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Meet and Join in the Weak Order Lattice

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CHAPTER 9

ON INFS AND SUPS IN THE WEAK ORDER LATTICE

Finally, we focus on the proof of Theorem 1.4.2. Before we prove what was stated there, we have a good deal in the way of preliminaries to take care of. The discussion below is inspired almost exclusively by material contained in the work [3].

9.1 A Connection with Complete, Directed, Acyclic Graphs

Given $\omega \in S_n$, recall the set of non-inversions of ω,

$$E(\omega) := \left\{(i, j) \,:\, i < j, \omega^{-1}(i) < \omega^{-1}(j)\right\},$$

and the set of inversions of ω,

$$E^*(\omega) := \left\{(i, j) \,:\, i > j, \omega^{-1}(i) < \omega^{-1}(j)\right\}.$$

Note that ω is uniquely determined by its $E(\omega)$ (equivalently, by its $E^*(\omega)$). We have seen that, given permutations $\pi, \sigma \in S_n$, we have $\pi \leq \sigma$ in the *weak order* (written $\pi \preceq \sigma$) if and only if $E(\pi) \supseteq E(\sigma)$ (equivalently $E^*(\pi) \subseteq E^*(\sigma)$). It is beneficial to consider the sets $E(\omega)$ and $E^*(\omega)$ as directed edges in a complete, simple, labelled digraph. Namely, we define

$$G(\omega) = ([n], E(\omega) \sqcup E^*(\omega))$$

by joining i and j with an arc directed from i to j if $(i,j) \in E(\omega)$ ($(i,j) \in E^*(\omega)$ resp.). Note that $G(\omega)$ is acyclic, where we are considering paths (hence cycles) in the sense of directed graphs, always moving in the direction specified by arcs.

Now consider an arbitrary complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where

$$E := \{(i,j) \,:\, i < j\},$$
$$E^* := \{(i,j) \,:\, i > j\}.$$

Given a subset $A \subseteq E \sqcup E^*$ of edges, we define the *transitive closure* $\overline{A}$ of A in G to be the set of ordered pairs (i,j) of vertices which are joined by a path consisting of A-edges in G directed from i to j. The *transitive part* of this closure $\overline{A}$ is defined to be

$$\mathcal{T}(A) := \overline{A} \backslash A$$

so that

$$\overline{A} = A \sqcup \mathcal{T}(A).$$

In particular, E and E^* are subsets of edges of G so we may consider their transitive closure in G. Note that E and E^* (equivalently G) coming from a permutation will be unchanged by this transitive closure operation, i.e. in this case we would have

$\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$. The following is a trivial, but important, observation about taking transitive closures:

Lemma 9.1.1. *Given a subset A of edges of G, we have $\overline{\overline{A}} = \overline{A}$. Equivalently, $\mathcal{T}\left(\overline{A}\right) = \emptyset$.*

Proof. Evidently $\overline{\overline{A}} \supseteq \overline{A}$. For the opposite containment, let $(i, j) \in \overline{\overline{A}}$. This means there is a path P consisting of edges $e_1, \ldots, e_k \in \overline{A}$ directed from i to j (if $k = 1$, this means $(i, j) = e_1 \in \overline{A}$). Here, we have indexed the edges $e_1, \ldots, e_k$ in the order they appear in P. Namely, e_1 has initial vertex i and terminal vertex equal to the initial vertex of e_2, and so on. Of course, e_k has terminal vertex j.

Note that each e_i is either an original edge of A, or else comes from a directed path P_i consisting of edges from A directed from the initial end to the terminal end of e_i. Hence, we can construct from P a path P' consisting only of A-edges in the following way: if $e_i \in A$, keep it; otherwise, replace e_i with the directed path P_i. Then P' is a directed path of A-edges from i to j, so $(i, j) \in \overline{A}$. □

In other words, Lemma 9.1.1 says that taking the transitive closure of a set of edges produces a set of edges which is transitively closed. We are ready to give some equivalent criteria which guarantee that G is induced by a permutation:

Lemma 9.1.2. *The following are equivalent:*

(i) $G = G(\omega)$ for some unique permutation $\omega \in S_n$.

(ii) G is acyclic.

(iii) $E = \overline{E}$ *and* $E^* = \overline{E^*}$ *(equivalently* $\mathcal{T}(E) = \emptyset = \mathcal{T}(E^*)$*).*

PROOF. (i)⇒(ii). This is obvious, as all edges of $G(\omega)$ are directed from $\omega(i)$ to $\omega(j)$ for each $1 \leq i < j \leq n$.

(ii)⇒(i). Suppose G is acyclic. We claim that there exists a unique vertex $v_1 \in [n]$ such that all edges incident there are inwardly-directed. Indeed, if there were no such vertex then we could enter and leave every vertex, eventually constructing a cycle as G is finite; contradiction. We get uniqueness of v_1 since, for any other vertex $v \neq v_1$, G complete implies there is an edge directed from v to v_1 (v_1 has all inwardly-directed incident edges) so that v has an outwardly-directed incident edge.

