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Algebraic entropy for Abelian groups *

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In Memoriam: Il Maestro, Adalberto Orsatti

Abstract

The theory of endomorphism rings of algebraic structures allows, in a natural way, a systematic approach based on the notion of entropy borrowed from dynamical systems. Here we study the algebraic entropy of the endomorphisms of Abelian groups, introduced in 1965 by Adler, Konheim and McAndrew. The so-called Addition Theorem is proved; this expresses the algebraic entropy of an endomorphism ϕ of a torsion group as the sum of the algebraic entropies of the restriction to a ϕ-invariant subgroup and of the endomorphism induced on the quotient group. Particular attention is paid to endomorphisms with zero algebraic entropy as well as to groups all whose endomorphisms have zero algebraic entropy. The significance of this class arises from the fact that any group not in this class can be shown to have endomorphisms of infinite algebraic

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entropy and we also investigate such groups. A uniqueness theorem for the algebraic entropy of endomorphisms of torsion Abelian groups is proved.

Introduction.

In their pioneering paper [AKM] of 1965, Adler, Konheim, and McAndrew introduced the notion of entropy for continuous self-maps of compact topological spaces. In the conclusion of [AKM] the authors wrote: "The notion of entropy has an abstract formulation which we have not dealt with here. It can be tailored to fit mappings on other mathematical structures". Actually, they sketched how to define the entropy for endomorphisms of abstract Abelian groups, remarking that "analogies to the general theorems can be established".

In 1975 Weiss [W] reconsidered the definition of entropy for endomorphisms of Abelian groups sketched in [AKM]. He called it "algebraic entropy", and gave detailed proofs of its basic properties. His main result was that the algebraic entropy of an endomorphism ϕ of the Abelian group G is equal to the topological entropy of the adjoint map Φ^* of Φ, where $\Phi = \phi\restriction_{t(G)}$ is the restriction of ϕ to the torsion subgroup $t(G)$ of G.

It is worth recalling that in 1979 Peters [P] gave a different definition of entropy for automorphisms of a discrete Abelian group G. After proving the basic properties, similar to those proved by Weiss, he generalized Weiss's main result to countable Abelian groups, relating the entropy of an automorphism of G to the measure-theoretic Kolmogorov-Sinai entropy of the adjoint automorphism of the dual of G. The definition of entropy of automorphisms given by Peters is easily adaptable to endomorphisms of Abelian groups. A fundamental difference in the two approaches is that Weiss bases his definition on the supremum, *over all finite subgroups*, of a certain function, while Peters consider the supremum *over all finite subsets*.

Comparing the two notions of entropy considered above, it can be seen immediately that Weiss's definition, based on [AKM], has the advantage of being intrinsically "algebraic". On the other hand, Weiss's definition has the disadvantage of being trivial for torsion-free groups, whereas Peters's definition gives rise to interesting questions in that case. Actually, the two definitions produce the same notion if one considers torsion groups, but for torsion-free and mixed groups they produce two different notions. In this paper we will make use of Weiss's definition and consequently we shall focus mainly on torsion groups. Thereby we take up the "challenge" in the conclusion of [AKM] quoted above, tailoring the notion of entropy to the specific algebraic structure with which we are dealing.

However in none of these cases was much progress made in actually determining the entropy of infinite Abelian groups, largely due to the fact that, at that time, the complicated structure of such groups was poorly understood other than by a small number of experts in algebra. Recent developments now make it an opportune time to revisit this important topic. Indeed, very recently Alcaraz, Dikranjan, and Sanchis [ADS] investigated the generalization given by Bowen

[B] of the notion of entropy introduced in [AKM] for uniformly continuous self-maps of uniform spaces. They showed the failure of the so-called "Completion Theorem" for endomorphisms of totally bounded groups. They also studied the class of topological groups without endomorphisms of infinite entropy, as well as the class of groups all of whose endomorphisms have zero entropy. This paper originated from some open questions posed for these classes in [ADS].

After giving in Section 1 the definition of the algebraic entropy of an endomorphism of an Abelian group, we recall Weiss's results and develop the basic facts; we furnish examples of endomorphisms of zero, positive and infinite algebraic entropy. A general result is also proved, which states that every infinite direct sum of non-zero Abelian groups admits an endomorphism of infinite algebraic entropy.

