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Unit Sum Numbers of Rings and Modules

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Dedicated to Graham Higman on his 80th Birthday

§0 Introduction

The relationship between the endomorphisms and automorphisms of algebraic objects has long been a subject of interest. In 1963 Fuchs [3] raised the question of when the automorphism group of an abelian group (additively) generates the endomorphism group. Further interest in a different direction on the relationship between automorphisms and endomorphisms of an abelian group was raised by Kaplansky's introduction [8] of the notions of transitivity and full transitivity. The problem of Fuchs and related generalizations have produced ongoing interest and there is an existing literature of which [1], [2], [6], [7], [11] are the principal results.

The primary focus of attention in this paper is the representation of arbitrary endomorphisms of a module as the sum of a fixed number of automorphisms. Our first focus is on linear transformations of vector spaces of arbitrary dimension. The result (Theorem 2.5) that, with one exception, every such transformation is the sum of two automorphisms can hardly be new but we have been unable to find any reference to it in the literature. It is worth remarking that the proof is constructive with an explicit algorithm for the construction of the automorphisms being given. In the remainder of the paper we exploit this result on vector spaces to derive similar results for a wider class of modules.

Our terminology is standard and may be found in the texts [4], [5] or [8]; an exception being that we write mappings on the right. It will also be useful to distinguish rings from groups or modules and to this end we adopt the notation of using bold face characters for rings; as usual $\mathbf{Z}$, $\mathbf{Z_k}$, $\mathbf{J_p}$ will denote the ring of integers, of integers modulo k and of p–adic integers while $\mathbf{GF(q)}$ and $\mathbf{Q}$ will denote the Galois field of q elements and the field of rational integers respectively.

§1 Unit sum numbers of rings

An associative unital ring $\mathbf{R}$ is said to have the *n–sum property* if every element of $\mathbf{R}$ can be written as the sum of exactly n units of $\mathbf{R}$. It is immediate that if $\mathbf{R}$ has the n–sum property then it has the k–sum property for every $k \geq n$. (It might seem more useful at first sight to confine the n–sum property to the non–zero elements of $\mathbf{R}$. However, by considering for example the field $\mathbf{GF(2)}$ which would have the 1–sum property in this new sense, one sees that the k–sum property for $k \geq 1$ does not hold.) Thus it makes sense to define the *unit sum number* of $\mathbf{R}$ by $usn(\mathbf{R}) = min\{n \mid \mathbf{R}\ has\ the\ n{-}sum\ property\}$. If there is an element of $\mathbf{R}$ which cannot be written as a sum of units we write $usn(\mathbf{R}) = \infty$; if every element of $\mathbf{R}$ is a sum of units but $\mathbf{R}$ does not have the n–sum property for any integer n, we write $usn(\mathbf{R}) = \omega$.

Example 1.1

(a) The ring $\mathbf{R}$ has $usn(\mathbf{R}) = 1$ if and only if $\mathbf{R}$ is the trivial ring with $0 = 1$.

(b) The rings $\mathbf{GF(2)}$ and $\mathbf{Z}$ have unit sum number equal to ω.

(c) If $\mathbf{R} = \mathbf{Q}[\mathbf{x}]$ the ring of rational polynomials, it is well known that the only units of $\mathbf{R}$ are the non–zero constant polynomials and so $usn(\mathbf{R}) = \infty$ in this case.

(d) The ring $\mathbf{R} = \mathbf{Q}$ has $usn(\mathbf{R}) = 2$.

(e) If $\mathbf{R} = \mathbf{Z_q}$, the ring of integers modulo the prime q, then direct calculation gives $usn(\mathbf{R}) = 2$; see example 1.4 for a generalization of this result.

The n–sum property is inherited in a number of simple ways:

Proposition 1.2

(a) *If the rings $\mathbf{R}_i$ $(i \in I)$ each have the n-sum property then so also has the ring direct product $\prod_{i \in I} \mathbf{R}_i$.*

(b) *If $\mathbf{I}$ is an ideal of the ring $\mathbf{R}$ which has the n–sum property then so also has the quotient ring $\mathbf{R}/\mathbf{I}$.*

(c) *If the ring* **R** *has the n-sum property then so also does the ring* $\mathbf{M_k(R)}$ *of* $k \times k$ *matrices over* **R**, *for any finite k.*

Proof: (a), (b) are straightforward. To establish (c) we proceed by induction on k. By hypothesis the result is true for k=1. Suppose $A = (a_{ij})$ is a $(k+1) \times (k+1)$ matrix over **R** and the result is true for $k \times k$ matrices. Then

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k+1} \\ a_{21} & & & \\ \vdots & & B & \\ a_{k+11} & & & \end{pmatrix}$$

and we may write $a_{11} = \sum_{i=1}^{n} u_i$ where each u_i is a unit of **R**. Since B is a $k \times k$ matrix it can be expressed as a sum of invertible $k \times k$ matrices, $B = B_1 + \cdots + B_n$. But then

$$A = \begin{pmatrix} u_1 & a_{12} & \dots & a_{1k+1} \\ 0 & & & \\ \vdots & & B_1 & \\ 0 & & & \end{pmatrix} + \begin{pmatrix} u_2 & 0 & \dots & 0 \\ a_{21} & & & \\ \vdots & & B_2 & \\ a_{k+11} & & & \end{pmatrix} +$$

$$\begin{pmatrix} u_3 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_3 & \\ 0 & & & \end{pmatrix} + \cdots + \begin{pmatrix} u_n & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_n & \\ 0 & & & \end{pmatrix}$$

and it is easy to see that each of the above matrices is a unit in $\mathbf{M_{k+1}(R)}$. □

There is a particularly useful partial converse to Proposition 1.2(b); we include the well–known proof for completeness.

