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Integrable Models of Internal Gravity Water Waves Beneath a Flat Surface

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Integrable Models of Internal Gravity Water Waves beneath a Flat Surface

Alan C. Compelli, Rossen I. Ivanov and Tony Lyons

Abstract. A two-layer fluid system separated by a pycnocline in the form of an internal wave is considered. The lower layer is bounded below by a flat bottom and the upper layer is bounded above by a flat surface. The fluids are incompressible and inviscid and Coriolis forces as well as currents are taken into consideration. A Hamiltonian formulation is presented and appropriate scaling leads to a KdV approximation. Additionally, considering the lower layer to be infinitely deep leads to a Benjamin-Ono approximation.

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1. Introduction

The presented material provides a review of some well-known long wave models: the KdV and Benjamin-Ono approximations. The context is an oceanic fluid system comprising of two layers separated by an internal wave, created by a sharp density gradient, bounded above and below by a flat surface and flat seabed respectively. Many irrotational studies of both single layered and stratified systems such as [2, 3, 4, 17, 18, 19, 27, 28] have followed on from Zakharov's determination in [32] of a canonical Hamiltonian structure for a deep fluid with gravitational surface waves. The consideration of vorticity, however, is necessary for the inclusion of currents. The interaction of waves and currents have been examined for single layer systems in [10, 11, 14, 15, 16, 30, 31] and for stratified systems in [5, 6, 7, 12, 13].

2. The set-up

Consider a fluid system consisting of two domains as shown in Figure 1. The

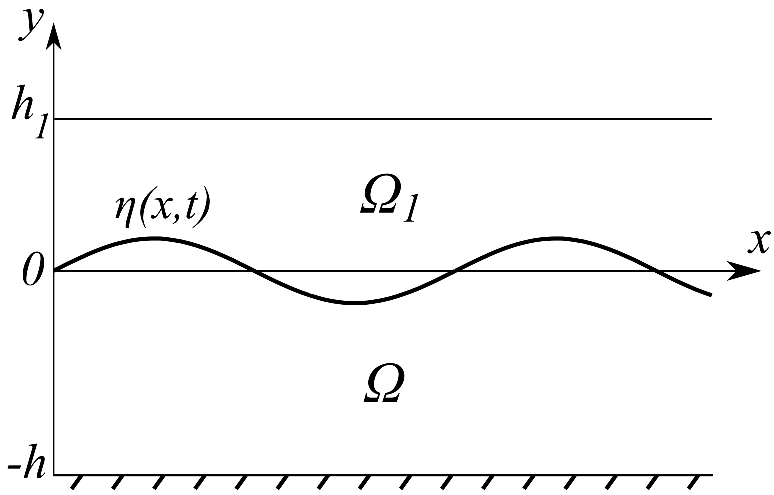

Figure 1. Set-up for the system.

lower medium is bounded underneath by a solid, stationary, impermeable layer of constant depth called the 'flatbed' at a depth h and the upper medium is bounded by a flat surface called the 'lid' at a height h_1. The physical reasoning is that the surface waves in the ocean have usually much smaller amplitudes in comparison to the internal waves. Typically h_1 may be of the order of hundreds of metres and h may be of the order of kilometres.

The system comprises of two separate fluids which have different densities due to different salinity levels and temperatures. Some prescribed flow has been generated by, perhaps, surface winds permeating downwards or due to tidal influences. However, at the interface the fluids do not mix and form a free common interface in the form of an internal wave. The wave is two-dimensional (in the x-y plane), propagating in the positive x-direction, due to the assumption that there is no lateral movement. This is a reasonable assumption for example, for oceanic waves of constant depth travelling along the equator [13, 22, 26]. The wave extends to infinity in both the positive and negative directions. The wave is characterised by the elevation function $\eta(x,t)$ with respect to the level $y = 0$. In other words the equation of the interface is

$$y = \eta(x,t). \tag{2.1}$$

The mean value of η is taken to be zero for convenience,

$$\int_{\mathbb{R}} \eta(x,t)dx = 0, \quad \text{for all } t. \tag{2.2}$$

The system is assumed to be on the surface of the Earth, that is on a rotating solid body. The wave is acted upon by the restorative action of gravity. The Earth's centre of gravity is considered to be in the negative y-direction.

The domains Ω and Ω_1 are defined as

$$\begin{aligned} \Omega &:= \{(x,y) \in \mathbb{R}^2 : -h < y < \eta(x,t)\} \\ \text{and} \quad \Omega_1 &:= \{(x,y) \in \mathbb{R}^2 : \eta(x,t) < y < h_1\}. \end{aligned}$$

Due to an assumption of incompressibility the constant densities are given by ρ and ρ_1 and stability is ensured by the assumption of immiscibility and that $\rho > \rho_1$.

The stream functions, ψ and ψ_1, are related to the velocity fields $\mathbf{u} = (u, v)$ and $\mathbf{u}_1 = (u_1, v_1)$ via the relations

$$u = \psi_y, \quad u_1 = \psi_{1,y}, \quad v = -\psi_x \quad \text{and} \quad v_1 = -\psi_{1,x} \tag{2.3}$$

due to the incompressibility assumption $\nabla \cdot \mathbf{u} = 0$, $\nabla \cdot \mathbf{u}_1 = 0$.

The velocity potentials, φ and φ_1, are introduced such that

$$u = \varphi_x + \gamma y, \quad u_1 = \varphi_{1,x} + \gamma_1 y, \quad v = \varphi_y \quad \text{and} \quad v_1 = \varphi_{1,y} \tag{2.4}$$

where γ and γ_1 are the constant vorticities, where the vorticities are defined as

$$\gamma = -v_x + u_y \quad \text{and} \quad \gamma_1 = -v_{1,x} + u_{1,y}. \tag{2.5}$$

This setup allows for modelling of an undercurrent, such as the Equatorial Undercurrent. A piecewise linear current profile can be represented by the velocity fields of the form (2.4), [12] by writing

$$u = \widetilde{\varphi}_x + \gamma y + \kappa, \quad u_1 = \widetilde{\varphi}_{1,x} + \gamma_1 y + \kappa_1, \quad v = \widetilde{\varphi}_y \quad \text{and} \quad v_1 = \widetilde{\varphi}_{1,y} \tag{2.6}$$

where κ and κ_1 are constants representing the current horizontal velocities at $y = 0$. The wave-only components have been separated out by introducing a tilde notation.

