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Classifying E-algebras over Dedekind domains

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Abstract

An R-algebra A is said to be a generalized E-algebra if A is isomorphic to the algebra $\mathrm{End}_R(A)$. Generalized E-algebras have been extensively investigated. In this work they are classified 'modulo cotorsion-free modules' when the underlying ring R is a Dedekind domain.

1. Introduction

The notion of an E-ring goes back to a seminal paper of Schultz [20] written in response to Problem 45 in the well-known book 'Abelian Groups' by Laszlo Fuchs [11]. In this paper Schultz distinguished between two possibly different approaches, the first we will continue to call an E-ring, while the second we shall refer to as a *generalized* E-ring. Thus a ring R is said to be an E-ring if R is isomorphic to the endomorphism ring of its underlying additive group, R^+, via the mapping sending an element $r \in R$ to the endomorphism given by left multiplication by r, whilst R is a generalized E-ring if some isomorphism, not necessarily left multiplication, exists between R and its endomorphism ring $\mathrm{End}(R^+)$.

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Since right multiplication is always an endomorphism, it is not difficult to see that E-rings are necessarily commutative. The existence of a non-commutative generalized E-ring has recently been established [15], and so it follows that the class of generalized E-rings is strictly larger than the class of E-rings.

Since Schultz's original paper there has been a great deal of interest in E-rings and some natural generalizations, see e.g. [1,2,4,6,8–10,17,19,21]. A notable feature of much of this recent work has been the use of so-called realization theorems, whereby a cotorsion-free ring is realized, using combinatorial ideas derived from Shelah's Black Box—see e.g. [7] for details of this technique—as the endomorphism ring of an Abelian group. This present work arose from an observation of the second author in response to a question from the first about the existence of generalized E-algebras over the ring J_p of p-adic integers; see [16] for further details. A natural question which arises, is to what extent is it necessary for a ring to be cotorsion-free in order to be a generalized E-ring and the principal objective of this work is to characterize generalized E-rings 'modulo cotorsion-free groups.' The characterization is quite elementary but seems to have been overlooked heretofore. It should be noted that Bowshell and Schultz showed in [2] that a reduced cotorsion E-ring has the form $\prod_{p\in U}\mathbb{Z}(p^{k_p})\oplus\prod_{p\in V}J_p$ where U, V are disjoint sets of primes.

It will be just as convenient to study generalized E(R)-algebras; if R is a commutative ring and M is an R-algebra, then M is said to be a generalized E(R)-algebra if M is isomorphic, as an algebra, to its own endomorphism algebra, $\mathrm{End}_R(M)$. Indeed many of our results could be stated in terms of the module structure only, i.e. we could restrict our attention to R-modules M with the property that the endomorphism *module* $\mathrm{Endo}_R(M)$ is isomorphic, as a module, to M. There is little to be gained from this distinction so we will refer in general to generalized E(R)-algebras hereafter, or simply generalized E-algebras if there is no possibility of confusion.

We finish this introduction by establishing some notation which will remain fixed throughout the sequel. So let R be a Dedekind domain with quotient field Q and let S be the set of prime ideals of R. A module M is said to be *S-divisible* if $M = PM$ for all $P \in S$, while it is said to be *S-reduced* if it has no non-trivial S-divisible submodules. A submodule N of M is said to be *S-pure* in M, denoted $N \leqslant_S M$, if $IN = N \cap IM$ for all ideals I of R which are products of prime ideals in the set S. By analogy with the standard notation used in Abelian group theory, we shall write $R(P^\infty)$ for the direct limit of the cyclic modules R/P^nR, or equivalently the injective hull of R/PR; similarly J_P will denote the completion in the P-adic topology of the discrete valuation ring R_P, the localization of R at P. Finally we note that if R is field then it is immediate that only R itself can be a generalized E-algebra and so we shall always assume that the ring R is not a field. At some points we shall also need to assume that R has a localization at a prime ideals $P \in S$ which is not a complete discrete valuation ring; we note that this is equivalent to saying that R itself is not a complete discrete valuation domain—see e.g. [18, Proposition 4].