The "local" case of the study of a single endomorphism ϕ of an Abelian group G can also be considered as another facet of the theory of modules over the polynomial ring $\mathbb{Z}[X]$ (or $J_p[X]$, if G is a p-group). Thus in Section 2 we investigate the $J_p[\phi]$-module structure of a p-group G, in connection with various questions related to the algebraic entropy of a fixed endomorphism ϕ of G. We characterize the endomorphisms with zero algebraic entropy as being point-wise integral over J_p in an appropriate sense. This allows us to dispense with the usual technically involved definition of the algebraic entropy. In the final subsection we borrow ideas from dynamical systems theory and show that a strong analogue of the Poincaré - Birkhoff Recurrence Theorem in ergodic theory holds for monomorphisms.

Section 3 is devoted to proving the so-called Addition Theorem, relating the algebraic entropy of an endomorphism ϕ of a torsion group G to the algebraic entropy of its restriction to a ϕ-invariant subgroup and to that of the induced endomorphism on the factor group. This theorem is crucial in the development of the whole theory, and allows us to calculate explicitly the algebraic entropy of an endomorphism ϕ of a bounded p-group G, using its structure as $J_p[\phi]$-module.

Both Section 4 and Section 5 make use of the fundamental results on endomorphism rings of p-groups obtained in the '60's and '70's by Pierce [Pi] and Corner [C1, 2]. In Section 4 it is shown that p-groups without endomorphisms of infinite algebraic entropy are necessarily *semi-standard* (i.e., their Ulm-Kaplansky invariants are finite) and essentially finitely indecomposabble. It is also proved that infinite bounded p-groups and many unbounded p-groups, including totally projective groups, torsion-complete groups and $p^{\omega+1}$-projective groups, have endomorphisms of infinite algebraic entropy. We also provide an example of a standard essentially indecomposable p-group admitting an endomorphism of infinite algebraic entropy.

Section 5 is devoted to the investigation of p-groups with zero algebraic entropy. We show that certain strictly quasi-complete p-groups (see [F, XI.74] and [S, Section 49]), investigated by Hill and Megibben [HM], [M] under the name of *quasi-closed* groups, and first constructed by Pierce [Pi], have zero algebraic entropy. The crucial point, which may be of some independent interest, is the result (Theorem 5.2) that small endomorphisms of semi-standard p-groups have zero algebraic entropy. We also show that even the endomorphisms which are integral over J_p modulo the ideal of the small endomorphisms have zero algebraic entropy. Using Corner's [C2] notion of thin endomorphisms, a generalization of the concept of small endomorphisms, we construct p-groups of any length strictly smaller than ω^2 with zero algebraic entropy. For both separable and non-separable groups of length $< \omega^2$, we derive from Corner's realization theorems in [C1, 2] the existence of a family of $2^{2^{\aleph_0}}$ groups, each with zero algebraic entropy, and having the property that homomorphisms between different members are small (respectively thin).

In Section 6 we prove a uniqueness theorem for the algebraic entropy of endomorphisms of torsion Abelian groups, stating that the algebraic entropy is the unique nonnegative real numerical invariant associated to the endomorphisms, that satisfies certain properties. Thus we obtain an axiomatic characterization of the algebraic entropy of endomorphisms for torsion groups, similar to that given by Stojanov in [St] for the topological entropy of compact groups.

We finish this introduction by noting that our terminology and notations are standard and any undefined term may be found in the texts [F] and [S]. In particular all groups are additively written and Abelian, so that if A, B are subgroups of the group G, $A + B$ will denote the subgroup of G generated by A and B. Finally the authors would like to express their thanks to the Referee for a number of useful suggestions which have been incorporated into the final text.

1 Algebraic entropy of endomorphisms and of groups.

1.1 The definition.

Let G be an Abelian group and denote by $\mathcal{F}(G)$ the family of its finite subgroups. If $\phi : G \to G$ is an endomorphism of G, for every positive integer n and every $F \in \mathcal{F}(G)$ we set

$$T_n(\phi, F) = F + \phi F + \phi^2 F + \ldots + \phi^{n-1} F.$$

For every $n > 0$, we have $T_{n+1}(\phi, F) = T_n(\phi, F) + \phi^n F$, so

$$\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)} \cong \frac{\phi^n F}{T_n(\phi, F) \cap \phi^n F}.$$

The subgroup of G, $T(\phi, F) = \sum_{n>0} T_n(\phi, F) = \sum_{n>0} \phi^n F$ will be called the *ϕ-trajectory* of F. The ϕ-trajectory of an element x is just the ϕ-trajectory of the cyclic subgroup $\mathbb{Z}x$, i.e., the smallest ϕ-invariant subgroup of G containing x, simply denoted by $T(\phi, x)$. It is clear that the ϕ-trajectory of the finite group F is finite if and only if the ϕ-trajectory of each $x \in F$ is such.

