2007-01-01

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Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 317 (2007) 510–518
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On cosmall Abelian groups

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Received 8 February 2007

Communicated by Kent R. Fuller

Abstract

It is a well-known homological fact that every Abelian group G has the property that $\mathrm{Hom}(G, -)$ commutes with direct products. Here we investigate the 'dual' property: an Abelian group G is said to be cosmall if $\mathrm{Hom}(-, G)$ commutes with direct products. We show that cosmall groups are cotorsion-free and that no group of cardinality less than a strongly compact cardinal can be cosmall. In particular, if there is a proper class of strongly compact cardinals, then there are no cosmall groups.

Keywords: Abelian groups; Homological algebra; Set theory

1. Introduction

There are two canonical isomorphisms in homological algebra that relate the homomorphism groups involving an Abelian group G and the direct sums and products of indexed families $\{A_i\colon\ i \in I\}$—see for example [6, Theorems 43.2, 43.1]:

$$\mathrm{Hom}\Big(G, \prod_{i\in I} A_i\Big) \cong \prod_{i\in I} \mathrm{Hom}(G, A_i), \tag{1}$$

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1 The author would like to acknowledge the support of a Dublin Institute of Technology travel grant to visit Dublin for this joint work.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.05.009

$$\mathrm{Hom}\Bigl(\bigoplus_{i\in I} A_i, G\Bigr) \cong \prod_{i\in I} \mathrm{Hom}(A_i, G). \tag{2}$$

This paper examines consequences of trying to 'dualize' (1) and (2) by interchanging direct sums and direct products in various ways.

In addition to the two homological isomorphisms above, there are six further possible simple isomorphism assertions involving the operations Hom, $\bigoplus$ and $\prod$. Three are evidently false except in the trivial case where $G = 0$, for simple reasons of cardinality: there is no non-trivial group G such that

$$\mathrm{Hom}\Bigl(G, \prod_{i\in I} A_i\Bigr) \cong \bigoplus_{i\in I} \mathrm{Hom}(G, A_i), \tag{3}$$

$$\mathrm{Hom}\Bigl(\bigoplus_{i\in I} A_i, G\Bigr) \cong \bigoplus_{i\in I} \mathrm{Hom}(A_i, G) \tag{4}$$

or such that

$$\mathrm{Hom}\Bigl(G, \bigoplus_{i\in I} A_i\Bigr) \cong \prod_{i\in I} \mathrm{Hom}(G, A_i) \tag{5}$$

for all indexed families $\{A_i\colon i \in I\}$.

Notable contributions concerning two of the remaining possibilities were made in 1975 by Arnold and Murley [2] and Göbel [8]. Arnold and Murley defined the concept of smallness without any explicit reference to the homological isomorphisms above: an Abelian group G is *small* if

$$\mathrm{Hom}\Bigl(G, \bigoplus_{i\in I} A_i\Bigr) \cong \bigoplus_{i\in I} \mathrm{Hom}(G, A_i) \tag{6}$$

for all indexed families $\{A_i\colon i \in I\}$. Note that the isomorphism is not required to be canonical in any sense. A group G is *self-small* if $\mathrm{Hom}(G, G^{(I)}) \cong (\mathrm{Hom}(G, G))^{(I)}$ for all I, where, as usual, $G^{(I)}$ denotes the direct sum of $|I|$ copies of G. Thus, the notion of smallness is obtained if the direct products are replaced by direct sums in (1) above, and the class of small groups is therefore defined by the dual of (1).

It is an easy, and well-known, exercise to show that the group of integers, $\mathbb{Z}$, is small and hence self-small. Indeed it is not difficult to modify an argument due to Rentschler [11], to show that a torsion-free group is small precisely if it has finite rank. Arnold and Murley have shown that self-small groups exist in reasonable abundance: for example if the endomorphism ring of a torsion-free group G, $\mathrm{End}(G)$, is countable, then G is self-small. There has been considerable interest in self-small groups since the appearance of the original paper by Arnold and Murley; a typical example of work in this area is the recent paper by U. Albrecht, S. Breaz and W. Wickless [1].