2. E(R)-algebras with cotorsion submodules

Suppose then that A is a generalized E(R)-algebra but that A is not cotorsion-free. It follows from an easy extension of the well-known classification of cotorsion-free groups—

see e.g. [7, V, Theorem 2.9]—that A is either (i) not reduced, (ii) not torsion-free or (iii) it is torsion-free reduced but contains a submodule isomorphic to J_P the algebra of P-adic integers. It is easy to handle case (i) and the possibility in case (ii) that A is torsion. We record this as

Proposition 2.1.

(i) *If A is not reduced, then A is a generalized E-algebra if and only if $A \cong Q \oplus C$ where C is a cyclic torsion R-module.*
(ii) *If A is torsion, then A is a generalized E-algebra if and only if A is cyclic.*

Proof. See Theorems 1 and 3 in [20] or see [3] for a discussion in the context of Dedekind domains. $\square$

The remaining part of case (ii) can now be dealt with. So assume that A is a reduced generalized E-algebra which is mixed and let T denote the torsion submodule of A. It follows from Lemma 2 of Schultz [20], that $T = \bigoplus_{P\in\mathbb{P}'} R/P^{k_P}R$ for some set of primes $\mathbb{P}' \subseteq S$ and integers k_P; let V denote the corresponding direct product $V = \prod_{P\in\mathbb{P}'} R/P^{k_P}R$. Also it follows from [20, Lemma 4] that A is an extension of an ideal I by a pure subalgebra of V which contains T; the ideal I consists of the elements in A which have infinite P-height for all $P \in \mathbb{P}'$.

It is well known—see e.g. [12, Section 58]—that A can be embedded as a pure submodule in its cotorsion completion $A^\bullet = \operatorname{Ext}(Q/R, A)$ and that this latter splits as $T^\bullet \oplus F^\bullet$ where, as before, T is the torsion submodule of A and F is the torsion-free quotient A/T—see e.g. [12, Lemma 55.2]. Since F is torsion-free, we know from Theorem 52.3 in [12] that $\operatorname{Ext}(Q/R, F) \cong \operatorname{Hom}(Q/R, D/F)$ where D is the injective hull of F. Moreover, $\operatorname{Hom}(Q/R, D/F) \cong \prod_P \operatorname{Hom}(R(P^\infty), D/F) \cong \prod_P \operatorname{Hom}(R(P^\infty), \bigoplus_m R(P^\infty))$ where m is the torsion-free rank of F. From Proposition 44.3 in [12], we conclude that $\operatorname{Hom}(Q/R, D/F) \cong \prod_P \widehat{\bigoplus_m J_P}$, a torsion-free pure injective module.

Lemma 2.2. *The cotorsion completion of T is isomorphic to a direct summand of the cotorsion completion of V, $V^\bullet \cong T^\bullet \oplus Y$ for some Y. Moreover, $V^\bullet \cong V$.*

Proof. It is easy to see that V/T is torsion-free and so if we consider the short exact sequence

$$0 \to T \to V \to X \to 0,$$

where $X = V/T$, we get an induced sequence

$$0 \to T^\bullet \to V^\bullet \to \operatorname{Ext}(Q/R, X) \to 0.$$

Since X is torsion-free, $\operatorname{Ext}(Q/R, X) \cong \operatorname{Hom}(Q/R, D/X)$ where D is the injective hull of X. But it follows easily from [12, Proposition 44.3] that this latter is isomorphic to a direct product of completions of direct sums of modules J_P, for various primes P. In particular it is torsion-free and since $T^\bullet$ is cotorsion, the extension $0 \to T^\bullet \to V^\bullet \to \operatorname{Ext}(Q/R, X) \to 0$ splits, i.e. $T^\bullet$ is a direct summand of $V^\bullet$ as required. Furthermore since $V = \prod_{P\in\mathbb{P}'} R/P^{k_P}R$, we have $V^\bullet = \operatorname{Ext}(Q/R, V) \cong \prod_{P\in\mathbb{P}'} \operatorname{Ext}(Q/R, R/P^{k_P}R) \cong \prod_{P\in\mathbb{P}'} \operatorname{Ext}(R/P^{k_P}R, R/P^{k_P}R) \cong V$. □

It is now possible to shed some light on the structure of the ideal I of elements of infinite P-height ($P \in \mathbb{P}'$).