Proposition 1.3 *If* **R** *is a ring with Jacobson radical* $\mathbf{J(R)}$ *and* $\mathbf{R/J(R)}$ *has the n–sum property, then* **R** *has the n–sum property.*

Proof: We show that units "lift" modulo the Jacobson radical. If $y + \mathbf{J(R)}$ is a unit in $\mathbf{R/J(R)}$ then there exists $z \in \mathbf{R}$ such that

$$zy + \mathbf{J}(\mathbf{R}) = 1 + \mathbf{J}(\mathbf{R}) = yz + \mathbf{J}(\mathbf{R}).$$

The first equality gives $zy = 1 - r$ for some $r \in \mathbf{J}(\mathbf{R})$ and so, by the standard properties of the Jacobson radical, zy is a unit of $\mathbf{R}$. Thus $((zy)^{-1}z)y = 1$ implying y has a left inverse in $\mathbf{R}$. But a similar argument using the second equality above shows that y also has a right inverse in $\mathbf{R}$ and hence y is a unit in $\mathbf{R}$. Since units "lift" from $\mathbf{R}/\mathbf{J}(\mathbf{R})$ to $\mathbf{R}$, it is immediate that the n–sum property "lifts" also. □

We conclude this section by investigating the unit sum numbers of some well–known rings.

Example 1.4

(a) Let $\mathbf{R} = \mathbf{Z_k}$, the ring of integers modulo k. If k is even then all units in $\mathbf{R}$ are necessarily odd and so a simple parity argument shows that $\mathbf{R}$ cannot have the n–sum property for any n. However it is immediate that every element of $\mathbf{R}$ is a sum of units. If k is odd we consider firstly the case in which k is a prime power. In this situation 2 is a unit of $\mathbf{R}$ so every unit can be expressed as a sum of two units. Moreover nonunits are precisely those integers divisible by the prime and so adding and subtracting 1 expresses each nonunit as a sum of two units. Returning to the general case of k odd we may express $\mathbf{R}$ as a direct product of rings of prime power order, and it follows from Proposition 1.2(a) that $\mathbf{R}$ has the 2–sum property in this case. In summary then $usn(\mathbf{Z_k}) = \begin{cases} 2 & : \ k \textit{ odd} \\ \omega & : \ k \textit{ even.} \end{cases}$

(b) If $\mathbf{R}$ is a field (not necessarily commutative) then $usn(\mathbf{R}) = 2$ unless $\mathbf{R} = \mathbf{GF(2)}$ in which case $usn(\mathbf{R}) = \omega$. To see this consider separately the cases where the characteristic of $\mathbf{R}$ $= 2$ and $\neq 2$.
The latter case is easily handled: since 2 is a unit we have $x = \frac{1}{2}x + \frac{1}{2}x \ (x \neq 0)$ and $0 = 1 - 1$, in each case a sum of two units. If $char(\mathbf{R}) = 2$ but $|\mathbf{R}| > 2$, consider any $x \in \mathbf{R} \setminus \{0, 1\}$. Then $x - 1$ is again a unit and $x = (x - 1) + 1$, a sum of two units. However we also have $0 = 1 + 1$ and $1 = (1 - a) + a$ for any $a \in \mathbf{R} \setminus \{0, 1\}$; in each case we have a sum of two units. Finally note that $usn(\mathbf{GF(2)}) = \omega$ as observed in Example 1.1(b).

(c) If $\mathbf{R} = \mathbf{J_p}$, the ring of p–adic integers, then $usn(\mathbf{R}) = 2$ unless $p = 2$ in which case $usn(\mathbf{R}) = \omega$. This can be seen by direct or alternatively by noting that $\mathbf{J}(\mathbf{R}) = p\mathbf{R}$ and $\mathbf{R}/\mathbf{J}(\mathbf{R}) \cong \mathbf{GF}(\mathbf{p})$, the field of p elements.
For $p \neq 2$ the result then follows from Proposition 1.3 and example (b) above. For $p = 2$ all units are congruent to 1 modulo 2 and a simple parity argument establishes the impossibility of having a finite unit sum number. However every 2–adic integer is the sum of at most two units so we deduce $usn(\mathbf{J_2}) = \omega$.

(d) If $\mathbf{R} = \mathbf{Q_p}$, the ring of rationals with denominators prime to the given prime p, then $usn(\mathbf{R}) = 2$. To see this observe that a rational $\frac{a}{b}$ is a unit in $\mathbf{Q_p}$ if, and only if, $(a, p) = 1 = (b, p)$. If $x \in \mathbf{Q_p}$ then $x = p^r(\frac{a}{b})$, where $\frac{a}{b}$ is a unit and so $x = (p^r - 1)(\frac{a}{b}) + (\frac{a}{b})$, a sum of two units.

§2 Unit sum numbers of modules

If M is a module over the ring $\mathbf{R}$, then the set of $\mathbf{R}$–endomorphisms of M form a ring $\mathbf{E_R}(M)$. We shall say that the module M has the n–sum property or has the unit sum number k if the ring $\mathbf{E_R}(M)$ has the corresponding property.

It follows immediately from Proposition 1.2(c) that a free $\mathbf{R}$–module of finite rank has the n–sum property if $\mathbf{R}$ has; in particular a finite dimensional vector space over a field $\mathbf{F} \neq \mathbf{GF}(\mathbf{2})$ has unit sum number equal to 2. Indeed finite dimensional vector spaces over $\mathbf{GF}(\mathbf{2})$ have the same property, with one exception, as established below.

Proposition 2.1 *If V is a vector space of finite dimension $n > 1$ over the field $\mathbf{GF}(\mathbf{2})$, then $usn(V) = 2$.*

Proof: We prove the result by induction on the dimension n of V. First consider $\dim(V) = 2$. Then $V = b_0F + b_1F$ where $F = \mathbf{GF}(\mathbf{2})$. Therefore any endomorphism of V may be described by a 2×2 – matrix with entries 0 or 1. An easy calculation shows that any of these 16 matrices can be written as a sum of two matrices whose determinants are units, i.e. a sum of two automorphisms of V, e.g.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now assume inductively that the result is true for all vector spaces over $\mathbf{GF(2)}$ of dimension less than or equal to n. Let V have dimension $n+1$ and let ϕ be an arbitrary endomorphism of V.

First we consider the case $\ker\phi \neq 0$. Then $V = \ker\phi \oplus V_1$ where $\dim(\mathrm{Im}\phi) = \dim V_1 \leq n$. So there exists an isomorphism $\theta\colon \mathrm{Im}\phi \longrightarrow V_1$. But the composite $\phi\restriction_{V_1} \theta$ is an endomorphism of V_1 and hence $\phi\restriction_{V_1} \theta = \alpha' + \beta'$ where α' and β' are automorphisms of V_1. Now we define $\alpha,\ \beta : V \longrightarrow V$ with respect to the decomposition $V = \ker\phi \oplus V_1$ by $\alpha = I_{\ker\phi} \oplus \alpha'\theta^{-1}$ and $\beta = -I_{\ker\phi} \oplus \beta'\theta^{-1}$ where I denotes the identity. Clearly, α and β are automorphisms of V and $\alpha + \beta = 0 \oplus (\alpha' + \beta')\theta^{-1} = 0 \oplus (\phi\restriction_{V_1} \theta)\theta^{-1} = 0 \oplus \phi\restriction_{V_1} = \phi$. Therefore α and β are the required automorphisms of V.

