma$



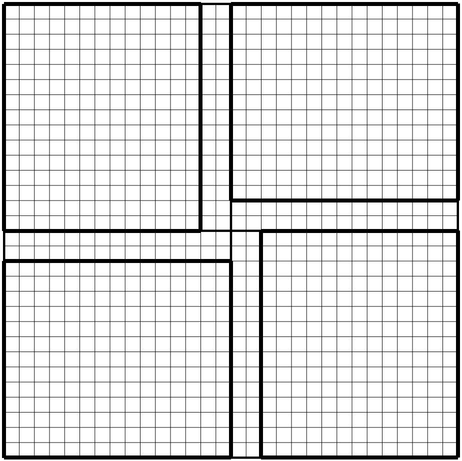
Finding a Necessary Condition for $\pi \leq \sigma$



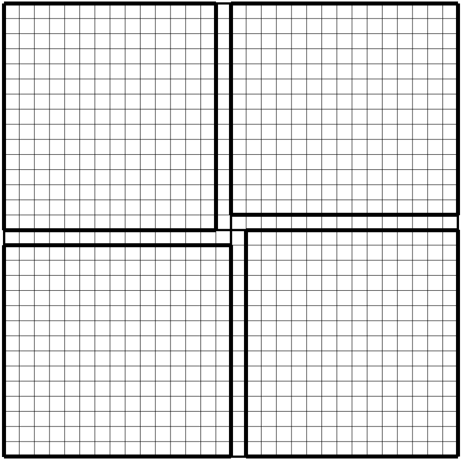
Finding a Necessary Condition for $\pi \leq \sigma$



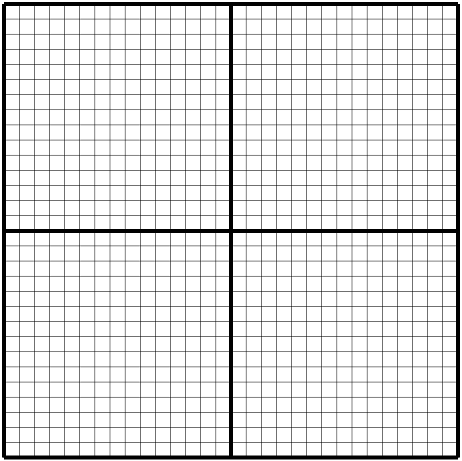
Finding a Necessary Condition for $\pi \leq \sigma$



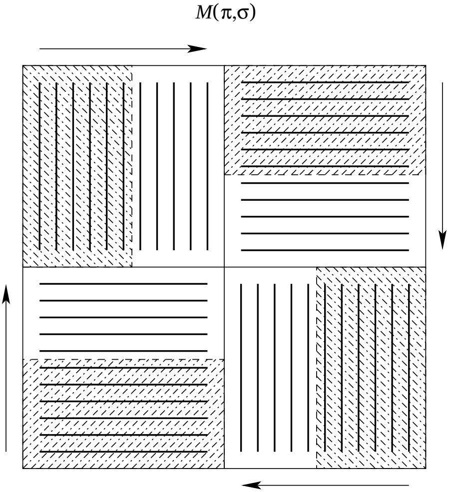
Finding a Necessary Condition for $\pi \leq \sigma$



Finding a Necessary Condition for $\pi \leq \sigma$



Finding a Necessary Condition for $\pi \leq \sigma$



Upper Bound Proof (final form)

- In a manner similar to the $O(1/n)$ proof, we obtain

$$
\begin{aligned}
&P(\pi \leq \sigma) \\
&\leq \sum_{m_1 \geq m_2} \frac{(m_1 - m_2 + 1)^4 (n/2 + 1)^4}{(m_1 + 1)^4 (n/2 - m_2 + 1)^4} \binom{n/2}{m_1}^4 \binom{n/2}{m_2}^4 \\
&\qquad\qquad \times \frac{m_1!^2 (n/2 - m_1)!^2 m_2!^2 (n/2 - m_2)!^2}{n!^2} \\
&= \sum_{m_1 \geq m_2} \frac{(m_1 - m_2 + 1)^4 (n/2 + 1)^4}{(m_1 + 1)^4 (n/2 - m_2 + 1)^4} \prod_{i=1}^{2} \frac{\binom{n/2}{m_i}\binom{n/2}{n/2 - m_i}}{\binom{n}{n/2}}.
\end{aligned}
$$

Upper Bound Proof (final form)

- Extending this last sum over all m_1, m_2 (not just $m_1 \geq m_2$), we see that the extended sum equals

$$E\left[\frac{(M_1 - M_2 + 1)^4 (n/2 + 1)^4}{(M_1 + 1)^4 (n/2 - M_2 + 1)^4}\right],$$

so that this expectation bounds our probability $P(\pi \leq \sigma)$ from above.