Now A is a pure submodule of $A^\bullet = T^\bullet \oplus F^\bullet$, which by Lemma 2.2 and the discussion immediately preceding it, is a pure submodule of

$$V \oplus \prod_P \widehat{\bigoplus_m J_P} = \left[V \oplus \prod_{P\in\mathbb{P}'} \widehat{\bigoplus_m J_P} \right] \bigoplus \prod_{P\in S\setminus\mathbb{P}'} \widehat{\bigoplus_m J_P}.$$

Let U denote the radical defined by $\mathrm{U}(X) = \bigcap_{P\in\mathbb{P}'} P^\omega X$ for any module X. Then $\mathrm{U}(A)$ is the set of elements in A which have infinite P-height for all $P \in \mathbb{P}'$. It follows from elementary properties of radicals that

$$\mathrm{U}(A) \leqslant \mathrm{U}\left(\left[V \oplus \prod_{P\in\mathbb{P}'} \widehat{\bigoplus_m J_P} \right] \right) \oplus \mathrm{U}\left(\prod_{P\in S\setminus\mathbb{P}'} \widehat{\bigoplus_m J_P} \right).$$

The former term however is clearly zero and so we have established a slight extension of [3, Theorem 1.1]:

Proposition 2.3. *If A is a reduced generalized E-algebra which is a mixed algebra and the P-primary component of the torsion submodule $t_P(A) \neq 0$ for each prime $P \in \mathbb{P}'$, then A is an extension of an ideal I by a P-pure (for each $P \in \mathbb{P}'$) subalgebra with identity of the algebra $\prod_{P\in\mathbb{P}'} R/P^{k_P}R$ containing $\bigoplus_{P\in\mathbb{P}'} R/P^{k_P}R$. Moreover, I is contained in $\prod_{P\in S\setminus\mathbb{P}'} \widehat{\bigoplus_m J_P}$ for a suitable cardinal m.*

The final case to be considered is when the generalized E-algebra A is torsion-free, reduced and contains a submodule isomorphic to the module J_P of P-adic integers. Since E-algebras of arbitrary large rank with many additional properties have been constructed—see e.g. [4–6]—there is no possibility of obtaining a characterization in this case. We can, however, obtain a complete characterization 'modulo cotorsion-free modules.'

Suppose that A is a generalized E-algebra which is torsion-free, reduced but not cotorsion-free. Then the set $\{P \in S \mid A \text{ has a pure submodule isomorphic to } J_P\} \neq \emptyset$. Let $\mathbb{P}'$ denote those primes P for which A has a submodule isomorphic to J_P; we refer to $\mathbb{P}'$ as the set of *relevant* primes.

Lemma 2.4. *A does not contain a pure submodule of the form $\bigoplus_{\aleph_0} J_P$ for any $P \in \mathbb{P}'$.*

Proof. Suppose for a contradiction that A did contain such a pure submodule, B say. Then we have a short exact sequence

$$0 \to A \to \hat{A} \to D \to 0,$$

where $\hat{A}$ is the completion of A in the natural topology. Since A is torsion-free reduced, the quotient $\hat{A}/A = D$ is torsion-free and divisible (possibly $= 0$)—see e.g. [13, Chapter VIII, Theorem 2.1]. Thus we get an induced sequence