It remains to consider the case $\ker\phi = 0$. In this case ϕ is an automorphism of V since V is finite dimensional. Let $\{b_0, b_1, \ldots, b_n\}$ be a basis of V. We define $\alpha : V \longrightarrow V$ by $b_0\alpha = b_0 - b_n,\ b_i\alpha = b_{i-1}$ for $i > 0$ and $\beta : V \longrightarrow V$ by $b_0\beta = b_n,\ b_i\beta = b_i - b_{i-1}$ for $i > 0$. Then $\alpha,\ \beta$ are automorphisms whose sum is the identity. Thus $\alpha\phi + \beta\phi = (\alpha + \beta)\phi = \phi$ and so $\alpha\phi,\ \beta\phi$ are the required automorphisms of V, each being a composition of automorphisms of V. □

Indeed the above proposition remains true if V is replaced by a finite rank (> 1) free module over any PID $\mathbf{R}$; see [11] for details using the Smith Normal Form.

The question naturally arises as to whether or not such results extend to the infinite dimensional/rank situation. We focus initially on vector spaces where we can give a complete answer to the problem. The proof of our next result Theorem 2.2 closely follows an argument outlined by Graham Higman to the first author and greatly simplifies our original proof.

Theorem 2.2 *If V is a vector space of countably infinite dimension over an arbitrary field $\mathbf{F}$, then $usn(V) = 2$.*

Proof: Let $V = \bigoplus_{i\in\omega} b_i F$ be a countable dimensional vector space over the field F and let ϕ be an endomorphism of V.

First we assume that the image of ϕ is finite dimensional. Then $V = \mathrm{Im}\phi \oplus W = \ker\phi \oplus U$ for some subspaces $U,\ W$ where $\dim U = \dim(\mathrm{Im}\phi)$ and hence $U \cong \mathrm{Im}\phi$. Let

$\{w_0, w_1, \ldots, w_n, \ldots\}$ be a basis of W, $\{x_0, x_1 \ldots, x_n, \ldots\}$ a basis of kerϕ, $\{u_0, u_1, \ldots, u_n\}$ a basis of U and $\{y_0, y_1, \ldots, y_n\}$ a basis of Imϕ. The set $\{x_{n+1}, x_{n+2}, \ldots\} \cup \{u_0, \ldots, u_n\}$ is countably infinite as is $\{w_0, w_1, \ldots\}$, so there exists a bijection
$f\colon \{w_0, w_1, \ldots\} \longrightarrow \{x_{n+1}, x_{n+2}, \ldots\} \cup \{u_0, u_1, \ldots, u_n\}$.
We define $\tilde{f} : \{y_0, \ldots, y_n\} \cup \{w_0, w_1, \ldots\} \longrightarrow \{x_0, x_1, \ldots\} \cup \{u_0, u_1, \ldots, u_n\}$ by $y_i\tilde{f} = x_i$ for $0 \leq i \leq n$ and $w_i\tilde{f} = w_i f$ for all $i \in \omega$; $\tilde{f}$ is a bijection from one basis of V onto another. Thus $\tilde{f}$ extends to an automorphism α of V. Also $(\mathrm{Im}\phi)\alpha \subseteq \ker\phi$ and so $\eta^2 = \phi\alpha\phi\alpha = 0$ for $\eta = \phi\alpha$. Hence $(I + \eta)(I - \eta) = I = (I - \eta)(I + \eta)$ where I denotes the identity. Therefore $I + \eta = I + \phi\alpha$ is an automorphism of V.
Finally, $\alpha^{-1} + \phi$ is an automorphism of V since $I + \alpha\phi = \alpha^{-1}\alpha + \phi\alpha = (\alpha^{-1} + \phi)\alpha$. Thus $\phi = (\alpha^{-1} + \phi) + (-\alpha^{-1})$ is a sum of two automorphisms.

Now let Imϕ be of countably infinite dimension. We construct inductively automorphisms α and β with $\phi = \alpha + \beta$ along the given basis $\{b_i \mid i \in \omega\}$ of V.
Assume that α and β have been defined on $\{b_i \mid i \in I_0\}$ for some finite set $I_0 \subseteq \omega$ such that $\{b_i\alpha \mid i \in I_0\}$ and $\{b_i\beta \mid i \in I_0\}$ are linearly independent sets and also $b_i(\alpha+\beta) = b_i\phi$ for all $i \in I_0$. ($I_0 = \emptyset$ may be taken as a staring point for the induction.)
We extend α, β enlarging the domains (Step 1) and images (Step 2/3) as follows:

Step 1: First pick the smallest integer $m \in \omega \setminus I_0$. Then pick the smallest integer $n \in \omega$ such that $b_n \notin \bigoplus_{i\in I_0}(b_i\alpha)F$ and $(b_m\phi - b_n) \notin \bigoplus_{i\in I_0}(b_i\beta)F$.
We define $b_m\alpha = b_n$ and $b_m\beta = b_m\phi - b_n$, i.e. b_m now belongs to the domains of α and β.
Let $I_1 = I_0 \cup \{m\}$, then $\{b_i\alpha \mid i \in I_1\}$ and $\{b_i\beta \mid i \in I_1\}$ are again linearly independent sets and $b_i(\alpha + \beta) = b_i\phi$ for all $i \in I_1$ where I_1 is finite.

Step 2: First pick the smallest integer q such that $b_q \notin \bigoplus_{i\in I_1}(b_i\alpha)F$. Then pick the smallest integer $p \in \omega \setminus I_1$ such that $b_p\phi - b_q \notin \bigoplus_{i\in I_1}(b_i\beta)F$.
We define $b_p\alpha = b_q$ and $b_p\beta = b_p\phi - b_q$, i.e. b_q now belongs to the image of α.
Let $I_2 = I_1 \cup \{p\}$, then $\{b_i\alpha \mid i \in I_2\}$ and $\{b_i\beta \mid i \in I_2\}$ are again linearly independent sets and $b_i(\alpha + \beta) = b_i\phi$ for all $i \in I_2$ where I_2 is finite.