Define $\omega(n) = v_1$, and delete v_1 from G, giving a new labelled, complete, simple digraph $G - \{v_1\}$ with vertex set $[n] \backslash \{v_1\}$. Of course $G - \{v_1\}$ is still acyclic, so we may repeat the above argument on this new digraph, giving a unique vertex $v_2 \in [n] \backslash \{v_1\}$ such that all edges incident there are inwardly-directed. We put $\omega(n-1) = v_2$ and continue in this way, finally arriving at a unique permutation $\omega \in S_n$ such that $G = G(\omega)$.

(ii)⇒(iii). Suppose, say, $E \neq \overline{E}$. Then there exists $(i, j) \in \overline{E} \backslash E$. Hence, we can find edges $e_1, \ldots, e_k \in E$, $k > 1$, that form a directed path from i to j in G (i.e., the terminal end of e_t is the initial end of e_{t+1} for each $1 \leq t \leq k-1$). Since $(i, j) \notin E$ and G is complete, we have $(j, i) \in E^*$. Therefore $C := (e_1, \ldots, e_k, (j, i))$ forms a cycle in G. By a similar argument we can show that $E^* \neq \overline{E^*}$ implies G contains a cycle.

(iii)⇒(ii). Suppose G contains a cycle. Since G is both antisymmetric and complete, it contains a cycle of length 3. Let a, b and c be the distinct vertices in $[n]$ that form this cycle. Re-labelling if necessary, we may assume $a < b < c$. If the cycle is (a, b, c), then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*$$

so that $(a, c) \in \overline{E} \backslash E$, i.e., $E \neq \overline{E}$. On the other hand, if (a, c, b) is the cycle, then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*$$

so that $(c, a) \in \overline{E^*} \backslash E^*$, i.e., $E^* \neq \overline{E^*}$. This completes the proof of Lemma 9.1.2. $\square$

9.2 Computing Infs and Sups in the Weak Order Lattice

With this machinery, we now show that the poset $(S_n, \preceq)$ is a lattice. What's more, we can say precisely how to compute $\inf\{\pi_1, \ldots, \pi_r\}$ ($\sup\{\pi_1, \ldots, \pi_r\}$ resp.), where $\pi_1, \ldots, \pi_r \in S_n$.

Lemma 9.2.1. $(S_n, \preceq)$ *is a lattice with*

$$E(\inf\{\pi_1, \ldots, \pi_r\}) = \overline{\cup_{i=1}^r E(\pi_i)}$$

and

$$E^*(\sup\{\pi_1, \ldots, \pi_r\}) = \overline{\cup_{i=1}^r E^*(\pi_i)}.$$

PROOF. We will prove this only for infimums; the proof for supremums is completely analogous. By Lemma 9.1.2, it is sufficient to prove that the complete, simple, labelled digraph $G = ([n], E \sqcup E^*)$, where $E = \overline{\cup_{i=1}^r E(\pi_i)}$, contains no cycle.

Suppose G does contain a cycle. Then, since G is both antisymmetric and complete, it contains a cycle of length 3, passing through the vertices a, b and c, say. We may assume $a < b < c$; otherwise just re-label the vertices. If the cycle is (a, b, c), then

$$(a, b), (b, c) \in E; \quad (c, a) \in E^*,$$

which violates the transitivity of E (note that E is transitively closed by Lemma 9.1.1). So this is impossible.

On the other hand, suppose the cycle is (a, c, b). Then

$$(a, c) \in E; \quad (c, b), (b, a) \in E^*.$$

Therefore $(a, b), (b, c) \notin \cup_{i=1}^r E(\pi_i)$, and hence

$$(c, b), (b, a) \in \cap_{i=1}^r E^*(\pi_i).$$

From transitivity, $(c, a) \in \cap_{i=1}^r E^*(\pi_i)$, and therefore

$$(a, c) \notin \cup_{i=1}^r E(\pi_i).$$

So, as $(a, c) \in E$, there exist indices $i_1, \ldots, i_k$ and vertices $a = x_1, x_2, \ldots, x_k, x_{k+1} = c$ with $x_j < x_{j+1}$, $x_j \neq b$ and

$$(x_j, x_{j+1}) \in E(\pi_{i_j}), \quad \forall\, j \leq k.$$

Let $1 \leq \ell \leq k$ be the index such that $x_\ell < b < x_{\ell+1}$. If it happens that $(b, x_\ell) \in E^*(\pi_{i_\ell})$, then as $(x_\ell, x_{\ell+1}) \in E(\pi_{i_\ell})$ we must have $(b, x_{\ell+1}) \in E(\pi_{i_\ell})$ by transitivity of the permutation π_{i_ℓ}. Hence $(b, x_{\ell+1}) \in E$, and since $(x_{\ell+1}, x_{\ell+2}) \in E$ we get $(b, x_{\ell+2}) \in E$ by transitivity of E. Using repeatedly the transitivity of E in this way, we eventually obtain $(b, c) \in E$, contradicting $(c, b) \in E^*$.