There is a harmonic conjugate relationship between ψ and $\widetilde{\varphi}$ (*cf.* [21, 25]) given by the complex analytic function

$$f(z) = \widetilde{\varphi}(x,y,t) + i\Big(\psi(x,y,t) - \frac{1}{2}\gamma y^2 - \kappa y\Big),$$

where $z = x + iy \in \Omega$, and similar for Ω_1. The fact that $f(z)$ is analytic in the corresponding domain allows the determination of the velocity potential $\widetilde{\varphi}(x,y,t)$ in Ω from its value $\phi(x,t)$ at the interface $y = \eta(x,t)$ (see (4.13) below, $\phi(x,t)$ can be expressed through the canonical Hamiltonian variables defined at the interface). Hence, the physical quantities in the body of the fluid can be determined from the variables at the interface as well.

We assume that the functions $\eta(x,t)$, $\widetilde{\varphi}(x,y,t)$ and $\widetilde{\varphi}_1(x,y,t)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R})$ (*cf.* [24]) with respect to x (for any y and t). The assumption of course implies that for large absolute values of x the internal wave attenuates, and is vanishing at infinity, and therefore

$$\lim_{|x|\to\infty} \eta(x,t) = \lim_{|x|\to\infty} \widetilde{\varphi}(x,y,t) = \lim_{|x|\to\infty} \widetilde{\varphi}_1(x,y,t) = 0. \tag{2.7}$$

Note that we have not specified the dynamics (the time-evolution) of our physical variables yet.

3. Governing equations

The fluid velocities and the net forces per unit mass for the inviscid media under study are related through the Euler equations

$$\mathbf{u}_t+(\mathbf{u}.\nabla)\mathbf{u}=-\frac{1}{\rho}\nabla p+\mathbf{g}+\mathbf{F} \quad \text{and} \quad \mathbf{u}_{1,t}+(\mathbf{u}_1.\nabla)\mathbf{u}_1=-\frac{1}{\rho_1}\nabla p_1+\mathbf{g}+\mathbf{F_1} \tag{3.1}$$

where

$$\mathbf{F}=2\omega\nabla\psi \quad \text{and} \quad \mathbf{F}_1=2\omega\nabla\psi_1 \tag{3.2}$$

are the Coriolis forces per unit mass with ω being the rotational speed of the Earth, $\mathbf{g}=(0,0,-g)$ is the Earth acceleration, i.e. g is the acceleration due to gravity, ρ and ρ_1 (due to the assumption of incompressibility) are the constant densities and p and p_1 are the corresponding pressures.

The pressure gradients are given as

$$\nabla p=-\rho\nabla\Big(\widetilde{\varphi}_t+\frac{1}{2}|\nabla\psi|^2-(\gamma+2\omega)\psi+gy\Big)$$
$$\text{and} \quad \nabla p_1=-\rho_1\nabla\Big(\widetilde{\varphi}_{1,t}+\frac{1}{2}|\nabla\psi_1|^2-(\gamma_1+2\omega)\psi_1+gy\Big).$$

We hence can establish a Bernoulli condition at the interface ($p=p_1$)

$$\rho\Big((\widetilde{\varphi}_t)_c+\frac{1}{2}|\nabla\psi|_c^2-(\gamma+2\omega)\chi+g\eta\Big) = \rho_1\Big((\widetilde{\varphi}_{1,t})_c+\frac{1}{2}|\nabla\psi_1|_c^2-(\gamma_1+2\omega)\chi_1+g\eta\Big) \tag{3.3}$$

where the subscript c signifies the evaluation at the common interface $y=\eta(x,t)$, $\chi=\psi(x,\eta,t)$ and $\chi_1=\psi_1(x,\eta,t)$. The equation (3.5) will eventually produce the evolution of the quantity

$$\xi:=\rho(\widetilde{\varphi})_c-\rho_1(\widetilde{\varphi}_1)_c$$

and this indicates that ξ can be chosen as a *momentum* variable in the Hamiltonian formulation of the problem. The obvious candidate for a counterpart *coordinate* variable is $\eta(x,t)$ and it evolves according to the so called *kinematic boundary condition* at the interface

$$\eta_t=v-u\eta_x=v_1-u_1\eta_x. \tag{3.4}$$

This can be expressed in terms of the stream functions, using (2.3), as

$$\eta_t=-(\psi_x)_c-(\psi_y)_c\eta_x=-(\psi_{1,x})_c-(\psi_{1,y})_c\eta_x, \tag{3.5}$$

and in terms of the velocity potentials, using (2.6), as

$$\eta_t=(\widetilde{\varphi}_y)_c-\big((\widetilde{\varphi}_x)_c+\gamma\eta+\kappa\big)\eta_x=(\widetilde{\varphi}_{1,y})_c-\big((\widetilde{\varphi}_{1,x})_c+\gamma_1\eta+\kappa_1\big)\eta_x. \tag{3.6}$$

The kinematic boundary condition at the bottom, requiring that there is no velocity component in the y-direction on the flat bed, is given by

$$\big(\widetilde{\varphi}(x,-h,t)\big)_y = 0 \quad \text{and} \quad \big(\psi(x,-h,t)\big)_x = 0 \tag{3.7}$$

and, additionally, there is a kinematic boundary condition at the top, requiring that there is no velocity component in the y-direction on the surface, given by

$$\big(\widetilde{\varphi}_1(x,h_1,t)\big)_y = 0 \quad \text{and} \quad \big(\psi_1(x,h_1,t)\big)_x = 0. \tag{3.8}$$

4. Hamiltonian formulation

The functional H, which describes the total energy of the system, can be written as the sum of the kinetic, $\mathcal{K}$, and potential energy, $\mathcal{V}$ contributions. The potential part is

$$V(\eta) = \rho g \int_{\mathbb{R}} \int_{-h}^{\eta} y\, dydx + \rho_1 g \int_{\mathbb{R}} \int_{\eta}^{h_1} y\, dydx.$$

However, the potential energy is always measured from some reference value, e.g. $V(\eta = 0)$ which is the potential energy of the current (without wave motion). Therefore, the relevant part of the potential energy, contributing to the wave motion is

$$\mathcal{V}(\eta) = V(\eta) - V(0) = \rho g \int_{\mathbb{R}} \int_{0}^{\eta} y\, dydx + \rho_1 g \int_{\mathbb{R}} \int_{\eta}^{0} y\, dydx = \frac{1}{2}(\rho - \rho_1) g \int_{\mathbb{R}} \eta^2 dx.$$

