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On Comparability of Random Permutations

Adam J. Hammett
Cedarville University, ahammett@cedarville.edu

ON COMPARABILITY OF RANDOM PERMUTATIONS

DISSERTATION

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By

Adam Hammett, B.S.

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Dissertation Committee:

Dr. Boris Pittel, Advisor

Dr. Gerald Edgar

Dr. Akos Seress

Approved by

Advisor
Graduate Program in Mathematics

ABSTRACT

Two permutations of $[n] := \{1, 2, \ldots, n\}$ are comparable in the *Bruhat order* if one can be obtained from the other by a sequence of transpositions decreasing the number of inversions. We show that the total number of pairs of permutations (π, σ) with $\pi \leq \sigma$ is of order $(n!)^2/n^2$ at most. Equivalently, if π, σ are chosen uniformly at random and independently of each other, then $P(\pi \leq \sigma)$ is of order n^{-2} at most. By a direct probabilistic argument we prove $P(\pi \leq \sigma)$ is of order $(0.708)^n$ at least, so that there is currently a wide qualitative gap between the upper and lower bounds.

Next, emboldened by a connection with Ferrers diagrams and plane partitions implicit in Bressoud's book [13], we return to the Bruhat order upper bound and show that for n-permutations $\pi_1, \ldots, \pi_r$ selected independently and uniformly at random,

$$P(\pi_1 \leq \cdots \leq \pi_r) = O\left(n^{-r(r-1)}\right),$$

thus providing an extension of our result for pairs of permutations to chains of length $r > 2$.

Turning to the related *weak order* "$\preceq$" – when only adjacent transpositions are admissible – we use a non-inversion set criterion to prove that $P_n^* := P(\pi \preceq \sigma)$ is submultiplicative, thus showing existence of $\rho = \lim \sqrt[n]{P_n^*}$. We demonstrate that ρ is 0.362 at most. Moreover, we prove the lower bound $\prod_{i=1}^{n} (H(i)/i)$ for P_n^*, where

$H(i) := \sum_{j=1}^{i} 1/j$. In light of numerical experiments, we conjecture that for each order the *upper* bounds for permutation-pairs are qualitatively close to the actual behavior. We believe that extensions to r-chains similar to that for the Bruhat order upper bound can be made for our other bounds in each order, and are presently working in this direction.

Finally, the weak order poset happens to be a lattice, and we study some properties of its infimums and supremums. Namely, we prove that the number of r-tuples $(\pi_1, \ldots, \pi_r)$ of n-permutations with minimal infimum, $12\cdots n$, asymptotically equals

$$-\frac{(n!)^r}{h_r'(z^*)(z^*)^{n+1}}, \quad r \geq 2, \quad n \to \infty. \tag{1}$$

Here, $z^* = z^*(r) \in (1, 2)$ is the unique (positive) root of the equation

$$h_r(z) := \sum_{j \geq 0} \frac{(-1)^j}{(j!)^r} z^j = 0$$

within the disk $|z| \leq 2$. Moreover, (1) is also the asymptotic number of r-tuples with maximal supremum, $n(n-1)\cdots 1$.

To my wife, Rachael, for her unending suppport.

ACKNOWLEDGMENTS

First and foremost, I wish to thank my advisor, Boris Pittel. Our collaboration was quite intensive on this research project, and without his countless suggestions and endless encouragement, none of these results would have materialized. He is truly an outstanding example of what a mentor should be.

This work was inspired in large part by a thought-provoking talk Mark Skandera gave at the MIT Combinatorics Conference honoring Richard Stanley's 60^{th} birthday. We are grateful to Mark for an enlightening follow-up discussion of comparability criteria for the Bruhat order. We thank Sergey Fomin for encouragement and for introducing us to an instrumental notion of the permutation-induced poset, and also for many insightful suggestions regarding the history of Bruhat order. Without Ed Overman's invaluable guidance we would not have been able to obtain our numerical results. Craig Lennon gave us an idea for proving an exponential lower bound in the case of Bruhat order. We thank Miklós Bóna for his interest in this work and for drawing our attention to the lower bound for the number of linear extensions in Richard Stanley's book.

VITA

April 28, 1979	Born - Santa Maria, CA
1997-2001	Undergraduate, Westmont College - Santa Barbara, CA
2001	B.S. in Mathematics, Westmont College
2001-Present	Graduate Teaching Associate, The Ohio State University

FIELDS OF STUDY

Major Field: Mathematics

Specialization: Combinatorial Probability

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CHAPTER 1

INTRODUCTION

In this chapter, we present the fundamental ideas necessary to understand the material in subsequent chapters. We also state the main results to be proved in this dissertation. It is assumed that the reader has some familiarity with probability theory and combinatorics. A good graduate-level introduction to these subjects can be found, for instance, in Billingsley [4] and Stanley's volumes on enumerative combinatorics, [46] and [47].

1.1 Bruhat Order, Preliminaries

Let $n \geq 1$ be an integer. Two permutations of $[n] := \{1, \ldots, n\}$ are comparable in the *Bruhat order* if one can be obtained from the other by a sequence of transpositions of pairs of elements forming an inversion. Here is a precise definition of the Bruhat order on the set of permutations S_n (see [46, p. 172, ex. 75. a.], Humphreys [30, p. 119]). If $\omega = \omega(1) \cdots \omega(n) \in S_n$, then a *reduction* of ω is a permutation obtained from ω by interchanging some $\omega(i)$ with some $\omega(j)$ provided $i < j$ and $\omega(i) > \omega(j)$. We say that $\pi \leq \sigma$ in the Bruhat order if there is a chain $\sigma = \omega_1 \to \omega_2 \to \cdots \to \omega_s = \pi$, where each ω_t is a reduction of ω_{t-1}. The number of inversions in ω_t strictly decreases with t.

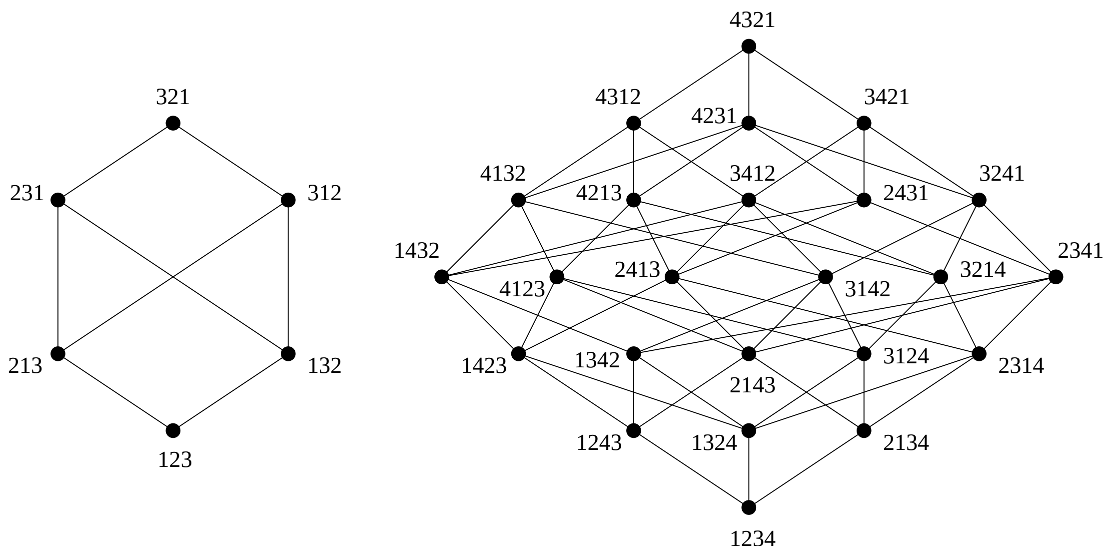

Figure 1.1: The Bruhat order on S_3 and S_4.

Indeed, one can show that if ω_2 is a reduction of ω_1 via the interchange $\omega_1(i) \leftrightarrow \omega_1(j)$, $i < j$, then

$$\mathrm{inv}(\omega_1) = \mathrm{inv}(\omega_2) + 2N(\omega_1) + 1,$$
$$N(\omega_1) := |\{k \,:\, i < k < j,\, \omega_1(i) > \omega_1(k) > \omega_1(j)\}|;$$

here $\mathrm{inv}(\omega_1)$, say, is the number of inversions in ω_1 (see Björner and Brenti [6]). Figure 1.1 illustrates this poset on S_3 and S_4. The Bruhat order notion can be extended to other Coxeter groups (see Björner [5], Deodhar [20], and [6, p. 63] for historical background), but we will be dealing with the symmetric group S_n only.

The definition of the Bruhat order is very transparent, and yet deciding for given π, σ

whether $\pi \le \sigma$ from the definition is computationally difficult, even for smallish n. Fortunately, there exist efficient algorithms for checking Bruhat comparability, which can all be traced back to an algorithmic comparability criterion due to Ehresmann (1934) [22] (see also Knuth [34], Björner and Brenti [6]). The Ehresmann "tableau criterion" states that $\pi \le \sigma$ if and only if $\pi_{i,j} \le \sigma_{i,j}$ for all $1 \le i \le j \le n-1$, where $\pi_{i,j}$ and $\sigma_{i,j}$ are the i-th entry in the increasing rearrangement of $\pi(1), \ldots, \pi(j)$ and of $\sigma(1), \ldots, \sigma(j)$. These arrangements form two staircase tableaux, hence the term "tableau criterion". For example, $41523 > 21534$ is verified by element-wise comparisons of the two tableaux

$$\begin{array}{cccc} 1 & 2 & 4 & 5 \\ 1 & 4 & 5 & \\ 1 & 4 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 5 \\ 1 & 2 & 5 & \\ 1 & 2 & & \\ 2 & & & \end{array}.$$

Also, it is well-known that Ehresmann's criterion is equivalent to the $(0,1)$-matrix criterion. It involves comparing the number of 1's contained in certain submatrices of the $(0,1)$-permutation matrices representing π and σ (see Bóna [10], [6]). Later, Björner and Brenti [7] were able to improve on the result of [22], giving a tableau criterion that requires fewer operations. Very recently, Drake, Gerrish and Skandera [21] have found two new comparability criteria, involving totally nonnegative polynomials and the Schur functions respectively. We are aware of other criteria (see [5], Fulton [25, pp. 173-177], Lascoux and Schützenberger [36], [20]), but we found the $(0,1)$-matrix and Ehresmann criteria most amenable to probabilistic study.

The $(0,1)$-matrix criterion for Bruhat order on S_n says that for $\pi, \sigma \in S_n$, $\pi \leq \sigma$ if and only if for all $i, j \leq n$, the number of $\pi(1), \ldots, \pi(i)$ that are at most j exceeds (or equals) the number of $\sigma(1), \ldots, \sigma(i)$ that are at most j (see [11] for this version). It is referred to as the $(0,1)$-matrix criterion because of the following recasting of this condition: let $M(\pi)$, $M(\sigma)$ be the permutation matrices corresponding to π, σ, so that for instance the (i,j)-entry of $M(\pi)$ is 1 if $\pi(j) = i$ and 0 otherwise. Here, we are labeling columns $1, 2, \ldots, n$ when reading from *left to right*, and rows are labeled $1, 2, \ldots, n$ when reading from *bottom to top* so that this interpretation is like placing ones at points $(i, \pi(i))$ of the $n \times n$ integer lattice and zeroes elsewhere. Denoting submatrices of $M(\cdot)$ corresponding to rows I and columns J by $M(\cdot)_{I,J}$, this criterion says that $\pi \leq \sigma$ if and only if for all $i, j \leq n$, the number of ones in $M(\pi)_{[i],[j]}$ is at least the number of ones in $M(\sigma)_{[i],[j]}$ (see [21] for this version).

An effective way of visualizing this criterion is to imagine the matrices $M(\pi)$ and $M(\sigma)$ as being superimposed on one another into a single matrix, $M(\pi, \sigma)$, with the ones for $M(\pi)$ represented by $\times$'s ("crosses"), the ones for $M(\sigma)$ by $\circ$'s ("balls") and the zeroes for both by empty entries. Note that some entries of $M(\pi, \sigma)$ may be occupied by both a cross and a ball. Then the $(0,1)$-matrix criterion says that $\pi \leq \sigma$ if and only if every southwest submatrix of $M(\pi, \sigma)$ contains at least as many crosses as balls. Here, in the notation above, a *southwest submatrix* is a submatrix $M(\pi, \sigma)_{[i],[j]}$ of $M(\pi, \sigma)$ for some $i, j \leq n$. It is clear that we could also check $\pi \leq \sigma$ by checking that crosses are at least as numerous as balls in every northeast submatrix of $M(\pi, \sigma)$. Likewise, $\pi \leq \sigma$ if and only if balls are at least as numerous as crosses in every northwest submatrix of $M(\pi, \sigma)$, or similarly balls are at least

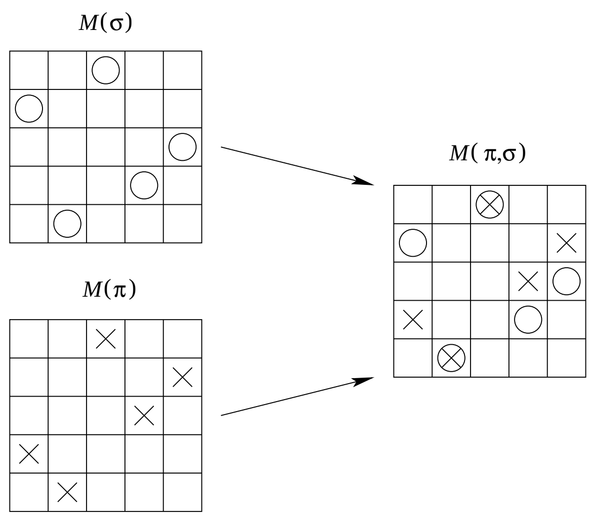

Figure 1.2: Superimposing $M(\pi)$ and $M(\sigma)$ to form $M(\pi, \sigma)$.

as numerous as crosses in every southeast submatrix of $M(\pi, \sigma)$. Parts of all four of these equivalent conditions will be used in our proofs. As a quick example, with $\pi = 21534$ and $\sigma = 41523$, $\pi < \sigma$ is checked by examining southwest submatrices of $M(\pi, \sigma)$ in Figure 1.2. Also, the superimposing of $M(\pi)$ with $M(\sigma)$ to form $M(\pi, \sigma)$ is illustrated in this figure.

1.2 Main Results Related to the Bruhat Order

In this dissertation, we use the $(0, 1)$-matrix and the Ehresmann criteria to obtain upper and lower bounds for the number of pairs (π, σ) with $\pi \leq \sigma$.

Theorem 1.2.1. *Let $n \geq 1$ be an integer, and let $\pi, \sigma \in S_n$ be selected independently and uniformly at random. Then there exist universal constants $c_1, c_2 > 0$ such that*

$$c_1 (0.708)^n \leq P(\pi \leq \sigma) \leq c_2/n^2.$$

Equivalently, the number of pairs (π, σ) with $\pi \leq \sigma$ is sandwiched between the counts $c_1(0.708)^n(n!)^2$ and $c_2 n^{-2}(n!)^2$. The lower bound follows from a sufficient condition derived from the $(0,1)$-matrix criterion, and a computer-aided tabulation of an attendant function of a smallish integer argument. Empirical estimates based on generating pairs of random permutations suggest that $P(\pi \leq \sigma)$ is of order $n^{-(2+\delta)}$, for δ close to 0.5 from above. So apparently it is the upper bound which comes close to the true proportion $P(\pi \leq \sigma)$. It is certain that the constant 0.708 can be further improved, but we do not know if our method could be extended to deliver a lower bound $(1-o(1))^n$. A lower bound n^{-a}, a qualitative match of the upper bound, seems out of sight presently.

A deeper insight reveals a more general result, related to chains of length r in Bruhat order, once we realize some connections with MacMahon's formula [13] for counting plane partitions contained in an $r \times s \times t$ box. Without going into much unnecessary detail here, one can visualize a *plane partition* as stacks of unit cubes pushed into a corner. The k-th Ehresmann condition contains a clear connection between Bruhat order on permutations and counting combinatorial objects related to plane partitions, namely *non-intersecting lattice paths*, a notion we will make precise later on. A closer look at our methods for permutation-pairs in the spirit of Gessel and Viennot's work

[26] implies an extension of Theorem 1.2.1, upper bound, from pairs of permutations to r-tuples:

Theorem 1.2.2. *Let $\pi_1, \ldots, \pi_r \in S_n$ be selected independently and uniformly at random. Then there exists a uniform constant $c > 0$ such that*

$$P(\pi_1 \leq \cdots \leq \pi_r) \leq c/n^{r(r-1)}.$$

Note that this result implies that there are at most $cn^{-r(r-1)}(n!)^r$ length r chains in the Bruhat order poset.

1.3 Weak Order, Preliminaries

Then we turn to the modified order on S_n, the *weak order* "$\preceq$". Here $\pi \preceq \sigma$ if there is a chain $\sigma = \omega_1 \to \omega_2 \to \cdots \to \omega_s = \pi$, where each ω_t is a *simple reduction* of ω_{t-1}, i.e. obtained from ω_{t-1} by transposing two *adjacent* elements $\omega_{t-1}(i)$, $\omega_{t-1}(i+1)$ with $\omega_{t-1}(i) > \omega_{t-1}(i+1)$. Since at each step the number of inversions decreases by 1, all chains connecting σ and π have the same length. Alternatively, there is a simple non-inversion (resp. inversion) set criterion, contained in Berge [3], we can use to check $\pi \preceq \sigma$. Indeed, given $\omega \in S_n$ introduce the set of non-inversions of ω:

$$E(\omega) = \left\{(i,j) \,:\, i < j,\, \omega^{-1}(i) < \omega^{-1}(j)\right\}.$$

Similarly, for $\omega \in S_n$ we introduce the set of inversions of ω:

$$E^*(\omega) = \left\{(i,j)\,:\, i > j,\, \omega^{-1}(i) < \omega^{-1}(j)\right\}.$$

Then, for given $\pi, \sigma \in S_n$, we have $\pi \preceq \sigma$ if and only if $E(\pi) \supseteq E(\sigma)$ (equivalently $E^*(\pi) \subseteq E^*(\sigma)$). Note that $\omega \in S_n$ is uniquely determined by its $E(\omega)$ (resp. its $E^*(\omega)$).

It turns out that the poset $(S_n, \preceq)$ is a lattice (see [3]). Indeed, given $\pi_1, \ldots, \pi_r \in S_n$, there is an efficient way to compute $E(\inf\{\pi_1, \ldots, \pi_r\})$ (resp. $E^*(\sup\{\pi_1, \ldots, \pi_r\})$) from the set $\cup_{i=1}^r E(\pi_i)$ (resp. $\cup_{i=1}^r E^*(\pi_i)$). We will see precisely how to do this later.

1.4 Main Results Related to the Weak Order

We prove the following probabilistic result for weak order comparability:

Theorem 1.4.1. *Let $\pi, \sigma \in S_n$ be selected independently and uniformly at random, and let $P_n^* := P(\pi \preceq \sigma)$. Then P_n^* is submultiplicative, i.e. $P_{n_1+n_2}^* \le P_{n_1}^* P_{n_2}^*$. Consequently there exists $\rho = \lim \sqrt[n]{P_n^*}$. Furthermore, there exists an absolute constant $c > 0$ such that*

$$\prod_{i=1}^{n} \left(H\left(i\right)/i\right) \le P_n^* \le c\left(0.362\right)^n,$$

where $H(i) := \sum_{j=1}^{i} 1/j$. Consequently, $\rho \le 0.362$.

The proof of the upper bound is parallel to that of Theorem 1.2.1, lower bound, while the lower bound follows from the non-inversion (resp. inversion) set criterion described last section. Empirical estimates indicate that ρ is close to 0.3. So here too, as in Theorem 1.2.1, the upper bound seems to be qualitatively close to the actual probability P_n^*. And our lower bound, though superior to the trivial bound $1/n!$, is decreasing superexponentially fast with n, which makes us believe that there ought to be a way to vastly improve it.

Paradoxically, it is the lower bound that required a deeper combinatorial insight. Clearly the number of π's below (or equal to) σ equals $e(\mathcal{P})$, the total number of linear extensions of $\mathcal{P} = \mathcal{P}(\sigma)$, the poset induced by σ. (The important notion of $\mathcal{P}(\sigma)$ was brought to our attention by Sergey Fomin [24].) We prove that for *any* poset $\mathcal{P}$ of cardinality n,

$$e\,(\mathcal{P}) \geq n! \Big/ \prod_{i \in \mathcal{P}} d\,(i)\,, \tag{1.1}$$

where $d\,(i) := |\,\{j \in \mathcal{P}\,:\, j \leq i \text{ in } \mathcal{P}\}\,|$. (This bound is an exact value of $e(\mathcal{P})$ if the Hasse diagram is a (directed) rooted tree, Knuth [34, sect. 5.1.4, ex. 20], or a forest of such trees, Björner and Wachs [8].) The bound (1.1) for $e(\mathcal{P}(\sigma))$ together with the independence of sequential ranks in the uniform permutation were the key ingredients in the proof of Theorem 1.4.1, lower bound.

Miklós Bóna [12] has informed us that this general lower bound for $e(\mathcal{P})$ had been stated by Richard Stanley as a level 2 exercise in [46, p. 312, ex. 1] without a solution. We have decided to keep the proof in the dissertation, since we could not find a published proof anywhere either. The classic hook formula provides an example of a poset $\mathcal{P}$ for which (1.1) is markedly below $e(\mathcal{P})$. It remains to be seen whether (1.1) can be strengthened in general, or at least for $\mathcal{P}(\sigma)$. As an illustration,

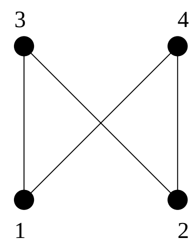

Figure 1.3: The permutation-induced poset $\mathcal{P}(2143)$.

$\mathcal{P} = \mathcal{P}(2143)$ has the Hasse diagram appearing in Figure 1.3. Then $e(\mathcal{P}) = 4$, but (1.1) delivers only

$$e(\mathcal{P}) \geq 24/9 \quad \Longrightarrow \quad e(\mathcal{P}) \geq 3.$$

Regarding the lattice properties of $(S_n, \preceq)$, note that the identity permutation $12\cdots n$ is the unique minimum, and $n(n-1)\cdots 1$ is the unique maximum. Let $\pi_1, \ldots, \pi_r \in S_n$ be selected independently and uniformly at random. It is natural to ask: "How likely is it that the infimum (resp. supremum) of $\{\pi_1, \ldots, \pi_r\}$ is the unique minimal (resp. maximal) element in the weak order lattice?" Equivalently, what is the asymptotic number of r-tuples $(\pi_1, \ldots, \pi_r)$ such that $\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n$ (resp. $\sup\{\pi_1, \ldots, \pi_r\} = n(n-1)\cdots 1$), $n \to \infty$? It turns out that the answer is the same whether we consider infs or sups, which allows us to focus only on infimums. We prove the following:

Theorem 1.4.2. *Let* $P_n^{(r)} = P\left(\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n\right)$. *Then*

1. *As a function of* n, $P_n^{(r)}$ *is submultiplicative. Hence, there exists*

$$p(r) = \lim \sqrt[n]{P_n^{(r)}} = \inf \sqrt[n]{P_n^{(r)}}, \quad r \geq 1.$$

2. *For each* $r \geq 1$*, put* $h_r(z) = \sum_{j\geq 0} \frac{(-1)^j}{(j!)^r} z^j$ *and* $H_r(z) = (h_r(z))^{-1}$*. Then, letting* $P_0^{(r)} = 1$*, we have*

$$H_r(z) = \sum_{n\geq 0} P_n^{(r)} z^n,$$

from which we obtain (Darboux theorem [2])

$$P_n^{(r)} \sim -\frac{1}{z^* h_r'(z^*)} \frac{1}{(z^*)^n}, \qquad n \to \infty.$$

Here, $z^* = z^*(r) \in (1, 2)$ *is the unique simple root of* $h_r(z) = 0$ *in the disc* $|z| \leq 2$*. Consequently,* $p(r) = 1/z^*$*.*

Unlike our results about comparability in the Bruhat and weak orderings, here we have a *sharp asymptotic formula.* The key to the proof of this theorem is establishing the *exact* formula

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r},$$

which follows from the principle of inclusion-exclusion. This formula for $P_n^{(r)}$ is, in some sense, an r-analog of that for the *Eulerian numbers* (Bóna [11], Knuth [34]). Indeed, it turns out that $P_n^{(r)}$ is the probability that the uniform, independent permutations $\pi_1^{-1}, \ldots, \pi_r^{-1}$ have *no common descents.* Introduce the random variable $S_n^{(r)}$, the number of these common descents, so that $P_n^{(r)} = P\left(S_n^{(r)} = 0\right)$. Another natural question here is:

Problem. *What is the limiting distribution of $S_n^{(r)}$?*

We believe that the answer here is "Gaussian", as it is in the case of the number of descents in a *single* uniformly random permutation (Sachkov [44]). Our feeling is that the proof will involve use of the bivariate generating function $F_r(x, y) = \sum_{n\geq 1} x^n E\left[(1+y)^{S_n^{(r)}}\right]$, which we prove has the simple form

$$F_r(x,y) = \frac{xf_r(xy)}{1 - xf_r(xy)}, \qquad f_r(z) := \sum_{j\geq 0} \frac{z^j}{(j+1)!^r}.$$

Interestingly, this generating function is a special case of a more general result proved by Richard Stanley [45], although he was probably unaware of the connections his work had with the weak ordering.