Hence, it must be that $(x_\ell, b) \in E(\pi_{i_\ell})$. So $(x_\ell, b) \in E$, and by the transitivity of E we have $(a, x_\ell) \in E$. Therefore, using transitivity once more, $(a, b) \in E$, contradicting $(b, a) \in E^*$. Therefore G must be acyclic, and hence (Lemma 9.1.2) $G = G(\pi)$ for some unique permutation $\pi \in S_n$. Finally, any permutation $\omega \in S_n$ that is a lower bound for all of $\pi_1, \ldots, \pi_r$ will have

$$E(\omega) \supseteq \cup_{i=1}^{r} E(\pi_i)$$

by definition of the weak order. Hence, since $E(\omega)$ is transitively closed, we have $E(\omega) \supseteq E$. We have just shown $E = E(\pi)$, and hence

$$E(\omega) \supseteq E(\pi) \supseteq \cup_{i=1}^{r} E(\pi_i)$$

so that $\omega \preceq \pi \preceq \pi_i$, $1 \leq i \leq r$. That is, $\pi = \inf\{\pi_1, \ldots, \pi_r\}$ and we are done. □

9.3 Some Equivalent Criteria for $\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n$

Let $\mathcal{T}(\mathcal{E}_r)$ denote the transitive part of the closure of $\mathcal{E}_r := \cup_{\ell=1}^{r} E(\pi_\ell)$. Note that any pair $(i, k) \in \mathcal{T}(\mathcal{E}_r)$ has $k \geq i + 2$ since we must be able to find j with $i < j < k$. Hence, no pair $(i, i+1)$, $1 \leq i \leq n-1$, could possibly belong to $\mathcal{T}(\mathcal{E}_r)$. By Lemma 9.2.1,

$$E(\inf\{\pi_1, \ldots, \pi_r\}) = \overline{\mathcal{E}_r} = \mathcal{E}_r \sqcup \mathcal{T}(\mathcal{E}_r).$$

So, if $\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n$, the unique minimum in this lattice, then every pair (i, j) with $i < j$ belongs to $E(\inf\{\pi_1, \ldots, \pi_r\})$ and hence every pair $(i, i+1)$, $1 \leq i \leq n-1$, must belong to $\mathcal{E}_r$. Thus, choosing $\pi_1, \ldots, \pi_r \in S_n$ independently and uniformly at random, we have proved the containment of events

$$\{\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n\} \subseteq \bigcap_{i=1}^{n-1} \{(i, i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}.$$

But the event on the right is also sufficient for $\{\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n\}$! Indeed, if every pair $(i, i+1)$, $1 \leq i \leq n-1$, belongs to $\mathcal{E}_r$, then taking the transitive closure of this set gives us *every* pair (i, j) with $i < j$! We have therefore proved

$$\{\inf\{\pi_1, \ldots, \pi_r\} = 12 \cdots n\} = \bigcap_{i=1}^{n-1} \{(i, i+1) \in \cup_{\ell=1}^{r} E(\pi_\ell)\}. \tag{9.1}$$

We can take this a step further. Given $\omega \in S_n$, introduce the set of *descents* of ω:

$$D(\omega) := \{i \,:\, \omega(i) > \omega(i+1)\}.$$

Consider the event on the right-hand side of (9.1). We have

$$
\begin{aligned}
(i,i+1)\in\cup_{\ell=1}^{r}E(\pi_\ell)\,\forall\, i\in[n-1] &\iff \forall\, i\in[n-1],\,\exists\ell\in[r],\,(i,i+1)\in E(\pi_\ell)\\
&\iff \forall\, i\in[n-1],\,\exists\ell\in[r],\, i\notin D(\pi_\ell^{-1})\\
&\iff \bigcap_{\ell=1}^{r}D(\pi_\ell^{-1})=\emptyset.
\end{aligned}
\tag{9.2}
$$

Moreover, observe that

$$
\begin{aligned}
i\in D\left(\inf\{\pi_1,\dots,\pi_r\}^{-1}\right) &\iff (i+1,i)\in E^*\left(\inf\{\pi_1,\dots,\pi_r\}\right)\\
&\iff (i,i+1)\notin E\left(\inf\{\pi_1,\dots,\pi_r\}\right)\\
&\iff (i,i+1)\notin E\left(\pi_j\right)\,\forall\, j\\
&\iff (i+1,i)\in E^*\left(\pi_j\right)\,\forall\, j\\
&\iff i\in D(\pi_j^{-1})\,\forall\, j.
\end{aligned}
$$

This shows that $D\left(\inf\{\pi_1,\dots,\pi_r\}^{-1}\right)=\bigcap_{\ell=1}^{r}D(\pi_\ell^{-1})$. Combining this with (9.1) and (9.2), we have therefore proved:

Lemma 9.3.1. *Let $\pi_1,\dots,\pi_r\in S_n$ be selected independently and uniformly at random, and let $P_n^{(r)}:=P\left(\inf\{\pi_1,\dots,\pi_r\}=12\cdots n\right)$. Then*

$$
\begin{aligned}
P_n^{(r)} &\overset{\text{(a)}}{=} P\left(\bigcap_{i=1}^{n-1}\{(i,i+1)\in\cup_{\ell=1}^{r}E(\pi_\ell)\}\right)\\
&\overset{\text{(b)}}{=} P\left(D\left(\inf\{\pi_1,\dots,\pi_r\}^{-1}\right)=\bigcap_{\ell=1}^{r}D(\pi_\ell^{-1})=\emptyset\right). \qquad \square
\end{aligned}
$$

This allows us to instead study the probabilities (a) and (b), whichever happens to be convenient for us.