The dual of the homological isomorphism (2) above is obtained by simultaneously replacing direct sums by direct products and direct products by direct sums. Indeed, Rüdiger Göbel [8], in his discussion of (2), explicitly invoked the terminology of duality, pointing out a parallel with the Riesz–Fischer theorem in functional analysis. The substitution yields the following:

$$\mathrm{Hom}\left(\prod_{i\in I} A_i, G\right) \cong \bigoplus_{i\in I} \mathrm{Hom}(A_i, G) \tag{7}$$

for all indexed families $\{A_i\colon i \in I\}$. At first glance, this concept is an apparent strengthening of the well-known concept of a slender group, and so we shall refer to a group G satisfying (7) as *strongly slender*. When each A_i is replaced by G itself, we shall say that G is *strongly self-slender*. This latter notion is studied in recent papers by Faticoni [5] and Göbel & the authors [9].

Applying the dual approach to that used to define small groups, one obtains the class of *co-small groups*, i.e. those groups G such that

$$\mathrm{Hom}\left(\prod_{i\in I} A_i, G\right) \cong \prod_{i\in I} \mathrm{Hom}(A_i, G) \tag{8}$$

for all indexed families $\{A_i\colon i \in I\}$. Replacing each A_i by G itself, we obtain the obvious analogue of self-small, which we shall call *self-co-small*: G is self-co-small if $\mathrm{Hom}(G^\lambda, G) \cong \mathrm{End}(G)^\lambda$ for every cardinal λ. We shall write these terms as cosmall and self-cosmall; clearly the trivial group is both cosmall and self-cosmall. Moreover it will follow from Theorem 2.5 and our remark above about small groups, that the only group which is both small and cosmall is the trivial group.

This paper studies the immediate natural question whether non-trivial cosmall or self-cosmall groups exist; *we shall assume throughout that we are dealing with non-trivial groups*.

We write G^I and $G^{(I)}$ for the Cartesian product $\prod_{i\in I} G$ and direct sum $\bigoplus_{i\in I} G$ except where the latter notation is easier to read. $\mathrm{End}(G)$ stands for $\mathrm{Hom}(G, G)$, the group of endomorphisms of G. Infinite cardinals are usually denoted by κ, λ, μ; all other notation is standard and may be found in [4,6,7]; in particular all groups shall be additively written Abelian groups. The symbols $\mathbb{Z}, \mathbb{Q}, J_p$ will be used to denote respectively the additive groups of integers, rationals and p-adic integers. Recall that a cardinal κ is *ω-measurable*, if there exists a countably complete non-principal ultrafilter over κ. An uncountable cardinal κ is *measurable* if there exists a κ-complete non-principal ultrafilter over κ. The least ω-measurable cardinal is measurable. Measurable cardinals are strongly inaccessible. If κ is measurable, then there are at least 2^κ κ-complete non-principal ultrafilters over κ. It is, however, consistent with ordinary set theory (*ZFC*) that no measurable cardinals exist. For example, $ZFC + V = L$ implies that there are no measurable cardinals (and hence no ω-measurable cardinals). We shall use κ^* to denote the first measurable cardinal, if there exist measurable cardinals; otherwise the condition $\alpha < \kappa^*$ is vacuously satisfied for every α. To avoid confusion, we point out that in Fuchs [7] the term measurable is used to refer to a non-trivial countably additive two-valued measure; this is what we call ω-measurable, as in [4]. Finally recall the beth function: for a cardinal κ and an ordinal α, define $\beth_\alpha(\kappa)$ by: $\beth_0(\kappa) = \kappa$, $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$, and for a limit ordinal α, $\beth_\alpha(\kappa) = \sup_{\beta<\alpha} \beth_\beta(\kappa)$.

2. Cosmall and self-cosmall groups

Let us note first a criterion.

Lemma 2.1. *Suppose that for some infinite cardinals κ, λ with $\kappa \geqslant |G|$, G^κ has a direct summand of the form $H^{(\lambda)}$, for some non-trivial $H \leqslant G$. Then if $2^\lambda > 2^\kappa$, G is not self-cosmall.*

Proof. By (2) above, $|\mathrm{Hom}(G^\kappa, G)| \geqslant |\mathrm{Hom}(H^{(\lambda)}, G)| = |\mathrm{Hom}(H, G)^\lambda| \geqslant 2^\lambda > 2^\kappa$. However $|\mathrm{End}(G)^\kappa| \leqslant 2^\kappa$ so $\mathrm{End}(G)^\kappa$ and $\mathrm{Hom}(G^\kappa, G)$ cannot be isomorphic. □

A group G is cotorsion-free if it does not contain any non-zero subgroups that are cotorsion. Recall that a necessary and sufficient condition for a group G to be cotorsion-free, is that G be reduced, torsion-free and not contain a direct summand isomorphic to J_p for any prime p.