In conclusion we mention several papers that are in the same spirit of this dissertation. First, [39] and [40] (both by Boris Pittel), where the "probability-of-comparability" problems were solved for the poset of integer partitions of n under dominance order, and for the poset of set partitions of $[n]$ ordered by refinement. Also, [41] (again by B. Pittel), where the "infimum/supremum" problem was solved for the lattice of set partitions of $[n]$ ordered by refinement. In [16], E. R. Canfield presents an enlightening extension of the inf/sup work done in [41]. Very recently, in [1], R. M. Adin and Y. Roichman explore the valency properties of a typical permutation in the Bruhat order Hasse diagram.

This work is in large part the result of an intensive collaborative effort with my doctoral advisor, Boris Pittel. Portions of this dissertation have been accepted for publication (2006) in the journal *Transactions of the American Mathematical Society* (see [28] for availability).

CHAPTER 2

THE PROOF OF THE BRUHAT ORDER UPPER BOUND

In this chapter, we focus on the proof of Theorem 1.2.1, upper bound. The proof divides naturally into three steps, hence the divisions of the sections that follow.

We need to show that

$$P\left(\pi \leq \sigma\right) = O\left(n^{-2}\right).$$

The argument is based on the $(0, 1)$-matrix criterion. We assume that n is even. Only minor modifications are necessary for n odd.

2.1 A Necessary Condition for Bruhat Comparability

The $(0, 1)$-matrix criterion requires that a set of n^2 conditions are met. The challenge is to select a subset of those conditions which meets two conflicting demands. It has to be sufficiently simple so that we can compute (estimate) the probability that the random pair (π, σ) satisfies all the chosen conditions. On the other hand, collectively these conditions need to be quite stringent for this probability to be $o(1)$. In our first advance we were able (via Ehresmann's criterion) to get a bound $O(n^{-1/2})$ by using about $2n^{1/2}$ conditions. We are about to describe a set of $2n$ conditions that does the job.

Let us split the matrices $M(\pi, \sigma)$, $M(\pi)$ and $M(\sigma)$ into 4 submatrices of equal size $n/2 \times n/2$ – the southwest, northeast, northwest and southeast corners, denoting them $M_{sw}(\cdot)$, $M_{ne}(\cdot)$, $M_{nw}(\cdot)$ and $M_{se}(\cdot)$ respectively. In the southwest corner $M_{sw}(\pi, \sigma)$, we restrict our attention to southwest submatrices of the form $i \times n/2$, $i = 1, \ldots, n/2$. If $\pi \leq \sigma$, then as we read off rows of $M_{sw}(\pi, \sigma)$ from bottom to top keeping track of the total number of balls and crosses encountered thus far, at any intermediate point we must have at least as many crosses as balls. Let us denote the set of pairs (π, σ) such that this occurs by $\mathcal{E}_{sw}$. We draw analogous conclusions for the northeast corner, reading rows from top to bottom, and we denote by $\mathcal{E}_{ne}$ the set of pairs (π, σ) satisfying this condition.

Similarly, we can read columns from left to right in the northwest corner, and here we must always have at least as many balls as crosses. Denote the set of these pairs (π, σ) by $\mathcal{E}_{nw}$. The same condition holds for the southeast corner when we read columns from right to left. Denote the set of these pairs (π, σ) by $\mathcal{E}_{se}$. Letting $\mathcal{E}$ denote the set of pairs (π, σ) satisfying all four of the conditions above, we get

$$\{\pi \leq \sigma\} \subseteq \mathcal{E} = \mathcal{E}_{sw} \cap \mathcal{E}_{ne} \cap \mathcal{E}_{nw} \cap \mathcal{E}_{se}.$$

Pairs of permutations in $\mathcal{E}$ satisfy $2n$ of the n^2 conditions required by the $(0, 1)$-matrix criterion. And unlike the set $\{\pi \leq \sigma\}$, we are able to compute $|\mathcal{E}|$, and to show that $P(\mathcal{E}) = (n!)^{-2}|\mathcal{E}| = O(n^{-2})$. Figure 2.1 is a graphical visualization of the reading-off process that generates the restrictions defining the set $\mathcal{E}$.

If a row (column) of a submatrix $M(\pi)_{I,J}$ ($M(\sigma)_{I,J}$ resp.) contains a marked entry, we say that it *supports* the submatrix. Clearly the number of supporting rows (columns)

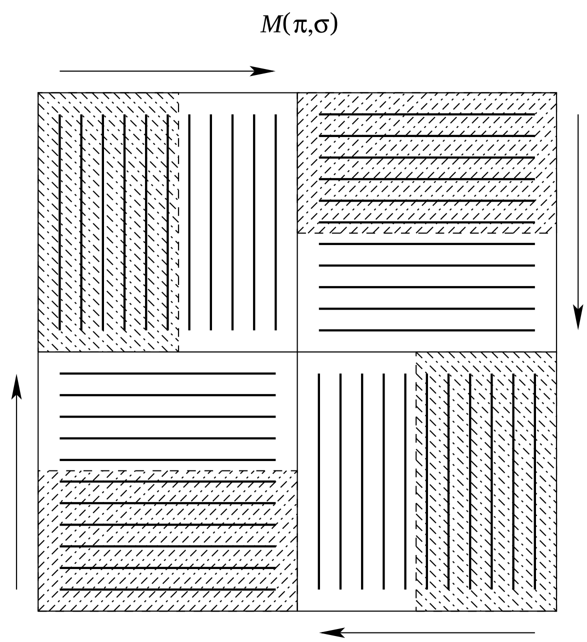

Figure 2.1: Finding a necessary condition for $\pi \leq \sigma$.

equals the number of marked entries in $M(\pi)_{I,J}$ ($M(\sigma)_{I,J}$ resp.). Now, given π, σ, let $M_1 = M_1(\pi)$, $M_2 = M_2(\sigma)$ denote the total number of rows that support $M_{sw}(\pi)$ and $M_{sw}(\sigma)$ respectively. Then $M_{nw}(\pi)$, $M_{nw}(\sigma)$ are supported by $M_3 = n/2 - M_1$ columns and by $M_4 = n/2 - M_2$ columns respectively. The same holds for the southeastern corners of $M(\pi)$ and $M(\sigma)$. Obviously the northeastern submatrices of $M(\pi)$ and $M(\sigma)$ are supported by M_1 rows and M_2 rows respectively. Then we have

$$P(\mathcal{E}) = \sum_{m_1, m_2} P(\mathcal{E} \cap \mathcal{A}(m_1, m_2)), \tag{2.1}$$
$$\mathcal{A}(m_1, m_2) := \{(\pi, \sigma) : M_1 = m_1, M_2 = m_2\}.$$

Clearly $\mathcal{E} \cap \mathcal{A}(m_1, m_2) = \emptyset$ if $m_1 < m_2$. We claim that, for $m_1 \geq m_2$,

$$P(\mathcal{E} \cap \mathcal{A}(m_1, m_2)) = \left[\frac{(m_1 - m_2 + 1)(n/2 + 1)}{(n/2 - m_2 + 1)(m_1 + 1)}\right]^4 \cdot \frac{\prod_{i=1}^4 \binom{n/2}{m_i}}{\binom{n}{n/2}^2}. \tag{$\dagger$}$$

Here and below $m_3 := n/2 - m_1$ and $m_4 := n/2 - m_2$ stand for generic values of M_3 and M_4 in the event $\mathcal{A}(m_1, m_2)$.

To prove ($\dagger$), let us count the number of pairs (π, σ) in $\mathcal{E} \cap \mathcal{A}(m_1, m_2)$. First consider the southwest corner, $M_{sw}(\pi, \sigma)$. Introduce $L_1 = L_1(\pi, \sigma)$, the number of rows supporting both $M_{sw}(\pi)$ and $M_{sw}(\sigma)$. So L_1 is the number of rows in the southwest corner $M_{sw}(\pi, \sigma)$ containing both a cross and a ball. Suppose that we are on the event $\{L_1 = \ell_1\}$. We choose ℓ_1 rows to support both $M_{sw}(\pi)$ and $M_{sw}(\sigma)$ from the $n/2$ first rows. Then, we choose $(m_1 - \ell_1 + m_2 - \ell_1)$ more rows from the remaining $(n/2 - \ell_1)$ rows. Each of these secondary rows is to support either $M_{sw}(\pi)$ or $M_{sw}(\sigma)$, but not both. This step can be done in

$$\binom{n/2}{\ell_1}\binom{n/2 - \ell_1}{m_1 - \ell_1 + m_2 - \ell_1}$$

ways. Next, we partition the set of $(m_1 - \ell_1 + m_2 - \ell_1)$ secondary rows into two row subsets of cardinality $(m_1 - \ell_1)$ (rows to contain crosses) and $(m_2 - \ell_1)$ (rows to contain balls) that will support $M_{sw}(\pi)$ and $M_{sw}(\sigma)$, accompanying the ℓ_1 primary rows supporting both submatrices. We can visualize each of the resulting row selections as a subsequence of $(1, \ldots, n/2)$ which is a disjoint union of two subsequences, one with ℓ_1 elements labeled by a ball and a cross, and another with $(m_1-\ell_1+m_2-\ell_1)$ elements, $(m_1-\ell_1)$ labeled by crosses and the remaining $(m_2-\ell_1)$ elements labeled by balls. The condition $\mathcal{E}_{sw}$ is equivalent to the restriction: moving along the subsequence from left to right, at each point the number of crosses is not to fall below the number of balls. Obviously, no double-marked element can cause violation of this condition. Thus, our task is reduced to determination of the number of $(m_1-\ell_1+m_2-\ell_1)$-long sequences of $m_1-\ell_1$ crosses and $m_2-\ell_1$ balls such that at no point the number of crosses is strictly less than the number of balls. By the classic ballot theorem (see Takacs [49, pp. 2-7]), the total number of such sequences equals

$$\frac{(m_1-\ell_1+1)-(m_2-\ell_1)}{(m_1-\ell_1+1)+(m_2-\ell_1)}\binom{m_1-\ell_1+m_2-\ell_1+1}{m_1-\ell_1+1}$$
$$=\frac{m_1-m_2+1}{m_1-\ell_1+1}\binom{m_1-\ell_1+m_2-\ell_1}{m_1-\ell_1}.$$

The second binomial coefficient is the total number of $(m_1-\ell_1+m_2-\ell_1)$-long sequences of $(m_1-\ell_1)$ crosses and $(m_2-\ell_1)$ balls. So the second fraction is the probability that the sequence chosen uniformly at random among all such sequences meets the ballot theorem condition. The total number of ways to designate the rows supporting $M_{sw}(\pi)$ and $M_{sw}(\sigma)$, subject to the condition $\mathcal{E}_{sw}$, is the product of two counts, namely

$$\binom{n/2}{\ell_1}\binom{n/2-\ell_1}{m_1-\ell_1+m_2-\ell_1}\binom{m_1-\ell_1+m_2-\ell_1}{m_1-\ell_1}\frac{m_1-m_2+1}{m_1-\ell_1+1}$$
$$=\frac{m_1-m_2+1}{n/2-m_2+1}\binom{n/2}{m_2}\binom{m_2}{\ell_1}\binom{n/2-m_2+1}{m_1-\ell_1+1}.$$

Summing this last expression over all $\ell_1 \le m_2$, we obtain

$$\begin{aligned}\frac{m_1-m_2+1}{n/2-m_2+1}\binom{n/2}{m_2}\sum_{\ell_1\le m_2}\binom{m_2}{\ell_1}\binom{n/2-m_2+1}{m_1-\ell_1+1}&\\ =\frac{m_1-m_2+1}{n/2-m_2+1}\binom{n/2}{m_2}\binom{n/2+1}{m_1+1}&\\ =\frac{(m_1-m_2+1)(n/2+1)}{(n/2-m_2+1)(m_1+1)}\binom{n/2}{m_1}\binom{n/2}{m_2}&.\end{aligned} \tag{2.2}$$

Here, in the first equality, we have used the binomial theorem. The product of the two binomial coefficients in the final count (2.2) is the total number of row selections from the first $n/2$ rows, m_1 to contain crosses and m_2 to contain balls. So the fraction preceding these two binomial factors is the probability that a particular row selection chosen uniformly at random from all such row selections satisfies our ballot condition "crosses never fall below balls". Equivalently, by the very derivation, the expression (2.2) is the total number of paths $(X(t), Y(t))_{0\le t\le n/2}$ on the square lattice connecting $(0,0)$ and (m_1, m_2) such that $X(t+1)-X(t)$, $Y(t+1)-Y(t) \in \{0,1\}$, and $X(t) \ge Y(t)$ for every t. (To be sure, if $X(t+1)-X(t) = 1$ and $Y(t+1)-Y(t) = 1$, the corresponding move is a combination of horizontal and vertical unit moves.)

Likewise, we consider the northeast corner, $M_{ne}(\pi,\sigma)$. We introduce $L_2 = L_2(\pi,\sigma)$, the number of rows in $M_{ne}(\pi,\sigma)$ containing both a cross and a ball. By initially restricting to the event $\{L_2 = \ell_2\}$, then later summing over all $\ell_2 \le m_2$, we obtain another factor (2.2). Analogously, a third and fourth factor (2.2) comes from considering *columns* in the northwest and southeast corners, $M_{nw}(\pi,\sigma)$ and $M_{se}(\pi,\sigma)$. Importantly, the row selections for the southwest and the northeast submatrices do not interfere with the column selections for the northwest and the southeast corners. So by multiplying these four factors (2.2) we obtain the total number of row and column selections on the event $\mathcal{A}(m_1, m_2)$ subject to all four restrictions defining $\mathcal{E}$! Once such a row-column selection has been made, we have determined which rows and columns support the four submatrices of $M(\pi)$ and $M(\sigma)$. Consider, for instance, the southwest corner of $M(\pi)$. We have selected m_1 rows (from the first $n/2$ rows) supporting $M_{sw}(\pi)$, and we have selected m_3 columns (from the first $n/2$ columns) supporting $M_{nw}(\pi)$. Then it is the remaining $n/2 - m_3 = m_1$ columns that support $M_{sw}(\pi)$. The number of ways to match these m_1 rows and m_1 columns, thus to determine $M_{sw}(\pi)$ completely, is $m_1!$. The northeast corner contributes another $m_1!$, while each of the two other corners contributes $m_3!$, whence the overall matching factor is $(m_1!m_3!)^2$. The matching factor for σ is $(m_2!m_4!)^2$. Multiplying the number of admissible row-column selections by the resulting $\prod_{i=1}^4 (m_i!)^2$ and dividing by $(n!)^2$, we obtain

$$P\left(\mathcal{E} \cap \mathcal{A}\left(m_1, m_2\right)\right) = \left[\frac{(m_1 - m_2 + 1)(n/2 + 1)}{(n/2 - m_2 + 1)(m_1 + 1)} \binom{n/2}{m_1} \binom{n/2}{m_2}\right]^4 \cdot \frac{\prod_{i=1}^4 (m_i!)^2}{(n!)^2},$$

which is equivalent to (†). Figure 2.2 is a graphical explanation of this matching factor. In it, we show the matrix $M(\pi)$ in a case when in the southwest and the

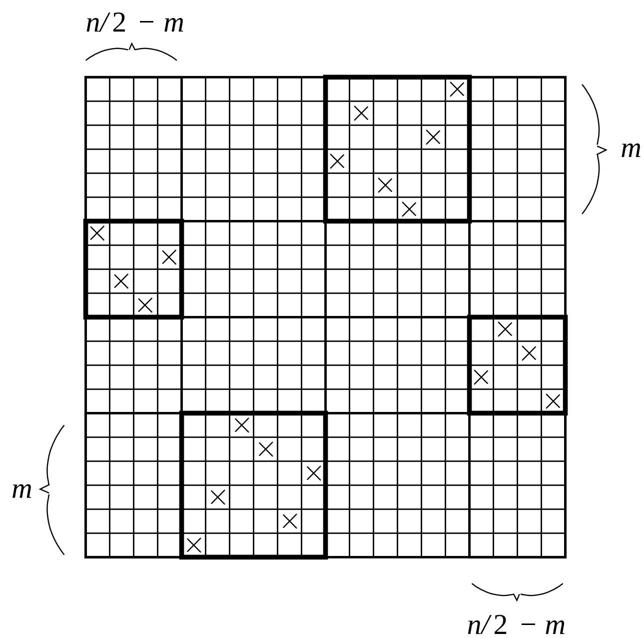

Figure 2.2: Selection of first $m = m_1$ ($n/2-m$ resp.) rows (columns resp.) in corners to support $M(\pi)$.

northeast squares π is supported by the bottom $m(= m_1)$ and the top m rows respectively; likewise, in the northwest and the southeast squares π is supported by the $n/2 - m$ leftmost and the $n/2 - m$ rightmost columns respectively.

2.2 A Probabilistic Simplification

Let us show that (2.1) and (†) imply

$$P(\mathcal{E}) \leq E\left[\frac{(M_1 - M_2 + 1)^4 (n/2 + 1)^4}{(n/2 - M_2 + 1)^4 (M_1 + 1)^4}\right]. \tag{‡}$$

First, M_1 and M_2 are independent with

$$P(M_i = m_i) = \frac{\binom{n/2}{m_i}^2}{\binom{n}{n/2}}, \quad i = 1, 2.$$

Indeed, M_i obviously equals the cardinality of the intersection with $[n/2]$ of a uniformly random subset of size $n/2$ from $[n]$, which directly implies these formulas. Thus, each M_i has the hypergeometric distribution with parameters $n/2, n/2, n/2$; in other words, M_i has the same distribution as the number of red balls in a uniformly random sample of $n/2$ balls from an urn containing $n/2$ red balls and $n/2$ white balls. By the independence of M_1 and M_2, we obtain

$$P(M_1 = m_1, M_2 = m_2) = \frac{\binom{n/2}{m_1}^2 \binom{n/2}{m_2}^2}{\binom{n}{n/2}^2}.$$

It remains to observe that (2.1) and (†) imply

$$\begin{aligned}
P(\mathcal{E}) &= \sum_{m_1 \geq m_2} \frac{(m_1 - m_2 + 1)^4 (n/2 + 1)^4}{(n/2 - m_2 + 1)^4 (m_1 + 1)^4} \cdot P(M_1 = m_1, M_2 = m_2) \\
&\leq \sum_{m_1, m_2} \frac{(m_1 - m_2 + 1)^4 (n/2 + 1)^4}{(n/2 - m_2 + 1)^4 (m_1 + 1)^4} \cdot P(M_1 = m_1, M_2 = m_2) \\
&= E\left[\frac{(M_1 - M_2 + 1)^4 (n/2 + 1)^4}{(n/2 - M_2 + 1)^4 (M_1 + 1)^4}\right],
\end{aligned}$$

and (‡) is proved.

2.3 Asymptotics

The advantage of (‡) is that it allows us to use probabilistic tools exploiting the independence of the random variables M_1 and M_2. Typically the M_i's are close to $n/4$, while $|M_1 - M_2|$ is of order $n^{1/2}$ at most. So, in view of (‡) we expect that $P(\mathcal{E}) = O(n^{-2})$.

We now make this argument rigorous. First of all, by the "sample-from-urn" interpretation of M_i,

$$E\left[M_i\right] = \frac{n}{2}\frac{\binom{n-1}{n/2-1}}{\binom{n}{n/2}} = n/4. \tag{2.3}$$

Then (see Janson et al. [31, p. 29]) the probability generating function of M_i is dominated by that of $\mathrm{Bin}(n, 1/4)$, and consequently for each $t \geq 0$ we have

$$P\left(|M_i - n/4| \geq t\right) = O\left(\exp\left(-4t^2/n\right)\right).$$

Hence, setting $t = n^{2/3}$ we see that

$$P\left(n/4 - n^{2/3} < M_i < n/4 + n^{2/3}\right) \geq 1 - e^{-cn^{1/3}},$$

for some absolute constant $c > 0$. Introduce the event

$$A_n = \bigcap_{i=1}^{2}\left\{n/4 - n^{2/3} < M_i < n/4 + n^{2/3}\right\}.$$

Combining the estimates for M_i, we see that for some absolute constant $c_1 > 0$,

$$P(A_n) \geq 1 - e^{-c_1 n^{1/3}}.$$

Now the random variable in (‡), call it X_n, is bounded by 1, and on the event A_n, within a factor of $1 + O(n^{-1/3})$,

$$X_n = \left(\frac{4}{n}\right)^8 (M_1 - M_2 + 1)^4 \, (n/2 + 1)^4 .$$

Therefore

$$P(\mathcal{E}) \leq \left(\frac{5}{n}\right)^8 (n/2+1)^4 \, E\left[(M_1 - M_2 + 1)^4\right] + O\left(e^{-c_1 n^{1/3}}\right).$$

It remains to prove that this expected value is $O(n^2)$. Introduce $\overline{M}_i = M_i - E[M_i]$, $i = 1, 2$. Then

$$(M_1 - M_2 + 1)^4 = (\overline{M}_1 - \overline{M}_2 + 1)^4 \leq 27(\overline{M}_1^4 + \overline{M}_2^4 + 1),$$

as

$$(a + b + c)^2 \leq 3(a^2 + b^2 + c^2).$$

We now demonstrate that $E[\overline{M}_i^4] = O(n^2)$. To this end, notice first that $E[\overline{M}_i^2]$ is of order n exactly. Indeed, extending the computation in (2.3),

$$\begin{aligned} E[M_i(M_i - 1)] &= \frac{n}{2}\left(\frac{n}{2} - 1\right) \frac{\binom{n-2}{n/2-2}}{\binom{n}{n/2}} \\ &= \frac{n(n-2)^2}{16(n-1)}. \end{aligned}$$

Therefore

$$
\begin{aligned}
E\left[\overline{M}_i^2\right] &= \mathrm{Var}[M_i] \\
&= E[M_i(M_i-1)] + E[M_i] - E^2[M_i] \\
&= \frac{n(n-2)^2}{16(n-1)} + \frac{n}{4} - \frac{n^2}{16} \qquad (2.4) \\
&= \frac{n}{16} + O(1).
\end{aligned}
$$

Furthermore, as a special instance of the hypergeometrically distributed random variable, M_i has the same distribution as the sum of $n/2$ independent Bernoulli variables $Y_j \in \{0, 1\}$ (see Vatutin and Mikhailov [38], alternatively [31, p. 30]). Therefore, (2.4) and the Lindeberg-Feller Central Limit Theorem imply

$$
\frac{\overline{M}_i}{\sqrt{n/16}} \Longrightarrow \mathcal{N}(0,1), \tag{2.5}
$$

where $\mathcal{N}(0, 1)$ is the standard normal random variable. In fact, since

$$
\frac{Y_j - E[Y_j]}{\sqrt{n/16}} \to 0, \quad n \to \infty,
$$

we can say more. Indeed, we have (2.5) together with convergence of all the moments (see Billingsley [4, p. 391]). Therefore, in particular

$$
\frac{E\left[\overline{M}_i^4\right]}{\left(\sqrt{n/16}\right)^4} \to E\left[\mathcal{N}(0,1)^4\right], \quad n \to \infty,
$$

i.e. $E[\overline{M}_i^4] = O(n^2)$. This completes the proof of Theorem 1.2.1 (upper bound). $\square$

CHAPTER 3

THE PROOF OF THE BRUHAT ORDER LOWER BOUND

In this chapter, we prove Theorem 1.2.1, lower bound. We will actually prove something better than what was stated there, showing that for each $\epsilon > 0$

$$P\left(\pi \leq \sigma\right) = \Omega\left(\left(\alpha - \epsilon\right)^n\right),$$

where

$$\alpha = \sqrt[11]{\frac{25497938851324213335}{22!}} = 0.70879\ldots.$$

First, some preliminaries.