Step 3: First pick the smallest integer r such that $b_r \notin \bigoplus_{i\in I_2}(b_i\beta)F$. Then pick the smallest integer $t \in \omega \setminus I_2$ such that $b_t\phi - b_r \notin \bigoplus_{i \in I_2}(b_i\alpha)F$.

We define $b_t\beta = b_r$ and $b_t\alpha = b_t\phi - b_r$, i.e. b_r now belongs to the image of β.

Finally let $I_3 = I_2 \cup \{t\}$, then $\{b_i\alpha \mid i \in I_3\}$ and $\{b_i\beta \mid i \in I_3\}$ are linearly independent sets and $b_i(\alpha + \beta) = b_i\phi$ for all $i \in I_3$ where I_3 is finite, i.e. we have the same conditions as at the beginning.

Note that, in each step, we can always choose such integers since the vector space V and also the image of the given endomorphism ϕ are of infinite dimension. Now we can repeat the three steps above taking I_3 as the new I_0. We continue this process as often as possible, i.e. a countably infinite number of times.

Now let α, β be the union of all these extensions constructed above. We show that α and β are the required automorphisms whose sum is ϕ.

It follows immediately, by our construction, that $\text{dom}\alpha = \text{dom}\beta = V$ and $\text{Im}\alpha = \text{Im}\beta = V$. Thus it remains to show that α, β are injective.

Consider any basis element $b = b_s$ of the given basis. On some stage s must have been the smallest integer not belonging to the finite set I_k $(k \in \{0, 1, 2\})$ since $s \in \omega$. Hence s was choosen in of the three steps.

If s was chosen in step 1 then $b = b_m$ and so $b\alpha = b_n \neq 0$ and $b\beta = b\phi - b_n \neq 0$ where the latter is true by the choice of n.

If s was chosen in step 2 then $b = b_p$ and so $b\alpha = b_q \neq 0$ and $b\beta = b\phi - b_q \neq 0$ where the latter is true by the choice of q.

If s was chosen in step 3 then $b = b_t$ and so $b\beta = b_r \neq 0$ and $b\alpha = b\phi - b_r \neq 0$ where the latter is true by the choice of r.

So, in either case, $b\alpha \neq 0$ and $b\beta \neq 0$ and thus all basis elements are mapped onto non–zero elements under α and β. But also $\{b_i\alpha \mid i < \omega\}$ and $\{b_i\beta \mid i < \omega\}$ are linearly independent sets. Therefore, $\ker\alpha = \ker\beta = \{0\}$ and hence α and β are automorphisms. Moreover $\phi = \alpha + \beta$ by our construction.

So any endomorphism ϕ is a sum of two automorphisms α, β, i.e. $usn(V) = 2$. □

The restriction of countable dimension can be easily removed by using an essentially set–

theoretic trick of Castagna [1], which we state more generally than is required for vector spaces.

Lemma 2.3 *Let M be a free $\mathbf{R}$–module of rank κ, an infinite cardinal, and ϕ an endomorphism of M. Then M can be written as $M = \bigcup_{\alpha<\kappa} M_\alpha$ such that*

(i) $M_\alpha \subseteq M_{\alpha+1}$ *for all* $\alpha < \kappa$,

(ii) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ *if α is a limit ordinal,*

(iii) $M_{\alpha+1} = M_\alpha \oplus C_\alpha$ *where* C_α *is of countable rank (C_α can be choosen to be of rank at least two), and*

(iv) $M_\alpha\phi \subseteq M_\alpha$ *for each* $\alpha < \kappa$.

Proof: See Theorem 2.2. in [1]. □

We can now easily deduce the result we were seeking:

Corollary 2.4 *If V is a vector space of arbitrary infinite dimension, then every linear transformation of V is a sum of two invertible transformations.*

Proof: Let V be a vector space of arbitrary infinite dimension and ψ an endomorphism of V. Then we may write $V = \bigcup_{\beta\in\kappa} V_\beta$ as in Lemma 2.3. We define automorphisms θ_β and θ'_β of V_β inductively such that , for each $\beta < \kappa$,

$$\psi\restriction_{V_\beta} = \theta_\beta + \theta'_\beta \text{ and } \theta_\beta\restriction_{V_\alpha} = \theta_\alpha,\ \theta'_\beta\restriction_{V_\alpha} = \theta'_\alpha \ \text{ for } \alpha < \beta. \qquad (*)$$

For $\beta = 0$ we may assume $V_0 = 0$ and so property (*) is satisfied taking $\theta_0 = 0 = \theta'_0$.
Now suppose that $\theta_\alpha,\ \theta'_\alpha$ have been defined satisfying (*) for all $\alpha < \beta$.
First assume that β is a limit ordinal. Then $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ and so we may define $\theta_\beta = \bigcup_{\alpha<\beta} \theta_\alpha$ and $\theta'_\beta = \bigcup_{\alpha<\beta} \theta'_\alpha$. Obviously, θ_β and θ'_β satisfy property (*) since the $\theta_\alpha,\ \theta'_\alpha$ do so by assumption.
Next let $\beta = \alpha+1$ be a successor ordinal. Then, by Lemma 2.3, $V_{\alpha+1} = V_\alpha \oplus C_\alpha$ for some subspace C_α of countable dimension. Let π_1 and π_2 be the projections of $V_{\alpha+1}$ onto V_α and C_α respectively. Then $(\psi\restriction_{C_\alpha})\pi_2$ is an endomorphism of C_α. Thus, by Lemma 2.2, there exist automorphisms ϕ and ϕ', of C_α such that $(\psi\restriction_{C_\alpha})\pi_2 = \phi + \phi'$. Note that there is no restriction for V being a vector space over $\mathbf{GF(2)}$ since we may assume $2 \leq \dim C_\alpha \leq \aleph_0$.