$$0 \to \mathrm{Hom}(\hat{A}, A) \to \mathrm{Hom}(A, A) \to \mathrm{Ext}(D, A)$$

and note that the final term is torsion-free, divisible since D is. (Since modules over a Dedekind domain have projective dimension at most 1, the standard argument for Abelian groups [12, 52(J)] carries over to this situation.) Hence $A \cong \mathrm{Hom}(A, A)$ contains a pure submodule isomorphic to $\mathrm{Hom}(\hat{A}, A)$. Now it follows from the standard classification of pure injective modules, which also carries over in this situation—see e.g. [13, Chapter XIII, Proposition 4.5]—that $\hat{A}$ has the form $\hat{A} = \widehat{\bigoplus_\kappa J_P} \oplus Y$, for some Y with $\mathrm{Hom}(J_P, Y) = 0$. Note that κ is an invariant of $\hat{A}$ and, since the closure of B in $\hat{A}$ is again pure and thus a summand, $\kappa \geqslant \aleph_0$. Since A has a summand isomorphic to J_P and $\hat{A}$ has a summand $\widehat{\bigoplus_\kappa J_P}$, it follows that $\mathrm{Hom}(\hat{A}, A)$ has a summand isomorphic to $\mathrm{Hom}(\widehat{\bigoplus_\kappa J_P}, J_P)$. But now $\mathrm{Hom}(\widehat{\bigoplus_\kappa J_P}, J_P) \cong \mathrm{Hom}(\bigoplus_\kappa J_P, J_P) \cong \prod_\kappa J_P$ and this latter is isomorphic to $\widehat{\bigoplus_{2^\kappa} J_P}$—see Fuchs [12, §40, Example 1]. Thus A has a pure submodule isomorphic to $\widehat{\bigoplus_{2^\kappa} J_P}$ and hence, so also has $\hat{A}$. Since $\widehat{\bigoplus_{2^\kappa} J_P}$ is pure injective, this would mean that $\hat{A}$ has a direct summand which is the completion of a direct sum of strictly more than κ copies of J_P—contradiction. $\square$

Recall that the P-cotorsion radical of an R-module G is defined by

$$\mathcal{R}(G) = \bigcap_{\phi : G \to J_P} \mathrm{Ker}\,\phi.$$

Lemma 2.5. *For each relevant prime $P \in \mathbb{P}'$, $A = J_P \oplus X_{n_P}$, where X_{n_P} is P-cotorsion-free and fully invariant.*

Proof. Firstly we show that A cannot have a sequence of summands A_i where $A_i \cong J_P^{(n_i)}$ with $n_1 < n_2 < \cdots$. Suppose such a sequence exists $A = A_i \oplus X_i$, say, where each X_i is P-cotorsion-free. Then taking P-cotorsion radicals we get $\mathcal{R}(A) = X_i = X_j$ for all i, j. This is impossible since it would imply that $A_i \cong A_j$ and this is not so. Hence we can construct inductively a sequence $B_1 = A_1, B_2, \ldots$ of summands of A with $B_1 < B_2 < \cdots$ and each B_i is a direct sum of a finite number of copies of J_P. The union of this sequence would then be a pure submodule of A isomorphic to $\bigoplus_{\aleph_0} J_P$; this last conclusion coming from the fact that successive terms in the sequence of submodules B_i split with factors isomorphic to direct sums of copies of J_P. This, however, is impossible since, by the previous lemma, A has no pure submodule isomorphic to $\bigoplus_{\aleph_0} J_P$ and so $A = J_P^{(n_P)} \oplus X_{n_P}$ for some X_{n_P} and finite integer n_P. However as $A \cong \mathrm{Hom}(A, A)$, it is immediate that $n_P = 1$. Clearly then, the complement X_{n_P} is P-cotorsion-free. If $\mathrm{Hom}(X_{n_P}, J_P) = 0$, then X_{n_P} is fully invariant. If not, there exists a non-zero mapping σ say: $X_{n_P} \to J_P$ and then the composition $\sigma\pi$ is a mapping: $X_{n_P} \to J_P$ for all P-adic integers π, i.e. $\mathrm{Hom}(X_{n_P}, J_P)$ contains a copy of J_P and so A has a summand with at least two copies of J_P—contradiction. Thus X_{n_P} is fully invariant as claimed. $\square$