3.1 A Sufficient Condition for Bruhat Comparability

Introduce π^* (σ^* resp.), the permutation π (σ resp.) with the element n deleted. More generally, for $k \leq n$, π^{k*} (σ^{k*} resp.) is the permutation of $[n-k]$ obtained from π (σ resp.) by deletion of the k largest elements, n, $n-1, \ldots, n-k+1$. The key to the proof is the following:

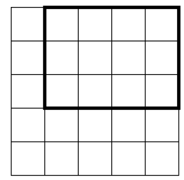

Figure 3.1: G_5 and an emboldened subgrid C.

Lemma 3.1.1. *Let $k \in [n]$. If every northeastern submatrix of $M(\pi, \sigma)$ with at most k rows contains at least as many crosses as balls, and $\pi^{k*} \le \sigma^{k*}$, then $\pi \le \sigma$.*

Before proceeding with the proof, we introduce one more bit of notation. Let G_n be the empty $n \times n$ grid depicted in the $M(\cdot)$'s of Figure 1.2. Figure 3.1 is a depiction of G_5 and an emboldened northeastern-corner 3×4 subgrid of it, denoted by C. If C is any subgrid of G_n, then $M(\cdot \,|\, C)$ denotes the submatrix of $M(\cdot)$ that "sits" on C. To repeat, the $(0,1)$-matrix criterion says that $\pi \le \sigma$ if and only if for each northeastern-corner subgrid C of G_n, we have at least as many crosses as balls in $M(\pi, \sigma \,|\, C)$.

Proof. By the assumption, it suffices to show that the balls do not outnumber crosses in $M(\pi, \sigma \,|\, C)$ for every such subgrid C with strictly more than k rows. Consider any such C. Let $C^{(k)}$ denote the subgrid formed by the top k rows of C. Given a submatrix A of $M(\pi)$ (of $M(\sigma)$ resp.), let $|A|$ denote the number of columns in A with a cross (a ball resp.). We need to show

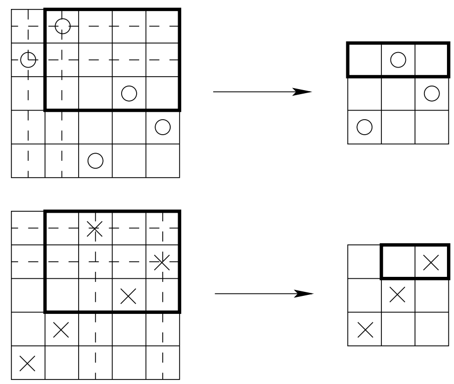

Figure 3.2: Deletion of 2 largest elements of π, σ, and its affect on C.

$$|M(\pi \,|\, C)| \geq |M(\sigma \,|\, C)|.$$

By the assumption, we have $|M(\pi \,|\, C^{(k)})| \geq |M(\sigma \,|\, C^{(k)})|$. Write $|M(\pi \,|\, C^{(k)})| = |M(\sigma \,|\, C^{(k)})| + \lambda$, $\lambda \geq 0$. We now delete the top k rows from $M(\pi), M(\sigma)$ together with the k columns that contain the top k crosses in the case of $M(\pi)$ and the k columns that contain the top k balls in the case of $M(\sigma)$. This produces the matrices $M(\pi^{k*})$ and $M(\sigma^{k*})$. In either case, we obtain the grid G_{n-k} together with a new northeastern subgrid: $C(\pi^{k*})$ in the case of $M(\pi)$ and $C(\sigma^{k*})$ in the case of $M(\sigma)$. Figure 3.2 is a graphical visualization of this deletion process in the special case $\pi = 12534$, $\sigma = 45132$, $k = 2$ and C the 3×4 northeastern subgrid of G_5. We have emboldened C in $M(\pi), M(\sigma)$, and the resulting $C(\pi^{2*}), C(\sigma^{2*})$ in $M(\pi^{2*}), M(\sigma^{2*})$ respectively.

Since we delete more columns in the case of π than σ, note that $C(\pi^{k*}) \subseteq C(\sigma^{k*})$ as northeastern subgrids of G_{n-k}. In fact, these grids have the same number of rows, but $C(\pi^{k*})$ has λ fewer columns. Hence, as $\pi^{k*} \leq \sigma^{k*}$, we have

$$|M(\pi^{k*} \,|\, C(\pi^{k*}))| \geq |M(\sigma^{k*} \,|\, C(\pi^{k*}))| \geq |M(\sigma^{k*} \,|\, C(\sigma^{k*}))| - \lambda.$$

So

$$\begin{aligned}
|M(\pi \,|\, C)| &= |M(\pi \,|\, C^{(k)})| + |M(\pi^{k*} \,|\, C(\pi^{k*}))| \\
&= |M(\sigma \,|\, C^{(k)})| + \lambda + |M(\pi^{k*} \,|\, C(\pi^{k*}))| \\
&\geq |M(\sigma \,|\, C^{(k)})| + |M(\sigma^{k*} \,|\, C(\sigma^{k*}))| \\
&= |M(\sigma \,|\, C)|,
\end{aligned}$$

which proves the lemma. □

3.2 A Reduction to Uniforms

For each $k \leq n$, let $\mathcal{E}_{n,k}$ denote the event "every northeast submatrix of the top k rows has at least as many crosses as balls". Then by Lemma 3.1.1,

$$\{\pi \leq \sigma\} \supseteq \mathcal{E}_{n,k} \cap \left\{\pi^{k*} \leq \sigma^{k*}\right\}.$$

Now the events $\mathcal{E}_{n,k}$ and $\left\{\pi^{k*} \leq \sigma^{k*}\right\}$ are independent! So we get

$$P\left(\pi \leq \sigma\right) \geq P\left(\mathcal{E}_{n,k}\right) P\left(\pi^{k*} \leq \sigma^{k*}\right). \tag{3.1}$$

For the permutation π (σ resp.) introduce $\ell_i(\pi) = \pi^{-1}(i)$ ($\ell_i(\sigma) = \sigma^{-1}(i)$ resp.), the index of a column that contains a cross (a ball resp.) at the intersection with row i. In terms of the $\ell_i(\cdot)$'s, $\mathcal{E}_{n,k}$ is the event: for each integer $j \le k$ and $m \le n$, the number of $\ell_n(\pi), \ell_{n-1}(\pi), \ldots, \ell_{n-j+1}(\pi)$ that are m at least is more than or equal to the number of $\ell_n(\sigma), \ell_{n-1}(\sigma), \ldots, \ell_{n-j+1}(\sigma)$ that are m at least. We could have replaced an integer $m \le n$ with a real number, which means that

$$\mathcal{E}_{n,k} = \{(\pi, \sigma) : (\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{C}_k\},$$

for some cone-shaped (Borel) set $\mathcal{C}_k \subset \mathbb{R}^{2k}$; here $\boldsymbol{\ell}(\pi) = \{\ell_{n-i+1}(\pi)\}_{1 \le i \le k}$, $\boldsymbol{\ell}(\sigma) = \{\ell_{n-i+1}(\sigma)\}_{1 \le i \le k}$.

Our task is to estimate sharply $P(\mathcal{E}_{n,k})$ for a fixed k, and $n \to \infty$. Observe first that $\boldsymbol{\ell}(\pi)$ and $\boldsymbol{\ell}(\sigma)$ are independent, and each uniformly distributed. For instance

$$P(\ell_n(\pi) = j_1, \ldots, \ell_{n-k+1}(\pi) = j_k) = \frac{1}{(n)_k}, \quad 1 \le j_1 \ne \cdots \ne j_k \le n,$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. Since $(n)_k \sim n^k$ as $n \to \infty$, $\ell_n(\pi), \ldots, \ell_{n-k+1}(\pi)$ are almost independent $[n]$-uniforms for large n, and fixed k. Let us make this asymptotic reduction rigorous. Let U be a uniform-$[0,1]$ random variable, and let $U_1, \ldots, U_n$ be independent copies of U. Then each $\lceil nU_i \rceil$ is uniform on $[n]$, and it is easy to show that

$$P(\lceil nU_1 \rceil = i_1, \ldots, \lceil nU_k \rceil = i_k \mid \lceil nU_1 \rceil \ne \cdots \ne \lceil nU_k \rceil) = \frac{1}{(n)_k}.$$

In other words, $\{\ell_{n-i+1}(\pi)\}_{1 \le i \le k}$ has the same distribution as the random vector $\lceil n\mathbf{U} \rceil := \{\lceil nU_i \rceil\}_{1 \le i \le k}$ conditioned on the event $\mathcal{A}_{n,k} = \{\lceil nU_1 \rceil \ne \cdots \ne \lceil nU_k \rceil\}$. Analogously $\{\ell_{n-i+1}(\sigma)\}_{1 \le i \le k}$ is distributed as $\lceil n\mathbf{V} \rceil := \{\lceil nV_i \rceil\}_{1 \le i \le k}$ conditioned on

$\mathcal{B}_{n,k} = \{\lceil nV_1 \rceil \neq \cdots \neq \lceil nV_k \rceil\}$, where $V_1, \ldots, V_k$ are independent $[0,1]$-uniforms, independent of $U_1, \ldots, U_k$. We will need yet another event $\mathcal{D}_{n,k}$ on which

$$\min\{\min_{i \neq j} |U_i - U_j|, \min_{i \neq j} |V_i - V_j|, \min_{i,j} |U_i - V_j|\} > 1/n.$$

Clearly on $\mathcal{D}_{n,k}$

$$(\lceil n\mathbf{U} \rceil, \lceil n\mathbf{V} \rceil) \in \mathcal{C}_k \Longleftrightarrow (\mathbf{U}, \mathbf{V}) \in \mathcal{C}_k;$$

here $\mathbf{U} := \{U_i\}_{1 \leq i \leq k}$, $\mathbf{V} := \{V_i\}_{1 \leq i \leq k}$. In addition $\mathcal{D}_{n,k} \subseteq \mathcal{A}_{n,k} \cap \mathcal{B}_{n,k}$, and

$$P(\mathcal{D}^c_{n,k}) \leq 2k^2 P(|U_1 - U_2| \leq 1/n) \leq 4k^2/n.$$

Therefore

$$\begin{aligned}
P(\mathcal{E}_{n,k}) &= P((\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{C}_k) \\
&= \frac{P\left(\{(\lceil n\mathbf{U} \rceil, \lceil n\mathbf{V} \rceil) \in \mathcal{C}_k\} \cap \{\mathcal{A}_{n,k} \cap \mathcal{B}_{n,k}\}\right)}{P(\mathcal{A}_{n,k} \cap \mathcal{B}_{n,k})} \\
&= \frac{P\left(\{(\lceil n\mathbf{U} \rceil, \lceil n\mathbf{V} \rceil) \in \mathcal{C}_k\} \cap \mathcal{D}_{n,k}\right) + O(P(\mathcal{D}^c_{n,k}))}{1 - O(P(\mathcal{D}^c_{n,k}))} \\
&= \frac{P\left((\mathbf{U}, \mathbf{V}) \in \mathcal{C}_k\right) + O(k^2/n)}{1 - O(k^2/n)} \\
&= Q_k + O(k^2/n),
\end{aligned}$$

where $Q_k = P\left((\mathbf{U}, \mathbf{V}) \in \mathcal{C}_k\right)$. Let us write $P_n = P(\pi \leq \sigma)$. Using (3.1) and the last estimate, we obtain then

$$P_n \geq Q_k P_{n-k} \left(1 + O(k^2/n)\right) = Q_k P_{n-k} \exp\left(O(k^2/n)\right), \quad n > k.$$

Iterating this inequality $\lfloor n/k \rfloor$ times gives

$$P_n \geq Q_k^{\lfloor n/k \rfloor} P_{n-\lfloor n/k \rfloor k} \exp\left[\sum_{j=0}^{\lfloor n/k \rfloor -1} O\left(\frac{k^2}{n-jk}\right)\right].$$

Since the sum in the exponent is of order $O(k^2 \log n)$, we get

$$\liminf \sqrt[n]{P_n} \geq \sqrt[k]{Q_k}, \quad \forall k \geq 1.$$

Thus

$$\liminf \sqrt[n]{P_n} \geq \sup_k \sqrt[k]{Q_k}.$$

Therefore, for each k and $\epsilon \in \left(0, \sqrt[k]{Q_k}\right)$, we have

$$P_n = \Omega\left(\left(\sqrt[k]{Q_k} - \epsilon\right)^n\right). \tag{3.2}$$

Next

Lemma 3.2.1. *As a function of k, Q_k is supermultiplicative, i.e. $Q_{k_1+k_2} \geq Q_{k_1} Q_{k_2}$ for all $k_1, k_2 \geq 1$. Consequently there exists $\lim_{k\to\infty} \sqrt[k]{Q_k}$, and moreover*

$$\lim_{k\to\infty} \sqrt[k]{Q_k} = \sup_{k\geq 1} \sqrt[k]{Q_k}.$$

Thus we expect that our lower bound would probably improve as k increases.

Proof. Q_k is the probability of the event $E_k = \{(\mathbf{U}^{(k)}, \mathbf{V}^{(k)}) \in \mathcal{C}_k\}$; here $\mathbf{U}^{(k)} := \{U_i\}_{1\leq i\leq k}$, $\mathbf{V}^{(k)} := \{V_i\}_{1\leq i\leq k}$. Explicitly, for each $j \leq k$ and each $c \in [0,1]$, the number of $U_1, \ldots, U_j$ not exceeding c is at most the number of $V_1, \ldots, V_j$ not exceeding c. So $Q_{k_1+k_2} = P(E_{k_1+k_2})$, $Q_{k_1} = P(E_{k_1})$, while $Q_{k_2} = P(E_{k_2}) = P(E^*_{k_2})$. Here the event $E^*_{k_2}$ means that for each $j^* \leq k_2$ and each $c \in [0,1]$, the number of U_i, $i = k_1+1, \ldots, k_1+j^*$, not exceeding c is at most the number of V_i, $i = k_1+1, \ldots, k_1+j^*$, not exceeding c. The events E_{k_1} and $E^*_{k_2}$ are independent. Consider the intersection of E_{k_1} and $E^*_{k_2}$. There are two cases:

1) $j \leq k_1$. Then the number of U_i, $i \leq j$ not exceeding c is at most the number of V_i, $i \leq j$ not exceeding c, as E_{k_1} holds.

2) $k_1 < j \leq k_1 + k_2$. Then the number of U_i, $i \leq j$, not exceeding c is at most the number of V_i, $i \leq k_1$ not exceeding c (as E_{k_1} holds), plus the number of V_i, $k_1 < i \leq j$, not exceeding c (as $E^*_{k_2}$ holds). The total number of these V_i is the number of all V_i, $i \leq j$, that are at most c, $c \in [0,1]$.

So $E_{k_1+k_2} \supseteq E_{k_1} \cap E^*_{k_2}$, and we get $Q_{k_1+k_2} \geq Q_{k_1}Q_{k_2}$. The rest of the statement follows from a well-known result about super(sub)multiplicative sequences (see Pólya and Szegö [43, p. 23, ex. 98]). $\square$

Given $1 \leq j \leq i \leq k$, let $U_{i,j}$ ($V_{i,j}$ resp.) denote the j-th element in the increasing rearrangement of $U_1, \ldots, U_i$ ($V_1, \ldots, V_i$ resp.). Then, to put it another way, Q_k is the probability that the k Ehresmann conditions are met by the independent k-dimensional random vectors $\mathbf{U}$ and $\mathbf{V}$, both of which have independent entries. That is, we check $U_{i,j} > V_{i,j}$ for each $1 \leq j \leq i \leq k$ by performing element-wise comparisons in the following tableaux:

$$
\begin{array}{ccccc}
U_{k,1} & U_{k,2} & U_{k,3} & \cdots & U_{k,k} \\
\vdots & \vdots & \vdots & \iddots & \\
U_{3,1} & U_{3,2} & U_{3,3} & & \\
U_{2,1} & U_{2,2} & & & \\
U_{1,1} & & & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
V_{k,1} & V_{k,2} & V_{k,3} & \cdots & V_{k,k} \\
\vdots & \vdots & \vdots & \iddots & \\
V_{3,1} & V_{3,2} & V_{3,3} & & \\
V_{2,1} & V_{2,2} & & & \\
V_{1,1} & & & &
\end{array}
.
$$

3.3 An Algorithm to Maximize the Bound

What's left is to explain how we determined $\alpha = 0.70879....$

It should be clear that whether or not $(\mathbf{U}^{(k)}, \mathbf{V}^{(k)})$ is in $\mathcal{C}_k$ depends only on the size ordering of $U_1, \ldots, U_k, V_1, \ldots, V_k$. There are $(2k)!$ possible orderings, all being equally likely. Thus $Q_k = N_k/(2k)!$, where N_k is the number of these linear orderings satisfying this modified Ehresmann criterion. Since the best constant in the lower exponential bound is probably $\lim_{k\to\infty} \sqrt[k]{Q_k}$, our task was to compute N_k for k as large as our computer could handle. ("Probably", because we do not know for certain that $\sqrt[k]{Q_k}$ increases with k.)

Here is how N_k was tabulated. Recursively, suppose we have determined all N_{k-1} orderings of $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$ such that $(\mathbf{x}^{(k-1)}, \mathbf{y}^{(k-1)}) \in \mathcal{C}_{k-1}$. Each such ordering can be assigned a $2(k-1)$-long sequence of 0's and 1's, 0's for x_i's and 1's for y_j's, $1 \le i, j \le k-1$. Each such sequence meets the ballot-theorem condition: as we read it from *left to right* the number of 1's never falls below the number of 0's. We also record the *multiplicity* of each sequence, which is the number of times it is encountered in the list of all N_{k-1} orderings. The knowledge of all $2(k-1)$-long ballot-sequences together with their multiplicities is all we need to compile the list of all $2k$-long ballot-sequences with their respective multiplicities.

For $k = 1$, there is only one ballot-sequence to consider, namely 10, and its multiplicity is 1. So $N_1 = 1$, and

$$Q_1 = 1/2!.$$

Passing to $k = 2$, we must count the number of ways to insert 1 and 0 into 10 so that we get a 4-long ballot-sequence of two 0's and two 1's. Inserting 1 at the beginning, giving 110, we can insert 0 into positions 2, 3 or 4, producing three ballot-sequences

$$1010, \quad 1100, \quad 1100,$$

respectively. (Inserting 0 into position 1 would have resulted in 0110 which is not a ballot-sequence.) Similarly, inserting 1 into position 2, we get 110, and inserting 0 under the ballot condition gives three ballot-sequences

$$1010, \quad 1100, \quad 1100.$$

Finally, inserting 1 at the end, giving 101, we can only insert 0 at the end, obtaining one ballot-sequence

$$1010.$$

Hence, starting from the ballot-sequence 10 of multiplicity 1, we have obtained two

4-long ballot-sequences, 1010 of multiplicity 3 and 1100 of multiplicity 4. Therefore $N_2 = 3 + 4 = 7$, and

$$Q_2 = 7/4!.$$

Pass to $k = 3$. Sequentially we insert 1 in each of 5 positions in the ballot-sequence 1010, and then determine all positions for the new 0 which would result in a 6-long ballot-sequence. While doing this we keep track of how many times each 6-long ballot-sequence is encountered. Multiplying these numbers by 3, the multiplicity of 1010, we obtain a list of 6-long ballot-sequences spawned by 1010 with the number of their occurrences. We do the same with the second sequence 1100. Adding the numbers of occurrences of each 6-long ballot-sequence for 1010 and 1100, we arrive at the following list of five 6-long ballot-sequences with their respective multiplicities:

$$111000 : 36,$$

$$110100 : 32,$$

$$110010 : 24,$$

$$101100 : 24,$$

$$101010 : 19.$$

Therefore $N_3 = 36 + 32 + 24 + 24 + 19 = 135$, and

$$Q_3 = 135/6!.$$

k	$N_k = (2k)!Q_k$	$Q_k = N_k/(2k)!$	$\sqrt[k]{Q_k}$
1	1	0.50000 ...	0.50000 ...
2	7	0.29166 ...	0.54006 ...
3	135	0.18750 ...	0.57235 ...
4	5193	0.12879 ...	0.59906 ...
5	336825	0.09281 ...	0.62162 ...
6	33229775	0.06937 ...	0.64101 ...
7	4651153871	0.05335 ...	0.65790 ...
8	878527273745	0.04198 ...	0.67280 ...
9	215641280371953	0.03368 ...	0.68608 ...
10	66791817776602071	0.02745 ...	0.69800 ...
11	25497938851324213335	0.02268 ...	0.70879 ...

Table 3.1: Exact computation of N_k for smallish k.

We wrote a computer program for this algorithm. Pushed to its limit, the computer delivered table 3.1.

Combining (3.2) and the value of $\sqrt[11]{Q_{11}}$ in this table, we see that for each $\epsilon > 0$,

$$P_n = \Omega\left(\left(\sqrt[11]{Q_{11}} - \epsilon\right)^n\right) = \Omega\left((0.708... - \epsilon)^n\right). \qquad \square$$

The numbers $\sqrt[k]{Q_k}$ increase steadily for $k < 12$, so at this moment we would not rule out the tantalizing possibility that $\sqrt[k]{Q_k} \to 1$ as $k \to \infty$. Determination of the actual limit is a challenging open problem. The proof just given only involves mention of the $(0, 1)$-matrix criterion, but it was the Ehresmann criterion that actually inspired our initial insights.

CHAPTER 4

AN EXTENSION TO CHAINS IN BRUHAT ORDER

Our goal in this chapter is to prove Theorem 1.2.2, which extends our upper bound result on Bruhat-comparability of permutation-pairs. Namely, we will show that for $\pi_1, \dots, \pi_r \in S_n$ selected independently and uniformly at random, we have

$$P(\pi_1 \leq \cdots \leq \pi_r) = O\left(n^{-r(r-1)}\right).$$

So the number of length r chains in the Bruhat poset is of order at most $n^{-r(r-1)}(n!)^r$. Our basic approach will be, at its core, the same as it was for permutation-pairs, but our enumerative techniques will mimic those established by Gessel and Viennot to count various classes of non-intersecting lattice paths (see [26]). These same techniques are also highlighted in Bressoud's book [13], which recounts the proof of the Alternating-Sign Matrix Conjecture.