For each $c \in C_\alpha$ choose ν_c in V_α such that $\nu_c = (c\psi)\pi_1$. Note that $(\psi \restriction_{C_\alpha})\pi_1$ is a mapping from C_α to V_α. Now we define $\theta_{\alpha+1}$ and $\theta'_{\alpha+1}$ on $V_{\alpha+1} = V_\alpha \oplus C_\alpha$ by $(x+c)\theta_{\alpha+1} = x\theta_\alpha + c\phi + \nu_c$ and $(x+c)\theta'_{\alpha+1} = x\theta'_\alpha + c\phi'$ for any $x + c \in V_{\alpha+1}$ where $x,\ c$ are the unique components in V_α and C_α respectively.

It follows immediately from the definition that $\theta_{\alpha+1}$ and $\theta'_{\alpha+1}$ are homomorphisms and also that $\theta_{\alpha+1}$, $\theta'_{\alpha+1}$ are extensions of θ_α, θ'_α respectively since $\nu_c = 0$ for $c = 0$.

Next we show that $\theta_{\alpha+1}$ is an automorphism. First consider an element $x+c \in \ker(\theta_{\alpha+1})$. Then $0 = (x+c)\theta_{\alpha+1} = x\theta_\alpha + c\phi + \nu_c = (x\theta_\alpha + \nu_c) + c\phi$ where $x\theta_\alpha + \nu_c \in V_\alpha$ and $c\phi \in C_\alpha$. Hence $x\theta_\alpha + \nu_c = 0$ and $c\phi = 0$. Since ϕ is an automorphism of C_α we get $c = 0$ and thus $\nu_c = c\psi\pi_1 = 0$. Therefore $x\theta_\alpha = 0$ which, by assumption, gives $x = 0$. So we have $\ker(\theta_{\alpha+1}) = 0$, i.e $\theta_{\alpha+1}$ is injective. Also $\theta_{\alpha+1}$ is surjective since, for any $a+b \in V_{\alpha+1}$ $(a \in V_\alpha,\ b \in C_\alpha)$, $a+b = (x+c)\theta_{\alpha+1} = x\theta_\alpha + c\phi + \nu_c$ for $c = b\phi^{-1},\ x = y\theta_\alpha^{-1}$ where $y = a - \nu_c$. Therefore $\theta_{\alpha+1}$ is an automorphism.

Now we show that $\theta'_{\alpha+1}$ is an automorphism of $V_{\alpha+1}$. Let $x+c \in \ker(\theta'_{\alpha+1})$ $(x \in V_\alpha,\ c \in C_\alpha)$. Then $x\theta'_\alpha + c\phi' = 0$ which implies $x\theta'_\alpha = -c\phi' \in V_\alpha \cap C_\alpha = 0$. Thus $x = 0 = c$ as θ'_α and ϕ' are automorphisms of V_α and C_α respectively. Hence $\theta'_{\alpha+1}$ is injective.

Also, for any $a+b \in V_{\alpha+1}$ $(a \in V_\alpha,\ b \in C_\alpha)$, defining $c = b(\phi')^{-1}$ and $x = a(\theta'_\alpha)^{-1}$ gives $(x+c)\theta'_{\alpha+1} = a+b$ and so $\theta'_{\alpha+1}$ is surjective.

We now have automorphisms $\theta_{\alpha+1}$ and $\theta'_{\alpha+1}$ of V and all that remains to show is that their sum is ψ. Let $x+c \in V_\alpha \oplus C_\alpha$, then $(x+c)(\theta_{\alpha+1}+\theta'_{\alpha+1}) = x(\theta_\alpha+\theta'_\alpha)+c(\phi+\phi')+\nu_c = x(\psi\restriction_{V_\alpha}) + (c\psi)\pi_2 + (c\psi)\pi_1 = x\psi + (c\psi)(\pi_1+\pi_2) = x\psi + c\psi = (x+c)\psi$.

Finally we get $\psi = \theta + \theta'$ taking $\theta = \bigcup_{\beta<\kappa} \theta_\beta$ and $\theta' = \bigcup_{\beta<\kappa} \theta'_\beta$ which are automorphisms of $V = \bigcup_{\beta<\kappa} V_\beta$. □

Collecting the above results and noting Example 1.1(b) we can give a complete solution to the problem of determining the unit sum number of a vector space:

Theorem 2.5 *If V is a vector space of arbitrary dimension over a field* $\mathbf{F}$ *then* $usn(V) = 2$ *unless V is one dimensional and* $\mathbf{F} = \mathbf{GF(2)}$*; in this case* $usn(V) = \omega$. □

For the remainder of this section we focus our attention on torsion–free modules over the ring $\mathbf{J_p}$ of p–adic integers. (It is worth noting that the arguments hold more generally for torsion–free modules over a complete discrete valuation ring.) The situation for free modules of finite rank is entirely analagous to the finite dimensional vector space situation. This result can be obtained by a direct argument as for vector spaces or it can be deduced as an easy consequence of our next result and Proposition 1.3 since free p–adic modules of finite rank are necessarily complete.

Proposition 2.6

(a) *If M is a complete torsion–free $\mathbf{J_p}$–module, then* $\mathbf{J}(\mathbf{E}(M)) = p\mathbf{E}(M)$.

(b) *If M is a free $\mathbf{J_p}$–module (of arbitrary rank), then* $\mathbf{E}(M)/p\mathbf{E}(M) \cong \mathbf{E}(M/pM)$.

Proof: For part (a) see (2.3) Theorem in [10].

To prove (b) define $\Delta : \mathbf{E}(M) \longrightarrow \mathbf{E}(M/pM)$ by $\psi\Delta = \overline{\psi}$ where $\overline{\psi}$ is the induced endomorphism. Obviously, Δ is a ring homomorphism.

For $M = \bigoplus_{i\in I} x_i\mathbf{J_p}$ and $\theta \in \mathbf{E}(M/pM)$ we define $\psi \in \mathbf{E}(M)$ by $x_i\psi = \sum_{j\in I} r_{ij}x_j$ ($r_{ij} = 0$ for almost all j) where the $r_{ij} \in \mathbf{J_p}$ are chosen corresponding to $(x_i + pM)\theta = \sum_{j\in I} r_{ij}(x_j + pM)$. Hence $\psi\Delta = \overline{\psi} = \theta$ and so Δ is surjective.