In order to determine the kinetic energy of the wave motion, from the total kinetic energy of the fluid

$$\frac{1}{2}\rho \int_{\mathbb{R}} \int_{-h}^{\eta} (u^2 + v^2) dydx + \frac{1}{2}\rho_1 \int_{\mathbb{R}} \int_{\eta}^{h_1} (u_1^2 + v_1^2) dydx \tag{4.1}$$

one should subtract again the constant, but infinite kinetic energy of the current which is

$$\frac{1}{2}\rho \int_{\mathbb{R}} \int_{-h}^{0} (\gamma y + \kappa)^2 dydx + \frac{1}{2}\rho_1 \int_{\mathbb{R}} \int_{0}^{h_1} (\gamma_1 y + \kappa_1)^2 dydx. \tag{4.2}$$

In terms of the dependent variables $\eta(x,t)$, $\widetilde{\varphi}(x,t)$ and $\widetilde{\varphi}_1(x,t)$ this kinetic energy is

$$\mathcal{K}(\eta,\widetilde{\varphi},\widetilde{\varphi}_1) = \frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}\big((\widetilde{\varphi}_x+\gamma y+\kappa)^2+(\widetilde{\varphi}_y)^2\big)\,dydx - \frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{0}(\gamma y+\kappa)^2 dydx$$
$$+\frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\big((\widetilde{\varphi}_{1,x}+\gamma_1 y+\kappa_1)^2+(\widetilde{\varphi}_{1,y})^2\big)\,dydx - \frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{0}^{h_1}(\gamma_1 y+\kappa_1)^2 dydx$$
$$=\frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}\big((\widetilde{\varphi}_x)^2+(\widetilde{\varphi}_y)^2+2\widetilde{\varphi}_x(\gamma y+\kappa)\big)\,dydx$$
$$+\frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\big((\widetilde{\varphi}_{1,x})^2+(\widetilde{\varphi}_{1,y})^2+2\widetilde{\varphi}_{1,x}(\gamma_1 y+\kappa_1)\big)\,dydx$$
$$+\frac{1}{6}(\rho\gamma^2-\rho_1\gamma_1^2)\int_{\mathbb{R}}\eta^3 dx+\frac{1}{2}\big(\rho\gamma\kappa-\rho_1\gamma_1\kappa_1\big)\int_{\mathbb{R}}\eta^2 dx. \tag{4.3}$$

The Hamiltonian is therefore

$$H(\eta,\widetilde{\varphi},\widetilde{\varphi}_1)=\mathcal{K}+\mathcal{V}=\frac{1}{2}\rho\int_{\mathbb{R}}\int_{-h}^{\eta}\big((\widetilde{\varphi}_x)^2+(\widetilde{\varphi}_y)^2+2\widetilde{\varphi}_x(\gamma y+\kappa)\big)\,dydx$$
$$+\frac{1}{2}\rho_1\int_{\mathbb{R}}\int_{\eta}^{h_1}\big((\widetilde{\varphi}_{1,x})^2+(\widetilde{\varphi}_{1,y})^2+2\widetilde{\varphi}_{1,x}(\gamma_1 y+\kappa_1)\big)\,dydx$$
$$+\frac{1}{6}(\rho\gamma^2-\rho_1\gamma_1^2)\int_{\mathbb{R}}\eta^3 dx+\frac{1}{2}\big((\rho\gamma\kappa-\rho_1\gamma_1\kappa_1)+(\rho-\rho_1)g\big)\int_{\mathbb{R}}\eta^2 dx. \tag{4.4}$$

The Dirichlet-Neumann operators $G(\eta)$ and $G_1(\eta)$ are introduced defined as [20]

$$G(\eta)\phi=(\widetilde{\varphi}_{\mathbf{n}})_c\sqrt{1+\eta_x^2}\quad\text{ and }\quad G_1(\eta)\phi_1=(\widetilde{\varphi}_{1_{\mathbf{n}_1}})_c\sqrt{1+\eta_x^2} \tag{4.5}$$

where $\mathbf{n}$ and $\mathbf{n}_1$ are the *unit* exterior normals, $\sqrt{1+(\eta_x)^2}$ is a normalisation factor and

$$\phi(x,t):=(\widetilde{\varphi})_c=\widetilde{\varphi}(x,\eta(x,t),t)\quad\text{ and }\quad\phi_1(x,t):=(\widetilde{\varphi}_1)_c=\widetilde{\varphi}_1(x,\eta(x,t),t) \tag{4.6}$$

have been introduced as the interface velocity potentials and also introduce the operator B [18] as

$$B:=\rho G_1(\eta)+\rho_1 G(\eta). \tag{4.7}$$

Using the boundary conditions

$$\begin{cases} G(\eta)\phi = -\eta_x(\tilde{\varphi}_x)_c + (\tilde{\varphi}_y)_c = \eta_t + (\gamma\eta + \kappa)\eta_x, \\ G_1(\eta)\phi_1 = \eta_x(\tilde{\varphi}_{1,x})_c - (\tilde{\varphi}_{1,y})_c = -\eta_t - (\gamma_1\eta + \kappa_1)\eta_x \end{cases} \tag{4.8}$$

we get

$$G(\eta)\phi + G_1(\eta)\phi_1 = \mu \tag{4.9}$$

where

$$\mu := \big((\gamma - \gamma_1)\eta + (\kappa - \kappa_1)\big)\eta_x. \tag{4.10}$$

Introducing the *momentum* variable [2, 3]

$$\xi(x,t) = \rho\phi(x,t) - \rho_1\phi_1(x,t) \tag{4.11}$$

we can show that

$$B\phi = \rho_1 G(\eta)\phi + \rho G_1(\eta)\phi = \rho_1\mu + G_1(\eta)\xi \tag{4.12}$$

and thus

$$\begin{cases} \phi = B^{-1}\big(\rho_1\mu + G_1(\eta)\xi\big) \\ \phi_1 = B^{-1}\big(\rho\mu - G(\eta)\xi\big) \end{cases} \tag{4.13}$$

gives the explicit expression of ϕ and ϕ_1 in terms of η and ξ. Due to the initial assumptions on the velocity potentials, $\xi(x,t)$ is a Schwartz class $\mathcal{S}(\mathbb{R})$ function in x (for any t).