4.1 The General Setup

First, some preliminaries. Recall the "superimposed" matrix $M(\pi, \sigma)$ of $\times$'s and $\circ$'s we introduced earlier ($\times$'s for π and $\circ$'s for σ). Let's introduce the analogous more general matrix $M(\pi_1, \dots, \pi_r)$, where we place $\times_j$'s at positions $(i, \pi_j(i))$, $1 \leq i \leq n$,

$1 \leq j \leq r$. Here, we read rows *bottom to top* and columns *left to right*. For instance, if $\pi_1 = 21543$, $\pi_2 = 25143$ and $\pi_3 = 54213$, we have

$$M(\pi_1, \pi_2, \pi_3) = \begin{array}{c|c|c|c|c|c|}
\cline{2-6}
5 & \times_3 & \times_2 & \times_1 & & \\
\cline{2-6}
4 & & \times_3 & & \times_1, \times_2 & \\
\cline{2-6}
3 & & & & & \times_1, \times_2, \times_3 \\
\cline{2-6}
2 & \times_1, \times_2 & & \times_3 & & \\
\cline{2-6}
1 & & \times_1 & \times_2 & \times_3 & \\
\cline{2-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5}
\end{array} .$$

Given a set of rows $I \subseteq [n]$ and columns $J \subseteq [n]$, let $M(\cdot)_{I,J}$ denote the submatrix of $M(\cdot)$ corresponding to rows I and columns J. Again, rows are labeled $1, 2, \ldots, n$ from *bottom to top*, and columns are labeled $1, 2, \ldots, n$ from *left to right*. The $(0, 1)$-matrix criterion says that $\pi_1 \leq \cdots \leq \pi_r$ if and only if for each southwest submatrix $M(\pi_1, \ldots, \pi_r)_{[\mu],[\nu]}$, $\mu, \nu \in [n]$, we have

$$\# \times_1 \text{'s} \geq \cdots \geq \# \times_r \text{'s}.$$

Note that this is the case in $M(\pi_1, \pi_2, \pi_3)$ above, so that we have $\pi_1 \leq \pi_2 \leq \pi_3$. One more bit of notation: let $M_j(\mu, \nu)$ denote the number of $\times_j$'s in $M(\pi_j)_{[\mu],[\nu]}$. In this notation,

$$\pi_1 \leq \cdots \leq \pi_r \quad \Longleftrightarrow \quad M_1(\mu, \nu) \geq \cdots \geq M_r(\mu, \nu) \quad \forall \mu, \nu \in [n].$$

4.2 A Tractable Necessary Condition

Now, as in the case of permutation-pairs, we need to find an event which contains

$$\{\pi_1 \leq \cdots \leq \pi_r\}$$

that is more amenable to enumerative techniques. For simplicity, let's assume n is even and fix $\mu = \nu = n/2$ (in what follows, only minor modifications are necessary for n odd). In our computations, we will primarily concentrate on the single southwest submatrix

$$M_{sw}(\pi_1, \ldots, \pi_r) := M(\pi_1, \ldots, \pi_r)_{[n/2],[n/2]}.$$

We similarly denote by

$$M_{nw}(\pi_1, \ldots, \pi_r), \quad M_{ne}(\pi_1, \ldots, \pi_r), \quad M_{se}(\pi_1, \ldots, \pi_r)$$

the northwest, northeast and southeast $n/2 \times n/2$ subsquares of $M(\pi_1, \ldots, \pi_r)$, respectively. If $\pi_1 \leq \cdots \leq \pi_r$, it is necessary that

$$M_1(i, n/2) \geq \cdots \geq M_r(i, n/2), \quad i = 1, \ldots, n/2. \tag{4.1}$$

This is analogous to the necessary condition we considered for permutation-pairs: we read off rows of $M_{sw}(\pi_1, \ldots, \pi_r)$ one at a time, keeping track of the total number of $\times_j$'s encountered thus far, $1 \leq j \leq r$. If $\pi_1 \leq \cdots \leq \pi_r$, at any intermediate point in this "reading-off" process, we must never have encountered more $\times_j$'s than $\times_{j-1}$'s, $1 < j \leq r$.

Let $\mathcal{E}_{sw}$ denote the event described in (4.1). Recalling our work with permutation-pairs, we extract similar events necessary for $\{\pi_1 \leq \cdots \leq \pi_r\}$ by considering columns in the northwest $n/2 \times n/2$ square (denote this event $\mathcal{E}_{nw}$), rows in the northeast square ($\mathcal{E}_{ne}$) and columns in the southeast square ($\mathcal{E}_{se}$). Then

$$\{\pi_1 \leq \cdots \leq \pi_r\} \subseteq \mathcal{E} := \mathcal{E}_{sw} \cap \mathcal{E}_{nw} \cap \mathcal{E}_{ne} \cap \mathcal{E}_{se},$$

and unlike the set $\{\pi_1 \leq \cdots \leq \pi_r\}$, we can compute $|\mathcal{E}|$! Namely, our task is reduced to showing

$$P(\mathcal{E}) = \frac{|\mathcal{E}|}{(n!)^r} = O\left(n^{-r(r-1)}\right).$$

We have seen Figure 4.1 before, but we show it again here to aid in visualizing the "reading-off" process used to generate the restrictions defining the event $\mathcal{E}$.

4.3 The Core Counting Problem

If a row (column) of a submatrix $M(\pi_j)_{I,J}$ contains a marked entry, we say that it *supports* the submatrix. Clearly the number of supporting rows (columns) equals the number of marked entries in $M(\pi_j)_{I,J}$. Now, given π_j, $1 \leq j \leq r$, let $M_j := M_j(n/2, n/2)$, the total number of rows that support $M_{sw}(\pi_j)$. Then $M_{nw}(\pi_j)$ is supported by $n/2 - M_1$ columns. The same holds for the southeastern corner, $M_{se}(\pi_j)$. Obviously the northeastern submatrix $M_{ne}(\pi_j)$ is supported by M_j rows. Then we have

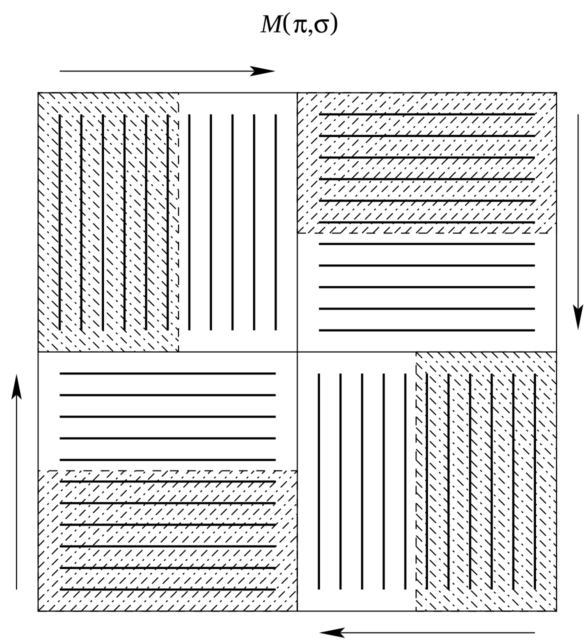

Figure 4.1: Finding a necessary condition for $\pi_1 \leq \cdots \leq \pi_r$.

$$P(\mathcal{E}) = \sum_{m_1,\ldots,m_r} P(\mathcal{E}(m_1,\ldots,m_r)), \tag{4.2}$$

$$\mathcal{E}(m_1,\ldots,m_r) := \mathcal{E} \cap \{(\pi_1,\ldots,\pi_r) \, : \, M_1 = m_1,\ldots,M_r = m_r\}.$$

Clearly, by the (0,1)-matrix criterion, if there is $1 \leq i < j \leq r$ such that $m_i < m_j$, then $\mathcal{E}(m_1,\ldots,m_r) = \emptyset$. Otherwise, we claim:

Theorem 4.3.1. *For* $m_1 \geq \cdots \geq m_r$,

$$P\left(\mathcal{E}\left(m_1,\ldots,m_r\right)\right)=\left[\prod_{1\le i<j\le r}\frac{(m_i-m_j+j-i)(n/2+j-i)}{(m_i+j-i)(n/2-m_j+j-i)}\right]^4\cdot\prod_{i=1}^{r}\frac{\binom{n/2}{m_i}\binom{n/2}{n/2-m_i}}{\binom{n}{n/2}}.$$

This result is really the crux of our argument. The proof, however, will take a little work. We present it in three steps.

Proof. Let us assume we are on the event $\mathcal{E}(m_1,\ldots,m_r)$ and that $m_1\ge\cdots\ge m_r$.

Step 1. As promised, we first concentrate on the southwest $n/2\times n/2$ subsquare $M_{sw}(\pi_1,\ldots,\pi_r)$. For each $1\le j\le r$, we need to choose m_j rows to support $M_{sw}(\pi_j)$ in such a way that $\mathcal{E}_{sw}$ is satisfied. We claim that the number of ways to choose these supporting rows, which we denote $N_{n/2}(m_1,\ldots,m_r)$, is given by the determinant-type formula

$$N_{n/2}(m_1,\ldots,m_r)=\det\left[\binom{n/2}{m_i-i+j}\right]_{i,j=1}^{r}; \tag{4.3}$$

here i is the row index, and j the column index. To prove (4.3), we will exploit a connection with non-intersecting lattice paths implicit in the restrictions defining $\mathcal{E}_{sw}$. As we have already mentioned, these enumerative techniques were first introduced by Gessel and Viennot [26], and were highlighted by Bressoud [13].

For each $j\in[r]$, we construct a lattice path associated with $M_{sw}(\pi_j)$ as follows: start at the point $(j,-j)$ in the plane. If the first row of $M_{sw}(\pi_j)$ contains a marked entry, execute the move $(0,1)$. Otherwise, execute the move $(1,0)$. In general, when looking at the i-th row of $M_{sw}(\pi_j)$, $1\le i\le n/2$, we move from our current position up 1 unit if the i-th row contains a marked entry, and move right 1 unit otherwise. This way, $M_{sw}(\pi_j)$ generates a lattice path consisting of unit moves $(0,1)$ or $(1,0)$, connecting the point $(j,-j)$ with $(n/2-m_j+j, m_j-j)$.

Now, by considering all r of these paths together in the plane, we get what is known as a *nest* of lattice paths. The restrictions defining $\mathcal{E}_{sw}$ imply that the nest of r lattice paths we have just constructed is *non-intersecting*, i.e. no two paths touch each other. So to prove the formula (4.3) we need only show that this determinant counts the total number of these nests of r non-intersecting lattice paths, with moves $(0,1)$ and $(1,0)$, joining the points $\mathcal{S} := \{(j,-j) : j \in [r]\}$ to the points $\mathcal{F} := \{(n/2-m_i+i, m_i-i) : i \in [r]\}$. Figure 4.2 is an illustration of one such nest of $r=7$ paths.

To count the number of these non-intersecting nests, we instead consider the collection of *all* nests of r lattice paths, with moves $(0,1)$ and $(1,0)$, joining the points of $\mathcal{S}$ to the points of $\mathcal{F}$. We require only that no two of the r paths in this nest begin at the same point or end at the same point, with no further restrictions. In particular, such a nest of r lattice paths uses every point from both $\mathcal{S}$ and $\mathcal{F}$. This allows for some *very* tangled nests, like the one shown in Figure 4.3.

To weed out the intersecting nests from the non-intersecting ones, we will employ a special inclusion-exclusion type argument which gives rise to a sum over permutations of $[r]$. Each nest gives rise to a permutation of $[r]$ as follows: define the i-th path to be the one that ends at $(n/2-m_i+i, m_i-i)$, $i \in [r]$. If the i-th path starts at $(j,-j)$, $j \in [r]$, then we define $\sigma(i) = j$. For instance, the tangled nest in Figure 4.3 corresponds to the permutation $\sigma = 3512476$.

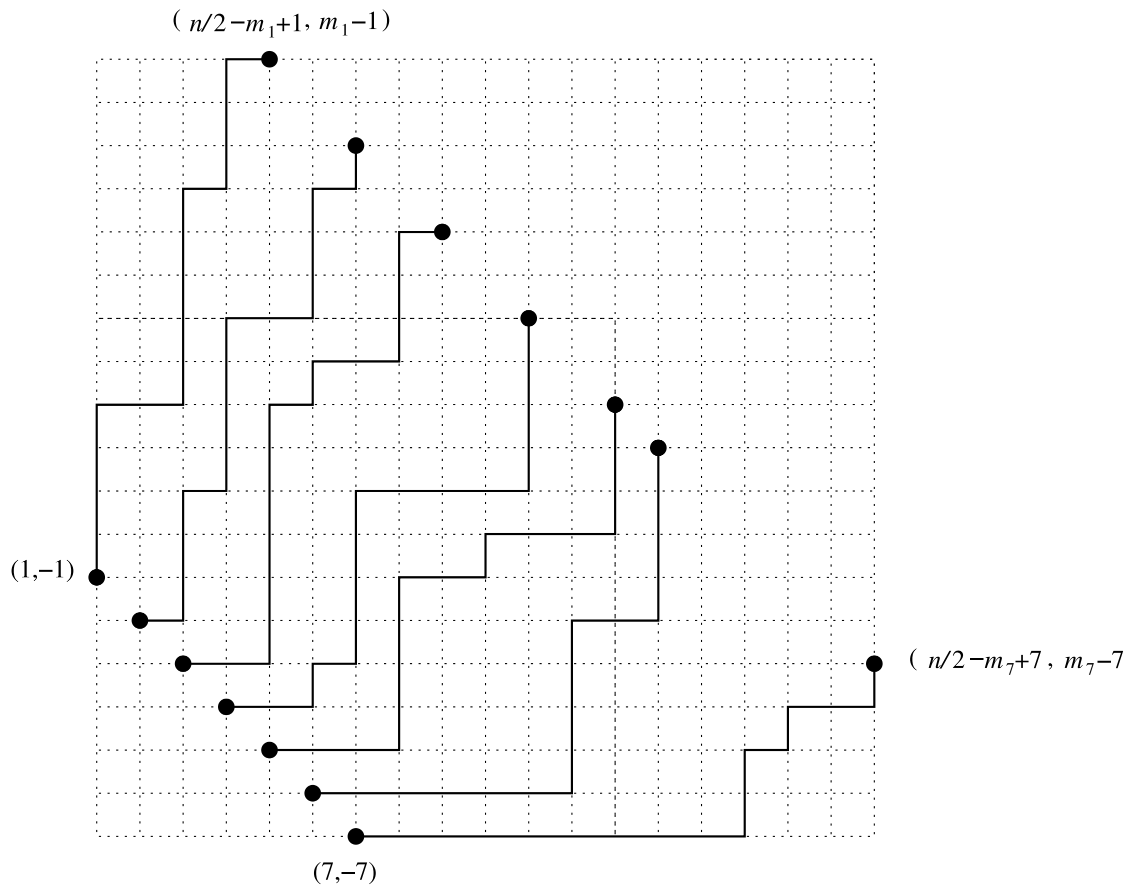

Figure 4.2: A non-intersecting nest of 7 lattice paths.

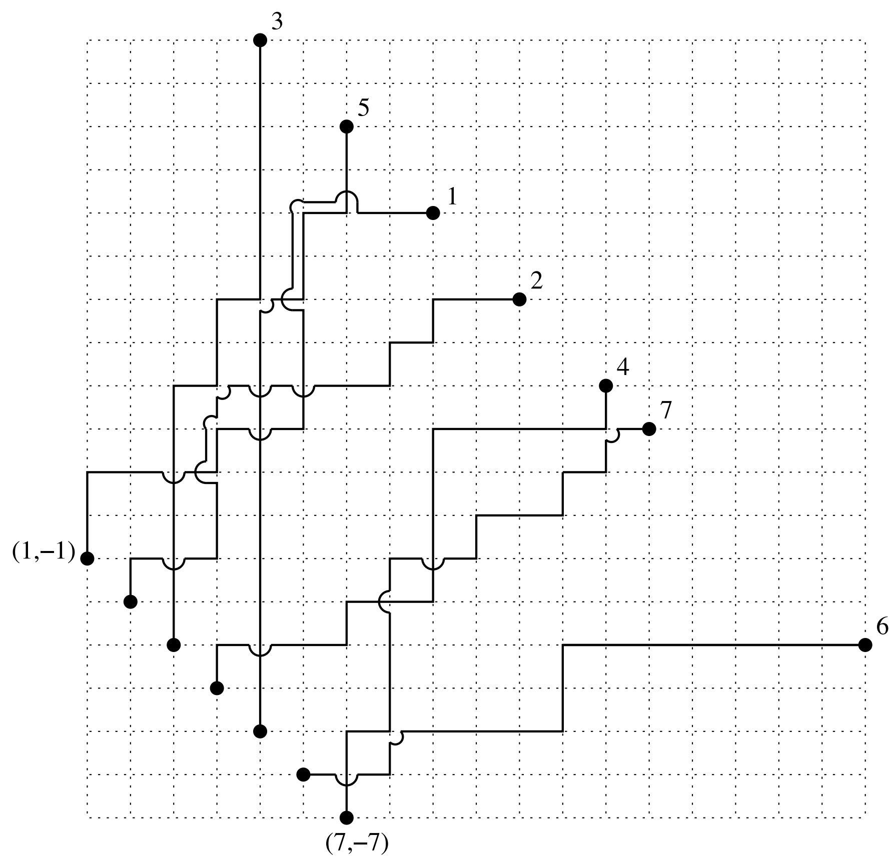

Figure 4.3: An intersecting nest of 7 lattice paths.

On the other hand, given $\sigma \in S_r$, in order that a nest give rise to σ in this correspondence the i-th lattice path must end at $(n/2 - m_i + i, m_i - i)$ and begin at $(\sigma(i), -\sigma(i))$, and so takes a total of $m_i - i + \sigma(i)$ steps northward and $n/2 - m_i + i - \sigma(i)$ steps eastward, $i \in [r]$. Hence, the total number of nests corresponding to the permutation σ equals

$$\prod_{i=1}^{r} \binom{n/2}{m_i - i + \sigma(i)}.$$

Introduce

$$\mathcal{I}(\sigma) := \left|\left\{(i,j) \,:\, 1 \le i < j \le r,\, \sigma^{-1}(i) > \sigma^{-1}(j)\right\}\right|,$$

the *inversion number* of σ. We claim that the number of nests of r non-intersecting lattice paths joining $\mathcal{S}$ to $\mathcal{F}$ equals

$$\sum_{\sigma \in S_r} (-1)^{\mathcal{I}(\sigma)} \prod_{i=1}^{r} \binom{n/2}{m_i - i + \sigma(i)} = \det\left[\binom{n/2}{m_i - i + j}\right]_{i,j=1}^{r}, \tag{4.4}$$

which proves the formula (4.3).

To prove (4.4), notice that this determinant sums over all possible nests, both intersecting and non-intersecting, where each nest is counted as $+1$ if the inversion number of the corresponding σ is even, and as -1 otherwise. If a nest happens to be non-intersecting, then the corresponding permutation is the identity, $12\cdots n$, which has inversion number 0, and so these nests are counted as $+1$. We need to show that everything else in this sum cancels. To do this, we will pair intersecting nests up, one corresponding to a permutation with an even inversion number, the other an odd inversion number.

Let a nest $\mathcal{N}$ with at least one intersection point be given, and let $\sigma \in S_r$ be its corresponding permutation. Consider the intersection point (x, y) furthest to the right in $\mathcal{N}$. If there is more than one intersection point in this column, let (x, y) be the one that is highest. In Figure 4.3, this is the point $(13, 2)$. We now “swap tails”

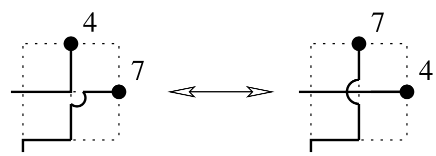

Figure 4.4: "Swapping" the tails in Figure 4.3.

at (x, y). Specifically, if the paths cross each other at (x, y), we swap the tails so that they just meet, and vice versa in the other situation. Figure 4.4 is a graphical visualization of this "swapping" process in the case of our running example Figure 4.3.

Doing this, we get a new intersecting nest $\mathcal{N}'$ that differs from $\mathcal{N}$ only at this "swapping point", (x, y). Let $\sigma' \in S_r$ denote the permutation corresponding to $\mathcal{N}'$. By our choice of intersection point (x, y), it is clear that σ' only differs from σ by a single adjacent swap of entries in σ. For instance, in our Figure 4.3 example, we have $\sigma = 3512476$ and $\sigma' = 3512746$. In general we will have $\mathcal{I}(\sigma') = \mathcal{I}(\sigma) \pm 1$, and so this pair of intersecting nests cancel each other out in the sum (4.4). Therefore, what we claimed in (4.4) (and hence (4.3) also) is proved.

Step 2. Next, we claim that formula (4.3) implies

$$P\left(\mathcal{E}(m_1, \ldots, m_r)\right) = \left(\det\left[\binom{n/2}{m_i - i + j}\right]_{i,j=1}^{r}\right)^4 \cdot \frac{\prod_{i=1}^{r}\left[(m_i!)^2 (n/2 - m_i)!^2\right]}{(n!)^r}. \quad (4.5)$$

As a first step to the proof of (4.5), we notice that we have shown something more general regarding the count $N_{n/2}(m_1, \ldots, m_r)$. Consider the following ballot-counting problem: suppose we have r canditates, $C_1, \ldots, C_r$, running for election, receiving a total of $\mu_1 \geq \cdots \geq \mu_r$ votes respectively. Suppose we count the votes in a rather peculiar way: we have a total of ν ballot boxes arranged in a row. Each box is allowed to have at most one vote for each candidate, with no further restrictions. In particular, a given box could possibly be empty, and may have at most r ballots in it, one cast for each candidate. We open the ballot boxes one at a time, keeping track of the cumulative total votes cast for each candidate at every intermediate point. We wish to know the total number of allocations of ballots in boxes so that at each of these intermediate points, we have C_1 with at least as many votes as C_2, who in turn has at least as many votes as C_3, and so on. By our very derivation above, this count is given by $N_\nu(\mu_1, \ldots, \mu_r)$. Namely, we have proved:

Lemma 4.3.2. *For the ballot-counting problem above, we have*

$$N_\nu(\mu_1, \ldots, \mu_r) = \det \left[\binom{\nu}{\mu_i - i + j} \right]_{i,j=1}^{r}. \qquad \square$$

What's more, we claim that

$$N_\nu(\nu - \mu_r, \ldots, \nu - \mu_1) = N_\nu(\mu_1, \ldots, \mu_r). \tag{4.6}$$

Indeed, by Lemma 4.3.2, the left-hand side is given by

$$N_\nu(\nu-\mu_r,\ldots,\nu-\mu_1) = \det\left[\binom{\nu}{\nu-\mu_{r-i+1}-i+j}\right]_{i,j=1}^{r}$$
$$= \det\left[\binom{\nu}{\mu_{r-i+1}+i-j}\right]_{i,j=1}^{r}.$$

We now switch row i with row $r-i+1$, and column j with column $r-j+1$, $i,j \in [r]$. This has no effect on the determinant. Hence

$$N_\nu(\nu-\mu_r,\ldots,\nu-\mu_1) = \det\left[\binom{\nu}{\mu_i-i+j}\right]_{i,j=1}^{r} = N_\nu(\mu_1,\ldots,\mu_r),$$

and formula (4.6) is proved.

We now prove (4.5). First of all, we have already seen that the number of allowable supporting-row selections in the southwest subsquare, subject to the restrictions defining $\mathcal{E}_{sw}$ is given by the count in (4.3). A second factor (4.3) comes from choosing supporting-rows subject to the restrictions defining $\mathcal{E}_{ne}$ in the northeast subsquare. By considering supporting-*column* selections in the northwest subsquare, subject to $\mathcal{E}_{nw}$, Lemma 4.3.2 together with equation (4.6) tell us that the total number of allowable supporting-column selections equals

$$N_{n/2}(n/2-m_r,\ldots,n/2-m_1) = N_{n/2}(m_1,\ldots,m_r) = \det\left[\binom{n/2}{m_i-i+j}\right]_{i,j=1}^{r}$$

also, thus giving a third factor. Analogously, a fourth factor comes from considering supporting-column selections in the southeast subsquare, subject to the restrictions defining $\mathcal{E}_{se}$. So by multiplying these four factors (4.3) we obtain the total number of row and column selections on the event $\mathcal{E}(m_1, \ldots, m_r)$ subject to all four restrictions defining $\mathcal{E}$!

Once such a row-column selection has been made, we have determined which rows and columns support the four submatrices of $M(\pi_i)$, $i \in [r]$. Consider, for instance, the southwest corner of $M(\pi_1)$. We have selected m_1 rows (from the first $n/2$ rows) supporting $M_{sw}(\pi_1)$, and we have selected $n/2 - m_1$ columns (from the first $n/2$ columns) supporting $M_{nw}(\pi_1)$. Then it is the remaining $n/2 - (n/2 - m_1) = m_1$ columns that support $M_{sw}(\pi_1)$. The number of ways to match these m_1 rows and m_1 columns, thus to determine $M_{sw}(\pi_1)$ completely, is $m_1!$. The northeast corner contributes another $m_1!$, while each of the two other corners contributes $(n/2 - m_1)!$, whence the overall matching factor is $(m_1!)^2(n/2 - m_1)!^2$. In general, the matching factor for π_i is $(m_i!)^2(n/2 - m_i)!^2$, $i \in [r]$. Multiplying the number of admissible row-column selections by the resulting total matching factor $\prod_{i=1}^{r}[(m_i!)^2(n/2 - m_i)!^2]$ and dividing by $(n!)^r$, we obtain the formula (4.5).