Theorem 2.2. *A self-cosmall group G is cotorsion-free; in particular it is torsion-free.*

Proof. Suppose not, then G has a summand isomorphic to one of $\mathbb{Z}(p^n), \mathbb{Z}(p^\infty), \mathbb{Q}$ or J_p for some prime p. We show that each of these possibilities leads to a contradiction. Let κ be a cardinal with $\kappa > |G|$. Consider firstly the possibility that G has a summand isomorphic to $\mathbb{Z}(p^n)$. Then G^κ has a summand isomorphic to $\prod_\kappa \mathbb{Z}(p^n) \cong \bigoplus_{2^\kappa} \mathbb{Z}(p^n)$ and it follows by the previous lemma that G is not self-cosmall. Since $\mathbb{Q}^\kappa \cong \mathbb{Q}^{(2^\kappa)}$ a similar argument shows that G cannot have a subgroup isomorphic to $\mathbb{Q}$. If G has a subgroup isomorphic to $\mathbb{Z}(p^\infty)$ then a minor variation of this argument works: $\mathbb{Z}(p^\infty)^\kappa$ is no longer isomorphic to $\bigoplus_{2^\kappa} \mathbb{Z}(p^\infty)$ but it has a summand isomorphic to the latter and this clearly suffices. Finally to see that G cannot have a subgroup isomorphic to J_p, note that in such circumstances G^κ would have a summand J_p^κ which is isomorphic to $\widehat{\bigoplus_{2^\kappa} J_p}$. Thus $|\mathrm{Hom}(G^\kappa, G)| \geqslant |\mathrm{Hom}(\widehat{\bigoplus_{2^\kappa} J_p}, G)| \geqslant |\mathrm{Hom}(\widehat{\bigoplus_{2^\kappa} J_p}, J_p)|$. However the fact that J_p is algebraically compact implies that $\mathrm{Hom}(\widehat{\bigoplus_{2^\kappa} J_p}, J_p) \cong \mathrm{Hom}(\bigoplus_{2^\kappa} J_p, J_p)$ and this then suffices, by an entirely analogous argument to the above, to dispose of the possibility that $J_p \leqslant G$. □

Proposition 2.3. *A product of cosmall groups is cosmall.*

Proof. Suppose that $\{G_k\colon k \in K\}$ is a family of cosmall groups and $\{A_i\colon i \in I\}$ is an arbitrary indexed family. Then, using (1) and the cosmallness of each G_k,

$$\mathrm{Hom}\Bigl(\prod_{i\in I} A_i, \prod_{k\in K} G_k\Bigr) \cong \prod_{k\in K} \mathrm{Hom}\Bigl(\prod_{i\in I} A_i, G_k\Bigr) \cong \prod_{k\in K}\prod_{i\in I} \mathrm{Hom}(A_i, G_k).$$

On the other hand,

$$\prod_{i\in I} \mathrm{Hom}\Bigl(A_i, \prod_{k\in K} G_k\Bigr) \cong \prod_{i\in I}\prod_{k\in K} \mathrm{Hom}(A_i, G_k) \cong \prod_{k\in K}\prod_{i\in I} \mathrm{Hom}(A_i, G_k),$$

since the repeated products are isomorphic. □

Proposition 2.4. *The class of cosmall groups is not closed under direct sums.*

Proof. Suppose that G is cosmall and let $\lambda = |G|$. Choose κ so that $\kappa^\lambda > \kappa \geqslant 2^\lambda$; for example take $\kappa = \beth_\omega(\lambda)$. Claim that $G^{(\kappa)}$ is not cosmall. If it were, then $\mathrm{Hom}(\mathbb{Z}^\lambda, G^{(\kappa)}) \cong \prod_\lambda \mathrm{Hom}(\mathbb{Z}, G^{(\kappa)})$ and since $\bigoplus_\kappa G$ has cardinality κ, $\mathrm{Hom}(\mathbb{Z}^\lambda, G^{(\kappa)})$ would have cardinality κ^λ. However if ϕ is any map in $\mathrm{Hom}(\mathbb{Z}^\lambda, G^{(\kappa)})$, then since G is torsion-free and reduced, it follows from the extension of a result of Chase—see [4, III, Theorem 3.9]—that there exists a finite subset J_ϕ of λ and a finite subset E of κ such that the image of $\phi \restriction \prod_{\lambda\setminus J_\phi}$ is contained in $G^{(E)}$.