Finally, ker$\Delta = \{\psi \in \mathbf{E}(M) \mid \overline{\psi} = 0\} = \{\psi \in \mathbf{E}(M) \mid M\psi \subseteq pM\} = p\mathbf{E}(M)$ and thus $\mathbf{E}(M)/p\mathbf{E}(M) \cong \mathbf{E}(M/pM)$. □

Theorem 2.7 *If M is a free $\mathbf{J_p}$–module of finite rank n then $usn(M) = 2$ unless $n = 1$ and $p = 2$; in this case $usn(M) = \omega$.*

Proof: By Proposition 2.6 (b), $\mathbf{E}(M)/p\mathbf{E}(M) \cong \mathbf{E}(M/pM)$. But M/pM is a vector space and hence, by Theorem 2.5 every endomorphism of M/pM can be written as a sum of two automorphisms unless $p = 2$ and $\dim(M/pM) = 1$; in this case $usn(M/pM) = \omega$. Therefore $usn(\mathbf{E}(M)/p\mathbf{E}(M) = 2$ unless $p = 2$ and $\mathrm{rk}(\mathbf{E}(M)) = 1$; in this case $usn(\mathbf{E}(M)/p\mathbf{E}(M)) = \omega$. But $p\mathbf{E}(M) = \mathbf{J}(\mathbf{E}(M))$, the Jacobson radical of $\mathbf{E}(M)$, by Proposition 2.6(a) since a free p–adic of finite rank is obviously complete. Hence, by Proposition 1.3, the unit sum number of $\mathbf{E}(M)$ and hence of M is 2 unless $p = 2$ and $\mathrm{rk}(\mathbf{E}(M)) = 1$ which implies $\mathrm{rk}(M) = 1$; in this case $usn(M) = \omega$. □

There is not, however, an easy transition to even countably infinite rank using the above arguments since unfortunately the Jacobson radical of a free p–adic module is much more complicated than that of a complete module as illustrated by our next result.

Proposition 2.8 *If M is a free $\mathbf{J_p}$–module of infinite rank then $\mathbf{J}(\mathbf{E}(M))$ is properly contained in $p\mathbf{E}(M)$.*

Proof: See (2.4) Proposition in [10].

Nor indeed is it immediately clear how to modify the algorithm used for countable dimensional vector spaces. The essence of the problem is that at each step of the process purity needs to be preserved and unfortunately preserving p–heights is not, in this situation, adequate to ensure purity. Consequently we adopt an approach reminiscent of arguments used by Goldsmith [6], which in turn are based on Castagna [1]. In [6] it was established that if M is a free p–adic module then $usn(M) \leq 4$. In light of our results on vector spaces it seems likely that $usn(M) = 2$ but the best we can achieve at present is $usn(M) \leq 3$ if $p \neq 2$.

Suppose $M = \bigoplus_{i<\omega} x_i\mathbf{R}$, with $\mathbf{R} = \mathbf{J_p}$, is a free p–adic module of countably infinte rank and $\eta \in \mathbf{E_R}(M)$. Then, extending some terminology introduced by Freedman [2], we say

(i) η is *locally nilpotent* if, for any $x \in M$, there is some $k < \omega$ such that $x\eta^k = 0$.

(ii) η is an *α–endomorphism* if $x_i\eta \in \bigoplus_{k>i} x_k\mathbf{R}$ for all $i < \omega$.

(iii) η is a *β–endomorphism* if $x_i\eta \in \bigoplus_{k=0}^{i-1} x_k\mathbf{R}$ for all $i < \omega$.

(iv) η is a *d–endomorphism* if $x_i\eta \in x_i\mathbf{R}$ for all $i < \omega$.

Theorem 2.9 *Let $\mathbf{R} = \mathbf{J_p}$ with $p \neq 2$ and let M be a free $\mathbf{R}$–module of countably infinite rank; $M = \bigoplus_{i<\omega} x_i\mathbf{R}$. Then every endomorphism of M can be expressed as a sum of three automorphisms.*

Proof: Let ψ be an endomorphism of M. Obviously we can write ψ as a sum of a d–endomorphism δ, a β–endomorphism ϕ and an α–endomorphism η.

Since we may write any p–adic integer as a sum of two and hence also of three units we can express δ as a sum of three d–automorphisms τ, θ_1, and θ_2, i.e. τ, θ_1, θ_2 are automorphisms as well as d–endomorphism.

First we show that $\tau+\phi$ is an automorphism of M. Certainly $\tau+\phi=\tau(I+\tau^{-1}\phi)$. Since τ and therefore τ^{-1} are d–automorphisms and ϕ is a β–endomorphism we get, for each basis element x_i, $x_i(\tau^{-1}\phi)\in(x_i\mathbf{R})\phi\subseteq\bigoplus_{j=0}^{i-1}x_j\mathbf{R}$, so $x_i(\tau^{-1}\phi)^2\in(\bigoplus_{j=0}^{i-1})\tau^{-1}\phi\subseteq\bigoplus_{j=0}^{i-2}x_j\mathbf{R}\ldots$ and finally $x_i(\tau^{-1}\phi)^{i+1}=0$, i.e. $\tau^{-1}\phi$ is locally nilpotent. Thus the expression $X=I-\tau^{-1}\phi+(\tau^{-1}\phi)^2-\ldots+(-1)^n(\tau^{-1}\phi)^n\ldots$ has a well–defined meaning considered as an endomorphism of M. In fact X is the inverse of $(I+\tau^{-1}\phi)$ and so $(I+\tau^{-1}\phi)$, and hence the product $\tau(I+\tau^{-1}\phi)$, are automorphisms of M, i.e. $\tau+\phi$ is an automorphism of M.

Next we consider the α–endomorphism. Since, for any $i<\omega$, there is a minimal $m<\omega$ such that $x_i\eta\in\bigoplus_{j=i+1}^{m}x_j\mathbf{R}$ there exists a strictly ascending sequence $0=r_0<r_1<r_2<\ldots$ of integers having the property that $x_i\eta\in\bigoplus_{j=i+1}^{r_{s+2}-1}x_j\mathbf{R}$ whenever $r_s\le i<r_{s+1}$. We define the mappings η_1 and η_2 by

$$x_i\eta_1=\begin{cases}x_i\eta & for \quad r_{2t}\le i<r_{2t+1}\\ 0 & for \quad r_{2t+1}\le i<r_{2t+2}\end{cases}\qquad\text{and}\qquad x_i\eta_2=\begin{cases}0 & for \quad r_{2t}\le i<r_{2t+1}\\ x_i\eta & for \quad r_{2t+1}\le i<r_{2t+2}\end{cases}$$

where $t=0,1,2,\ldots$. It follows immediately that η_1 and η_2 are again α–endomorphisms and that their sum is η. Moreover, an easy calculation shows, using the definition and the sequence $r_0<r_1<\ldots$ above, that η_1 and η_2 are locally nilpotent. (See Lemma 2 in [2].) Thus we can write the α-endomorphism η as a sum of two locally nilpotent α–endomorphisms.