Upper Bound Proof (final form)

- Extending this last sum over all m_1, m_2 (not just $m_1 \geq m_2$), we see that the extended sum equals

$$E\left[\frac{(M_1 - M_2 + 1)^4 (n/2 + 1)^4}{(M_1 + 1)^4 (n/2 - M_2 + 1)^4}\right],$$

  so that this expectation bounds our probability $P(\pi \leq \sigma)$ from above.
- Here, M_1, M_2 are independent copies of the Hypergeometric random variable with parameters $n/2, n/2, n/2$. So M_i is equal in distribution to the number of red balls in a uniformly random sample of size $n/2$ from a bin containing $n/2$ red and $n/2$ white balls.

Upper Bound Proof (final form)

- To finish our bound on $P(\pi \leq \sigma)$, it remains to estimate this expectation from above.

Upper Bound Proof (final form)

- To finish our bound on $P(\pi \leq \sigma)$, it remains to estimate this expectation from above.
- The M_i are sharply concentrated around their mean, $n/4$, with exponentially high probability. Further, the difference $|M_1 - M_2 + 1|$ has expectation of order $\sqrt{n}$ at most.

Upper Bound Proof (final form)

- To finish our bound on $P(\pi \leq \sigma)$, it remains to estimate this expectation from above.
- The M_i are sharply concentrated around their mean, $n/4$, with exponentially high probability. Further, the difference $|M_1 - M_2 + 1|$ has expectation of order $\sqrt{n}$ at most.
- So, roughly speaking, we conclude that

$$P(\pi \leq \sigma) \leq E\left[\frac{(M_1 - M_2 + 1)^4 (n/2 + 1)^4}{(M_1 + 1)^4 (n/2 - M_2 + 1)^4}\right]$$
$$= O\left(\frac{(\sqrt{n})^4 \cdot n^4}{n^4 \cdot n^4}\right) = O\left(n^{-2}\right).$$

Conjectures

Conjectures

- Write $P_n = P(\pi \leq \sigma)$, $P_n^* = P(\pi \preceq \sigma)$.

Conjectures

- Write $P_n = P(\pi \le \sigma)$, $P_n^* = P(\pi \preceq \sigma)$.

(1) There is $\delta \in [0.5, 1]$ and $C > 0$ such that $P_n \sim Cn^{-(2+\delta)}$.

Conjectures

- Write $P_n = P(\pi \leq \sigma)$, $P_n^* = P(\pi \preceq \sigma)$.

(1) There is $\delta \in [0.5, 1]$ and $C > 0$ such that $P_n \sim Cn^{-(2+\delta)}$.

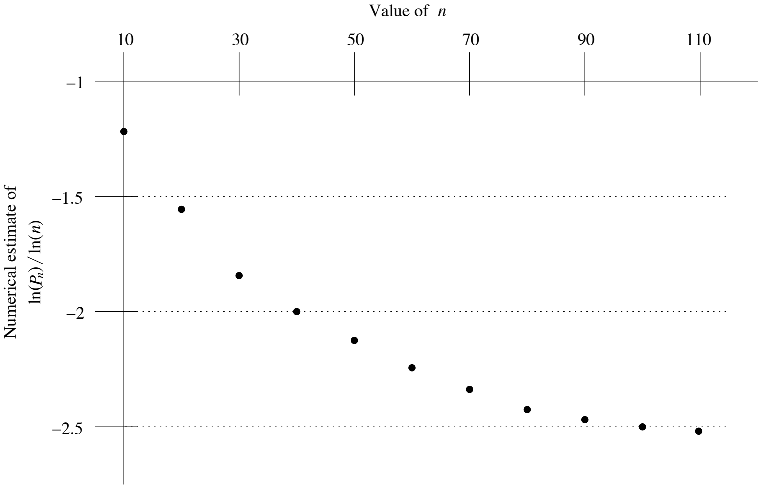
(2) There is $\rho \in [0.3, 1/3]$ and $C > 0$ such that $P_n^* \sim C\rho^n$. Here

$$\rho = \lim_{n\to\infty} \sqrt[n]{P_n^*}.$$

Bruhat Order Numerics

n	R_n	Estimate of $P_n \approx \frac{R_n}{10^9}$	Estimate of $\ln(P_n)/\ln n$
10	61589126	$0.0615891\ldots$	$-1.21049\ldots$
30	1892634	$0.0018926\ldots$	$-1.84340\ldots$
50	233915	$0.0002339\ldots$	$-2.13714\ldots$
70	50468	$0.0000504\ldots$	$-2.32886\ldots$
90	14686	$0.0000146\ldots$	$-2.47313\ldots$
110	5174	$0.0000051\ldots$	$-2.58949\ldots$