Usually there is no jump in the current velocity, hence in what follows we take $\kappa = \kappa_1$. The Hamiltonian of the system can be expressed in terms of variables defined on the interface only, η and ξ:

$$\begin{aligned} H(\eta,\xi) = \frac{1}{2}\int_{\mathbb{R}} \xi G(\eta)B^{-1}G_1(\eta)\xi\, dx - \frac{1}{2}\rho\rho_1(\gamma-\gamma_1)^2 \int_{\mathbb{R}} \eta\eta_x B^{-1}\eta\eta_x dx \\ -\gamma\int_{\mathbb{R}} \xi\eta\eta_x dx - \kappa\int_{\mathbb{R}} \xi\eta_x dx + \rho_1(\gamma-\gamma_1)\int_{\mathbb{R}} \eta\eta_x B^{-1}G(\eta)\xi\, dx + \frac{1}{6}(\rho\gamma^2 - \rho_1\gamma_1^2)\int_{\mathbb{R}} \eta^3 dx \\ + \frac{1}{2}\big((\rho\gamma - \rho_1\gamma_1)\kappa + g(\rho-\rho_1)\big)\int_{\mathbb{R}} \eta^2 dx. \end{aligned} \tag{4.14}$$

It is a natural physical fact that there is no flow through the common interface and therefore the stream functions $\chi = \psi(x,\eta,t)$ and $\chi_1 = \psi_1(x,\eta,t)$ at the interface coincide,

$$\chi = \chi_1 = -\int_{-\infty}^{x} \eta_t(x',t)dx' = -\partial_x^{-1}\eta_t \tag{4.15}$$

noting that due to (3.7)

$$\frac{d}{dx}\psi(x,\eta,t) = \psi_x + \psi_y(x,\eta,t)\eta_x = -\eta_t.$$

By evaluating the variations of the Hamiltonian one can show that (3.8) and (3.5) can be written in the form of a non-canonical Hamiltonian system [16]

$$\eta_t = \frac{\delta H}{\delta \xi} \quad \text{and} \quad \xi_t = -\frac{\delta H}{\delta \eta} + \Gamma\chi = -\frac{\delta H}{\delta \eta} - \Gamma \partial_x^{-1}\eta_t, \tag{4.16}$$

where

$$\Gamma := \rho\gamma - \rho_1\gamma_1 + 2\omega(\rho - \rho_1) \tag{4.17}$$

is a constant. Canonical equations of motion can be achieved by transforming the velocity potential at the interface, ξ, to a new variable, ζ, via the transformation (*cf.* [31])

$$\xi \quad \to \quad \zeta = \xi + \frac{\Gamma}{2}\int_{-\infty}^{x} \eta(x', t)dx, \tag{4.18}$$

and due to (2.2) the variable $\zeta \in \mathcal{S}(\mathbb{R})$ (for any t). For our further convenience however the equations (4.16) will be written in terms of the variable

$$\mathfrak{u} = \xi_x$$

and hence for a Hamiltonian in $\mathfrak{u}$ and η

$$\eta_t = -\Big(\frac{\delta H}{\delta \mathfrak{u}}\Big)_x \quad \text{and} \quad \mathfrak{u}_t = -\Big(\frac{\delta H}{\delta \eta}\Big)_x - \Gamma\eta_t. \tag{4.19}$$

5. Expanding the Dirichlet-Neumann operators

The Dirichlet-Neumann operators can be expanded in terms of powers of η as

$$G(\eta) = \sum_{j=0}^{\infty} G^{(j)}(\eta) \text{ and } G_1(\eta) = \sum_{j=0}^{\infty} G_1^{(j)}(\eta), \tag{5.1}$$

where $G^{(j)}(\eta)$ is a homogeneous expression in η of degree j, that is $G^{(j)}(b\eta) = b^j G^{(j)}(\eta)$ for any constant b. The explicit expansion is [18]

$$G(\eta) = DT(D) + D\eta D - DT(D)\eta DT(D) + \mathcal{O}(\eta^2) \tag{5.2}$$

$$\text{and} \quad G_1(\eta) = DT_1(D) - D\eta D + DT_1(D)\eta DT_1(D) + \mathcal{O}(\eta^2) \tag{5.3}$$

where

$$D := -i\partial_x \tag{5.4}$$

is a differential operator and

$$T(D) := \tanh(hD) \quad \text{and} \quad T_1(D) := \tanh(h_1 D) \tag{5.5}$$

have been introduced.

The operator B, as defined in (4.7), which is a function of the Dirichlet-Neumann operators, can therefore be expressed as

$$B = \rho\sum_{j=0}^{\infty} G_1^{(j)}(\eta) + \rho_1 \sum_{j=0}^{\infty} G^{(j)}(\eta).$$

It is noted that the leading (zeroth order in η) term in the expansion of B^{-1}, represented by $[B^{-1}]^{(0)}$, is

$$[B^{-1}]^{(0)} = \frac{1}{\rho D T_1(D) + \rho_1 D T(D)}. \tag{5.6}$$

6. Approximations

6.1. The KdV approximation

A KdV-type approximation will be derived (*cf.* [8]). This family of equations are characterised as having weakly nonlinear and dispersive components.

Small parameters associated to the physical scales

$$\varepsilon = \frac{a}{h_1} \quad \text{and} \quad \delta = \frac{h_1}{\lambda} \tag{6.1}$$

are introduced where λ is the wavelength of the internal wave and a is the average wave amplitude. Indeed, $\delta \ll 1$ is small for long waves $\lambda \gg h_1$. This approximation therefore is for the long-wave regime. The quantity $h_1 k$ where $k = 2\pi/\lambda$ is the wave number is therefore scaled as

$$\mathcal{O}(h_1 k) = \delta,$$

and therefore for the operator D (which on monochromatic waves has an eigenvalue equal to the wave number) clearly

$$\mathcal{O}(h_1 D) = \delta. \tag{6.2}$$

To keep track of the order of the variables we replace $h_1 D$ with $\delta h_1 D$ and further assume that $h_1 D$ itself is of order 1. Since h and h_1 are fixed constants, then their ratio is of order 1. The wave elevation function is scaled according to

$$\eta \to \varepsilon \eta. \tag{6.3}$$

It can be shown as in [8] that the scaling of ξ, leading to the KdV approximation is

$$\xi \to \delta \xi. \tag{6.4}$$

The expansion of the Dirichlet-Neumann operators, given in (5.2) and (5.3), can be scaled as

$$\begin{aligned} G(\eta) \;&\to\; \delta\big(D\tanh(\delta h D)\big) + \varepsilon\delta^2\big(D\eta D - D\tanh(\delta h D)\eta D\tanh(\delta h D)\big) + \mathcal{O}(\varepsilon^2\delta^4) \\ G_1(\eta) \;&\to\; \delta\big(D\tanh(\delta h_1 D)\big) - \varepsilon\delta^2\big(D\eta D - D\tanh(\delta h_1 D)\eta D\tanh(\delta h_1 D)\big) \\ &\qquad + \mathcal{O}(\varepsilon^2\delta^4). \end{aligned}$$