Theorem 2.6. *If A is a generalized E-algebra, then for each relevant prime $P \in \mathbb{P}'$, A has the form $A = A_P \oplus X_P$, where $A_P = J_P$ and the complement X_P, which is P-cotorsion-free is unique. Moreover, X_P is itself a generalized E-algebra.*

Proof. The complement X_P is unique since it is fully invariant—see e.g. [12, Corollary 9.7]. Since X_P is P-cotorsion-free, $\mathrm{Hom}(A_P, X_P) = 0$ and since X_P is fully invariant, $\mathrm{Hom}(X_P, A_P) = 0$. Thus $A_P \oplus X_P = A \cong \mathrm{Hom}(A, A) \cong J_P \oplus \mathrm{Hom}(X_P, X_P)$, and then, taking P-cotorsion radicals, we get that $X_P \cong \mathrm{Hom}(X_P, X_P)$. Thus X_P is a generalized E-algebra as claimed. $\square$

Corollary 2.7. *If A is a generalized E-algebra with a finite set $\mathbb{P}'$ of relevant primes, then A is an extension of a cotorsion-free ideal C, which is itself an E-algebra, by the algebra $\prod_{P \in \mathbb{P}'} J_P$.*

Proof. The result follows immediately by finite repetition from Theorem 2.6. $\square$

We can also recover the result noted in the introduction, that over a complete discrete valuation ring, generalized E-algebras exist in only the trivial way—see [16] for extensions of this result.

Corollary 2.8. *If A is a reduced torsion-free module over a complete discrete valuation ring R, then A is a generalized E-algebra if, and only if $A \cong R$.*

Proof. In one direction the proof is immediate. Conversely suppose that A is an R-algebra. Then the set of relevant primes has exactly one member. Moreover, the only cotorsion-free R-module is the trivial module 0. The result follows immediately from Corollary 2.7. $\square$

Lemma 2.9. *A contains a submodule of the form $B = \bigoplus_{P \in \mathbb{P}'} J_P$.*

Proof. Clearly the submodule B generated by the A_P will have this form if we can show that the sum is direct, or equivalently that $A_P \cap \sum_{I \neq P} A_I = \{0\}$. However if $x \in A_P$ and $x = x_1 + \cdots + x_k$, where $x_i \in A_{I_i}$ with $I_i \neq P$, then as each x_i is P-divisible, $x \in P^\omega A \cap A_P = P^\omega A_P = 0$. $\square$

Theorem 2.10. *If A is a reduced, torsion-free generalized E-algebra and $\mathbb{P}' \subseteq S$ denotes the set of primes P for which A contains a submodule isomorphic to J_P, then A is an extension of a cotorsion-free ideal X by a subalgebra C of $\prod_{P \in \mathbb{P}'} J_P$ and C contains an isomorphic copy of $B = \bigoplus_{P \in \mathbb{P}'} J_P$. Moreover,*

(i) *X is the intersection of a family of generalized E-algebras.*
(ii) *C contains the identity of $\prod_{P \in \mathbb{P}'} J_P$ and is hence an E-algebra.*

Proof. It follows from Theorem 2.6 that for $P \in \mathbb{P}'$, each element a of A can be expressed uniquely as $a = j_P + x_P$, where $j_P \in A_P$, $x_P \in X_P$, and so the mapping sending a to the vector $(\ldots, j_P, \ldots)$, is a well-defined homomorphism ϕ of A into $\prod_{P \in \mathbb{P}'} J_P$. The kernel of this mapping ϕ is precisely $X = \bigcap_{P \in \mathbb{P}'} X_P$. Since each X_P is P-cotorsion-free, the intersection X is $\mathbb{P}'$-cotorsion-free and hence, as no other primes are relevant, is cotorsion-free. Note that $\phi \restriction B$ acts as the identity, since an element of the form j_P when expressed as $j_I + x_I$, must have $j_I = 0$ because $j_I = j_P - x_I$ is I-divisible. Thus we have $B \cong B\phi \leqslant C = A/\operatorname{Ker}\phi \leqslant \prod_{P \in \mathbb{P}'} J_P$. Since ϕ is an algebra homomorphism, X is an ideal which is clearly the intersection of a family of generalized E-algebras, and C is a subalgebra. The final claim that C is an E-algebra follows from the general fact proved in Proposition 2.11 below. $\square$