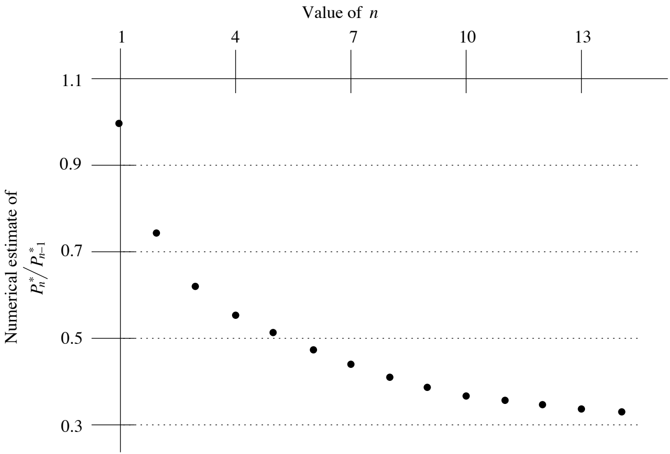
Bruhat Order Numerics



Weak Order Numerics

n	R_n^*	Estimate of $P_n^* \approx \frac{R_n^*}{10^9}$	Estimate of P_n^*/P_{n-1}^*
10	1538639	$0.0015386\ldots$	$0.368718\ldots$
11	541488	$0.0005414\ldots$	$0.351926\ldots$
12	184273	$0.0001842\ldots$	$0.340308\ldots$
13	59917	$0.0000599\ldots$	$0.325153\ldots$
14	18721	$0.0000187\ldots$	$0.312448\ldots$
15	5714	$0.0000057\ldots$	$0.305218\ldots$
16	1724	$0.0000017\ldots$	$0.301715\ldots$

Weak Order Numerics



Open Problems

An alternating sign matrix is a square matrix of 0's, 1's and -1's for which

Open Problems

An alternating sign matrix is a square matrix of 0's, 1's and -1's for which

- the sum of the entries in each row and column is 1,

Open Problems

An alternating sign matrix is a square matrix of 0's, 1's and -1's for which

- the sum of the entries in each row and column is 1,
- the non-zero entries of each row and column alternate in sign.

Open Problems

An alternating sign matrix is a square matrix of 0's, 1's and -1's for which

- the sum of the entries in each row and column is 1,
- the non-zero entries of each row and column alternate in sign.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Open Problems

An alternating sign matrix is a square matrix of 0's, 1's and -1's for which

- the sum of the entries in each row and column is 1,
- the non-zero entries of each row and column alternate in sign.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- Any permutation matrix is also an alternating sign matrix.

Open Problems

The set of Monotone Triangles of order n, $\mathfrak{M}_n$:

Open Problems

The set of Monotone Triangles of order n, $\mathfrak{M}_n$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Open Problems

The set of Monotone Triangles of order n, $\mathfrak{M}_n$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\longleftrightarrow \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 2 & & 3 & & & \\ & & 1 & & 3 & & 5 & & \\ & 1 & & 2 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Open Problems

- $(\mathfrak{M}_n, \leq)$, defined entry-wise, is the unique (MacNeille) completion of $(\mathfrak{S}_n, \leq)$ to a lattice (Stanley, “Enumerative Comb., Vol. II”).

Open Problems

- $(\mathfrak{M}_n, \leq)$, defined entry-wise, is the unique (MacNeille) completion of $(\mathfrak{S}_n, \leq)$ to a lattice (Stanley, “Enumerative Comb., Vol. II”).

(1) What about comparability probability for this lattice? Recent work has only focused on enumeration of these objects (Zeilberger, Kuperberg).

Open Problems

- $(\mathfrak{M}_n, \leq)$, defined entry-wise, is the unique (MacNeille) completion of $(\mathfrak{S}_n, \leq)$ to a lattice (Stanley, "Enumerative Comb., Vol. II").

(1) What about comparability probability for this lattice? Recent work has only focused on enumeration of these objects (Zeilberger, Kuperberg).

(2) What about the size of the largest *anti*-chain in weak order? This is closed for Bruhat order (it has the Sperner property; Engel, "Sperner Theory").

For Further Reading

A. Hammett, B. Pittel.
How often are two permutations comparable?
Trans. of the Amer. Math. Soc., 2009.