STEP 3. As a final step in the proof of Theorem 4.3.1, we show that

$$\det\left[\binom{n/2}{m_i - i + j}\right]_{i,j=1}^{r} = \binom{n/2}{m_1} \cdots \binom{n/2}{m_r} \prod_{1 \le i < j \le r} \frac{(m_i - m_j + j - i)(n/2 + j - i)}{(m_i + j - i)(n/2 - m_j + j - i)}. \tag{4.7}$$

By putting (4.7) into equation (4.5), we leave it to the interested reader to verify that we get the formula stated in the theorem.

First of all, we note that, for $j > i$,

$$
\begin{aligned}
\binom{n/2}{m_i-i+j} &= \frac{(n/2)!}{(m_i+j-i)\cdots(m_i+1)m_i!(n/2-m_i+i-j)!} \\
&= \frac{(n/2-m_i)(n/2-m_i-1)\cdots(n/2-m_i+i+1-j)}{(m_i+j-i)(m_i+j-i-1)\cdots(m_i+1)}\binom{n/2}{m_i} \\
&= \frac{\binom{n/2}{m_i}}{(m_i+r-i)\cdots(m_i+1)(n/2-m_i+i-1)\cdots(n/2-m_i+1)} \\
&\quad \times (n/2-m_i+i-1)(n/2-m_i+i-2)\cdots(n/2-m_i+i+1-j) \\
&\qquad \times (m_i+j-i+1)(m_i+j-i+2)\cdots(m_i+r-i) \\
&= \frac{\binom{n/2}{m_i}}{(m_i+r-i)\cdots(m_i+1)(n/2-m_i+i-1)\cdots(n/2-m_i+1)} \\
&\quad \times [-(x_i+b_2)][-(x_i+b_3)]\cdots[-(x_i+b_j)] \\
&\qquad \times (x_i+a_{j+1})(x_i+a_{j+2})\cdots(x_i+a_r),
\end{aligned} \tag{4.8}
$$

where $x_s := m_s - (s-1)$, $1 \le s \le r$, $a_t := t-1$ and $b_t := -n/2+t-2$, $2 \le t \le r$.

Similarly, for $j < i$,

$$\begin{aligned}
\binom{n/2}{m_i-i+j} &= \frac{(n/2)!}{(m_i+j-i)!(n/2-m_i+i-j)\cdots(n/2-m_i+1)(n/2-m_i)!}\\
&= \frac{m_i(m_i-1)\cdots(m_i+j-i+1)}{(n/2-m_i+i-j)(n/2-m_i+i-j-1)\cdots(n/2-m_i+1)}\binom{n/2}{m_i}\\
&= \frac{\binom{n/2}{m_i}}{(m_i+r-i)\cdots(m_i+1)(n/2-m_i+i-1)\cdots(n/2-m_i+1)}\\
&\quad\times (n/2-m_i+i-1)(n/2-m_i+i-2)\cdots(n/2-m_i+i+1-j)\\
&\quad\quad\times (m_i+j-i+1)(m_i+j-i+2)\cdots(m_i+r-i)\\
&= \frac{\binom{n/2}{m_i}}{(m_i+r-i)\cdots(m_i+1)(n/2-m_i+i-1)\cdots(n/2-m_i+1)}\\
&\quad\times [-(x_i+b_2)][-(x_i+b_3)]\cdots[-(x_i+b_j)]\\
&\quad\quad\times (x_i+a_{j+1})(x_i+a_{j+2})\cdots(x_i+a_r),
\end{aligned}\tag{4.9}$$

Obviously, for $j=i$ the identities (4.8) and (4.9) hold also. Hence, we obtain

$$\begin{aligned}
&\det\left[\binom{n/2}{m_i-i+j}\right]_{i,j=1}^{r}\\
&\quad= \prod_{i=1}^{r}\frac{\binom{n/2}{m_i}}{(m_i+r-i)\cdots(m_i+1)(n/2-m_i+i-1)\cdots(n/2-m_i+1)}\\
&\qquad\times \det\Big[[-(x_i+b_2)]\cdots[-(x_i+b_j)](x_i+a_{j+1})\cdots(x_i+a_r)\Big]_{i,j=1}^{r}\\
&\quad= \binom{n/2}{m_1}\cdots\binom{n/2}{m_r}\prod_{1\le i<j\le r}\frac{1}{(m_i+j-i)(n/2-m_j+j-i)}\\
&\qquad\times \det\Big[[-(x_i+b_2)]\cdots[-(x_i+b_j)](x_i+a_{j+1})\cdots(x_i+a_r)\Big]_{i,j=1}^{r},
\end{aligned}\tag{4.10}$$

so our task is reduced to computing the last determinant in (4.10). For this, we apply the following result of Krattenthaler [35], which extends the Vandermonde determinant:

Theorem 4.3.3. *(Krattenthaler's formula) Given arbitrary values for* $x_1, \ldots, x_r$, $a_2, \ldots, a_r$, *and* $b_2, \ldots, b_r$, *we have*

$$\det \left[(x_i + b_2) \cdots (x_i + b_j)(x_i + a_{j+1}) \cdots (x_i + a_r) \right]_{i,j=1}^{r}$$
$$= \prod_{1 \le i < j \le r} (x_i - x_j) \prod_{2 \le i \le j \le r} (b_i - a_j). \qquad \square$$

In order to use this result, we must factor $(-1)^{j-1}$ out of column j, $1 \le j \le r$, in the last determinant in (4.10). Doing this, we obtain

$$\begin{aligned}
&\det \left[[-(x_i + b_2)] \cdots [-(x_i + b_j)](x_i + a_{j+1}) \cdots (x_i + a_r) \right]_{i,j=1}^{r} \\
&\qquad = (-1)^{\binom{r}{2}} \det \left[(x_i + b_2) \cdots (x_i + b_j)(x_i + a_{j+1}) \cdots (x_i + a_r) \right]_{i,j=1}^{r} \\
&\qquad = (-1)^{\binom{r}{2}} \prod_{1 \le i < j \le r} (x_i - x_j) \prod_{2 \le i \le j \le r} (b_i - a_j) \\
&\qquad = \prod_{1 \le i < j \le r} (x_i - x_j) \prod_{2 \le i \le j \le r} (a_j - b_i), \qquad (4.11)
\end{aligned}$$

where the second to last equality follows from Krattenthaler's formula. By our definition of x_s, a_t and b_t, (4.11) implies

$$\det\left[[-(x_i+b_2)]\cdots[-(x_i+b_j)](x_i+a_{j+1})\cdots(x_i+a_r)\right]_{i,j=1}^{r}$$
$$= \prod_{1\le i<j\le r}(x_i-x_j)\prod_{2\le i\le j\le r}(a_j-b_i)$$
$$= \prod_{1\le i<j\le r}(m_i-m_j+j-i)\prod_{2\le i\le j\le r}(n/2+j-i+1)$$
$$= \prod_{1\le i<j\le r}\Big[(m_i-m_j+j-i)(n/2+j-i)\Big]. \tag{4.12}$$

Combining (4.10) with (4.12), formula (4.7) is proved and hence so is Theorem 4.3.1. □

4.4 A Probabilistic Simplification

Armed with Theorem 4.3.1, and with the combinatorial part behind us, the rest is relatively straightforward. First, we claim that

$$P(\mathcal{E}) \le E\left[\prod_{1\le i<j\le r}\frac{(M_i-M_j+j-i)^4(n/2+j-i)^4}{(M_i+j-i)^4(n/2-M_j+j-i)^4}\right], \tag{4.13}$$

where, to repeat, $M_i := M_i(n/2, n/2)$, the total number of rows that support $M_{sw}(\pi_j)$. It should be clear that the M_i are independent, with $M_i \overset{\mathcal{D}}{=} M$, a hypergeometric random variables with parameters $n/2, n/2, n/2$. That is, the M_i are indepedent copies of M, which in turn equals the number of red balls in a uniformly random sample of size $n/2$ from an urn containing a total of n balls, $n/2$ of them red and $n/2$ white. In particular

$$P\left(M_i = m_i,\, i \in [r]\right) = \prod_{i=1}^{r} P\left(M_i = m_i\right) = \prod_{i=1}^{r} \frac{\binom{n/2}{m_i}\binom{n/2}{n/2-m_i}}{\binom{n}{n/2}}. \tag{4.14}$$

Now (4.13) follows easily from (4.2), Theorem 4.3.1 and (4.14):

$$\begin{aligned}
P(\mathcal{E}) &= \sum_{m_1 \geq \cdots \geq m_r} \prod_{1 \leq i < j \leq r} \frac{\left(m_i - m_j + j - i\right)^4 \left(n/2 + j - i\right)^4}{\left(m_i + j - i\right)^4 \left(n/2 - m_j + j - i\right)^4} \cdot \prod_{i=1}^{r} \frac{\binom{n/2}{m_i}\binom{n/2}{n/2-m_i}}{\binom{n}{n/2}} \\
&\leq \sum_{m_1, \ldots, m_r} \prod_{1 \leq i < j \leq r} \frac{\left(m_i - m_j + j - i\right)^4 \left(n/2 + j - i\right)^4}{\left(m_i + j - i\right)^4 \left(n/2 - m_j + j - i\right)^4} \cdot P\left(M_i = m_i,\, i \in [r]\right) \\
&= E\left[\prod_{1 \leq i < j \leq r} \frac{(M_i - M_j + j - i)^4 (n/2 + j - i)^4}{(M_i + j - i)^4 (n/2 - M_j + j - i)^4}\right].
\end{aligned}$$

Of course, this runs parallel to what we did for permutation-pairs.

4.5 Asymptotics

Next, as we did in the case $r = 2$ also, we finish the argument by using known properties of the random variables M_i. Namely, it remains to prove that this expectation is $O\left(n^{-r(r-1)}\right)$. To avoid unnecessarily repeating things we did for the case $r = 2$, suffice it to say that the M_i's are close to their expectation, $n/4$, with exponentially high probability (see Janson et al. [31, p. 29]). In particular, there is an absolute constant $c > 0$ such that

$$E\left[\prod_{1\le i<j\le r}\frac{(M_i-M_j+j-i)^4(n/2+j-i)^4}{(M_i+j-i)^4(n/2-M_j+j-i)^4}\right]$$
$$=O\left(E\left[\prod_{1\le i<j\le r}\frac{(M_i-M_j+j-i)^4(n/2+j-i)^4}{(n/4+j-i)^4(n/2-n/4+j-i)^4}\right]+e^{-cn^{1/3}}\right)$$
$$=O\left(n^{-4\binom{r}{2}}E\left[\prod_{1\le i<j\le r}(M_i-M_j+j-i)^4\right]+e^{-cn^{1/3}}\right),$$

so we will be done if we can prove that

$$E\left[\prod_{1\le i<j\le r}(M_i-M_j+j-i)^4\right]=O\left(n^{2\binom{r}{2}}\right). \tag{4.15}$$

This will not be difficult, given our careful approach to the similar problem for permutation-pairs. As we did there, introduce $\mu_i = M_i - E[M_i]$, $i \in [r]$. Then

$$\begin{aligned}\prod_{1\le i<j\le r}(M_i-M_j+j-i)^4 &= \prod_{1\le i<j\le r}(\mu_i-\mu_j+j-i)^4\\ &\le \prod_{1\le i<j\le r}\left[27\left(\mu_i^4+\mu_j^4+(j-i)^4\right)\right]\\ &\le 27^{\binom{r}{2}}\prod_{1\le i<j\le r}\left(\mu_i^4+\mu_j^4+r^4\right)\\ &= 27^{\binom{r}{2}}\sum r^{4e_0}\mu_1^{4e_1}\cdots\mu_r^{4e_r},\end{aligned} \tag{4.16}$$

where the first inequality follows from

$$(a+b+c)^2 \le 3(a^2+b^2+c^2).$$

Here, the sum ranges over some set of exponents $e_0, e_1, \ldots, e_r \in \{0, 1, \ldots, \binom{r}{2}\}$ with $e_0 + \cdots + e_r = \binom{r}{2}$. Removing the dependencies among these exponents implied by the product range only increases this sum. Therefore, from (4.16) follows

$$\begin{aligned}\prod_{1 \le i < j \le r} (M_i - M_j + j - i)^4 &\le 27^{\binom{r}{2}} \sum r^{4e_0} \mu_1^{4e_1} \cdots \mu_r^{4e_r} \\ &\le 27^{\binom{r}{2}} \times \sum_{\substack{e_0, \ldots, e_r \ge 0 \\ e_0 + \cdots + e_r = \binom{r}{2}}} r^{4e_0} \mu_1^{4e_1} \cdots \mu_r^{4e_r} \\ &\le \left(27 r^4\right)^{\binom{r}{2}} \times \sum_{\substack{e_1, \ldots, e_r \ge 0 \\ e_1 + \cdots + e_r \le \binom{r}{2}}} \mu_1^{4e_1} \cdots \mu_r^{4e_r}.\end{aligned}$$

Hence, as the M_i (hence the μ_i) are independent,

$$E\left[\prod_{1 \le i < j \le r} (M_i - M_j + j - i)^4\right] = O\left(\sum_{\substack{e_1, \ldots, e_r \ge 0 \\ e_1 + \cdots + e_r \le \binom{r}{2}}} E\left[\mu_1^{4e_1}\right] \cdots E\left[\mu_r^{4e_r}\right]\right).$$

So, since the total number of terms in this sum is $\binom{r + \binom{r}{2}}{r}$, (4.15) will be proved if we demonstrate that

$$E\left[\mu_1^{4e_1}\right] \cdots E\left[\mu_r^{4e_r}\right] = O\left(n^{2\binom{r}{2}}\right) \tag{4.17}$$

for some fixed one of these r-tuples $(e_1, \ldots, e_r)$. To this end, notice first that $E[\mu_i^2]$ is of order n exactly. Indeed, recall that

$$\begin{aligned} E[M_i(M_i-1)] &= \frac{n}{2}\left(\frac{n}{2}-1\right)\frac{\binom{n-2}{n/2-2}}{\binom{n}{n/2}} \\ &= \frac{n(n-2)^2}{16(n-1)}. \end{aligned}$$

Therefore

$$\begin{aligned} E\left[\mu_i^2\right] &= \mathrm{Var}[M_i] \\ &= E[M_i(M_i-1)] + E[M_i] - E^2[M_i] \\ &= \frac{n(n-2)^2}{16(n-1)} + \frac{n}{4} - \frac{n^2}{16} \qquad (4.18) \\ &= \frac{n}{16} + O(1). \end{aligned}$$

Furthermore, as a special instance of the hypergeometrically distributed random variable, M_i has the same distribution as the sum of $n/2$ independent Bernoulli variables $Y_j \in \{0,1\}$ (see Vatutin and Mikhailov [38], alternatively [31, p. 30]). Therefore, (4.18) and the Lindeberg-Feller Central Limit Theorem imply

$$\frac{\mu_i}{\sqrt{n/16}} \Longrightarrow \mathcal{N}(0,1), \qquad (4.19)$$

where $\mathcal{N}(0,1)$ is the standard normal random variable. In fact, since

$$\frac{Y_j - E[Y_j]}{\sqrt{n/16}} \to 0, \quad n \to \infty,$$

we can say more. Indeed, we have (4.19) together with convergence of all the moments (see Billingsley [4, p. 391]). Therefore, in particular

$$\frac{E\left[\mu_i^{4e_i}\right]}{\left(\sqrt{n/16}\right)^{4e_i}} \to E\left[\mathcal{N}(0,1)^{4e_i}\right], \quad n \to \infty,$$

i.e.

$$E[\mu_1^{4e_1}] \cdots E[\mu_r^{4e_r}] = O\left(n^{2(e_1+\cdots+e_r)}\right) = O\left(n^{2\binom{r}{2}}\right)$$

as $e_1 + \cdots + e_r \leq \binom{r}{2}$. This completes the proof of (4.17), and thus of Theorem 1.2.2.

□

CHAPTER 5

SOME PROPERTIES OF THE WEAK ORDERING

We now move away from the Bruhat order to focus on its more restrictive counterpart, namely the *weak ordering.* In anticipation of the proof of Theorem 1.4.1, this chapter is devoted to various properties of this order.

5.1 A Criterion for Weak Comparability

Recall that π precedes σ in the *weak order* ($\pi \preceq \sigma$) if and only if there is a chain $\sigma = \omega_1 \to \cdots \to \omega_s = \pi$ where each ω_t is a simple reduction of ω_{t-1}, i.e. obtained by transposing two adjacent elements $\omega_{t-1}(i)$, $\omega_{t-1}(i+1)$ such that $\omega_{t-1}(i) > \omega_{t-1}(i+1)$. Clearly the weak order is more restrictive than the Bruhat order, so that $\pi \preceq \sigma$ impies $\pi \leq \sigma$. In particular, $P(\pi \preceq \sigma) \leq P(\pi \leq \sigma)$, hence (Theorem 1.2.1) $P(\pi \preceq \sigma) = O(n^{-2})$. We will show that, in fact, this probability is exponentially small. The proof is based on an inversion set criterion for $\pi \preceq \sigma$ implicit in [3, pp. 135-139].

Lemma 5.1.1. *Given $\omega \in S_n$, recall the set of non-inversions of ω:*

$$E(\omega) := \left\{ (i,j) \ : \ i < j, \, \omega^{-1}(i) < \omega^{-1}(j) \right\}.$$

$\pi \preceq \sigma$ if and only if $E(\pi) \supseteq E(\sigma)$.

Proof. Assume $\pi \preceq \sigma$. Then there exists a chain of simple reductions ω_t, $1 \le t \le s$, connecting $\sigma = \omega_1$ and $\pi = \omega_s$. By the definition of a simple reduction, for each $t > 1$ there is $i = i_t < n$ such that $E(\omega_t) = E(\omega_{t-1}) \cup \{(\omega_t(i), \omega_t(i+1))\}$, where $\omega_t(i) = \omega_{t-1}(i+1)$, $\omega_t(i+1) = \omega_{t-1}(i)$, and $\omega_{t-1}(i) > \omega_{t-1}(i+1)$. So the set $E(\omega_t)$ increases with t, hence $E(\pi) \supseteq E(\sigma)$.

Conversely, suppose $E(\pi) \supseteq E(\sigma)$. Since a permutation ω is uniquely determined by its $E(\omega)$, we may assume $E(\pi) \supsetneq E(\sigma)$.

Claim If $E(\pi) \supsetneq E(\sigma)$, then there exists $u < v \le n$ such that (v, u) is an adjacent inversion of σ, but $(u, v) \in E(\pi)$.

Assuming validity of the claim, we ascertain existence of an adjacent inversion (v, u) in σ with $(u, v) \in E(\pi)$. Interchanging the adjacent elements u and v in $\sigma = \omega_1$, we obtain a simple reduction ω_2, with $E(\omega_1) \subset E(\omega_2) \subseteq E(\pi)$. If $E(\omega_2) = E(\pi)$ then $\omega_2 = \pi$, and we stop. Otherwise we determine ω_3, a simple reduction of ω_2, with $E(\omega_2) \subset E(\omega_3) \subseteq E(\pi)$ and so on. Eventually we determine a chain of simple reductions connecting σ and π, which proves that $\pi \preceq \sigma$.

Proof of Claim. The claim is obvious for $n = 1, 2$. Assume inductively that the claim holds for permutations of length $n - 1 \ge 2$. Let $\pi, \sigma \in S_n$ and $E(\pi) \supsetneq E(\sigma)$. As in the proof of Theorem 1.2.1, let $\ell_n(\pi) = \pi^{-1}(n)$, $\ell_n(\sigma) = \sigma^{-1}(n)$, and π^*, σ^* are obtained by deletion of n from π and σ. Since $E(\pi) \supseteq E(\sigma)$, we have $E(\pi^*) \supseteq E(\sigma^*)$. Suppose first that $E(\pi^*) = E(\sigma^*)$. Then $\pi^* = \sigma^*$, and as $E(\pi) \supsetneq E(\sigma)$, we must have $\ell_n(\pi) > \ell_n(\sigma)$, i.e. $\ell_n(\sigma) < n$. Setting $v = n$ and $u = \sigma(\ell_n(\sigma) + 1)$, we obtain an adjacent inversion (v, u) in σ with $(u, v) \in E(\pi)$.

Alternatively, $E(\pi^*) \supsetneq E(\sigma^*)$. By inductive hypothesis, there exists $u < v \leq n-1$ such that (v,u) is an adjacent inversion of σ^*, but $(u,v) \in E(\pi^*)$. Now insert n back into π^*, σ^*, recovering π and σ. If n sits to the right of u or to the left of v in σ, then (v,u) is still an adjacent inversion of σ. Otherwise n is sandwiched between v on the left and u on the right. Therefore (n,u) is an adjacent inversion in σ. On the other hand $(v,n) \in E(\sigma)$, so since $E(\pi) \supseteq E(\sigma)$, we have $(v,n) \in E(\pi)$ also. Hence, the triple (u,v,n) are in exactly this order (not necessarily adjacent) in π. Therefore the adjacent inversion (n,u) in σ is such that $(u,n) \in E(\pi)$, and this proves the inductive step. □

Denote by $\bar{\omega}$ the permutation ω reversed in rank. For example, with $\omega = 13254$ we have $\bar{\omega} = 53412$. Then it is easy to see that

$$E(\pi) \supseteq E(\sigma) \Longleftrightarrow E(\bar{\pi}) \subseteq E(\bar{\sigma}).$$

By Lemma 5.1.1, these statements are equivalent to

$$\pi \preceq \sigma \Longleftrightarrow \bar{\sigma} \preceq \bar{\pi}.$$

We immediately obtain the following corollary to Lemma 5.1.1:

Corollary 5.1.2. *For $\omega \in S_n$, define*

$$E_i(\omega) := \{j < i \,:\, (j,i) \in E(\omega)\}, \quad 1 \leq i \leq n.$$

Then

$$E(\omega) = \bigsqcup_{i=1}^{n} \{(j,i) \,:\, j \in E_i(\omega)\},$$

and consequently

$$\pi \preceq \sigma \Longleftrightarrow E(\pi) \supseteq E(\sigma) \Longleftrightarrow E_i(\pi) \supseteq E_i(\sigma), \quad \forall i \leq n. \qquad \square$$

5.2 Submultiplicativity of P_n^*

Next, we establish one of the claims of Theorem 1.4.1, namely that $P_n^* := P(\pi \preceq \sigma)$ is submultiplicative. Of course ([43, p. 23, ex. 98] again) this implies that there exists $\lim \sqrt[n]{P_n^*}$.

Lemma 5.2.1. *Let $\pi, \sigma \in S_n$ be selected independently and uniformly at random. As a function of n, P_n^* is submultiplicative, i.e. for all $n_1, n_2 \geq 1$*

$$P_{n_1+n_2}^* \leq P_{n_1}^* P_{n_2}^*.$$

Consequently there exists $\lim_{n\to\infty} \sqrt[n]{P_n^*} = \inf_{n\geq 1} \sqrt[n]{P_n^*}$.

Proof. Let π, σ be two permutations of $[n_1 + n_2]$. Then $\pi \preceq \sigma$ if and only if

$$E_i(\pi) \supseteq E_i(\sigma), \quad 1 \leq i \leq n_1 + n_2.$$

Using these conditions for $i \leq n_1$, we see that

$$\pi[1, 2, \ldots, n_1] \preceq \sigma[1, 2, \ldots, n_1].$$

Here $\pi[1,2,\ldots,n_1]$, say, is what is left of the permutation π when the elements $n_1+1,\ldots,n_1+n_2$ are deleted.