Using the expansion for the hyperbolic tangent the Dirichlet-Neumann operators can be represented as

$$\begin{aligned} G(\eta) = \delta^2\Big(hD^2 + \varepsilon D\eta D\Big) - \delta^4\Big(\frac{1}{3}h^3 D^4 + \varepsilon h^2 D^2 \eta D^2\Big) \\ + \delta^6\Big(\frac{2}{15}h^5 D^6\Big) + \mathcal{O}(\delta^8, \varepsilon\delta^6, \varepsilon^2\delta^4) \end{aligned} \tag{6.5}$$

and

$$G_1(\eta) = \delta^2\Big(h_1 D^2 - \varepsilon D\eta D\Big) + \delta^4\Big(-\frac{1}{3}h_1^3 D^4 + \varepsilon h_1^2 D^2\eta D^2\Big) + \delta^6\Big(\frac{2}{15}h_1^5 D^6\Big) + \mathcal{O}(\delta^8, \varepsilon\delta^6, \varepsilon^2\delta^4). \quad (6.6)$$

and so the inverse of the operator B is given by

$$B^{-1} = \frac{1}{\delta^2(\rho_1 h + \rho h_1)} D^{-1}\Bigg\{1 - \varepsilon\frac{\rho_1 - \rho}{\rho_1 h + \rho h_1}\eta + \varepsilon^2\frac{(\rho_1 - \rho)^2}{(\rho_1 h + \rho h_1)^2}\eta^2 + \delta^2\Bigg(\frac{1}{3}\frac{\rho_1 h^3 + \rho h_1^3}{\rho_1 h + \rho h_1}D^2 - \frac{1}{3}\varepsilon\frac{(\rho_1 - \rho)(\rho_1 h^3 + \rho h_1^3)}{(\rho_1 h + \rho h_1)^2}\eta D^2 - \frac{1}{3}\varepsilon\frac{(\rho_1 - \rho)(\rho_1 h^3 + \rho h_1^3)}{(\rho_1 h + \rho h_1)^2}D^2\eta + \varepsilon\frac{\rho_1 h^2 - \rho h_1^2}{\rho_1 h + \rho h_1}D\eta D\Bigg) - \delta^4\Bigg(\frac{2}{15}\frac{\rho_1 h^5 + \rho h_1^5}{\rho_1 h + \rho h_1}D^4 - \frac{1}{9}\frac{(\rho_1 h^3 + \rho h_1^3)^2}{(\rho_1 h + \rho h_1)^2}D^4\Bigg) + \mathcal{O}(\delta^6, \varepsilon\delta^4, \varepsilon^2\delta^2, \varepsilon^3)\Bigg\}D^{-1}. \quad (6.7)$$

By assuming that ε and δ^2 are of the same order, so as to permit a balancing between nonlinearity and dispersion, the Hamiltonian to $\mathcal{O}(\delta^6)$ is therefore

$$H(\eta, \xi) = \frac{1}{2}\delta^4\alpha_1\int_{\mathbb{R}}\xi D^2\xi dx + \frac{1}{2}\delta^6\alpha_3\int_{\mathbb{R}}\xi D\eta D\xi dx - \frac{1}{2}\delta^6\alpha_2\int_{\mathbb{R}}\xi D^4\xi dx - \delta^4\kappa\int_{\mathbb{R}}\xi\eta_x dx - \delta^6\alpha_4\int_{\mathbb{R}}\xi\eta\eta_x dx + \frac{1}{6}\delta^6\alpha_6\int_{\mathbb{R}}\eta^3 dx + \frac{1}{2}\delta^4\alpha_5\int_{\mathbb{R}}\eta^2 dx \quad (6.8)$$

or

$$H(\eta, \mathfrak{u}) = \frac{1}{2}\delta^4\alpha_1\int_{\mathbb{R}}\mathfrak{u}^2 dx + \frac{1}{2}\delta^6\alpha_3\int_{\mathbb{R}}\eta\mathfrak{u}^2 dx - \frac{1}{2}\delta^6\alpha_2\int_{\mathbb{R}}\mathfrak{u}_x^2 dx + \delta^4\kappa\int_{\mathbb{R}}\eta\mathfrak{u} dx + \delta^6\frac{1}{2}\alpha_4\int_{\mathbb{R}}\mathfrak{u}\eta^2 dx + \frac{1}{6}\delta^6\alpha_6\int_{\mathbb{R}}\eta^3 dx + \frac{1}{2}\delta^4\alpha_5\int_{\mathbb{R}}\eta^2 dx \quad (6.9)$$

where the following constants have been introduced

$$\alpha_1 = \frac{hh_1}{\rho_1 h + \rho h_1}, \tag{6.10}$$

$$\alpha_2 = \frac{1}{3}\frac{h^2h_1^2(\rho_1 h_1 + \rho h)}{(\rho_1 h + \rho h_1)^2}, \tag{6.11}$$

$$\alpha_3 = \frac{\rho h_1^2 - \rho_1 h^2}{(\rho_1 h + \rho h_1)^2}, \tag{6.12}$$

$$\alpha_4 = \frac{\gamma\rho h_1 + \gamma_1\rho_1 h}{\rho_1 h + \rho h_1}, \tag{6.13}$$

$$\alpha_5 = (\rho\gamma - \rho_1\gamma_1)\kappa + g(\rho - \rho_1), \tag{6.14}$$

$$\alpha_6 = \rho\gamma^2 - \rho_1\gamma_1^2. \tag{6.15}$$

The equations of motion (4.19) are now written in terms of η and $\mathfrak{u}$ as

$$\eta_t + \kappa\eta_x + \alpha_1\mathfrak{u}_x + \delta^2\alpha_3(\mathfrak{u}\eta)_x + \delta^2\alpha_2\mathfrak{u}_{xxx} + \delta^2\alpha_4\eta\eta_x = 0 \tag{6.16}$$

$$\text{and}\quad \mathfrak{u}_t + \kappa\mathfrak{u}_x + \delta^2\alpha_3\mathfrak{u}\mathfrak{u}_x + \delta^2\alpha_4(\mathfrak{u}\eta)_x + \delta^2\alpha_6\eta\eta_x + \alpha_5\eta_x + \Gamma\eta_t = 0, \tag{6.17}$$

with an appropriate scaling of t. Noting the assumption that $g \gg 2\omega\kappa$ and introducing a Galilean shift

$$X = x - \kappa t, \quad T = t, \quad \partial_X = \partial_x \quad \text{and} \quad \partial_T = \partial_t + \kappa\partial_x \tag{6.18}$$

the equations of motion can be written as

$$\eta_T + \alpha_1\mathfrak{u}_X + \delta^2\big(\alpha_2\mathfrak{u}_{XXX} + \alpha_3(\mathfrak{u}\eta)_X + \alpha_4\eta\eta_X\big) = 0 \tag{6.19}$$

$$\begin{aligned}\text{and}\quad &\mathfrak{u}_T - \Gamma\alpha_1\mathfrak{u}_X + g(\rho-\rho_1)\eta_X + \delta^2\big(- \Gamma\alpha_2\mathfrak{u}_{XXX}\\ &+ \alpha_3\mathfrak{u}\mathfrak{u}_X + \alpha_4(\mathfrak{u}\eta)_X - \Gamma\alpha_3(\mathfrak{u}\eta)_X + \alpha_6\eta\eta_X - \Gamma\alpha_4\eta\eta_X\big) = 0.\end{aligned} \tag{6.20}$$