Given $\omega \in S_n$, let ω' denote $\omega = \omega(1)\cdots\omega(n)$ reversed in order, so that $\omega' = \omega(n)\cdots\omega(1)$, i.e. $\omega'(j) = \omega(n-j+1)$, $1 \le j \le n$. For example, if $\omega = 45123$ then $\omega' = 32154$. It is trivial to check that

$$\inf\{\pi_1,\ldots,\pi_r\} = \tau \quad \Longleftrightarrow \quad \sup\{\pi_1',\ldots,\pi_r'\} = \tau'.$$

Indeed, this only requires the observation

$$\cup_{\ell=1}^r E^*(\pi_\ell') = \{(j,i) \,:\, (i,j) \in \cup_{\ell=1}^r E(\pi_\ell)\}$$

followed by an application of Lemma 9.2.1. So we have

Lemma 9.3.2. *Let $\pi_1,\ldots,\pi_r \in S_n$ be selected independently and uniformly at random. Then*

$$P_n^{(r)} = P(\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots n) = P(\sup\{\pi_1,\ldots,\pi_r\} = n(n-1)\cdots 1).$$

Proof. We need only observe that $\pi_1,\ldots,\pi_r \in S_n$ independent and uniformly random implies that the permutations $\pi_1',\ldots,\pi_r'$ are as well. □

Hence, when answering the question "How likely is it that r independent and uniformly random permutations have infimum (supremum resp.) equal to the unique minimum (maximum resp.)?", Lemma 9.3.2 allows us to restrict our attention to infimums. We are now in a position to prove Theorem 1.4.2, part 1.

9.4 Submultiplicativity Again

We wish to prove the submultiplicativity of $P_n^{(r)}$ as a function of n, thus proving existence of

$$\lim_{n\to\infty} \sqrt[n]{P_n^{(r)}} = \inf_{n\geq 1} \sqrt[n]{P_n^{(r)}}$$

([43, p. 23, ex. 98] again). For this, we make use of Lemma 9.3.1.

Let $\pi_1, \ldots, \pi_r$ be independent and uniformly random permutations of $[n_1 + n_2]$. Introduce

$$\pi_i[1, 2, \ldots, n_1], \quad 1 \leq i \leq r,$$

the permutation of $[n_1]$ left after deletion of the elements $n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$ from π_i. Similarly

$$\pi_i[n_1 + 1, n_1 + 2, \ldots, n_1 + n_2], \quad 1 \leq i \leq r,$$

is the permutation of $\{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}$ left after deletion of the elements $1, 2, \ldots, n_1$ from π_i. Then the permutations

$$\pi_1[1, \ldots, n_1], \ldots, \pi_r[1, \ldots, n_1], \pi_1[n_1 + 1, \ldots, n_1 + n_2], \ldots, \pi_r[n_1 + 1, \ldots, n_1 + n_2]$$

are all uniform on their respective sets of permutations, and are mutually independent. By Lemma 9.3.1,

$$\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots(n_1+n_2) \iff (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell), \quad 1 \le i \le n_1+n_2-1,$$

and hence

$$\begin{aligned}
\inf\{\pi_1,\ldots,\pi_r\} &= 12\cdots(n_1+n_2) \\
&\implies (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell[1,\ldots,n_1]), \quad 1 \le i \le n_1-1 \\
&\iff \inf\{\pi_1[1,\ldots,n_1],\ldots,\pi_r[1,\ldots,n_1]\} = 12\cdots n_1.
\end{aligned}$$

Denote this first event by $\mathcal{E}_{n_1+n_2}$, and the last by $\mathcal{E}_{n_1}$. Thus we have proved the containment of events $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1}$. Similarly, we have

$$\begin{aligned}
&\inf\{\pi_1,\ldots,\pi_r\} = 12\cdots(n_1+n_2) \\
&\implies (i,i+1) \in \cup_{\ell=1}^r E(\pi_\ell[n_1+1,\ldots,n_1+n_2]), \quad n_1+1 \le i \le n_1+n_2-1 \\
&\iff \inf\{\pi_1[n_1+1,\ldots,n_1+n_2],\ldots,\pi_r[n_1+1,\ldots,n_1+n_2]\} \\
&\qquad\qquad = (n_1+1)(n_1+2)\cdots(n_1+n_2).
\end{aligned}$$

Denote the last event by $\mathcal{E}^*_{n_2}$, so that we have the containment $\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}^*_{n_2}$. Consequently

$$\mathcal{E}_{n_1+n_2} \subseteq \mathcal{E}_{n_1} \cap \mathcal{E}^*_{n_2},$$

and since the events on the right are independent, this implies $P^{(r)}_{n_1+n_2} \le P^{(r)}_{n_1} P^{(r)}_{n_2}$. Of course, the rest of the statement follows from the (by now familiar) classical Fekete lemma concerning sub(super)multiplicative sequences [43, p. 23, ex. 98]. ∎