Likewise, $\pi \preceq \sigma$ if and only if

$$E_i(\bar{\pi}) \subseteq E_i(\bar{\sigma}), \quad 1 \le i \le n_1+n_2.$$

Using these conditions for $i \le n_2$, we see that

$$\pi[n_1+1,\ldots,n_1+n_2] \preceq \sigma[n_1+1,\ldots,n_1+n_2].$$

Now, since π and σ are uniformly random and mutually independent, so are the four permutations

$$\pi[1,\ldots,n_1], \quad \pi[n_1+1,\ldots,n_1+n_2], \quad \sigma[1,\ldots,n_1], \quad \sigma[n_1+1,\ldots,n_1+n_2].$$

Hence,

$$P(\pi \preceq \sigma) \le P(\pi[1,\ldots,n_1] \preceq \sigma[1,\ldots,n_1]) \times P(\pi[n_1+1,\ldots,n_1+n_2] \preceq \sigma[n_1+1,\ldots,n_1+n_2]),$$

so that

$$P^*_{n_1+n_2} \le P^*_{n_1} P^*_{n_2}. \qquad \square$$

CHAPTER 6

THE PROOF OF THE WEAK ORDER UPPER BOUND

We now present the proof of Theorem 1.4.1, upper bound. We will prove something better than what was stated there, showing that for each $\epsilon > 0$,

$$P_n^* = O((\beta + \epsilon)^n),$$

where

$$\beta = \sqrt[6]{\frac{1065317}{12!}} = 0.36129\ldots.$$

6.1 A Necessary Condition for Weak Comparability

The proof of this upper bound for P_n^* parallels the proof of the lower bound for P_n in Theorem 1.2.1. As in that proof, given $k \geq 1$, let π^{k*} and σ^{k*} be obtained by deletion of the elements $n, \ldots, n-k+1$ from π and σ, and let $\ell_i(\pi) = \pi^{-1}(i)$, $\ell_i(\sigma) = \sigma^{-1}(i)$, $n-k+1 \leq i \leq n$. In the notations of the proof of Lemma 5.2.1, $\pi^{k*} = \pi[1, \ldots, n-k]$ and $\sigma^{k*} = \sigma[1, \ldots, n-k]$, and we saw that $\pi^{k*} \preceq \sigma^{k*}$ if $\pi \preceq \sigma$. Our task is to find the conditions these $\ell_i(\cdot)$'s must satisfy if $\pi \preceq \sigma$ holds.

To start, notice that

$$\pi \preceq \sigma \Longrightarrow |E_n(\pi)| \geq |E_n(\sigma)| \Longleftrightarrow \ell_n(\pi) \geq \ell_n(\sigma).$$

Next

$$\pi \preceq \sigma \Longrightarrow \pi^* \preceq \sigma^* \Longrightarrow \ell_{n-1}(\pi) \geq \ell_{n-1}(\pi^*) \geq \ell_{n-1}(\sigma^*) \geq \ell_{n-1}(\sigma) - 1,$$

as deletion of n from π, σ decreases the location of $n-1$ in each permutation by at most one. In general, for $0 < j < k$ we get

$$\pi \preceq \sigma \Longrightarrow \pi^{j*} \preceq \sigma^{j*} \Longrightarrow \ell_{n-j}(\pi) \geq \ell_{n-j}(\sigma) - j.$$

So, introducing $\boldsymbol{\ell}(\pi) = \{\ell_{n-i+1}(\pi)\}_{1 \leq i \leq k}$ and $\boldsymbol{\ell}(\sigma) = \{\ell_{n-i+1}(\sigma)\}_{1 \leq i \leq k}$,

$$\{\pi \preceq \sigma\} \subseteq \{(\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma) \in \mathcal{S}_k\}, \tag{6.1}$$
$$\mathcal{S}_k := \left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2k} : x_j \geq y_j - j + 1,\ 1 \leq j \leq k\right\}.$$

In addition, on $\{\pi \preceq \sigma\}$ every pair of elements, which forms an inversion in π, also forms an inversion in σ. Applying this to the elements $n-k+1, \ldots, n$, we have then

$$\{\pi \preceq \sigma\} \subseteq \{(\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{T}_k\}, \tag{6.2}$$
$$\mathcal{T}_k := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2k} : \forall 1 \leq i < j \leq k\,, x_i < x_j \Longrightarrow y_i < y_j\}.$$

Combining (6.1) and (6.2), we get

$$\{\pi \preceq \sigma\} \subseteq \{(\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{S}_k \cap \mathcal{T}_k\} \cap \left\{\pi^{k*} \preceq \sigma^{k*}\right\}.$$

So, since the two events on the right are independent,

$$P_n^* \leq P((\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{S}_k \cap \mathcal{T}_k) P_{n-k}^*. \tag{6.3}$$

6.2 A Reduction to Uniforms

It remains to estimate

$$P((\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{S}_k \cap \mathcal{T}_k).$$

As in the proof of Theorem 1.2.1 (lower bound), we observe that $(\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma))$ has the same distribution as $(\lceil n\mathbf{U}\rceil, \lceil n\mathbf{V}\rceil)$, conditioned on

$$\mathcal{A}_{n,k} \cap \mathcal{B}_{n,k} = \{\lceil nU_1\rceil \neq \cdots \neq \lceil nU_k\rceil\} \cap \{\lceil nV_1\rceil \neq \cdots \neq \lceil nV_k\rceil\}.$$

Here $U_1, \dots, U_k, V_1, \dots, V_k$ are independent $[0,1]$-uniforms. Then

$$P((\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{S}_k \cap \mathcal{T}_k) = \frac{P(\{(\lceil n\mathbf{U}\rceil, \lceil n\mathbf{V}\rceil) \in \mathcal{S}_k \cap \mathcal{T}_k\} \cap \mathcal{C}_{n,k})}{P(\mathcal{C}_{n,k})}, \quad \mathcal{C}_{n,k} = \mathcal{A}_{n,k} \cap \mathcal{B}_{n,k}.$$

Introduce the event $\tilde{\mathcal{D}}_{n,k}$ on which

$$\min\{\min_{i\neq j}|U_i - U_j|, \min_{i\neq j}|V_i - V_j|, \min_{i,j}|U_i - V_j|, k^{-1}\min_j |U_j - V_j|\} > 1/n.$$

Certainly $\tilde{\mathcal{D}}_{n,k} \subseteq \mathcal{C}_{n,k}$ and, thanks to the factor $1/k$ by $\min_j |U_j - V_j|$, on $\tilde{\mathcal{D}}_{n,k}$

$$\lceil nU_j\rceil \geq \lceil nV_j\rceil - j + 1 \Longrightarrow U_j \geq V_j - k/n \Longrightarrow U_j > V_j.$$

Therefore, on $\tilde{\mathcal{D}}_{n,k}$,

$$(\lceil n\mathbf{U}\rceil, \lceil n\mathbf{V}\rceil) \in \mathcal{S}_k \cap \mathcal{T}_k \Longrightarrow (\mathbf{U}, \mathbf{V}) \in \tilde{\mathcal{S}}_k \cap \mathcal{T}_k,$$
$$\tilde{\mathcal{S}}_k := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2k} : x_j > y_j,\, 1 \le j \le k\}.$$

Clearly $\tilde{\mathcal{S}}_k \cap \mathcal{T}_k$ is a cone-shaped subset of $\mathbb{R}^{2k}$. In addition, $P(\tilde{\mathcal{D}}^c_{n,k}) = O(k^2/n)$. Hence

$$P((\boldsymbol{\ell}(\pi), \boldsymbol{\ell}(\sigma)) \in \mathcal{S}_k \cap \mathcal{T}_k) \le \frac{P((\mathbf{U}, \mathbf{V}) \in \tilde{\mathcal{S}}_k \cap \mathcal{T}_k) + O(P(\tilde{\mathcal{D}}^c_{n,k}))}{1 - O(P(\tilde{\mathcal{D}}^c_{n,k}))}$$
$$= Q^*_k(1 + O(k^2/n)), \quad Q^*_k := P((\mathbf{U}, \mathbf{V}) \in \tilde{\mathcal{S}}_k \cap \mathcal{T}_k).$$

This and (6.3) imply

$$P^*_n \le Q^*_k P^*_{n-k} \exp(O(k^2/n)).$$

Hence, as in the proof of Theorem 1.2.1 (lower bound),

$$\limsup \sqrt[n]{P^*_n} \le \sqrt[k]{Q^*_k}, \quad \forall\, k \ge 1,$$

and so

$$P^*_n = O((\sqrt[k]{Q^*_k} + \epsilon)^n), \quad k \ge 1,\, \epsilon > 0. \tag{6.4}$$

Furthermore, from the definition of Q^*_k, it follows directly that Q^*_k is *submultiplicative*, i.e.

$$Q^*_{k_1+k_2} \le Q^*_{k_1} Q^*_{k_2}, \quad k_1, k_2 \ge 1.$$

Therefore ([43, p. 23, ex. 98] again)

$$\lim_{k\to\infty} \sqrt[k]{Q^*_k} = \inf_{k\ge 1} \sqrt[k]{Q^*_k}.$$

So the further we can push tabulation of Q^*_k, the better our exponential upper bound for P^*_n would probably be. ("Probably", because we do not have a proof that $\sqrt[k]{Q^*_k}$ decreases with k.)

6.3 An Algorithm to Minimize the Bound

As in the case of Q_k, $Q^*_k = N^*_k/(2k)!$. Here, by the definition of the sets $\tilde{\mathcal{S}}_k$ and $\mathcal{T}_k$, N^*_k is the total number of ways to order $x_1, \dots, x_k, y_1, \dots, y_k$ so that two conditions are met: (1) for each j, x_j is to the right of y_j; (2) for all $i < j$, if x_i is to the left of x_j then y_i is to the left of y_j.

It is instructive first to evaluate N^*_k by hand for $k = 1, 2$. $N^*_1 = 1$ as there is only one sequence, y_1x_1, meeting the conditions (1), (2). Passing to N^*_2, we must decide how to insert y_2 and x_2 into the sequence y_1x_1 in compliance with conditions (1), (2). First of all, y_2 has to precede x_2. If we insert x_2 at the beginning of y_1x_1, giving $x_2y_1x_1$, then we can only insert y_2 at the beginning of this triple, giving

$$y_2x_2y_1x_1.$$

Alternatively, inserting x_2 in the middle of y_1x_1, we have 2 possibilities for insertion of y_2, and we get two admissible orderings,

$$y_2y_1x_2x_1, \qquad y_1y_2x_2x_1.$$

Finally, insertion of x_2 at the end of y_1x_1 brings the condition (2) into play as we now have x_1 preceding x_2, and so y_1 must precede y_2. Consequently, we get two admissible orderings,

$$y_1y_2x_1x_2, \qquad y_1x_1y_2x_2.$$

Hence $N_2^* = 1 + 2 + 2 = 5$. Easy so far! However, passing to $k = 3$ is considerably more time-consuming than it was for computation of N_3 in the proof of the lower bound in Theorem 1.2.1. There, once we had determined the N_2 admissible orderings, we could afford not to keep track of relative orderings of $x_1, \dots, x_{k-1}$, and of $y_1, \dots, y_{k-1}$, whence the coding by 1's and 0's. All we needed for passing from $k-1$ to k was the list of all binary ballot-sequences of length $2(k-1)$ together with their multiplicities. Here the nature of the conditions (1), (2) does not allow lumping various sequences together, and we have to preserve the information of relative orderings of x's, and relative orderings of y's. This substantial complication seriously inhibits the computer's ability to compute N_k^* for k as large as in the case of N_k.

To get a feeling for how sharply the amount of computation increases for $k = 3$, let us consider one of the $N_2^* = 5$ admissible sequences, namely $y_2x_2y_1x_1$. As above, we write down all possible ways to insert y_3 and x_3 into this sequence so that (1) and (2) hold. Doing this, we produce the 10 sequences:

$$\begin{array}{ll} y_3x_3y_2x_2y_1x_1, & y_3y_2x_3x_2y_1x_1, \\ y_2y_3x_3x_2y_1x_1, & y_2y_3x_2x_3y_1x_1, \\ y_2x_2y_3x_3y_1x_1, & y_2y_3x_2y_1x_3x_1, \\ y_2x_2y_3y_1x_3x_1, & y_2x_2y_1y_3x_3x_1, \\ y_2x_2y_1y_3x_1x_3, & y_2x_2y_1x_1y_3x_3. \end{array}$$

We treat similarly the other four sequences from the $k = 2$ case, eventually arriving at $N_3^* = 55$. We wouldn't even think of computing N_4^* by hand.

Once again the computer programming to the rescue! Table 6.1 was produced by the computer after a substantial running time.

Using (6.4) with the value $k = 6$ from this table, we get for each $\epsilon > 0$

$$P_n^* = \left(\left(\sqrt[6]{Q_6^*} + \epsilon\right)^n\right) = \left((0.361... + \epsilon)^n\right). \qquad \square$$

k	$N_k^* = (2k)!Q_k^*$	$Q_k^* = N_k^*/(2k)!$	$\sqrt[k]{Q_k^*}$
1	1	$0.50000\ldots$	$0.50000\ldots$
2	5	$0.20833\ldots$	$0.45643\ldots$
3	55	$0.07638\ldots$	$0.42430\ldots$
4	1023	$0.02537\ldots$	$0.39910\ldots$
5	28207	$0.00777\ldots$	$0.37854\ldots$
6	1065317	$0.00222\ldots$	$0.36129\ldots$

Table 6.1: Exact computation of N_k^* for smallish k.

CHAPTER 7

THE PROOF OF THE WEAK ORDER LOWER BOUND

We now prove the lower bound stated in Theorem 1.4.1. Despite its sharp qualitative contrast to the upper bound in this same theorem, its proof requires a much deeper combinatorial insight. In particular, we will get as a consequence a lower bound (which is known [46, p. 312, ex. 1]) for the number of linear extensions of an arbitrary poset $\mathcal{P}$ of cardinality n.

7.1 A Formula for $P\left(E_i(\pi) \supseteq E_i(\sigma)\right)$

To bound $P\left(\pi \preceq \sigma\right)$ from below we will use the criterion (Corollary 5.1.2)

$$\pi \preceq \sigma \Longleftrightarrow E_i\left(\pi\right) \supseteq E_i\left(\sigma\right), \quad \forall\, i \leq n.$$

First of all,

Lemma 7.1.1. *Let $i \in [n]$, $B \subseteq [i-1]$ ($[0] = \emptyset$). If $\pi \in S_n$ is chosen uniformly at random, then*

$$P\left(E_i\left(\pi\right) \supseteq B\right) = \frac{1}{|B|+1}.$$

Proof. By the definition of $E_i(\pi)$,

$$\{E_i(\pi) \supseteq B\} = \{\pi^{-1}(j) < \pi^{-1}(i), \forall j \in B\}.$$

It remains to observe that π^{-1} is also uniformly random. $\square$

Lemma 7.1.1 implies the following key statement:

Lemma 7.1.2. *Let $\pi, \sigma \in S_n$ be selected independently and uniformly at random. Then, for $i \in [n]$,*

$$P(E_i(\pi) \supseteq E_i(\sigma)) = H(i)/i, \quad H(i) := \sum_{j=1}^{i} \frac{1}{j}.$$

Proof. By Lemma 7.1.1,

$$\begin{aligned} P(E_i(\pi) \supseteq E_i(\sigma)) &= \sum_{B \subseteq [i-1]} P(E_i(\pi) \supseteq B)\, P(E_i(\sigma) = B) \\ &= \sum_{B \subseteq [i-1]} \frac{P(E_i(\sigma) = B)}{|B| + 1} \\ &= E\left[\frac{1}{|E_i(\sigma)| + 1}\right] \\ &= \sum_{j=0}^{i-1} \frac{1}{i(j+1)} = \frac{H(i)}{i}. \end{aligned}$$ $\square$

Note. In the second to last equality, we have used the fact that $|E_i(\sigma)|$ is distributed uniformly on $\{0, 1, \ldots, i-1\}$. In addition, $|E_1(\sigma)|, \ldots, |E_n(\sigma)|$ are independent, a property we will use later. For completeness, here is a bijective proof of these facts.

By induction, the numbers $|E_i(\sigma)|$, $i \le t$, determine uniquely the relative ordering of elements $1, \ldots, t$ in the permutation σ. Hence the numbers $|E_i(\sigma)|$, $i \in [n]$, determine σ uniquely. Since the range of $|E_i(\sigma)|$ is the set $\{0, \ldots, i-1\}$ of cardinality i, and $|S_n| = n!$, it follows that the numbers $|E_i(\sigma)|$, $i \in [n]$, are uniformly distributed, and independent of each other.

7.2 Positive Correlation of the Events $\{E_i(\pi) \supseteq E_i(\sigma)\}$

Needless to say we are interested in $P(\pi \preceq \sigma) = P\left(\cap_{i=1}^n \{E_i(\pi) \supseteq E_i(\sigma)\}\right)$. Fortunately, the events $\{E_i(\pi) \supseteq E_i(\sigma)\}$ turn out to be positively correlated, and the product of the marginals $P(E_i(\pi) \supseteq E_i(\sigma))$ bounds that probability from below.

Theorem 7.2.1. *Let $\pi, \sigma \in S_n$ be selected independently and uniformly at random. Then*

$$P\left(\pi \preceq \sigma\right) \ge \prod_{i=1}^{n} P\left(E_i\left(\pi\right) \supseteq E_i\left(\sigma\right)\right) = \prod_{i=1}^{n} \frac{H\left(i\right)}{i}.$$

Proof. First notice that, conditioning on σ and using the independence of π and σ,

$$\begin{aligned} &P\big(E_i\left(\pi\right) \supseteq E_i\left(\sigma\right), \forall\, i \le n\big) \\ &\qquad = E\left[\, P\left(E_i\left(\pi\right) \supseteq E_i\left(\sigma\right), \forall\, i \le n \,|\, \sigma\,\right)\right] \\ &\qquad = E\left[\, P\left(E_i\left(\pi\right) \supseteq B_i, \forall\, i \le n\right)|_{B_i = E_i(\sigma)}\right]. \end{aligned}$$

So our task is to bound $P\left(E_i\left(\pi\right) \supseteq B_i, \forall\, i \le n\right)$, where these B_i's inherit the following property from the $E_i\left(\sigma\right)$'s:

$$i \in E_j(\sigma) \text{ and } j \in E_k(\sigma) \Longrightarrow i \in E_k(\sigma).$$

Lemma 7.2.2. *Let $n \geq 1$ be an integer, and let $B_i \subseteq [n]$, $i = 1, \ldots, n$, be such that*

$$i \notin B_i \text{ and } i \in B_j,\ j \in B_k \Longrightarrow i \in B_k, \quad \forall\, i, j, k \in [n].$$

Then, for $\pi \in S_n$ selected uniformly at random,

$$P\left(E_i(\pi) \supseteq B_i,\ \forall\, i \leq n\right) \geq \prod_{i=1}^{n} \frac{1}{|B_i| + 1}.$$

Proof of Lemma 7.2.2. Notice upfront that $\cup_i B_i \neq [n]$. Otherwise there would exist $i_1, \ldots, i_s$ such that $i_t \in B_{i_{t+1}}$, $1 \leq t \leq s$, $(i_{s+1} = i_1)$, and – using repeatedly the property of the sets B_i – we would get that, say, $i_1 \in B_{i_2}$ and $i_2 \in B_{i_1}$, hence $i_2 \in B_{i_2}$; contradiction.

Let $U_1, \ldots, U_n$ be independent uniform-$[0,1]$ random variables. Let a random permutation ω be defined by

$$\omega(i) = k \Longleftrightarrow U_i \text{ is } k^{\text{th}} \text{ smallest amongst } U_1, \ldots, U_n.$$

Clearly ω is distributed uniformly, and then so is $\pi := \omega^{-1}$. With π so defined, we obtain

$$\begin{aligned}\left\{E_i(\pi) \supseteq B_i,\ \forall\, i \leq n\right\} &= \left\{\pi^{-1}(i) > \pi^{-1}(j),\ \forall\, j \in B_i,\ i \leq n\right\} \\ &= \left\{U_i > U_j,\ \forall\, j \in B_i,\ i \leq n\right\}.\end{aligned}$$

Hence, the probability in question equals

$$P\left(U_i > U_j,\, \forall\, j \in B_i,\, i \leq n\right).$$

We write this probability as the n-dimensional integral

$$P\left(U_i > U_j,\, \forall\, j \in B_i,\, i \leq n\right) = \underset{D}{\int \cdots \int} dx_1 \cdots dx_n,$$
$$D = \left\{(x_1, \ldots, x_n) \in [0,1]^n \,:\, x_i > x_j,\, \forall\, j \in B_i,\, i \leq n\right\}.$$

Since $\cup_i B_i \neq [n]$, we can choose an index $k \in [n]$ such that $k \notin B_i$ for all i. Then we may rewrite the integral above as

$$\int_0^1 \left(\underset{D(x_k)}{\int \cdots \int} dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \right) dx_k,$$
$$D\left(x_k\right) = \left\{(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in [0,1]^{n-1} \,:\, x_i > x_j,\, \forall\, j \in B_i,\, i \leq n\right\}.$$

On $D\left(x_k\right)$, the only inequalities involving x_k are of the form $x_k > x_j$, $j \in B_k$. This suggests scaling those x_j by x_k, i.e. introducing new variables $t_j := x_j/x_k$, so that $t_j \in [0,1]$, $j \in B_k$. To keep notation uniform, let us also replace the remaining x_i, $i \notin B_k \cup \{k\}$, with t_i. Let $\mathfrak{D}\left(x_k\right)$ denote the integration region for the new variables t_i, $i \neq k$. Explicitly, the constraints $x_j < x_k$, $j \in B_k$, become $t_j < 1$, $j \in B_k$. Obviously each listed constraint $x_a < x_b$ $(a, b \in B_k)$ is replaced, upon scaling, with $t_a < t_b$. We only rename the other variables, so every constraint $x_a < x_b$ $(a, b \notin B_k)$ similarly becomes $t_a < t_b$. By the property of the sets B_i, there are no inequalities $x_a > x_b$, $a \in B_k$, $b \notin B_k$ (since the presence of this inequality implies $b \in B_a$). The only remaining inequalities are all of the type $x_a < x_b$, $a \in B_k$, $b \notin B_k$. In the new variables, such a constraint becomes $x_k t_a < t_b$, and it is certainly satified if $t_a < t_b$, as $x_k \leq 1$. Hence, $\mathfrak{D}\left(x_k\right) \supseteq D^*$, where

$$D^* := \left\{(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) \in [0,1]^{n-1} \,:\, t_i > t_j,\, \forall\, j \in B_i,\, i \neq k\right\},$$

and D^* does not depend on x_k! Observing that the constraints that determine D^* are those for D with the constraints $x_i < x_k$, $i \in B_k$, removed, we conclude that the innermost integral over $D(x_k)$ is bounded below by

$$x_k^{|B_k|} P(U_i > U_j,\, \forall\, j \in B_i,\, i \neq k).$$

($x_k^{|B_k|}$ is the Jacobian of the linear transformation $\{x_i\}_{i \neq k} \to \{t_i\}_{i \neq k}$.) Integrating with respect to x_k, we arrive at

$$P\left(U_i > U_j,\, \forall\, j \in B_i,\, i \leq n\right) \geq \frac{1}{|B_k| + 1} \cdot P\left(U_i > U_j,\, \forall\, j \in B_i,\, i \neq k\right). \qquad (7.1)$$

By induction on the number of sets B_i, with Lemma 7.1.1 providing basis of induction and (7.1) – the inductive step, we get

$$P\left(U_i > U_j,\, \forall\, j \in B_i,\, i \leq n\right) \geq \prod_{i=1}^{n} \frac{1}{|B_i| + 1}. \qquad \square$$

The rest is short. First, by Lemma 7.2.2,

$$\begin{aligned} P(E_i(\pi) \supseteq E_i(\sigma), \forall\, i \leq n) &= E\left[P(E_i(\pi) \supseteq B_i, \forall\, i \leq n)|_{B_i = E_i(\sigma)} \right] \\ &\geq E\left[\prod_{i=1}^{n} \frac{1}{|E_i\,(\sigma)\,| + 1} \right]. \end{aligned}$$

Since the cardinalities $|E_i\,(\sigma)\,|$ are independent, the last expected value equals

$$\prod_{i=1}^{n} E\left[\frac{1}{|E_i\,(\sigma)\,| + 1} \right] = \prod_{i=1}^{n} \left(\frac{1}{i} \sum_{j=0}^{i-1} \frac{1}{j+1} \right) = \prod_{i=1}^{n} \frac{H\,(i)}{i};$$

for the second to last equality see the proof of Lemma 7.1.2. □

7.3 Linear Extensions of Arbitrary Finite Posets

NOTE. Let $\mathcal{P}$ be a poset on $[n]$, and put $B_i := \left\{ j \in \mathcal{P} \,:\, j < i \text{ in } \mathcal{P} \right\}$. $B_i \cup \{i\}$ is called the *order ideal at* i. By the properties of $\mathcal{P}$, the B_i's satisfy the hypotheses of Lemma 7.2.2, so letting $e\,(\mathcal{P})$ denote the number of linear extensions of $\mathcal{P}$ we get

$$\begin{aligned} P\,(E_i\,(\pi) \supseteq B_i, \forall\, i \leq n) &= \frac{|\,\{\omega \,:\, \omega\,(i) > \omega\,(j)\,, \forall\, j \in B_i,\, i \leq n\}\,|}{n!} \\ &= \frac{e\,(\mathcal{P})}{n!} \geq \prod_{i=1}^{n} \frac{1}{|B_i| + 1}. \end{aligned}$$

Thus we have proved

Corollary 7.3.1. *For a poset $\mathcal{P}$ with n elements,*

$$e\,(\mathcal{P}) \geq n! \Big/ \prod_{i=1}^{n} d\,(i)\,, \quad d\,(i) := |\,\{j \in \mathcal{P} \,:\, j \leq i \;\; in\;\; \mathcal{P}\}\,|. \qquad \square$$

In a very special case of $\mathcal{P}$, whose Hasse diagram is a forest of rooted trees with edges directed away from the roots, this simple bound is actually the value of $e(\mathcal{P})$ ([46, p. 312, ex. 1], [34, sect. 5.1.4, ex. 20], [8]). There exist better bounds for the number of linear extensions in the case of the Boolean lattice (see Brightwell and Tetali [14], Kleitman and Sha [33]), but nothing has been done in the way of improving this bound for $\mathcal{P} = \mathcal{P}(\sigma)$, the permutation-induced poset. Indeed, our proof of the lower bound for P_n^* used only the universal bound of Corollary 7.3.1, and not one specific to this special poset. So this begs the question of whether we might improve the bound in this case, and consequently improve on our lower estimate for P_n^*. We are presently working in this direction.