Next we show that $\delta_1+\eta_1$ is an automorphism. We investigate $\delta_1^{-1}\eta_1$ for local nilpotence and clearly need only consider $x_i(\delta_1^{-1}\eta_1)$ for $r_{2t}\le i<r_{2t+1}$. Since δ_1^{-1} is a d–automorphism it only affects coefficients of x_i and so $x_i(\delta_1^{-1}\eta_1)\in\bigoplus_{j=i+1}^{r_{2t+2}-1}x_j\mathbf{R}$. Thus $x_i(\delta_1^{-1}\eta_1)^m\in\bigoplus_{j=r_{2t+1}}^{r_{2t+2}-1}x_j\mathbf{R}$ for some m and hence $x_i(\delta_1^{-1}\eta_1)^{m+1}=0$. So $\delta_1^{-1}\eta_1$ is locally nilpotent. By arguments similar to those above we get that $I+\delta_1^{-1}\eta_1$, and hence

$\delta_1 + \eta_1 = \delta_1(I + \delta_1^{-1}\eta_1)$, are automorphisms of M.
In the same manner, by considering the interval $r_{2t+1} \leq i < r_{2t+2}$ we get that $\delta_2^{-1}\eta_2$ is locally nilpotent and so $\delta_2 + \eta_2 = \delta_2(I + \delta_2^{-1}\eta_2)$ is also an automorphism of M.

Finally $\psi = (\tau + \phi) + (\delta_1 + \eta_1) + (\delta_2 + \eta_2)$, a sum of three automorphisms of M. □

We can extend this result to free modules of arbitrary rank by the same arguments as in Corollary 2.4.

Corollary 2.10 *If M is a reduced torsion–free p–adic module of infinite rank ($p \neq 2$) then each endomorphism of M is a sum of three automorphisms of M.* □

We remark that an analagous result has been obtained by Wans [11] with $\mathbf{R} = \mathbf{Z}$. Rather surprisingly if M is a complete torsion–free p–adic module then one can show that with one exception $usn(M) = 2$. This was observed in [6] but only a sketch of the proof was given. We present here the detailed argument. First we need:

Lemma 2.11 *If M is the completion of the free $\mathbf{J_p}$–module B then $\mathbf{E}(M)/p\mathbf{E}(M)$ is ring isomorphic to $\mathbf{E}(B)/p\mathbf{E}(B)$.*

Proof: Note first that we may consider $\mathbf{E}(B)$ as a subset of $\mathbf{E}(M)$ identifying an endomorphism of B with its unique extension to an endomorphism of M.
An easy argument gives $\mathbf{E}(M)/\mathbf{E}(B)$ is torsion–free and divisible using that M/B is torsion–free and divisible. Thus, for any $\psi \in \mathbf{E}(M)$, there exist $\psi' \in \mathbf{E}(M)$ and $\theta_\psi \in \mathbf{E}(B)$ such that $\psi = p\psi' + \theta_\psi$ where ψ' and θ_ψ are unique modulo $p\mathbf{E}(M)$ and $p\mathbf{E}(B)$ respectively since $\mathbf{E}(B)$ is pure in $\mathbf{E}(M)$.
We define $\chi\colon \mathbf{E}(M) \longrightarrow \mathbf{E}(B)/p\mathbf{E}(B)$ by $\psi\chi = \theta_\psi + p\mathbf{E}(B)$. Clearly χ is a ring homomorphism and it is surjective. Furthermore $\ker\chi = \{\psi \in \mathbf{E}(M) \mid \theta_\psi \in p\mathbf{E}(B)\}$. But $\theta_\psi \in p\mathbf{E}(B)$ implies $\theta_\psi = p\theta'$ and so $\psi = p\psi' + p\theta' = p(\psi' + \theta') \in p\mathbf{E}(M)$,
i.e. $\ker\chi = p\mathbf{E}(M)$. Therefore $\mathbf{E}(M)/p\mathbf{E}(M) \cong \mathbf{E}(B)/p\mathbf{E}(B)$ by the isomorphism theorem. □

Theorem 2.12 *If M is a complete torsion–free p–adic module of infinite rank then $usn(M) = 2$.*

Proof: Since M is complete it is the completion of any its basic submodules, $M = \widehat{B}$ say where B is a free p–adic module of infinite rank. Now it follows from Lemma 2.11 that $\mathbf{E}(M)/p\mathbf{E}(M)$ is ring isomorphic to $\mathbf{E}(B)/p\mathbf{E}(B)$ and since the latter is, by Proposition 2.6 (b), isomorphic to $\mathbf{E}(B/pB)$, we can deduce from Proposition 2.6 (a) that $\mathbf{E}(M)/\mathbf{J}(\mathbf{E}(M))$ is isomorphic to the endomorphism ring of the vector space B/pB. The desired result follows from Proposition 1.3 and Theorem 2.5. □

Corollary 2.13 *If M is a complete torsion–free $\mathbf{J_p}$–module then $usn(M) = 2$ unless M is $\mathbf{J_p}$ itself and $p = 2$; in this case $usn(M) = \omega$.* □

§3 Unit sum numbers of p–groups

In this final section we focus on the unit sum number of p–groups, particularly direct sums of cyclic groups and the torsion completions (=torsion subgroup of the p–adic completion) of such groups; see [5] for further details of torsion completions. The unit sum numbers of direct sums of countable p–groups and of totally projective p–groups, with $p \neq 2$, have been obtained by Castagna [1] and Hill [7] respectively. Their methods are based on extensions of the proof of Ulm's theorem for such classes and use difficult techniques. Our approach is based more on the arguments we have developed for vector spaces and is considerably more elementary. The price to be paid for this simplification is that we do not obtain sharp results.