9.5 Sharp Asymptotics of $P_n^{(r)}$

We are now ready to finish the proof of Theorem 1.4.2. The proof divides naturally into three steps. First, we will establish the exact formula

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k \sum_{\substack{b_1,\ldots,b_{n-k} \geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}. \tag{9.3}$$

which in turn facilitates computation of a bivariate generating function related to $P_n^{(r)}$. Finally, analytical techniques applied to a special case of this generating function yields the asymptotic result stated in Theorem 1.4.2:

$$P_n^{(r)} \sim -\frac{1}{z^* h_r'(z^*)} \frac{1}{(z^*)^n}, \quad r \geq 2, \quad n \to \infty,$$

where $z^* = z^*(r) \in (1,2)$ is the unique (positive) root of the equation

$$h_r(z) := \sum_{j \geq 0} \frac{(-1)^j}{(j!)^r} z^j = 0$$

within the disk $|z| \leq 2$.

Specifically, we will use this exact formula for $P_n^{(r)}$ to show that

$$P_n^{(r)} = [z^n] \frac{1}{h_r(z)}, \quad r \geq 1, \tag{9.4}$$

followed by some asymptotic analysis. As a partial check, for $r = 1$ we obtain

$$P_n^{(1)} = [z^n] \frac{1}{e^{-z}} = \frac{1}{n!},$$

as we should! Also, we immediately see that for $r \geq 2$, the limit $\lim_{n\to\infty} \sqrt[n]{P_n^{(r)}}$, whose existence we established last section, equals $1/z^*$.

9.5.1 Step 1: An Exact Formula for $P_n^{(r)}$

Here, we establish formula (9.3). Notice that, if $\pi_1, \ldots, \pi_r \in S_n$ are independent and uniformly random, then so are the n-permutations $\pi_1^{-1}, \ldots, \pi_r^{-1}$. Hence, the probability (b) in Lemma 9.3.1 is the same as

$$P\left(\bigcap_{i=1}^{r} D(\omega_i) = \emptyset\right),$$

where $\omega_1, \ldots, \omega_r \in S_n$ are independent and uniformly random. That is, we need to compute the probability that r independent and uniformly random permutations have no common descents.

Now, given $I \subseteq [n-1]$, let $\mathcal{E}_I$ denote the event "I belongs to $D(\omega_j)$, $1 \leq j \leq r$". So $\mathcal{E}_I$ is the event that I is common to all of the $D(\omega_j)$'s. Then, by Lemma 9.3.1,

$$1 - P_n^{(r)} = P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right).$$

By the principle of inclusion-exclusion,

$$P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right) = \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I\subseteq[n-1] \\ |I|=k}} P\left(\bigcap_{i\in I} \mathcal{E}_{\{i\}}\right). \tag{9.5}$$

But notice that, given $I \subseteq [n-1]$,

$$\bigcap_{i\in I}\mathcal{E}_{\{i\}} = \mathcal{E}_I.$$

Hence, (9.5) becomes

$$P\left(\bigcup_{i\in[n-1]}\mathcal{E}_{\{i\}}\right) = \sum_{k=1}^{n-1}(-1)^{k-1}\sum_{\substack{I\subseteq[n-1]\\|I|=k}}P\left(\mathcal{E}_I\right). \tag{9.6}$$

So it only remains to compute $P\left(\mathcal{E}_I\right)$ for a fixed $I\subseteq[n-1]$, $|I|=k$, $k\in[n-1]$. This computation is an r-analog of the formula in Boná's book [11, pg. 4]. We present a modification of his argument.

Observe that

$$P\left(\mathcal{E}_I\right) = \frac{|\mathcal{E}_I|}{(n!)^r},$$

so we need to count the number of r-tuples $(\omega_1,\dots,\omega_r)\in\mathcal{E}_I$.

Write $I=\{i_1<\cdots<i_k\}$, and $J:=[n-1]\backslash I=\{j_1<\cdots<j_{(n-1)-k}\}$. For $\omega\in S_n$, let $\bar{\omega}$ denote ω reversed in *rank*. So if $\omega=45123$, then $\bar{\omega}=21543$. Formally, $\bar{\omega}(j)=n-\omega(j)+1$, $1\le j\le n$. Notice that $D(\omega)\sqcup D(\bar{\omega})=[n-1]$. Hence

$$D(\omega_j)\supseteq I,\quad 1\le j\le r \quad\Longleftrightarrow\quad D(\bar{\omega_j})\subseteq J,\quad 1\le j\le r.$$

Again, $\omega_1,\dots,\omega_r$ independent and uniformly random implies that so are the permutations $\bar{\omega_1},\dots,\bar{\omega_r}$, so our task becomes to count the number of r-tuples of permutations $(\tau_1,\dots,\tau_r)$ such that $D(\tau_j)\subseteq J$ for every j. As the τ_j are independent, this is just

$$|\{\omega \in S_n \,:\, D(\omega) \subseteq J\}|^r.$$

To count $|\{\omega \in S_n \,:\, D(\omega) \subseteq J\}|$, we arrange the n entries of ω into $n-k$ segments so that the first i segments together have j_i entries for each i. Then, within each segment, we put the entries into increasing order. Then the only places where the resulting ω could possibly have a descent is where two segments meet, i.e., at entries $j_1, \ldots, j_{(n-1)-k}$, and hence $D(\omega) \subseteq J$.