Thus each ϕ may be expressed as a sum $\phi = \phi_0 + \phi_1$ where ϕ_1 is the restriction of ϕ to a product over a finite subset J_ϕ of λ. Moreover every such ϕ_0 may be regarded as an element of $\bigoplus_\kappa \mathrm{Hom}(\mathbb{Z}^\lambda, G) \cong \bigoplus_\kappa G^\lambda$ since G is cosmall by assumption. Consequently the cardinality of $\mathrm{Hom}(\mathbb{Z}^\lambda, G^{(\kappa)})$ is at most $\max\{\lambda^\lambda, \kappa\} = \kappa$ by the choice of κ—contradiction. Thus $G^{(\kappa)}$ is not cosmall, as required. □

Theorem 2.5. *A slender group is never cosmall; in particular free groups, completely decomposable groups and groups of power less than the continuum are never cosmall.*

Proof. If G is slender, we show that the groups $\mathrm{Hom}(\mathbb{Z}^\omega, G)$ and $\mathrm{Hom}(\mathbb{Z}, G)^\omega$ cannot be isomorphic. Now $\mathrm{Hom}(\mathbb{Z}^\omega, G) \cong \mathrm{Hom}(\mathbb{Z}, G)^{(\omega)} \cong G^{(\omega)}$ since G is slender. However, by a well-known theorem of Fuchs [7, Theorem 94.3], direct sums of slender groups are slender. However $\mathrm{Hom}(\mathbb{Z}, G)^\omega \cong G^\omega$ is not slender since it contains an isomorphic copy of $\mathbb{Z}^\omega$. Thus $\mathrm{Hom}(\mathbb{Z}^\omega, G)$ and $\mathrm{Hom}(\mathbb{Z}, G)^\omega$ are not isomorphic. The final assertion is immediate since direct sums of countable torsion-free groups are slender, as are groups of power less than the continuum. □

Corollary 2.6. *No subgroup of J_p is cosmall.*

Proof. By an observation of G.A. Reid—see [7, Exer. 3, p. 166]—cotorsion-free subgroups of J_p are slender. □

In the next sequence of results, we shall appeal frequently to the following theorem concerning $\mathrm{Hom}(A^\kappa, G)$, due, in various degrees of generality, to Balcerzyk, Łoś and Eda—see [4, III, Corollary 3.7].

Theorem. *Suppose that A is slender and that the cardinality of G is not ω-measurable. Then for any cardinal κ, there exists a cardinal $\lambda \geqslant \kappa$ such that* $\mathrm{Hom}(G^\kappa, A) \cong \mathrm{Hom}(G, A)^{(\lambda)}$. *If κ is not ω-measurable, then $\lambda = \kappa$.*

Proposition 2.7. *Suppose G is a subgroup of a Cartesian product $\prod_{i \in I} A_i$ of slender groups, and suppose $\lambda = |I| + \sup\{|\mathrm{Hom}(G, A_i)|\colon i \in I\} < \kappa^*$. Then G is not self-cosmall.*

Proof. Let $\kappa = 2^\lambda$. Note that $\kappa < \kappa^*$ since the latter is strongly inaccessible and hence κ is not ω-measurable. Note also that $\kappa^{|I|} = \kappa$. Now

$$\mathrm{Hom}\big(G^\kappa, G\big) \leqslant \mathrm{Hom}\Bigg(G^\kappa, \prod_{i \in I} A_i\Bigg) \cong \prod_{i \in I} \mathrm{Hom}\big(G^\kappa, A_i\big) \cong \prod_{i \in I} \bigoplus_\kappa \mathrm{Hom}(G, A_i)$$

where the last isomorphisms are obtained from the previous theorem. Thus

$$\big|\mathrm{Hom}\big(G^\kappa, G\big)\big| \leqslant \left|\prod_{i \in I} \bigoplus_\kappa \mathrm{Hom}(G, A_i)\right| \leqslant \kappa^{|I|} = \kappa.$$