For each $n \geq 1$ set $\tau_n = |T_n(\phi, F)|$, so that

$$0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n \leq \ldots$$

is an increasing sequence of positive integers, each one dividing the next one. For each $n \geq 1$ we set:

$$\alpha_{n+1} = \frac{\tau_{n+1}}{\tau_n} = \left|\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)}\right| = \left|\frac{\phi^n F}{T_n(\phi, F) \cap \phi^n F}\right|.$$

Lemma 1.1. *For each $n > 1$, α_{n+1} divides α_n in $\mathbb{N}$.*

Proof. Since ϕ and F remain unchanged during the proof, we write T_n in place of $T_n(\phi, F)$. The group $\phi^n F/(T_n \cap \phi^n F)$ is a quotient of the group $B_n = \phi^n F/(\phi T_{n-1} \cap \phi^n F)$, since $\phi T_{n-1} \cap \phi^n F$ is contained in $T_n \cap \phi^n F$. So α_{n+1} divides $\beta_{n+1} = |B_n|$. From $\phi T_n = \phi T_{n-1} + \phi^n F$ we conclude that

$$B_n \cong \frac{\phi T_n}{\phi T_{n-1}} \cong \frac{T_n}{T_{n-1} + (T_n \cap \operatorname{Ker} \phi)}.$$

Since the latter group is a quotient of T_n/T_{n-1}, we conclude that β_{n+1} divides α_n. Therefore α_{n+1} divides α_n. ∎

From the preceding lemma we immediately derive the following

Corollary 1.2. *Either the sequence $0 < \tau_1 \leq \tau_2 \leq \ldots$ is stationary, or $\tau_{n+1} = \tau_n \alpha$ for some integer $\alpha > 1$, for all n large enough. In particular, $|T_n(\phi, F)| = a_0 \alpha^{n-k}$ for all sufficiently large n, where a_0 and k depend only on F, not on n.*

Proof. The sequence of positive integers $\alpha_2, \alpha_3, \ldots$ is decreasing, hence it is eventually equal to some $\alpha \geq 1$. The first case happens when $\alpha = 1$, the latter when $\alpha > 1$. ∎

Given the finite subgroup F and the endomorphism ϕ of G, for each $n \geq 1$ we define the real number:

$$H_n(\phi, F) = \log |T_n(\phi, F)|.$$

Clearly we have the increasing sequence of real numbers

$$0 < H_1(\phi, F) \leq H_2(\phi, F) \leq H_3(\phi, F) \leq \ldots.$$

Now define

$$H(\phi, F) = \lim_{n\to\infty} \frac{H_n(\phi, F)}{n}.$$

In the next Proposition we show that this is a good definition, namely, the limit exists and it is finite. We calculate its exact value.

Proposition 1.3. *Given the endomorphism $\phi : G \to G$ and a finite subgroup F of G, either*

(i) $H(\phi, F) = 0$, which happens exactly if the ϕ-trajectory $T(\phi, F)$ of F is finite; or

(ii) $H(\phi, F) = \log(\alpha)$, *where* $\alpha = \left|\frac{T_{n+1}(\phi,F)}{T_n(\phi,F)}\right|$ *for all* n *large enough, which happens exactly if the* ϕ*-trajectory* $T(\phi, F)$ *of* F *is infinite.*

Proof. By Corollary 1.2, there exist integers n_0 and $\alpha \geq 1$ such that $\tau_{n+1} = \tau_n \alpha$ for all $n \geq n_0$. Case (i) happens when $\alpha = 1$, namely, the sequence $0 < \tau_1 \leq \tau_2 \leq \dots$ is stationary, since in this case $H_{n_0+k}(\phi, F) = H_{n_0}(\phi, F)$ for all k. The second case happens when $\alpha > 1$. In fact, using the equalities:

$$H_{n_0+k}(\phi, F) = \log(\tau_{n_0+k}) = \log(\tau_{n_0}\alpha^k) =$$
$$= \log \tau_{n_0} + k \log \alpha = H_{n_0}(\phi, F) + k \log \alpha$$

we have:

$$H(\phi, F) = \lim_{k\to\infty} \frac{H_{n_0+k}(\phi, F)}{n_0 + k} = \lim_{k\to\infty} \frac{H_{n_0}(\phi, F) + k \log \alpha}{n_0 + k} = \log \alpha.$$