The first segment of ω has to have length j_1, and therefore can be chosen in $\binom{n}{j_1}$ ways. The second segment has to be of length $j_2 - j_1$, and must be disjoint from the first one, so may be chosen in $\binom{n-j_1}{j_2-j_1}$ ways. In general, segment i must have length $j_i - j_{i-1}$ if $1 < i < n-k$, and has to be chosen from the remaining $n - j_{i-1}$ entries, in $\binom{n-j_{i-1}}{j_i-j_{i-1}}$ ways. There is only one choice for the last segment, as all remaining $n - j_{(n-1)-k}$ entries must go there. Therefore

$$\begin{aligned} |\{\omega \in S_n \,:\, D(\omega) \subseteq J\}| &= \binom{n}{j_1}\binom{n-j_1}{j_2-j_1}\binom{n-j_2}{j_3-j_2}\cdots\binom{n-j_{(n-1)-k}}{n-j_{(n-1)-k}} \\ &= \frac{n!}{j_1!(j_2-j_1)!\cdots(n-j_{(n-1)-k})!}, \end{aligned}$$

and consequently

$$P\left(\mathcal{E}_I\right) = \frac{|\mathcal{E}_I|}{(n!)^r} = \frac{1}{j_1!^r(j_2-j_1)!^r\cdots(n-j_{(n-1)-k})!^r}.$$

Putting this into (9.6), we obtain

$$
\begin{aligned}
1 - P_n^{(r)} &= P\left(\bigcup_{i\in[n-1]} \mathcal{E}_{\{i\}}\right) \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{I\subseteq[n-1] \\ |I|=k}} \frac{1}{j_1!^r (j_2 - j_1)!^r \cdots (n - j_{(n-1)-k})!^r} \\
&= \sum_{k=1}^{n-1} (-1)^{k-1} \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r},
\end{aligned}
$$

where $b_1 = j_1$, $b_i = j_i - j_{i-1}, 1 < i < n-k$, and $b_{n-k} = n - j_{(n-1)-k}$. This is clearly equivalent to (9.3).

9.5.2 Step 2: A Generating Function for $P_n^{(r)}$

Let us next use the formula (9.3) to establish the relation (9.4). Recall that we have defined $\mathcal{E}_{\{i\}}$ as the event "i belongs to every $D(\pi_j^{-1})$, $1 \leq j \leq n-1$", and that

$$
1 - P_n^{(r)} = P\left(\bigcup_{i=1}^{n-1} \mathcal{E}_{\{i\}}\right).
$$

Introduce the random variable $S_n^{(r)} = S_n^{(r)}(\pi_1, \ldots, \pi_r)$, the number of events $\mathcal{E}_{\{i\}}$ that are satisfied. As we have seen (Lemma 9.3.1), $S_n^{(r)}$ is also the number of descents in $\inf\{\pi_1, \ldots, \pi_r\}^{-1}$. Formally, $S_n^{(r)}$ is the sum of indicators

$$
S_n^{(r)} = \sum_{i=1}^{n-1} I_{\mathcal{E}_{\{i\}}}.
$$

Observe that

$$P_n^{(r)} = P\left(S_n^{(r)} = 0\right),$$

so the formula (9.3) gives the probability $P\left(S_n^{(r)} = 0\right)$. But, in fact, this formula tells us even more about $S_n^{(r)}$. Indeed, consider the k-th (unsigned) term in this expression

$$\sum_{\substack{I \subseteq [n-1] \\ |I| = k}} P\left(\bigcap_{i \in I} \mathcal{E}_{\{i\}}\right) = \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \cdots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}.$$

This is the expected number of k-sets of the events $\mathcal{E}_{\{i\}}$ that occur simultaneously. That is,

$$E\left[\binom{S_n^{(r)}}{k}\right] = \sum_{\substack{b_1, \dots, b_{n-k} \geq 1 \\ b_1 + \cdots + b_{n-k} = n}} \frac{1}{(b_1!)^r \cdots (b_{n-k}!)^r}, \quad 0 \leq k \leq n-1. \tag{9.7}$$

This produces the simple expression

$$P_n^{(r)} = \sum_{k=0}^{n-1} (-1)^k E\left[\binom{S_n^{(r)}}{k}\right].$$

We could have seen this another way, by observing that

$$P_n^{(r)} = P\left(S_n^{(r)} = 0\right) = E\left[(1-1)^{S_n^{(r)}}\right] = E\left[\sum_{k=0}^{n-1} (-1)^k \binom{S_n^{(r)}}{k}\right],$$

and using the linearity of expectation.