CHAPTER 8

NUMERICS

We now present some numerical results we have generated in hopes of determining how close our present bounds are to being sharp. These computer simulations were only done for comparability of permutation-pairs.

8.1 Bruhat Order Numerics

From computer-generated data we have collected, it appears that our $O(n^{-2})$ upper bound given in Theorem 1.2.1 correctly predicts the qualitative behavior of $P(\pi \le \sigma)$. The data suggests that $P(\pi \le \sigma)$ is of exact order $n^{-(2+\delta)}$ for some $\delta \in [0.5, 1]$, which begs the question of how to improve on our current bound. Writing $P_n = P(\pi \le \sigma)$, Figure 8.1 is a graph (based on this numerical experimentation) exhibiting convergence to the exponent $-a$ in the asymptotic equation $P_n \sim cn^{-a}$, $c > 0$ a constant, and $-a$ appears to be near -2.5. In table 8.1, we also provide a portion of the accompanying data used to generate this graph.

In this table, R_n represents the number of pairs (π, σ) out of 10^9 randomly-generated pairs such that we had $\pi \le \sigma$. We have also utilized the computer to find the actual probability P_n for $n = 1, 2, \ldots, 9$. Table 8.2 lists these true proportions.

n	R_n	Estimate of $P_n \approx \frac{R_n}{10^9}$	Estimate of $\ln(P_n)/\ln n$
10	61589126	0.0615891...	−1.21049...
30	1892634	0.0018926...	−1.84340...
50	233915	0.0002339...	−2.13714...
70	50468	0.0000504...	−2.32886...
90	14686	0.0000146...	−2.47313...
110	5174	0.0000051...	−2.58949...

Table 8.1: Computer simulation data for P_n.

n	$(n!)^2 P_n$	P_n
1	1	1.00000...
2	3	0.75000...
3	19	0.52777...
4	213	0.36979...
5	3781	0.26256...
6	98407	0.18982...
7	3550919	0.13979...
8	170288585	0.10474...
9	10501351657	0.07974...

Table 8.2: Exact computation of P_n for smallish n.

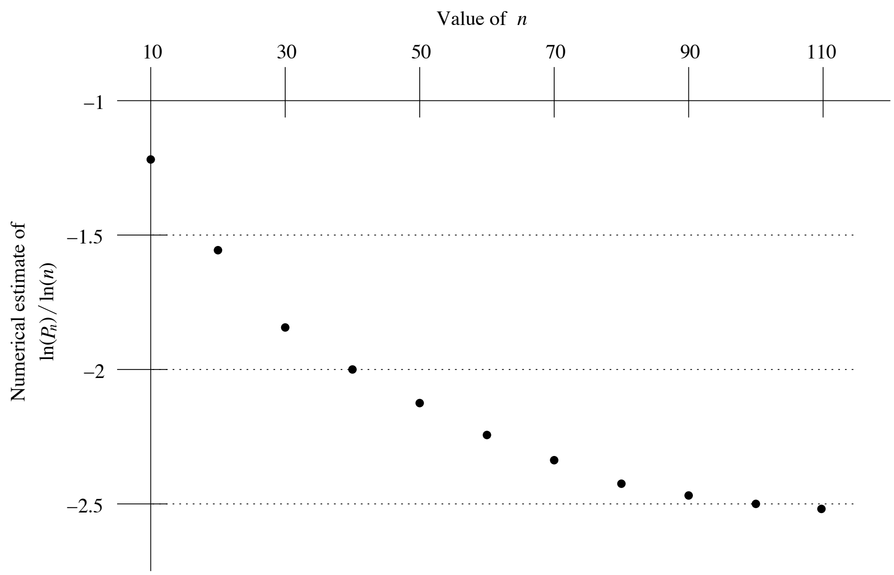

Figure 8.1: Experimental determination of the exponent $-a$ in the asymptotic equation $P_n \sim cn^{-a}$.

8.2 Weak Order Numerics

Concerning the weak order, computer-generated data suggests that $P(\pi \preceq \sigma)$ is of exact order $(0.3)^n$. So our current upper bound $O((0.362)^n)$ is a qualitative match for $P(\pi \preceq \sigma)$, but it appears that improvements are possible here also. Writing $P_n^* = P(\pi \preceq \sigma)$, Figure 8.2 is a graph (based on our numerical experiments) exhibiting convergence to the ratio ρ in the asymptotic equation $P_n^* \sim c\rho^n$, $c > 0$ a constant, and ρ appears to be near 0.3. In table 8.3, we also provide a portion of the accompanying data used to generate this graph.

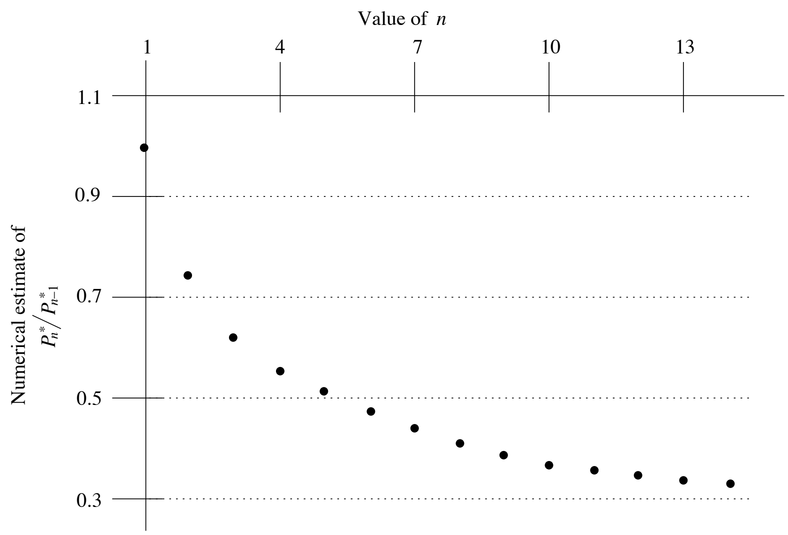

Figure 8.2: Experimental determination of the ratio ρ in the asymptotic equation $P_n^* \sim c\rho^n$.

In this table, R_n^* is defined analogously to R_n above. Table 8.4 lists the true proportions P_n^* for $n = 1, 2, \ldots, 9$.

Surprisingly, our Theorem 1.4.1 lower bound for P_n^* is quite good for these smallish values of n, as is seen in table 8.5.

n	R_n^*	Estimate of $P_n^* \approx \frac{R_n^*}{10^9}$	Estimate of P_n^*/P_{n-1}^*
10	1538639	0.0015386...	0.368718...
11	541488	0.0005414...	0.351926...
12	184273	0.0001842...	0.340308...
13	59917	0.0000599...	0.325153...
14	18721	0.0000187...	0.312448...
15	5714	0.0000057...	0.305218...
16	1724	0.0000017...	0.301715...

Table 8.3: Computer simulation data for P_n^*.

n	$(n!)^2 P_n^*$	P_n^*
1	1	1.00000...
2	3	0.75000...
3	17	0.47222...
4	151	0.26215...
5	1899	0.13187...
6	31711	0.06117...
7	672697	0.02648...
8	17551323	0.01079...
9	549500451	0.00417...

Table 8.4: Exact computation of P_n^* for smallish n.

n	$(n!)^2 \prod_{i=1}^{n} (H(i)/i)$	$\prod_{i=1}^{n} (H(i)/i)$
1	1.0...	1.00000...
2	3.0...	0.75000...
3	16.5...	0.45833...
4	137.5...	0.23871...
5	1569.8...	0.10901...
6	23075.9...	0.04451...
7	418828.3...	0.01648...
8	9106523.1...	0.00560...
9	231858583.9...	0.00176...

Table 8.5: Our theoretical lower bound for P_n^* applied for smallish n.

CHAPTER 9

ON INFS AND SUPS IN THE WEAK ORDER LATTICE

Finally, we focus on the proof of Theorem 1.4.2. Before we prove what was stated there, we have a good deal in the way of preliminaries to take care of. The discussion below is inspired almost exclusively by material contained in the work [3].

9.1 A Connection with Complete, Directed, Acyclic Graphs

Given $\omega \in S_n$, recall the set of non-inversions of ω,

$$E(\omega) := \left\{(i,j) \,:\, i < j, \omega^{-1}(i) < \omega^{-1}(j)\right\},$$

and the set of inversions of ω,

$$E^*(\omega) := \left\{(i,j) \,:\, i > j, \omega^{-1}(i) < \omega^{-1}(j)\right\}.$$

Note that ω is uniquely determined by its $E(\omega)$ (equivalently, by its $E^*(\omega)$). We have seen that, given permutations $\pi, \sigma \in S_n$, we have $\pi \leq \sigma$ in the *weak order* (written $\pi \preceq \sigma$) if and only if $E(\pi) \supseteq E(\sigma)$ (equivalently $E^*(\pi) \subseteq E^*(\sigma)$). It is beneficial to consider the sets $E(\omega)$ and $E^*(\omega)$ as directed edges in a complete, simple, labelled digraph. Namely, we define

$$G(\omega) = ([n], E(\omega) \sqcup E^*(\omega))$$

by joining i and j with an arc directed from i to j if $(i,j) \in E(\omega)$ ($(i,j) \in E^*(\omega)$ resp.). Note that $G(\omega)$ is acyclic, where we are considering paths (hence cycles) in the sense of directed graphs, always moving in the direction specified by arcs.

Now consider an arbitrary complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where

$$E := \{(i,j) \,:\, i < j\},$$
$$E^* := \{(i,j) \,:\, i > j\}.$$

Given a subset $A \subseteq E \sqcup E^*$ of edges, we define the *transitive closure* $\overline{A}$ of A in G to be the set of ordered pairs (i,j) of vertices which are joined by a path consisting of A-edges in G directed from i to j. The *transitive part* of this closure $\overline{A}$ is defined to be

$$\mathcal{T}(A) := \overline{A} \backslash A$$

so that

$$\overline{A} = A \sqcup \mathcal{T}(A).$$

In particular, E and E^* are subsets of edges of G so we may consider their transitive closure in G. Note that E and E^* (equivalently G) coming from a permutation will be unchanged by this transitive closure operation, i.e. in this case we would have

$\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$. The following is a trivial, but important, observation about taking transitive closures:

Lemma 9.1.1. *Given a subset A of edges of G, we have $\overline{\overline{A}} = \overline{A}$. Equivalently, $\mathcal{T}\left(\overline{A}\right) = \emptyset$.*

Proof. Evidently $\overline{\overline{A}} \supseteq \overline{A}$. For the opposite containment, let $(i, j) \in \overline{\overline{A}}$. This means there is a path P consisting of edges $e_1, \ldots, e_k \in \overline{A}$ directed from i to j (if $k = 1$, this means $(i, j) = e_1 \in \overline{A}$). Here, we have indexed the edges $e_1, \ldots, e_k$ in the order they appear in P. Namely, e_1 has initial vertex i and terminal vertex equal to the initial vertex of e_2, and so on. Of course, e_k has terminal vertex j.
Note that each e_i is either an original edge of A, or else comes from a directed path P_i consisting of edges from A directed from the initial end to the terminal end of e_i. Hence, we can construct from P a path P' consisting only of A-edges in the following way: if $e_i \in A$, keep it; otherwise, replace e_i with the directed path P_i. Then P' is a directed path of A-edges from i to j, so $(i, j) \in \overline{A}$. □

In other words, Lemma 9.1.1 says that taking the transitive closure of a set of edges produces a set of edges which is transitively closed. We are ready to give some equivalent criteria which guarantee that G is induced by a permutation:

Lemma 9.1.2. *The following are equivalent:*

(i) $G = G(\omega)$ for some unique permutation $\omega \in S_n$.

(ii) G is acyclic.

(iii) $E = \overline{E}$ *and* $E^* = \overline{E^*}$ *(equivalently* $\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$*).*

Proof. (i)⇒(ii). This is obvious, as all edges of $G(\omega)$ are directed from $\omega(i)$ to $\omega(j)$ for each $1 \leq i < j \leq n$.

(ii)⇒(i). Suppose G is acyclic. We claim that there exists a unique vertex $v_1 \in [n]$ such that all edges incident there are inwardly-directed. Indeed, if there were no such vertex then we could enter and leave every vertex, eventually constructing a cycle as G is finite; contradiction. We get uniqueness of v_1 since, for any other vertex $v \neq v_1$, G complete implies there is an edge directed from v to v_1 (v_1 has all inwardly-directed incident edges) so that v has an outwardly-directed incident edge.

Define $\omega(n) = v_1$, and delete v_1 from G, giving a new labelled, complete, simple digraph $G - \{v_1\}$ with vertex set $[n] \backslash \{v_1\}$. Of course $G - \{v_1\}$ is still acyclic, so we may repeat the above argument on this new digraph, giving a unique vertex $v_2 \in [n] \backslash \{v_1\}$ such that all edges incident there are inwardly-directed. We put $\omega(n-1) = v_2$ and continue in this way, finally arriving at a unique permutation $\omega \in S_n$ such that $G = G(\omega)$.

(ii)⇒(iii). Suppose, say, $E \neq \overline{E}$. Then there exists $(i, j) \in \overline{E} \backslash E$. Hence, we can find edges $e_1, \ldots, e_k \in E$, $k > 1$, that form a directed path from i to j in G (i.e., the terminal end of e_t is the initial end of e_{t+1} for each $1 \leq t \leq k-1$). Since $(i, j) \notin E$ and G is complete, we have $(j, i) \in E^*$. Therefore $C := (e_1, \ldots, e_k, (j, i))$ forms a cycle in G. By a similar argument we can show that $E^* \neq \overline{E^*}$ implies G contains a cycle.

(iii)⇒(ii). Suppose G contains a cycle. Since G is both antisymmetric and complete, it contains a cycle of length 3. Let a, b and c be the distinct vertices in $[n]$ that form this cycle. Re-labelling if necessary, we may assume $a < b < c$. If the cycle is (a, b, c), then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*$$

so that $(a, c) \in \overline{E} \backslash E$, i.e., $E \neq \overline{E}$. On the other hand, if (a, c, b) is the cycle, then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*$$

so that $(c, a) \in \overline{E^*} \backslash E^*$, i.e., $E^* \neq \overline{E^*}$. This completes the proof of Lemma 9.1.2. $\square$

9.2 Computing Infs and Sups in the Weak Order Lattice

With this machinery, we now show that the poset $(S_n, \preceq)$ is a lattice. What's more, we can say precisely how to compute $\inf\{\pi_1, \ldots, \pi_r\}$ ($\sup\{\pi_1, \ldots, \pi_r\}$ resp.), where $\pi_1, \ldots, \pi_r \in S_n$.

Lemma 9.2.1. $(S_n, \preceq)$ *is a lattice with*

$$E(\inf\{\pi_1, \ldots, \pi_r\}) = \overline{\cup_{i=1}^r E(\pi_i)}$$

and

$$E^*(\sup\{\pi_1, \ldots, \pi_r\}) = \overline{\cup_{i=1}^r E^*(\pi_i)}.$$

PROOF. We will prove this only for infimums; the proof for supremums is completely analogous. By Lemma 9.1.2, it is sufficient to prove that the complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where $E = \overline{\cup_{i=1}^r E(\pi_i)}$, contains no cycle.

Suppose G does contain a cycle. Then, since G is both antisymmetric and complete, it contains a cycle of length 3, passing through the vertices a, b and c, say. We may assume $a < b < c$; otherwise just re-label the vertices. If the cycle is (a, b, c), then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*,$$

which violates the transitivity of E (note that E is transitively closed by Lemma 9.1.1). So this is impossible.

On the other hand, suppose the cycle is (a, c, b). Then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*.$$

Therefore $(a, b), (b, c) \notin \cup_{i=1}^r E(\pi_i)$, and hence

$$(c, b), (b, a) \in \cap_{i=1}^r E^*(\pi_i).$$

From transitivity, $(c, a) \in \cap_{i=1}^r E^*(\pi_i)$, and therefore

$$(a, c) \notin \cup_{i=1}^r E(\pi_i).$$

So, as $(a, c) \in E$, there exist indices $i_1, \ldots, i_k$ and vertices $a = x_1, x_2, \ldots, x_k, x_{k+1} = c$ with $x_j < x_{j+1}$, $x_j \neq b$ and

$$(x_j, x_{j+1}) \in E(\pi_{i_j}), \quad \forall\, j \leq k.$$

Let $1 \leq \ell \leq k$ be the index such that $x_\ell < b < x_{\ell+1}$. If it happens that $(b, x_\ell) \in E^*(\pi_{i_\ell})$, then as $(x_\ell, x_{\ell+1}) \in E(\pi_{i_\ell})$ we must have $(b, x_{\ell+1}) \in E(\pi_{i_\ell})$ by transitivity of the permutation π_{i_ℓ}. Hence $(b, x_{\ell+1}) \in E$, and since $(x_{\ell+1}, x_{\ell+2}) \in E$ we get $(b, x_{\ell+2}) \in E$ by transitivity of E. Using repeatedly the transitivity of E in this way, we eventually obtain $(b, c) \in E$, contradicting $(c, b) \in E^*$.

Hence, it must be that $(x_\ell, b) \in E(\pi_{i_\ell})$. So $(x_\ell, b) \in E$, and by the transitivity of E we have $(a, x_\ell) \in E$. Therefore, using transitivity once more, $(a, b) \in E$, contradicting $(b, a) \in E^*$. Therefore G must be acyclic, and hence (Lemma 9.1.2) $G = G(\pi)$ for some unique permutation $\pi \in S_n$. Finally, any permutation $\omega \in S_n$ that is a lower bound for all of $\pi_1, \ldots, \pi_r$ will have

$$E(\omega) \supseteq \cup_{i=1}^r E(\pi_i)$$

by definition of the weak order. Hence, since $E(\omega)$ is transitively closed, we have $E(\omega) \supseteq E$. We have just shown $E = E(\pi)$, and hence

$$E(\omega) \supseteq E(\pi) \supseteq \cup_{i=1}^r E(\pi_i)$$

so that $\omega \preceq \pi \preceq \pi_i$, $1 \leq i \leq r$. That is, $\pi = \inf\{\pi_1, \ldots, \pi_r\}$ and we are done. □

9.3 Some Equivalent Criteria for $\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n$

Let $\mathcal{T}(\mathcal{E}_r)$ denote the transitive part of the closure of $\mathcal{E}_r := \cup_{\ell=1}^{r} E(\pi_\ell)$. Note that any pair $(i,k) \in \mathcal{T}(\mathcal{E}_r)$ has $k \geq i+2$ since we must be able to find j with $i < j < k$. Hence, no pair $(i, i+1)$, $1 \leq i \leq n-1$, could possibly belong to $\mathcal{T}(\mathcal{E}_r)$. By Lemma 9.2.1,

$$E(\inf\{\pi_1, \ldots, \pi_r\}) = \overline{\mathcal{E}_r} = \mathcal{E}_r \sqcup \mathcal{T}(\mathcal{E}_r).$$

So, if $\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n$, the unique minimum in this lattice, then every pair (i,j) with $i < j$ belongs to $E(\inf\{\pi_1, \ldots, \pi_r\})$ and hence every pair $(i, i+1)$, $1 \leq i \leq n-1$, must belong to $\mathcal{E}_r$. Thus, choosing $\pi_1, \ldots, \pi_r \in S_n$ independently and uniformly at random, we have proved the containment of events

$$\{\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n\} \subseteq \bigcap_{i=1}^{n-1} \{(i, i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}.$$

But the event on the right is also sufficient for $\{\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n\}$! Indeed, if every pair $(i, i+1)$, $1 \leq i \leq n-1$, belongs to $\mathcal{E}_r$, then taking the transitive closure of this set gives us *every* pair (i,j) with $i < j$! We have therefore proved

$$\{\inf\{\pi_1, \ldots, \pi_r\} = 12\cdots n\} = \bigcap_{i=1}^{n-1} \{(i, i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}. \tag{9.1}$$

We can take this a step further. Given $\omega \in S_n$, introduce the set of *descents* of ω:

$$D(\omega) := \{i \,:\, \omega(i) > \omega(i+1)\}.$$

Consider the event on the right-hand side of (9.1). We have

$$
\begin{aligned}
(i,i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\,\forall\, i \in [n-1] &\iff \forall\, i \in [n-1],\, \exists \ell \in [r],\, (i,i+1) \in E(\pi_\ell) \\
&\iff \forall\, i \in [n-1],\, \exists \ell \in [r],\, i \notin D(\pi_\ell^{-1}) \\
&\iff \bigcap_{\ell=1}^{r} D(\pi_\ell^{-1}) = \emptyset.
\end{aligned}
\tag{9.2}
$$

Moreover, observe that

$$
\begin{aligned}
i \in D\left(\inf\{\pi_1,\ldots,\pi_r\}^{-1}\right) &\iff (i+1,i) \in E^*\left(\inf\{\pi_1,\ldots,\pi_r\}\right) \\
&\iff (i,i+1) \notin E\left(\inf\{\pi_1,\ldots,\pi_r\}\right) \\
&\iff (i,i+1) \notin E\left(\pi_j\right)\,\forall\, j \\
&\iff (i+1,i) \in E^*\left(\pi_j\right)\,\forall\, j \\
&\iff i \in D(\pi_j^{-1})\,\forall\, j.
\end{aligned}
$$

This shows that $D\left(\inf\{\pi_1,\ldots,\pi_r\}^{-1}\right) = \bigcap_{\ell=1}^{r} D(\pi_\ell^{-1})$. Combining this with (9.1) and (9.2), we have therefore proved:

Lemma 9.3.1. *Let $\pi_1,\ldots,\pi_r \in S_n$ be selected independently and uniformly at random, and let $P_n^{(r)} := P\left(\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots n\right)$. Then*

$$
\begin{aligned}
P_n^{(r)} &\overset{\text{(a)}}{=} P\left(\bigcap_{i=1}^{n-1} \{(i,i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}\right) \\
&\overset{\text{(b)}}{=} P\left(D\left(\inf\{\pi_1,\ldots,\pi_r\}^{-1}\right) = \bigcap_{\ell=1}^{r} D(\pi_\ell^{-1}) = \emptyset\right). \qquad \square
\end{aligned}
$$

This allows us to instead study the probabilities (a) and (b), whichever happens to be convenient for us.