However the cardinality of $\prod_\kappa \mathrm{End}(G)$ is at least 2^κ and so this product cannot be isomorphic to $\mathrm{Hom}(G^\kappa, G)$. Thus G is not self-cosmall. □

Corollary 2.8. *A product G of slender groups that has cardinality less than κ^*, is never self-cosmall. In particular the higher Baer–Specker groups $\mathbb{Z}^\lambda (\lambda < \kappa^*)$ are not self-cosmall.*

Proof. This follows immediately from the previous proposition: $|\mathrm{End}(G)| \leqslant 2^{|G|} < \kappa^*$ since measurable cardinals are strongly inaccessible. □

If we work in a universe where there are no measurable cardinals, then the above results can be restated as:

Corollary 2.9. *Assume that no measurable cardinals exist. Then*

(i) *no subgroup of a Cartesian product of slender groups is self-cosmall*; *in particular, torsion-less groups are not cosmall*;
(ii) *for each cardinal κ, the higher Baer–Specker group $\mathbb{Z}^\kappa$ is not self-cosmall.*

By making use of more algebraic arguments we can remove restrictions on the size of products, obtaining results for cosmallness only:

Proposition 2.10. *For every infinite cardinal κ, the higher Baer–Specker group $\mathbb{Z}^\kappa$ is not cosmall.*

Proof. By (1) and the slenderness of $\mathbb{Z}$, $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}^\kappa) \cong \prod_\kappa \mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \prod_\kappa \bigoplus_\omega \mathbb{Z}$. However $\prod_\omega \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^\kappa) \cong \mathbb{Z}^\kappa$. Thus $\mathrm{Hom}(\mathbb{Z}^\omega, \mathbb{Z}^\kappa)$ and $\prod_\omega \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}^\kappa)$ belong to different Reid classes and hence are non-isomorphic—see [4, Chapter X]. □

It is in fact possible to extend Proposition 2.10 to a much larger class of groups. Let $\mathcal{C}$ be a class of non-zero slender groups, each of non-ω-measurable cardinality. Then, as observed in [4, p. 323], it is possible to carry through an analysis by generalized Reid classes, replacing $\mathbb{Z}$ with the class $\mathcal{C}$. Consequently one can easily extend the above proposition to obtain:

Corollary 2.11. *If each group $G_\alpha (\alpha < \kappa)$ is slender and of non-ω-measurable cardinality, then for every infinite cardinal κ, the product $\prod_{\alpha<\kappa} G_\alpha$ is not cosmall.*

We can extend the scope of Proposition 2.7 but again we require cardinality restrictions involving κ^*.

Proposition 2.12. *Suppose that G is an extension of a product $\prod_{\alpha<\kappa} A_\alpha$ of slender groups by a product $\prod_{\beta<\lambda} B_\beta$ of slender groups and $|G| < \kappa^*$, then G is not self-cosmall.*

Proof. Since G is an extension of the above form, we obtain, on taking homomorphisms from G^μ (μ arbitrary), a sequence

$$0 \to \mathrm{Hom}\Bigl(G^\mu, \prod_{\alpha<\kappa} A_\alpha\Bigr) \to \mathrm{Hom}\bigl(G^\mu, G\bigr) \to \mathrm{Hom}\Bigl(G^\mu, \prod_{\beta<\lambda} B_\beta\Bigr).$$

It follows that $|\mathrm{Hom}(G^\mu, G)| \leqslant |\mathrm{Hom}(G^\mu, \prod_{\alpha<\kappa} A_\alpha)|.|\mathrm{Hom}(G^\mu, \prod_{\beta<\lambda} B_\beta)|$. Now choose $\mu = 2^{|G|}$ and note that $\mu^\kappa = \mu^\lambda = \mu < \kappa^*$.