We will use these observations about $S_n^{(r)}$ to get a compact generating function related to this random variable, which happens to be amenable to asymptotic analysis. Introduce the bivariate generating function

$$F_r(x,y) := \sum_{n\geq 1} x^n E\left[(1+y)^{S_n^{(r)}}\right],$$

and let

$$f_r(z) := \sum_{\beta\geq 0} \frac{z^\beta}{(\beta+1)!^r}.$$

Using what we know about $S_n^{(r)}$, we can simplify $F_r(x,y)$:

$$\begin{aligned}
F_r(x,y) &= \sum_{n\geq 1} x^n E\left[(1+y)^{S_n^{(r)}}\right] \\
&= \sum_{n\geq 1} x^n \sum_{k=0}^{n-1} y^k E\left[\binom{S_n^{(r)}}{k}\right] \\
&= \sum_{n\geq 1} x^n \sum_{k=0}^{n-1} y^k \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r\cdots(b_{n-k}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{n>k} x^{n-k} \sum_{\substack{b_1,\ldots,b_{n-k}\geq 1 \\ b_1+\cdots+b_{n-k}=n}} \frac{1}{(b_1!)^r\cdots(b_{n-k}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{\nu\geq 1} x^{\nu} \sum_{\substack{b_1,\ldots,b_{\nu}\geq 1 \\ b_1+\cdots+b_{\nu}=\nu+k}} \frac{1}{(b_1!)^r\cdots(b_{\nu}!)^r} \\
&= \sum_{k\geq 0} (xy)^k \sum_{\nu\geq 1} x^{\nu} \sum_{\substack{\beta_1,\ldots,\beta_{\nu}\geq 0 \\ \beta_1+\cdots+\beta_{\nu}=k}} \frac{1}{(\beta_1+1)!^r\cdots(\beta_{\nu}+1)!^r}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\geq 0}(xy)^k[z^k]\sum_{\nu\geq 1}(xf_r(z))^\nu \\
&= \sum_{k\geq 0}(xy)^k[z^k]\frac{xf_r(z)}{1-xf_r(z)} \\
&= \frac{xf_r(xy)}{1-xf_r(xy)} \\
&= \frac{1}{1-xf_r(xy)} - 1.
\end{aligned}
$$

Therefore

$$
E\left[(1+y)^{S_n^{(r)}}\right] = [x^n]\frac{1}{1-xf_r(xy)}, \quad n \geq 1. \tag{9.8}
$$

Plugging $y = -1$ into this expression, we obtain

$$
\begin{aligned}
P_n^{(r)} = P\left(S_n^{(r)} = 0\right) = E\left[(1-1)^{S_n^{(r)}}\right] &= [x^n]\frac{1}{1-xf_r(-x)} \\
&= [x^n]\frac{1}{h_r(x)}, \quad n \geq 1,
\end{aligned} \tag{9.9}
$$

where $h_r(x) = \sum_{j\geq 0}\left((-1)^j/(j!)^r\right)x^j$, and this is (9.4). It should be duly noted that this generating function is a special case of one found by Richard Stanley [45], but it is probably safe to say that he was unaware of any connection with the weak ordering.

9.5.3 Step 3: Asymptotics

We are about to finish the proof; all of the combinatorial insights are behind us, and only some asymptotic analysis remains. Armed with formula (9.9), our goal is to use Darboux's theorem [2] to estimate

$$[z^n]\frac{1}{h_r(z)}, \quad h_r(z) = \sum_{j\geq 0} \frac{(-1)^j}{(j!)^r} z^j, \quad r \geq 2.$$

First of all, notice that for $z > 0$ we have

$$1 - z < h_r(z) < 1 - z + z^2/(2!)^r.$$

Hence, we get

$$0 = 1 - (1) < h_r(1); \quad h_r(2) < 1 - (2) + (2)^2/(2!)^r \leq 0, \quad r \geq 2.$$

So $h_r(z) = 0$ has a root in $(1, 2)$ by the intermediate value theorem.

Now, consider the circle $|z| = u$, where $u > 1$ will be specified later. Let

$$g(z) = 1 - z, \quad G(z) = \sum_{j\geq 2} \frac{(-1)^j}{(j!)^r} z^j.$$

$g(z) = 0$ has a single root, of multiplicity 1, within the circle $|z| = u$. For $|z| = u$,

$$|g(z)| \geq \min_{t\in[0,2\pi)} |1 - ue^{it}| = u - 1,$$

and

$$\begin{aligned} |G(z)| &\leq \frac{u^2}{2^r}\left(1 + \frac{u}{3^r} + \frac{u}{3^r}\frac{u}{4^r} + \cdots\right) \\ &\leq \frac{u^2}{2^r} \cdot \frac{1}{1 - \frac{u}{3^r}}, \quad u < 3^r. \end{aligned}$$

If we can find $u \in (1, 3^r)$ such that

$$u - 1 > \frac{\frac{u^2}{2^r}}{1 - \frac{u}{3^r}}, \tag{9.10}$$

then, by Rouché's theorem [48], $h_r(z) = g(z) + G(z)$ also has a unique, whence real positive, root z^* within the circle $|z| = u$. The inequality (9.10) is equivalent to

$$F(u) := u^2(2^{-r} + 3^{-r}) - u(1 + 3^{-r}) + 1 < 0.$$

$F(u)$ attains its minimum at

$$\bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (1, 3^r),$$

and

$$F(\bar{u}) = 1 - \frac{(1 + 3^{-r})^2}{4(2^{-r} + 3^{-r})}.$$

For $r > 2$,

$$4(2^{-r} + 3^{-r}) \leq 8 \cdot 2^{-3} = 1,$$

and so $F(\bar{u}) < 0$ in this case, and we are done. Actually, notice that our choice of circle radius

$$|z| = \bar{u} = \frac{1 + 3^{-r}}{2(2^{-r} + 3^{-r})} \in (2, 3^r), \quad r > 2.$$

So we have proved $h_r(z) = 0$ has a unique (positive) root $z^* = z^*(r) \in (1, 2)$ within the disk $|z| \leq 2$, $r > 2$, which is what we wanted.