For completeness let us record

Theorem 3.1 *If G is a totally projective p–group $(p \neq 2)$ then $usn(G) = 2$.*

Proof: See Theorem 4.1 in [7]. □

Note that the class of direct sums of reduced countable p–groups is contained in the class of totally projective p–groups.

Before developing our result let us remark that the restriction $p \neq 2$ is, in some senses, unavoidable here. There exist finite abelian 2–groups which even have unit sum number ∞as demonstrated below.

Example 3.2

Let $G =< a > \oplus < b >$ such that $o(a) = 2,\ o(b) = 8$. Then the elements $a \pm 2b$ have the same Ulm sequence $(0, 2, \infty, \infty, \ldots)$ and they are the only elements of G with an Ulm sequences of this form. Thus $(a \pm 2b)\theta \in \{a \pm 2b\}$ for any automorphism θ of G. In fact, for $\theta_i \in \mathrm{Aut}(G)$ $(i \leq k)$, we get $(a \pm 2b) \sum_{i=1}^{k} \theta_i \in \langle a \pm 2b \rangle = \{0, a + 2b, a - 2b, 4b\} = H$, the subgroup of G generated by $a \pm 2b$.

We define $\phi\colon G \longrightarrow G$ by $a\phi = 0$ and $b\phi = b$. If ϕ were a sum of automorphisms then $(a \pm 2b)\phi = \pm 2b$ would be an element of H contradicting $2b \notin H$. Therefore $E(G)$ is not even generated by $\mathrm{Aut}(G)$, i.e. $usn(G) = \infty$.

Our first result closely mirrors the corresponding one for free p–adic modules.

Theorem 3.3 *If G is a direct sum of countably many cyclic p–groups $(p \neq 2)$ then every endomorphism of G can be expressed as a sum of three automorphisms, i.e. $usn(G) \leq 3$.*

Proof: Let $G = \bigoplus_{i<\omega} \langle x_i \rangle$ with $o(x_i) = p^{n_i}$ $(n_i \in \omega)$. The proof follows that of Theorem 2.9 with only the d–endomorphism δ needing particular attention. Since δ is a d–endomorphism $x_i\delta = d_i x_i$ for some $d_i \in \mathbb{Z}$.

If $(d_i, p) = 1$ then $d_i = d_i + 1 - 1$, while if p divides d_i then $d_i = (2 + d_i) - 1 - 1$ where $(p, 2 + d_i) = 1$ since $p \neq 2$. So in either case we can express d_i as a sum of three units (in the ring $\mathbb{Z}_{p^{n_i}}$) and hence δ can be written as a sum of three d–automorphisms of G. □

In light of our results for complete p–adic modules it is no surprise that we can achieve a finer result for torsion–complete groups; a similar result has been obtained by different methods by Castagna [1].

Lemma 3.4 *Let $B = \bigoplus_{n\in I} B_n$ be a direct sum of cyclic groups, where each B_n is a direct sum of cyclic groups of order p^{n+1} and $I \subseteq \omega$ such that $B_n \neq 0$ for $n \in I$, and let $\overline{B}$ be the torsion completion of B. Then $\mathbf{E}(\overline{B})/\mathbf{J}(\mathbf{E}(\overline{B})) \cong \prod_{n\in I} \mathbf{E}(B_n[p])$.*

Proof: See Theorem 3.4 in [1]. □

Since each $B_n[p]$ is a vector space we have, except where $p = 2$ and $\dim(B_n[p]) = 1$, that every element of $\mathbf{E}(B_n[p])$ is a sum of **two** units (Theorem 2.5) and thus the ring direct product $\prod_{n\in I}\mathbf{E}(B_n[p])$ has this property by Proposition 1.2. Hence the quotient ring $\mathbf{E}(\overline{B})/\mathbf{J}(\mathbf{E}(\overline{B}))$ has this property and so also does $\mathbf{E}(\overline{B})$ as observed in Proposition 1.3. The exceptional case where $p = 2$ is easily handled and we now separate the cases $p = 2$ and $p \neq 2$ for clarity of presentation. Thus we have established the following result:

Theorem 3.5 *If G is a torsion–complete p–group ($p \neq 2$) then $usn(G) = 2$.* □

For $p = 2$ we have the following characterization which could not be obtained by Castagna's methods:

Theorem 3.6 *If G is a torsion–complete 2–group then every endomorphism of G is a sum of two automorphisms if and only if every non–zero Ulm invariant of G, $f_n(G)$, is at least 2.*

Proof: We note that $f_n(G) = f_n(B)$ for any basic subgroup B of G and so $f_n(G) = k$ if and only if in the decomposition $B = \bigoplus_{n\in I} B_n$, B_n is a direct sum of k cyclic groups of order 2^{n+1}.

Suppose now that $f_n(G) \geq 2$ if it is not zero, i.e. $f_n(G) \geq 2$ for $n \in I$. Then by Lemma 3.4, $\mathbf{E}(G)/\mathbf{J}(\mathbf{E}(G) \cong \prod_{n\in I}\mathbf{E}(B_n[2])$ and each vector space $B_n[2]$ has dimension at least 2. Thus every endomorphism of $B_n[2]$ is a sum of two automorphisms (Theorem 2.5) and a similar argument to that in Theorem 3.5 completes the proof.

Conversely, if $\mathbf{E}(G)$ has the property that every endomorphism is a sum of two automorphisms then so also has every (ring) homomorphic image of $\mathbf{E}(G)$. In particular it follows from Lemma 3.4 that $\mathbf{E}(B_n[2])$ is such an image and so $\mathbf{E}(B_n[2])$, the endomorphism ring of a vector space over $\mathbf{GF(2)}(2)$ has the property. We have already seen that this forces $\dim(B_n[2]) \geq 2$ and consequently $f_n(G) \geq 2$ for $n \in J$. □

Concluding Remarks: It would be interesting to know the range of the unit sum number function; in particular for each finite integer n does there exist a p-group with $usn(G) = n$? In this context it is worth noting that although every separable p–group G occurs as a pure subgroup of the torsion completion $\overline{B}$ of any of its basic, and hence direct sum of cyclics, subgroup, it is easy to use a realization theorem (see e.g. [2], [6]) to exhibit a separable p–group G ($p \neq 2$) of cardinality $2^{\aleph_0}$ for which $usn(G) = \infty$.

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