Since each A_α is slender and $|G| < \kappa^*$, $\mathrm{Hom}(G^\mu, \prod_{\alpha<\kappa} A_\alpha) \cong \prod_{\alpha<\kappa} \bigoplus_\mu \mathrm{Hom}(G, A_\alpha)$. Clearly $|\mathrm{Hom}(G, A_\alpha)| \leqslant 2^{|G|} = \mu$ and so it follows that $|\mathrm{Hom}(G^\mu, \prod_{\alpha<\kappa} A_\alpha)| = \mu^\kappa = \mu$. A similar argument shows that $|\mathrm{Hom}(G^\mu, \prod_{\beta<\lambda} B_\beta)| = \mu^\lambda = \mu$, and so we conclude that $|\mathrm{Hom}(G^\mu, G)| \leqslant \mu.\mu = \mu$. However it is immediate that $|\prod_\mu \mathrm{Hom}(G, G)| \geqslant 2^\mu$ and so G cannot be self-cosmall. □

Corollary 2.13. *If G is an extension of a higher Baer–Specker group by a higher Baer–Specker group and $|G| < \kappa^*$, then G is not self-cosmall.*

An immediate question arising from Proposition 2.12 is whether it is possible to drop the requirement that we have a *product* of slender groups B_β. Our next result shows that this is possible in the case where the product is replaced by a free group.

Proposition 2.14. *If G is self-cosmall and $|G| < \kappa^*$, then G has no slender summands.*

Proof. Suppose for a contradiction that $G = H \oplus K$, where K is a non-trivial slender group. Since K is slender and $|G| < \kappa^*$, $\mathrm{Hom}(G^\kappa, G) \cong \mathrm{Hom}(G^\kappa, H) \oplus \bigoplus_\kappa \mathrm{Hom}(G, K)$ provided that κ is also $< \kappa^*$; it follows immediately that, subject to the restriction on κ, we can write $\mathrm{Hom}(G^\kappa, G) \cong A \oplus \bigoplus_\kappa \mathrm{Hom}(K, K)$ for a suitable A. Since G is self-cosmall, we also have $\mathrm{Hom}(G^\kappa, G) \cong \prod_\kappa \mathrm{Hom}(G, G)$. Writing $X = \mathrm{Hom}(G, G)$ and $W = \mathrm{Hom}(K, K)$, this yields $X^\kappa \cong A \oplus \bigoplus_\kappa W$. Now take homomorphisms into K and choose $\kappa = |\mathrm{Hom}(X, K)|^+$; note that $\kappa < \kappa^*$. Then $\mathrm{Hom}(X^\kappa, K) \cong \bigoplus_\kappa \mathrm{Hom}(X, K)$ since K is slender and $\kappa < \kappa^*$. The choice of κ now ensures that $|\mathrm{Hom}(X^\kappa, K)| \leqslant \kappa$.

We also have that $\mathrm{Hom}(X^\kappa, K) \cong \mathrm{Hom}(A, K) \oplus \mathrm{Hom}(\bigoplus_\kappa W, K) \cong \mathrm{Hom}(A, K) \oplus \prod_\kappa \mathrm{Hom}(W, K)$. This last term will have cardinality $> \kappa$, a contradiction, provided that $\mathrm{Hom}(W, K) \neq 0$. Thus it suffices to show that $\mathrm{Hom}(W, K) \neq 0$. However $W = \mathrm{Hom}(K, K)$ and for each $0 \neq k \in K$, the evaluation map $\chi_k : \mathrm{Hom}(K, K) \to K$ given by $\chi_k(\phi) = \phi(k)$ is a non-zero homomorphism. □

3. Large cardinals and cosmall groups

With the exception of Corollary 2.9, all the results of Section 2 are theorems of ordinary set theory with choice (*ZFC*). Yet we have been unable to prove that there exist any cosmall or self-cosmall groups. In this section, we shall show that certain large cardinal axioms imply that in fact there are no cosmall or self-cosmall groups. Recall that an uncountable cardinal λ is *strongly compact* if for every set I, every λ-complete filter over I can be extended to a λ-complete ultrafilter over I. Strongly compact cardinals are measurable. It is an important theorem due to Kunen [10] and Comfort and Negrepontis [3] that if λ is strongly compact and $\kappa \geqslant \lambda$ is such that $\kappa^{<\lambda} = \kappa$, then there are 2^{2^κ} λ-complete ultrafilters on κ.

A simple observation, which is certainly well known and does not require the cardinal λ to be strongly compact, will be of fundamental use; a proof is included for completeness.