On the other hand, for $r = 2$,

$$F(\bar{u}) = 1 - \frac{(1+1/9)^2}{1+4/9} > 0,$$

so this case requires a bit more attention. Instead, consider

$$g(z) = 1 - z + \frac{z^2}{(2!)^2} - \frac{z^3}{(3!)^2}, \quad G(z) = \sum_{j\geq 4} \frac{(-1)^j}{(j!)^2} z^j,$$

and our strategy will be analogous to the above. First,

$$g'(z) = -1 + z/2 - z^2/12 = -\frac{(z-3)^2+3}{12} < 0, \quad z \in \mathbb{R},$$

so $g(z) = 0$ has one real root, z_1. Since $g(1) = 2/9 > 0$ and $g(2) = -2/9 < 0$, we have $z_1 \in (1, 2)$.

Let $z_2 = a + ib, \bar{z}_2 = a - ib$ denote the two complex roots of $g(z) = 0$. Then (Vieta's relations [48])

$$2a + z_1 = 9, \quad (a^2 + b^2)z_1 = 36.$$

In particular

$$a = \frac{9 - z_1}{2} > 3.5,$$

hence $|z_2| = |\bar{z}_2| > 3.5$. So, if we can find $u \in (z_1, 3.5)$ with

$$|g(z)| > |G(z)|, \quad |z| = u,$$

we will be done once again by Rouché's theorem. For $|z| = u$,

$$
\begin{aligned}
|G(z)| &\le \frac{u^4}{(4!)^2}\left(1 + \frac{u}{5^2} + \frac{u}{5^2}\frac{u}{6^2} + \cdots\right) \\
&\le \frac{u^4}{(4!)^2} \cdot \frac{1}{1-\frac{u}{5^2}}, \quad u < 5^2.
\end{aligned}
\tag{9.11}
$$

Take $u = 2$. Let us show that

$$
\min_{|z|=2} |g(z)| = |g(2)| = \frac{2}{9}.
$$

To this end, we bound

$$
\begin{aligned}
|g(z)| &= \frac{1}{36}|(z-z_1)(z-z_2)(z-\bar{z}_2)| \\
&\ge \frac{1}{36}(2-z_1)\min_{|z|=2}|z-z_2||z-\bar{z}_2|.
\end{aligned}
$$

Setting $z = 2e^{it}$, we obtain

$$
\begin{aligned}
|z-z_2|^2|z-\bar{z}_2|^2 &= \left[(2\cos t - a)^2 + (2\sin t - b)^2\right] \cdot \left[(2\cos t - a)^2 + (2\sin t + b)^2\right] \\
&= (4 - 4a\cos t + a^2 + b^2 - 4b\sin t)(4 - 4a\cos t + a^2 + b^2 + 4b\sin t) \\
&= (4 - 4a\cos t + a^2 + b^2)^2 - 16b^2\sin^2 t \\
&:= F(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
F'(t) &= 8a\sin t(4 - 4a\cos t + a^2 + b^2) - 32b^2\sin t\cos t \\
&= 8\sin t\left[a(4 + a^2 + b^2) - 4(a^2 + b^2)\cos t\right].
\end{aligned}
$$

So $F'(t) = 0$ if and only if $t = 0, \pi$, since

$$\begin{aligned}\frac{a(4+a^2+b^2)}{4(a^2+b^2)} &= \frac{a}{4} + \frac{a}{a^2+b^2} \\ &= \frac{9-z_1}{8} + \frac{z_1(9-z_1)}{72} \\ &= \frac{81-z_1^2}{72} > \frac{77}{72} > 1.\end{aligned}$$

This inequality also shows that $F'(t)$ always has the same sign as $\sin t$, hence $F'(t) > 0$ for $t \in (0, \pi)$ and $F'(t) < 0$ for $t \in (\pi, 2\pi)$. So $F(t)$ attains its minimum at $t = 0$, and consequently on $|z| = 2$

$$\begin{aligned}|g(z)| \geq (2-z_1)\sqrt{F(0)} &= (2-z_1)(4-4a+a^2+b^2) \\ &= (2-z_1)(2-z_2)(2-\bar{z}_2) \\ &= g(2) = \frac{2}{9}.\end{aligned}$$

Combining this with (9.11), we are done since

$$|g(z)| \geq \frac{2}{9} > \frac{\frac{2^4}{(4!)^2}}{1-\frac{2}{5^2}} \geq |G(z)|, \quad |z| = 2. \qquad \square$$