The linearised equations are therefore

$$\eta_T + \alpha_1\mathfrak{u}_X = 0 \tag{6.21}$$

$$\text{and}\quad \mathfrak{u}_T - \Gamma\alpha_1\mathfrak{u}_X + g(\rho-\rho_1)\eta_X = 0. \tag{6.22}$$

The variables, η and $\mathfrak{u}$ can be represented as

$$\eta(X,T) = \eta_0 e^{i(kX - \Omega(k)T)} \tag{6.23}$$

$$\text{and}\quad \mathfrak{u}(X,T) = \mathfrak{u}_0 e^{i(kX - \Omega(k)T)}. \tag{6.24}$$

Noting that the wave number, angular frequency and wave speed are related via $c(k) = \Omega(k)/k$ means it can be written that

$$-ick\eta + i\alpha_1 k\mathfrak{u} = 0 \tag{6.25}$$

$$\text{and}\quad -ick\mathfrak{u} + ig(\rho-\rho_1)k\eta - i\Gamma\alpha_1 k\mathfrak{u} = 0. \tag{6.26}$$

This has solutions for observers moving with the flow as

$$c = \frac{1}{2}\Big(- \Gamma\alpha_1 \pm \sqrt{\alpha_1^2\Gamma^2 + 4\alpha_1 g(\rho-\rho_1)}\Big). \tag{6.27}$$

From (6.25) in the leading order $\mathfrak{u} = \frac{c}{\alpha_1}\eta$. Considering a relation that goes to the next order

$$\mathfrak{u} = \frac{c}{\alpha_1}\eta + \delta^2\left(\sigma\eta_{XX} + \mu\eta^2\right) \tag{6.28}$$

for some constants μ and σ we can exclude $\mathfrak{u}$ from the system (6.19)-(6.20) and write both equations in terms of η. Of course they should coincide for the special choice of the constants μ and σ which is

$$\sigma = -\frac{c\alpha_2(c+\Gamma\alpha_1)}{\alpha_1^2(2c+\Gamma\alpha_1)} \tag{6.29}$$

and

$$\mu = \frac{\alpha_1\alpha_4(c-\Gamma\alpha_1) - \alpha_3 c(c+2\Gamma\alpha_1) + \alpha_1^2\alpha_6}{2\alpha_1^2(2c+\Gamma\alpha_1)} \tag{6.30}$$

giving the KdV equation

$$\eta_T + c\eta_X + \delta^2\left(\frac{c^2\alpha_2}{\alpha_1(2c+\Gamma\alpha_1)}\right)\eta_{XXX} + \delta^2\left(\frac{\alpha_1^2\alpha_6 + 3\alpha_3c^2 + 3\alpha_1\alpha_4 c}{\alpha_1(2c+\Gamma\alpha_1)}\right)\eta\eta_X = 0. \tag{6.31}$$

Recalling the constants (6.10)–(6.15) when $\gamma = \gamma_1 = \omega = 0$ this becomes

$$\eta_T + c\eta_X + \delta^2\frac{chh_1(\rho_1h_1+\rho h)}{6(\rho_1h+\rho h_1)}\eta_{XXX} + \frac{3}{2}\delta^2 c\frac{\rho h_1^2 - \rho_1 h^2}{hh_1(\rho_1h+\rho h_1)}\eta\eta_X = 0, \tag{6.32}$$

where

$$c = \pm\sqrt{\frac{hh_1(\rho-\rho_1)g}{\rho_1h+\rho h_1}} = \pm\sqrt{\frac{(\rho-\rho_1)g}{\rho_1/h_1+\rho/h}}.$$

In the case $h\to\infty$ we have

$$c_\infty = \pm\sqrt{\frac{h_1(\rho-\rho_1)g}{\rho_1}}. \tag{6.33}$$

Next, we recall fact that the canonical KdV equation

$$E_T + E_{XXX} + 6EE_X = 0 \tag{6.34}$$

has a one-soliton solution

$$E(X,T) = 2\nu^2\mathrm{sech}^2\nu(X - 4\nu^2T - X_0)$$

where ν, X_0 are constants, related to the soliton's initial position and velocity.

Let us now introduce

$$\mathcal{A} = \delta^2\frac{\alpha_1^2\alpha_6 + 3\alpha_3c^2 + 3\alpha_1\alpha_4c}{\alpha_1(2c+\Gamma\alpha_1)}$$

$$\mathcal{B} = \delta^2\frac{c^2\alpha_2}{\alpha_1(2c+\Gamma\alpha_1)}$$

and rescale the variables

$$\eta = \alpha E, \qquad X\to\beta X, \qquad T\to\beta T$$

in order to match the coefficients of (6.34). This gives $\alpha = 6\beta^2\mathcal{B}/\mathcal{A}$. Applying further a Galilean shift we obtain the one-soliton solution of (6.31) as

$$\eta(X,T) = \frac{12\mathcal{B}}{\mathcal{A}}\nu^2\beta^2 \mathrm{sech}^2\left(\nu\beta(X - X_0 - (c + 4\nu^2\beta^2\mathcal{B})T)\right).$$

Introducing the constant $K = \nu\beta$ which has a dimensionality (length)$^{-1}$ and the meaning of an analogue of a wave number, the above formula becomes

$$\eta(X,T) = \frac{12\mathcal{B}}{\mathcal{A}}K^2 \mathrm{sech}^2\left(K(X - X_0 - (c + 4K^2\mathcal{B})T)\right). \tag{6.35}$$

The maximal amplitude of the solitary wave is therefore

$$\eta_0 = \frac{12\mathcal{B}}{\mathcal{A}}K^2$$

and it is related to the constant K. The propagation speed is

$$V = c + 4K^2\mathcal{B}$$

which is represented from the component of the leading order linear wave c and the soliton speed $4K^2\mathcal{B}$ which is proportional to the amplitude η_0 due to the K^2 factor.