Proposition 2.11. *If X is a subalgebra with* 1 *of the direct product $\prod_{P \in \mathbb{P}'} J_P$ and X contains the corresponding direct sum $\bigoplus_{P \in \mathbb{P}'} J_P$, then X is an E-algebra.*

Proof. If $\phi \in \operatorname{End}(X)$ then $1\phi = x_0 \in X$. Now x_0 acts by multiplication componentwise on X and the difference $\phi - x_0$ acts as the zero map on $\bigoplus_{P \in \mathbb{P}'} J_P$. But this direct sum is dense in the product, and since maps are continuous, ϕ acts as multiplication by x_0 on X. $\square$

If the ideal X actually splits then we can deduce a good deal more:

Corollary 2.12. *If the ideal X splits then A is the split extension of a cotorsion-free generalized E-algebra X by an E-algebra C where $\bigoplus_{P \in \mathbb{P}'} J_P \leqslant C \leqslant \prod_{P \in \mathbb{P}'} J_P$.*

Proof. If $A = X \oplus C$ then $\operatorname{Hom}(C, X) = 0$ since X is cotorsion-free. Moreover, $\operatorname{Hom}(X, C)$ is a direct summand of A and if it is not zero it would be isomorphic to a product of copies of J_P, some $P \in \mathbb{P}'$. This is impossible and so we have $X \oplus C = A \cong \operatorname{Hom}(A, A) = \operatorname{Hom}(X, X) \oplus \operatorname{Hom}(C, C)$. By applying each of the P-cotorsion radicals ($P \in \mathbb{P}'$) and noting that $C \cong \operatorname{Hom}(C, C)$, we conclude, as in the proof of Theorem 2.6, that $X \cong \operatorname{Hom}(X, X)$ as required. $\square$

Our final result is a partial converse to Theorem 2.10; we show the existence of many non-splitting E-algebras.

Theorem 2.13. *Let R be a Dedekind domain with prime spectrum S and let $\mathbb{P}'$ be an infinite proper subset of S, $P_0 \in S \setminus \mathbb{P}'$ and λ a cardinal with $\lambda^{\aleph_0} = \lambda$. Then, provided that R_{P_0} is not a complete discrete valuation domain, there exists a generalized E-algebra A of cardinality λ such that A is an extension of a cotorsion-free ideal D by a subalgebra B, where $T = \bigoplus_{P \in \mathbb{P}'} J_P \leqslant B \leqslant \prod_{P \in \mathbb{P}'} J_P$.*

We note that in this situation where $\mathbb{P}'$ is an infinite set, A does not split.

The proof is similar to that used by Braun and the first author in [3]. First we need a lemma:

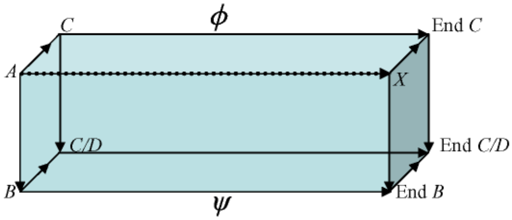

Fig. 1.