Lemma 3.1. *Let K have cardinality less than κ and suppose that $K \leqslant H$. Then $|\mathrm{Hom}(K^\lambda, H)|$ is at least as large as the number of κ-complete ultrafilters on λ.*

Proof. Suppose U is a κ-complete ultrafilter on λ. For $a \in K^\lambda$, let $\phi_U(a) = k$ iff for $k \in K$, $a^{-1}(k) = \{\alpha < \lambda \colon a(\alpha) = k\} \in U$. The κ-completeness of U ensures the map ϕ_U is well defined since the sets $a^{-1}(k)$ $(k \in K)$, are a partition of λ into fewer than κ subsets—see, for example, [4, Lemma II 2.6]. It is straightforward to verify that the map $\phi_U : K^\lambda \to K$ is a homomorphism. If U and V are different κ-complete ultrafilters, then $\phi_U \neq \phi_V$: to see this choose a set $X \in U \setminus V$. Fix a non-zero element $x \in K$ and define an element $g \in K^\lambda$ by setting $g(\alpha) = x$ for $\alpha \in X$ and $g(\alpha) = 0$ otherwise. From the definition of ϕ_U and the fact that $X \in U$, it follows that $\phi_U(g) = x \neq 0$. However $\phi_V(g) = 0$ since $\lambda \setminus X \in V$. Thus $\phi_U \neq \phi_V$. Since $K \leqslant H$, the result follows immediately. □

Our next result re-derives an earlier property of cosmall groups from this new standpoint. It is a useful introduction to the more important Theorem 3.3 below.

Proposition 3.2. *A cosmall group is torsion-free.*

Proof. Suppose G is cosmall and contains a torsion element, g say, of order n. Let $\kappa = \omega$ and set $\lambda = |G| + \omega$ in Lemma 3.1 with $K = \langle g \rangle$, and $H = G$. Since ultrafilters are always ω-complete, we see from that lemma that $|\mathrm{Hom}(\langle g \rangle^\lambda, G)| \geqslant 2^{2^\lambda}$ since there are 2^{2^λ} ultrafilters on λ. However $|\prod_\lambda \mathrm{Hom}(\langle g \rangle, G)| \leqslant (|G|^n)^\lambda \leqslant 2^\lambda < 2^{2^\lambda}$ and so G cannot be cosmall. □

Exactly the same idea works to show that under the assumption of a large cardinal axiom, there are no cosmall groups.

Theorem 3.3. *If there exists a strongly compact cardinal λ, then there are no cosmall groups.*

Proof. Suppose G is cosmall; note then that $\mathbb{Z} \leqslant G$. Set $\kappa = 2^{|G| \times \lambda} > \lambda, |G|$. Now apply Lemma 3.1 with $K = \mathbb{Z}$ and $H = G$. Thus $|\mathrm{Hom}(\mathbb{Z}^\kappa, G)| \geqslant$ the number of κ-complete ultrafilters on λ. Recall that $\kappa^{<\lambda} = \sup\{\kappa^\mu\colon \mu < \lambda\}$, so that in this case $\kappa^{<\lambda} = \kappa$. Hence it follows from the results of Kunen, and Comfort and Negrepontis mentioned previously, that $|\mathrm{Hom}(\mathbb{Z}^\kappa, G)| \geqslant 2^{2^\kappa}$. However $|\prod_\kappa \mathrm{Hom}(\mathbb{Z}, G)| = |G|^\kappa \leqslant 2^{|G| \times \kappa} = 2^\kappa < 2^{2^\kappa}$—contradiction. Thus G is not self-cosmall. □

Theorem 3.4. *Suppose that λ is a strongly compact cardinal, then no group of cardinality less than λ is self-cosmall.*

Proof. Suppose that $|G| < \lambda$. Since λ is strongly inaccessible, it follows that $|\mathrm{Hom}(G, G)^\lambda| \leqslant 2^\lambda$. Now apply Lemma 3.1 with $K = H = G$: since there are 2^{2^λ} many λ-complete ultrafilters on λ, it follows that $|\mathrm{Hom}(G^\lambda, G)| \geqslant 2^{2^\lambda} > 2^\lambda \geqslant |\mathrm{Hom}(G, G)|^\lambda$. Thus G is not self-cosmall. □

One can then deduce:

Corollary 3.5. *If there exists a proper class of strongly compact cardinals, then there are no self-cosmall groups.*

Theorem 3.4 renders the existence of self-cosmall groups at least as unlikely as the non-existence of many strongly compact cardinals. Hence we ask the following question:

Question. (i) Does *ZFC* (or $ZFC + V = L$) suffice to prove that there are no cosmall groups?
(ii) Does *ZFC* (or $ZFC + V = L$) suffice to prove that there are no self-cosmall groups?

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