Let us now analyse the irrotational case where

$$\eta_0 = \frac{4K^2h^2h_1^2(\rho_1 h_1 + \rho h)}{3(\rho h_1^2 - \rho_1 h^2)}.$$

Since ρ and ρ_1 are very close, and usually h is much bigger than h_1, then $\eta_0 < 0$ and the soliton is a depression wave. The velocity is

$$V = c\left(1 + \delta^2\frac{2}{3}K^2hh_1\right) = \pm\sqrt{\frac{hh_1(\rho - \rho_1)g}{\rho_1 h + \rho h_1}}\left(1 + \delta^2\frac{2}{3}K^2hh_1\frac{\rho_1 h_1 + \rho h}{\rho_1 h + \rho h_1}\right).$$

The plus and minus signs are for the right and left running waves respectively. Therefore the bigger wave travels faster.

6.2. The Benjamin-Ono approximation

For the Benjamin-Ono approximation we consider the system with an infinitely deep lower layer $h \to \infty$ (*cf.* [9]). The Hamiltonian is (4.14) with the following scaling

$$\eta \to \delta\eta, \quad \xi \to \xi \quad \text{and} \quad D \to \delta D. \tag{6.36}$$

The Dirichlet-Neumann operators, given in (5.2) and (5.3), can be expanded, taking into account that

$$\lim_{h\to\infty} \tanh(hD) = \mathrm{sgn}(D), \qquad \lim_{h\to\infty} D\tanh(hD) = |D|.$$

In order to explain the meaning of $|D|$, we introduce the Fourier transform

$$\hat{u}(k) := \mathcal{F}\{u(x)\}(k), \qquad u(x) = \mathcal{F}^{-1}\{\hat{u}(k)\}(x).$$

Then

$$|D|u(x) := \mathcal{F}^{-1}\{|k|\hat{u}(k)\}(x)$$

and similarly

$$\operatorname{sgn}(D)u(x) := \mathcal{F}^{-1}\{\operatorname{sgn}(k)\hat{u}(k)\}(x).$$

There is a relation between the Hilbert transform, $\mathcal{H}$

$$\mathcal{H}\{u\}(x) := \text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x')dx'}{x-x'}$$

and the Fourier transforms, namely

$$\mathcal{F}\{\mathcal{H}\{u\}(x)\}(k) = -i\operatorname{sgn}(k)\hat{u}(k)$$

or

$$\mathcal{H}\{u\}(x) = -i\mathcal{F}^{-1}\{\operatorname{sgn}(k)\hat{u}(k)\}(x).$$

Hence

$$\mathcal{H}\{Du\}(x) = -i\mathcal{F}^{-1}\{|k|\hat{u}(k)\}(x) = -i|D|u(x),$$

or

$$|D| = i\mathcal{H}D = \mathcal{H}\partial_x.$$

The expansion is

$$\begin{aligned} G(\eta) &= \delta|D| + \delta^3\big(D\eta D - |D|\eta|D|\big) + \mathcal{O}(\delta^5) \\ \text{and}\quad G_1(\eta) &= \delta D\tanh(\delta h_1 D) \\ &\quad -\delta^3\big(D\eta D - D\tanh(\delta h_1 D)\eta D\tanh(\delta h_1 D)\big) + \mathcal{O}(\delta^6) \end{aligned}$$

noting from [18] that the leading term for the infinite lower layer is $|D|$. Using the expansion for the tanh, the Dirichlet-Neumann operators can be represented further as

$$\begin{aligned} G(\eta) &= \delta|D| + \delta^3 D\eta D - \delta^3|D|\eta|D| + \mathcal{O}(\delta^5) \\ \text{and}\quad G_1(\eta) &= \delta^2 h_1 D^2 - \delta^3 D\eta D + \mathcal{O}(\delta^4) \end{aligned}$$

and so the inverse of the operator B is given by

$$B^{-1} = \frac{1}{\delta\rho_1}|D|D^{-1}\left\{1 - \delta\frac{\rho}{\rho_1}h_1|D| + \mathcal{O}(\delta^2)\right\}D^{-1}.$$

The Hamiltonian can therefore be written, using components of the expanded operators as (see the notations (6.14), (6.15))

$$\begin{aligned} H(\eta,\xi) = {} & \frac{1}{2}\delta^2\frac{h_1}{\rho_1}\int_{\mathbb{R}}\xi D^2\xi\,dx - \frac{1}{2}\delta^3\frac{h_1^2\rho}{\rho_1^2}\int_{\mathbb{R}}\xi|D|D^2\xi\,dx - \frac{1}{2}\delta^3\frac{1}{\rho_1}\int_{\mathbb{R}}\xi D\eta D\xi\,dx \\ & - \delta^3\gamma_1\int_{\mathbb{R}}\xi\eta\eta_x dx - \delta^2\kappa\int_{\mathbb{R}}\xi\eta_x dx + \frac{1}{6}\delta^3\alpha_6\int_{\mathbb{R}}\eta^3 dx + \frac{1}{2}\delta^2\alpha_5\int_{\mathbb{R}}\eta^2 dx + \mathcal{O}(\delta^4) \end{aligned} \tag{6.37}$$

and in terms of η, $\mathfrak{u}$

$$H(\eta,\mathfrak{u})=\frac{1}{2}\delta^2\frac{h_1}{\rho_1}\int_{\mathbb{R}}\mathfrak{u}^2\,dx-\frac{1}{2}\delta^3\frac{h_1^2\rho}{\rho_1^2}\int_{\mathbb{R}}\mathfrak{u}|D|\mathfrak{u}\,dx-\frac{1}{2}\delta^3\frac{1}{\rho_1}\int_{\mathbb{R}}\eta\mathfrak{u}^2\,dx$$
$$+\delta^3\frac{\gamma_1}{2}\int_{\mathbb{R}}\mathfrak{u}\eta^2\,dx+\delta^2\kappa\int_{\mathbb{R}}\mathfrak{u}\eta\,dx+\frac{1}{6}\delta^3\alpha_6\int_{\mathbb{R}}\eta^3dx+\frac{1}{2}\delta^2\alpha_5\int_{\mathbb{R}}\eta^2dx+\mathcal{O}(\delta^4). \tag{6.38}$$