Given $\omega \in S_n$, let ω' denote $\omega = \omega(1)\cdots\omega(n)$ reversed in order, so that $\omega' = \omega(n)\cdots\omega(1)$, i.e. $\omega'(j) = \omega(n-j+1)$, $1 \le j \le n$. For example, if $\omega = 45123$ then $\omega' = 32154$. It is trivial to check that

$$\inf\{\pi_1, \dots, \pi_r\} = \tau \quad \Longleftrightarrow \quad \sup\{\pi'_1, \dots, \pi'_r\} = \tau'.$$

Indeed, this only requires the observation

$$\cup_{\ell=1}^{r} E^*(\pi'_\ell) = \{(j,i) \,:\, (i,j) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}$$

followed by an application of Lemma 9.2.1. So we have

Lemma 9.3.2. *Let $\pi_1, \dots, \pi_r \in S_n$ be selected independently and uniformly at random. Then*

$$P_n^{(r)} = P(\inf\{\pi_1, \dots, \pi_r\} = 12\cdots n) = P(\sup\{\pi_1, \dots, \pi_r\} = n(n-1)\cdots 1).$$

PROOF. We need only observe that $\pi_1, \dots, \pi_r \in S_n$ independent and uniformly random implies that the permutations $\pi'_1, \dots, \pi'_r$ are as well. □

Hence, when answering the question "How likely is it that r independent and uniformly random permutations have infimum (supremum resp.) equal to the unique minimum (maximum resp.)?", Lemma 9.3.2 allows us to restrict our attention to infimums. We are now in a position to prove Theorem 1.4.2, part 1.

9.4 Submultiplicativity Again

We wish to prove the submultiplicativity of $P_n^{(r)}$ as a function of n, thus proving existence of

$$\lim_{n\to\infty} \sqrt[n]{P_n^{(r)}} = \inf_{n\geq 1} \sqrt[n]{P_n^{(r)}}$$

([43, p. 23, ex. 98] again). For this, we make use of Lemma 9.3.1.

Let $\pi_1, \ldots, \pi_r$ be independent and uniformly random permutations of $[n_1 + n_2]$. Introduce

$$\pi_i[1, 2, \ldots, n_1], \quad 1 \leq i \leq r,$$

the permutation of $[n_1]$ left after deletion of the elements $n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$ from π_i. Similarly

$$\pi_i[n_1 + 1, n_1 + 2, \ldots, n_1 + n_2], \quad 1 \leq i \leq r,$$

is the permutation of $\{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}$ left after deletion of the elements $1, 2, \ldots, n_1$ from π_i. Then the permutations

$$\pi_1[1, \ldots, n_1], \ldots, \pi_r[1, \ldots, n_1], \pi_1[n_1 + 1, \ldots, n_1 + n_2], \ldots, \pi_r[n_1 + 1, \ldots, n_1 + n_2]$$

are all uniform on their respective sets of permutations, and are mutually independent. By Lemma 9.3.1,

$$\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots(n_1+n_2) \Longleftrightarrow (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell), \quad 1 \le i \le n_1+n_2-1,$$

and hence

$$\begin{aligned}\inf\{\pi_1,\ldots,\pi_r\} &= 12\cdots(n_1+n_2)\\ &\Longrightarrow (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell[1,\ldots,n_1]), \quad 1 \le i \le n_1-1\\ &\Longleftrightarrow \inf\{\pi_1[1,\ldots,n_1],\ldots,\pi_r[1,\ldots,n_1]\} = 12\cdots n_1.\end{aligned}$$

Denote this first event by $\mathcal{E}_{n_1+n_2}$, and the last by $\mathcal{E}_{n_1}$. Thus we have proved the containment of events $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1}$. Similarly, we have

$$\begin{aligned}&\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots(n_1+n_2)\\ &\Longrightarrow (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell[n_1+1,\ldots,n_1+n_2]), \quad n_1+1 \le i \le n_1+n_2-1\\ &\Longleftrightarrow \inf\{\pi_1[n_1+1,\ldots,n_1+n_2],\ldots,\pi_r[n_1+1,\ldots,n_1+n_2]\}\\ &\qquad\qquad = (n_1+1)(n_1+2)\cdots(n_1+n_2).\end{aligned}$$

Denote the last event by $\mathcal{E}^*_{n_2}$, so that we have the containment $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}^*_{n_2}$. Consequently

$$\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1} \cap \mathcal{E}^*_{n_2},$$

and since the events on the right are independent, this implies $P^{(r)}_{n_1+n_2} \le P^{(r)}_{n_1} P^{(r)}_{n_2}$. Of course, the rest of the statement follows from the (by now familiar) classical Fekete lemma concerning sub(super)multiplicative sequences [43, p. 23, ex. 98]. $\square$

9.5 Sharp Asymptotics of $P_n^{(r)}$

We are now ready to finish the proof of Theorem 1.4.2. The proof divides naturally into three steps. First, we will establish the exact formula

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}. \tag{9.3}$$

which in turn facilitates computation of a bivariate generating function related to $P_n^{(r)}$. Finally, analytical techniques applied to a special case of this generating function yields the asymptotic result stated in Theorem 1.4.2:

$$P_n^{(r)} \sim -\frac{1}{z^* h_r'(z^*)} \frac{1}{(z^*)^n}, \quad r \geq 2, \quad n \to \infty,$$

where $z^* = z^*(r) \in (1, 2)$ is the unique (positive) root of the equation

$$h_r(z) := \sum_{j\geq 0} \frac{(-1)^j}{(j!)^r} z^j = 0$$

within the disk $|z| \leq 2$.

Specifically, we will use this exact formula for $P_n^{(r)}$ to show that

$$P_n^{(r)} = [z^n] \frac{1}{h_r(z)}, \quad r \geq 1, \tag{9.4}$$

followed by some asymptotic analysis. As a partial check, for $r = 1$ we obtain

$$P_n^{(1)} = [z^n] \frac{1}{e^{-z}} = \frac{1}{n!},$$

as we should! Also, we immediately see that for $r \geq 2$, the limit $\lim_{n\to\infty} \sqrt[n]{P_n^{(r)}}$, whose existence we established last section, equals $1/z^*$.

9.5.1 Step 1: An Exact Formula for $P_n^{(r)}$

Here, we establish formula (9.3). Notice that, if $\pi_1, \ldots, \pi_r \in S_n$ are independent and uniformly random, then so are the n-permutations $\pi_1^{-1}, \ldots, \pi_r^{-1}$. Hence, the probability (b) in Lemma 9.3.1 is the same as

$$P\left(\bigcap_{i=1}^{r} D(\omega_i) = \emptyset\right),$$

where $\omega_1, \ldots, \omega_r \in S_n$ are independent and uniformly random. That is, we need to compute the probability that r independent and uniformly random permutations have no common descents.

Now, given $I \subseteq [n-1]$, let $\mathcal{E}_I$ denote the event "I belongs to $D(\omega_j)$, $1 \leq j \leq r$". So $\mathcal{E}_I$ is the event that I is common to all of the $D(\omega_j)$'s. Then, by Lemma 9.3.1,

$$1 - P_n^{(r)} = P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right).$$

By the principle of inclusion-exclusion,

$$P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I\subseteq[n-1]\\|I|=k}} P\left(\bigcap_{i\in I} \mathcal{E}_{\{i\}}\right). \tag{9.5}$$

But notice that, given $I \subseteq [n-1]$,

$$\bigcap_{i\in I} \mathcal{E}_{\{i\}} = \mathcal{E}_I.$$

Hence, (9.5) becomes

$$P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right) = \sum_{k=1}^{n-1}(-1)^{k-1} \sum_{\substack{I\subseteq[n-1]\\|I|=k}} P\left(\mathcal{E}_I\right). \tag{9.6}$$

So it only remains to compute $P(\mathcal{E}_I)$ for a fixed $I \subseteq [n-1]$, $|I| = k$, $k \in [n-1]$. This computation is an r-analog of the formula in Boná's book [11, pg. 4]. We present a modification of his argument.

Observe that

$$P\left(\mathcal{E}_I\right) = \frac{|\mathcal{E}_I|}{(n!)^r},$$

so we need to count the number of r-tuples $(\omega_1, \ldots, \omega_r) \in \mathcal{E}_I$.

Write $I = \{i_1 < \cdots < i_k\}$, and $J := [n-1]\backslash I = \{j_1 < \cdots < j_{(n-1)-k}\}$. For $\omega \in S_n$, let $\bar{\omega}$ denote ω reversed in *rank*. So if $\omega = 45123$, then $\bar{\omega} = 21543$. Formally, $\bar{\omega}(j) = n - \omega(j) + 1$, $1 \le j \le n$. Notice that $D(\omega) \sqcup D(\bar{\omega}) = [n-1]$. Hence

$$D(\omega_j) \supseteq I, \quad 1 \le j \le r \quad \Longleftrightarrow \quad D(\bar{\omega_j}) \subseteq J, \quad 1 \le j \le r.$$

Again, $\omega_1, \ldots, \omega_r$ independent and uniformly random implies that so are the permutations $\bar{\omega_1}, \ldots, \bar{\omega_r}$, so our task becomes to count the number of r-tuples of permutations $(\tau_1, \ldots, \tau_r)$ such that $D(\tau_j) \subseteq J$ for every j. As the τ_j are independent, this is just

$$|\{\omega \in S_n \,:\, D(\omega) \subseteq J\}|^r.$$

To count $|\{\omega \in S_n \,:\, D(\omega) \subseteq J\}|$, we arrange the n entries of ω into $n-k$ segments so that the first i segments together have j_i entries for each i. Then, within each segment, we put the entries into increasing order. Then the only places where the resulting ω could possibly have a descent is where two segments meet, i.e., at entries $j_1, \dots, j_{(n-1)-k}$, and hence $D(\omega) \subseteq J$.

The first segment of ω has to have length j_1, and therefore can be chosen in $\binom{n}{j_1}$ ways. The second segment has to be of length $j_2 - j_1$, and must be disjoint from the first one, so may be chosen in $\binom{n-j_1}{j_2-j_1}$ ways. In general, segment i must have length $j_i - j_{i-1}$ if $1 < i < n-k$, and has to be chosen from the remaining $n - j_{i-1}$ entries, in $\binom{n-j_{i-1}}{j_i-j_{i-1}}$ ways. There is only one choice for the last segment, as all remaining $n - j_{(n-1)-k}$ entries must go there. Therefore

$$\begin{aligned} |\{\omega \in S_n \,:\, D(\omega) \subseteq J\}| &= \binom{n}{j_1}\binom{n-j_1}{j_2-j_1}\binom{n-j_2}{j_3-j_2}\cdots\binom{n-j_{(n-1)-k}}{n-j_{(n-1)-k}} \\ &= \frac{n!}{j_1!(j_2-j_1)!\cdots(n-j_{(n-1)-k})!}, \end{aligned}$$

and consequently

$$P(\mathcal{E}_I) = \frac{|\mathcal{E}_I|}{(n!)^r} = \frac{1}{j_1!^r(j_2-j_1)!^r\cdots(n-j_{(n-1)-k})!^r}.$$

Putting this into (9.6), we obtain

$$
\begin{aligned}
1 - P_n^{(r)} &= P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right) \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I\subseteq[n-1] \\ |I|=k}} \frac{1}{j_1!^r (j_2 - j_1)!^r \cdots (n - j_{(n-1)-k})!^r} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{b_1,\dots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r},
\end{aligned}
$$

where $b_1 = j_1$, $b_i = j_i - j_{i-1}, 1 < i < n-k$, and $b_{n-k} = n - j_{(n-1)-k}$. This is clearly equivalent to (9.3).

9.5.2 Step 2: A Generating Function for $P_n^{(r)}$

Let us next use the formula (9.3) to establish the relation (9.4). Recall that we have defined $\mathcal{E}_{\{i\}}$ as the event "i belongs to every $D(\pi_j^{-1})$, $1 \leq j \leq n-1$", and that

$$
1 - P_n^{(r)} = P\left(\bigcup_{i=1}^{n-1} \mathcal{E}_{\{i\}}\right).
$$

Introduce the random variable $S_n^{(r)} = S_n^{(r)}(\pi_1, \dots, \pi_r)$, the number of events $\mathcal{E}_{\{i\}}$ that are satisfied. As we have seen (Lemma 9.3.1), $S_n^{(r)}$ is also the number of descents in $\inf\{\pi_1, \dots, \pi_r\}^{-1}$. Formally, $S_n^{(r)}$ is the sum of indicators

$$
S_n^{(r)} = \sum_{i=1}^{n-1} I_{\mathcal{E}_{\{i\}}}.
$$

Observe that

$$P_n^{(r)} = P\left(S_n^{(r)} = 0\right),$$

so the formula (9.3) gives the probability $P\left(S_n^{(r)} = 0\right)$. But, in fact, this formula tells us even more about $S_n^{(r)}$. Indeed, consider the k-th (unsigned) term in this expression

$$\sum_{\substack{I \subseteq [n-1] \\ |I|=k}} P\left(\bigcap_{i \in I} \mathcal{E}_{\{i\}}\right) = \sum_{\substack{b_1,\ldots,b_{n-k} \geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}.$$

This is the expected number of k-sets of the events $\mathcal{E}_{\{i\}}$ that occur simultaneously. That is,

$$E\left[\binom{S_n^{(r)}}{k}\right] = \sum_{\substack{b_1,\ldots,b_{n-k} \geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}, \quad 0 \leq k \leq n-1. \tag{9.7}$$

This produces the simple expression

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k E\left[\binom{S_n^{(r)}}{k}\right].$$

We could have seen this another way, by observing that

$$P_n^{(r)} = P\left(S_n^{(r)} = 0\right) = E\left[(1-1)^{S_n^{(r)}}\right] = E\left[\sum_{k=0}^{n-1} (-1)^k \binom{S_n^{(r)}}{k}\right],$$

and using the linearity of expectation.

We will use these observations about $S_n^{(r)}$ to get a compact generating function related to this random variable, which happens to be amenable to asymptotic analysis. Introduce the bivariate generating function

$$F_r(x,y) := \sum_{n\geq 1} x^n E\left[(1+y)^{S_n^{(r)}}\right],$$

and let

$$f_r(z) := \sum_{\beta\geq 0} \frac{z^\beta}{(\beta+1)!^r}.$$

Using what we know about $S_n^{(r)}$, we can simplify $F_r(x,y)$:

$$\begin{aligned}
F_r(x,y) &= \sum_{n\geq 1} x^n E\left[(1+y)^{S_n^{(r)}}\right] \\
&= \sum_{n\geq 1} x^n \sum_{k=0}^{n-1} y^k E\left[\binom{S_n^{(r)}}{k}\right] \\
&= \sum_{n\geq 1} x^n \sum_{k=0}^{n-1} y^k \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1\\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r\cdots(b_{n-k}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{n>k} x^{n-k} \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1\\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r\cdots(b_{n-k}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{\nu\geq 1} x^{\nu} \sum_{\substack{b_1,\ldots,b_{\nu}\geq 1\\ b_1+\cdots+b_{\nu}=\nu+k}} \frac{1}{(b_1!)^r\cdots(b_{\nu}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{\nu\geq 1} x^{\nu} \sum_{\substack{\beta_1,\ldots,\beta_{\nu}\geq 0\\ \beta_1+\cdots+\beta_{\nu}=k}} \frac{1}{(\beta_1+1)!^r\cdots(\beta_{\nu}+1)!^r}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\geq 0} (xy)^k [z^k] \sum_{\nu\geq 1} (xf_r(z))^\nu \\
&= \sum_{k\geq 0} (xy)^k [z^k] \frac{xf_r(z)}{1 - xf_r(z)} \\
&= \frac{xf_r(xy)}{1 - xf_r(xy)} \\
&= \frac{1}{1 - xf_r(xy)} - 1.
\end{aligned}
$$

Therefore

$$
E\left[(1+y)^{S_n^{(r)}}\right] = [x^n]\frac{1}{1 - xf_r(xy)}, \quad n \geq 1. \tag{9.8}
$$

Plugging $y = -1$ into this expression, we obtain

$$
\begin{aligned}
P_n^{(r)} = P\left(S_n^{(r)} = 0\right) = E\left[(1-1)^{S_n^{(r)}}\right] &= [x^n]\frac{1}{1 - xf_r(-x)} \\
&= [x^n]\frac{1}{h_r(x)}, \quad n \geq 1,
\end{aligned} \tag{9.9}
$$

where $h_r(x) = \sum_{j\geq 0} \left((-1)^j/(j!)^r\right) x^j$, and this is (9.4). It should be duly noted that this generating function is a special case of one found by Richard Stanley [45], but it is probably safe to say that he was unaware of any connection with the weak ordering.

9.5.3 Step 3: Asymptotics

We are about to finish the proof; all of the combinatorial insights are behind us, and only some asymptotic analysis remains. Armed with formula (9.9), our goal is to use Darboux's theorem [2] to estimate

$$[z^n]\frac{1}{h_r(z)}, \quad h_r(z) = \sum_{j\geq 0} \frac{(-1)^j}{(j!)^r} z^j, \quad r \geq 2.$$

First of all, notice that for $z > 0$ we have

$$1 - z < h_r(z) < 1 - z + z^2/(2!)^r.$$

Hence, we get

$$0 = 1 - (1) < h_r(1); \quad h_r(2) < 1 - (2) + (2)^2/(2!)^r \leq 0, \quad r \geq 2.$$

So $h_r(z) = 0$ has a root in $(1, 2)$ by the intermediate value theorem.

Now, consider the circle $|z| = u$, where $u > 1$ will be specified later. Let

$$g(z) = 1 - z, \quad G(z) = \sum_{j\geq 2} \frac{(-1)^j}{(j!)^r} z^j.$$

$g(z) = 0$ has a single root, of multiplicity 1, within the circle $|z| = u$. For $|z| = u$,

$$|g(z)| \geq \min_{t\in[0,2\pi)} |1 - ue^{it}| = u - 1,$$

and

$$\begin{aligned}|G(z)| &\leq \frac{u^2}{2^r}\left(1 + \frac{u}{3^r} + \frac{u}{3^r}\frac{u}{4^r} + \cdots\right)\\ &\leq \frac{u^2}{2^r} \cdot \frac{1}{1 - \frac{u}{3^r}}, \quad u < 3^r.\end{aligned}$$

If we can find $u \in (1, 3^r)$ such that

$$u - 1 > \frac{\frac{u^2}{2^r}}{1 - \frac{u}{3^r}}, \tag{9.10}$$

then, by Rouché's theorem [48], $h_r(z) = g(z) + G(z)$ also has a unique, whence real positive, root z^* within the circle $|z| = u$. The inequality (9.10) is equivalent to

$$F(u) := u^2(2^{-r} + 3^{-r}) - u(1 + 3^{-r}) + 1 < 0.$$

$F(u)$ attains its minimum at

$$\bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (1, 3^r),$$

and

$$F(\bar{u}) = 1 - \frac{(1 + 3^{-r})^2}{4(2^{-r} + 3^{-r})}.$$

For $r > 2$,

$$4(2^{-r} + 3^{-r}) \leq 8 \cdot 2^{-3} = 1,$$

and so $F(\bar{u}) < 0$ in this case, and we are done. Actually, notice that our choice of circle radius

$$|z| = \bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (2, 3^r), \quad r > 2.$$

So we have proved $h_r(z) = 0$ has a unique (positive) root $z^* = z^*(r) \in (1, 2)$ within the disk $|z| \leq 2$, $r > 2$, which is what we wanted.

On the other hand, for $r = 2$,

$$F(\bar{u}) = 1 - \frac{(1+1/9)^2}{1+4/9} > 0,$$

so this case requires a bit more attention. Instead, consider

$$g(z) = 1 - z + \frac{z^2}{(2!)^2} - \frac{z^3}{(3!)^2}, \quad G(z) = \sum_{j \geq 4} \frac{(-1)^j}{(j!)^2} z^j,$$

and our strategy will be analogous to the above. First,

$$g'(z) = -1 + z/2 - z^2/12 = -\frac{(z-3)^2 + 3}{12} < 0, \quad z \in \mathbb{R},$$

so $g(z) = 0$ has one real root, z_1. Since $g(1) = 2/9 > 0$ and $g(2) = -2/9 < 0$, we have $z_1 \in (1, 2)$.

Let $z_2 = a + ib, \bar{z}_2 = a - ib$ denote the two complex roots of $g(z) = 0$. Then (Vieta's relations [48])

$$2a + z_1 = 9, \quad (a^2 + b^2) z_1 = 36.$$

In particular

$$a = \frac{9 - z_1}{2} > 3.5,$$

hence $|z_2| = |\bar{z}_2| > 3.5$. So, if we can find $u \in (z_1, 3.5)$ with

$$|g(z)| > |G(z)|, \quad |z| = u,$$

we will be done once again by Rouché's theorem. For $|z| = u$,

$$\begin{aligned} |G(z)| &\le \frac{u^4}{(4!)^2}\left(1 + \frac{u}{5^2} + \frac{u}{5^2}\frac{u}{6^2} + \cdots\right) \\ &\le \frac{u^4}{(4!)^2} \cdot \frac{1}{1 - \frac{u}{5^2}}, \quad u < 5^2. \end{aligned} \tag{9.11}$$

Take $u = 2$. Let us show that

$$\min_{|z|=2} |g(z)| = |g(2)| = \frac{2}{9}.$$

To this end, we bound

$$\begin{aligned} |g(z)| &= \frac{1}{36}|(z - z_1)(z - z_2)(z - \bar{z}_2)| \\ &\ge \frac{1}{36}(2 - z_1) \min_{|z|=2} |z - z_2||z - \bar{z}_2|. \end{aligned}$$

Setting $z = 2e^{it}$, we obtain

$$\begin{aligned} |z - z_2|^2|z - \bar{z}_2|^2 &= \left[(2\cos t - a)^2 + (2\sin t - b)^2\right] \cdot \left[(2\cos t - a)^2 + (2\sin t + b)^2\right] \\ &= (4 - 4a\cos t + a^2 + b^2 - 4b\sin t)(4 - 4a\cos t + a^2 + b^2 + 4b\sin t) \\ &= (4 - 4a\cos t + a^2 + b^2)^2 - 16b^2\sin^2 t \\ &:= F(t). \end{aligned}$$

Then

$$\begin{aligned} F'(t) &= 8a\sin t(4 - 4a\cos t + a^2 + b^2) - 32b^2\sin t\cos t \\ &= 8\sin t\left[a(4 + a^2 + b^2) - 4(a^2 + b^2)\cos t\right]. \end{aligned}$$

So $F'(t) = 0$ if and only if $t = 0, \pi$, since

$$\begin{aligned}\frac{a(4+a^2+b^2)}{4(a^2+b^2)} &= \frac{a}{4} + \frac{a}{a^2+b^2} \\ &= \frac{9-z_1}{8} + \frac{z_1(9-z_1)}{72} \\ &= \frac{81-z_1^2}{72} > \frac{77}{72} > 1.\end{aligned}$$

This inequality also shows that $F'(t)$ always has the same sign as $\sin t$, hence $F'(t) > 0$ for $t \in (0, \pi)$ and $F'(t) < 0$ for $t \in (\pi, 2\pi)$. So $F(t)$ attains its minimum at $t = 0$, and consequently on $|z| = 2$

$$\begin{aligned}|g(z)| \geq (2-z_1)\sqrt{F(0)} &= (2-z_1)(4-4a+a^2+b^2) \\ &= (2-z_1)(2-z_2)(2-\bar{z}_2) \\ &= g(2) = \frac{2}{9}.\end{aligned}$$

Combining this with (9.11), we are done since

$$|g(z)| \geq \frac{2}{9} > \frac{\frac{2^4}{(4!)^2}}{1-\frac{2}{5^2}} \geq |G(z)|, \quad |z| = 2. \qquad \square$$

CHAPTER 10

OPEN PROBLEMS

In this final chapter, we present some problems that we find important and/or interesting, and which we intend to pursue in future research.

10.1 The Problems

Problem 10.1.1. *Compute exactly the limit* $lim_{n\to\infty} \sqrt[n]{Q_n}$ *in the proof of Theorem 1.2.1, lower bound.*

Problem 10.1.2. *Compute exactly the limit* $lim_{n\to\infty} \sqrt[n]{Q_n^*}$ *in the proof of Theorem 1.4.1, upper bound.*

Problem 10.1.3. *Find an argument that improves the lower bound in Theorem 1.2.1 to something on the order of* n^{-a} *for some* $a > 0$*.*

Problem 10.1.4. *Find an argument that improves the lower bound in Theorem 1.4.1 to something of exponentially small order.*

Problem 10.1.5. *Find extensions to chains of length* r *for the other bounds in both orders, similar to that in the case of Bruhat order (upper bound).*

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