Lemma 2.14. *Let $T = \bigoplus_{P\in\mathbb{P}'} J_P \leqslant B \leqslant \prod_{P\in\mathbb{P}'} J_P = \Pi$, where B is a $\mathbb{P}'$-pure subalgebra with 1 of Π and suppose that C is a $\mathbb{P}'$-divisible, cotorsion-free generalized E-algebra with a fixed isomorphism $\phi : C \to \mathrm{Hom}(C, C)$ satisfying*:

(i) *there is a fully invariant ideal D of C and $C/D \cong B/T$,*
(ii) *the induced map $C \cong \mathrm{Hom}(C, C) \to \mathrm{Hom}(C/D, C/D)$ maps each $c \in C$ to multiplication by the residue class of c.*

Then the pullback A below is a generalized E-ring.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & D \arrow[r, equal] \arrow[d] & D \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & B \arrow[r] \arrow[d] & B/T \cong C/D \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Proof. Note that B is automatically an E-algebra so that there exists an isomorphism $\psi : B \to \mathrm{End}(B)$ and that T is fully invariant in B. By hypothesis every endomorphism of C induces an endomorphism on C/D; similarly for B and B/T. Hence we may form another pullback, X say, of $\mathrm{End}(C)$ and $\mathrm{End}(B)$ and we have a composite diagram—see Fig. 1—where the mapping from $C/D \to \mathrm{End}(C/D)$ is the induced map and the map from A to X is the mapping coming from the universal property of pullbacks.

A straightforward check using the pullback definition shows that X may be identified with those endomorphisms of A which induce endomorphisms on both C and B. Moreover, our hypotheses ensure that the composite diagram in Fig. 1 is commutative and so, by the pullback property, A is isomorphic to X and so may be identified with the same set of endomorphisms of A. As can be seen from the first diagram, $A/D \cong B$ and $A/T \cong C$, and so the set of endomorphisms of A inducing endomorphisms on C, B is precisely the set of endomorphisms of A leaving D and T invariant. Now B/T is torsion-free and isomorphic to C/D, thus we may conclude that D is pure in the $\mathbb{P}'$-divisible module C. Since $A/D \cong$

B, and B is $\mathbb{P}'$-reduced, we conclude that D is the maximal $\mathbb{P}'$-divisible submodule of A and hence is invariant under all endomorphisms of A. Moreover, $A/T \cong C$, and C is cotorsion-free. We claim that T must be fully invariant. To see this recall the notion of the hyper-cotorsion radical of a module: a module M is said to be hyper-cotorsion if every non-trivial epimorphic image contains a non-trivial cotorsion submodule and the submodule hM is the hyper-cotorsion radical of M if hM is hyper-cotorsion and the quotient M/hM is cotorsion-free. (Further details of this notion may be found in [14].) Hence we see that T is the hyper-cotorsion radical of A and, as a radical, is fully invariant in A.

Hence every endomorphism of A leaves D and T invariant and so A is identified with the full endomorphism algebra of A, i.e. A is a generalized E-algebra. $\square$

Proof of Theorem 2.13. Since $\mathbb{P}'$ is an infinite set, $T \neq \Pi$ and so we may choose $0 \neq E := B_0/T \cong R_{P_0}$ where $P_0 \notin \mathbb{P}'$. Then, since the localization at P_0 is not a complete discrete valuation ring, E is P_0-cotorsion-free and so we can apply the realization theorems for cotorsion-free E-algebras—see e.g. [6]. Thus we may obtain an E-algebra C with $C \subseteq E[Y]_{P_0} = E_{P_0}[Y]$, a polynomial ring over the set Y of cardinal λ with coefficients from the localization of E at the prime P_0. If D is taken as the polynomials with constant term equal to zero, then $E \subseteq C/D \subseteq E_{P_0}$. Since Π/T is divisible, torsion-free we can identify E_{P_0}, and hence C/D as a subalgebra B/T of P/T for some B. With this choice of B, T, C and D, apply Lemma 2.14. This yields the desired generalized E-ring A.

Finally observe that A does not split over D for if it did, the corresponding projection onto D would be multiplication by a non-zero idempotent in A. However the image of 1 under this idempotent must lie in D and so there would exist a non-zero idempotent polynomial with zero constant term; this is clearly impossible and so we conclude that A does not split, as required. $\square$

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