The equations of motion (4.19) are now written in terms of η and $\mathfrak{u}$ as

$$\eta_t+\kappa\eta_x+\frac{h_1}{\rho_1}\mathfrak{u}_x-\delta\frac{h_1^2\rho}{\rho_1^2}|D|\mathfrak{u}_x-\delta\frac{1}{\rho_1}(\eta\mathfrak{u})_x+\delta\gamma_1\eta\eta_x=0 \tag{6.39}$$

$$\text{and}\quad \mathfrak{u}_t+\kappa\mathfrak{u}_x-\delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_x+\delta\gamma_1(\eta\mathfrak{u})_x+\delta\alpha_6\eta\eta_x+\alpha_5\eta_x+\Gamma\eta_t=0. \tag{6.40}$$

Again we perform the Galilean shift (6.18) noting that $g\gg 2\omega\kappa$ and $\alpha_5-\Gamma\kappa\approx g(\rho-\rho_1)$ to obtain

$$\eta_T+\frac{h_1}{\rho_1}\mathfrak{u}_X-\delta\frac{h_1^2\rho}{\rho_1^2}|D|\mathfrak{u}_X-\delta\frac{1}{\rho_1}(\eta\mathfrak{u})_X+\delta\gamma_1\eta\eta_X=0 \tag{6.41}$$

$$\text{and}\quad \mathfrak{u}_T-\delta\frac{1}{\rho_1}\mathfrak{u}\mathfrak{u}_X+\delta\gamma_1(\eta\mathfrak{u})_X+\delta\alpha_6\eta\eta_X+g(\rho-\rho_1)\eta_X+\Gamma\eta_T=0. \tag{6.42}$$

In the leading order

$$\eta_T=-\frac{h_1}{\rho_1}\mathfrak{u}_X\quad\text{and}\quad \mathfrak{u}_T=-g(\rho-\rho_1)\eta_X-\Gamma\eta_T.$$

Again using exponential representations (6.23) the above equations give

$$-c\eta=-\frac{h_1}{\rho_1}\mathfrak{u} \tag{6.43}$$

$$\text{and}\quad -c\mathfrak{u}=\big(-g(\rho-\rho_1)+c\Gamma\big)\eta. \tag{6.44}$$

This gives an equation $c^2=-h_1\big(-g(\rho-\rho_1)+c\Gamma\big)/\rho_1$ with solutions

$$c=-\frac{h_1}{2\rho_1}\Gamma\pm\frac{1}{2}\sqrt{\frac{h_1^2}{\rho_1^2}\Gamma^2+4\frac{h_1}{\rho_1}g(\rho-\rho_1)}. \tag{6.45}$$

Considering an expansion of the type of (6.28)

$$\mathfrak{u}=\frac{\rho_1}{h_1}c\eta+\delta\alpha\eta^2+\delta\beta|D|\eta,$$

we can determine that

$$\alpha=\frac{\rho_1(\rho_1c^2+2h_1\Gamma c-\gamma_1h_1^2\Gamma+\rho_1\gamma_1h_1c+h_1^2\alpha_6)}{2h_1^2\big(2\rho_1c+h_1\Gamma\big)} \tag{6.46}$$

and

$$\beta=\frac{\rho(\rho_1c^2+h_1\Gamma c)}{2\rho_1c+h_1\Gamma}. \tag{6.47}$$

The Benjamin-Ono equation is therefore given by

$$\eta_T + c\eta_X - \delta\frac{\rho h_1 c^2}{2\rho_1 c + h_1\Gamma}|\partial_X|\eta_X + \delta\frac{-3\rho_1 c^2 + 3\rho_1\gamma_1 h_1 c + h_1^2\alpha_6}{h_1(2\rho_1 c + h_1\Gamma)}\eta\eta_x = 0. \quad (6.48)$$

The obtained equation is the well known Benjamin-Ono (BO) equation [1, 29] which is an integrable equation whose solutions can be obtained by the Inverse Scattering method [23].

The Benjamin-Ono equation in the irrotational case ($\gamma = \gamma_1 = \omega = 0$, $\alpha_6 = \Gamma = 0$) becomes (*cf.* [5])

$$\eta_t + c\eta_x - \frac{1}{2}\delta\frac{\rho h_1 c}{\rho_1}|D|\eta_x - \frac{3}{2}\delta\frac{c}{h_1}\eta\eta_x = 0, \quad (6.49)$$

where, from (6.45)

$$c = \pm\sqrt{\frac{h_1}{\rho_1}g(\rho - \rho_1)}.$$

This wavespeed of course coincides with (6.33).

The BO equation in the form

$$\eta_T + c\eta_X + \mathcal{A}\eta\eta_X + \mathcal{B}|\partial_X|\eta_X = 0 \quad (6.50)$$

has a one-soliton solution

$$\eta(X,T) = \frac{\eta_0}{1 + \left(\frac{\mathcal{A}\eta_0}{4\mathcal{B}}\right)^2\left[X - X_0 - \left(c + \frac{1}{4}\mathcal{A}\eta_0\right)T\right]^2} \quad (6.51)$$

where the amplitude η_0 and the initial displacement X_0 are arbitrary constants. From (6.48) for the internal wave equation

$$\mathcal{A} := \delta\frac{-3\rho_1 c^2 + 3\rho_1\gamma_1 h_1 c + h_1^2\alpha_6}{h_1(2\rho_1 c + h_1\Gamma)} \quad (6.52)$$

and

$$\mathcal{B} := \delta\frac{\rho h_1 c^2}{2\rho_1 c + h_1\Gamma}. \quad (6.53)$$

We note that (6.51) shows that the wavespeed of the soliton $c + \frac{1}{4}\mathcal{A}\eta_0$ depends on its amplitude η_0 and on the parameters of the system.

7. Discussion

The illustrative one-soliton solutions of the KdV (6.35) and the BO equation (6.51) suffers, however, from the following disadvantages. First, the BO soliton is not in the Schwartz class in the x-variable, which is not a very serious disadvantage from the physical point of view. Second, the assumption (2.2) for η is violated since for the one-soliton solutions have finite “mass” proportional to $\int_{\mathbb{R}}\eta(X,T)dX$, which for the KdV model is $24\mathcal{B}K/\mathcal{A}$ and for the BO model is $\pi\mathcal{B}/\mathcal{A}$. One can argue again that this does not change the physical setup. Indeed, the average value of η would be

$$\langle \eta \rangle = \frac{\int_{\mathbb{R}} \eta(X,T)dX}{\int_{\mathbb{R}} dX} = 0$$

since the nominator is finite and the denominator is infinite. We note also that the "mass" $\int_{\mathbb{R}} \eta(X,T)dX$, is always a conserved quantity due to (4.19). Therefore the extra condition (2.2) can be properly relaxed, allowing for solitary waves with a finite "mass".

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