■

Observe that for any endomorphism ϕ and finite subgroup F, one has the equality $H(\phi, F) = H(\phi, \phi F)$ since $T_n(\phi, F) \leq F + T_n(\phi, \phi F)$ and $T_n(\phi, \phi F) \leq T_{n+1}(\phi, F)$.

Following Weiss, we define the *algebraic entropy* of an endomorphism ϕ of G as

$$\mathrm{ent}(\phi) = \sup_{F\in\mathcal{F}(G)} H(\phi, F)$$

and the *algebraic entropy* of G as

$$\mathrm{ent}(G) = \sup_{\phi\in\mathrm{End}(G)} \mathrm{ent}(\phi).$$

Henceforth the term “entropy” will always mean “algebraic entropy”, unless explicitly stated to the contrary.

Of course, Proposition 1.3 (i) implies that $\mathrm{ent}(G) = 0$ whenever G is finite. Moreover, by Proposition 1.3, both the algebraic entropy of ϕ and the algebraic entropy of G are either the logarithm of a positive integer, or the symbol ∞. In particular, when G is a p-group, the entropies of G and of its endomorphisms are either ∞ or an integral multiple of $\log p$.

The following three results will be very useful.

Lemma 1.4. *For every finite subgroup* F *of the Abelian group* G*, let* $G_F = T(\phi, F)$ *be the* ϕ*-trajectory of* F*. Then* $\mathrm{ent}(\phi) = \sup_{F\in\mathcal{F}(\mathcal{G})} \mathrm{ent}(\phi\restriction_{G_F})$.

Furthermore, if $\mathrm{ent}(\phi)$ *is finite, then* $\mathrm{ent}(\phi) = H(\phi, F_0) = \mathrm{ent}(\phi\restriction_{G_{F_0}})$*, for a suitable finite subgroup* F_0 *of* G*.*

Proof. From $\mathrm{ent}(\phi) = \sup_F H(\phi, F)$ and

$$\mathrm{ent}(\phi\restriction_{G_F}) = \sup_{F'} H(\phi\restriction_{G_F}, F') \geq H(\phi, F),$$

where F' ranges over the finite subgroups of G_F, we deduce that $\mathrm{ent}(\phi) \leq \sup_F \mathrm{ent}(\phi\restriction_{G_F})$. The converse inequality is obvious, hence we have equality. The final claim of the lemma is clear since the supremum must be attained in this situation. ■

Lemma 1.5. *Let $\phi : G \to G$ be an endomorphism of a torsion group G, H a ϕ-invariant subgroup of G, and $\bar{\phi} : G/H \to G/H$ the induced endomorphism. If* $\mathrm{ent}(\bar{\phi}) = 0$*, then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_H)$.

Proof. The definition of entropy easily yields $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_H)$ - this is recorded as property (g) in §1.2. Then it suffices to show that $\mathrm{ent}(\phi) \leq \mathrm{ent}(\phi\restriction_H)$, whenever $\mathrm{ent}(\bar{\phi}) = 0$. To this end it suffices to check that $H(\phi, F) \leq \mathrm{ent}(\phi\restriction_H)$ for an arbitrary finite subgroup F of G.

Let $\pi : G \to G/H$ be the canonical homomorphism and let $F_1 = \pi(F)$. Since $\bar{\phi}$ has zero entropy, there exists $m > 0$ such that the subgroup $T_m(\bar{\phi}, F_1)$ of G/H is $\bar{\phi}$-invariant (see Proposition 1.3). As F is finite, there exists a finite subgroup F_2 of H such that $\phi^m F \leq T_m(\phi, F) + F_2$. This gives

$$\phi T_m(\phi, F) \leq T_m(\phi, F) + F_2,$$

from which, by induction on $k > 0$, we get

$$\phi^k T_m(\phi, F) \leq T_m(\phi, F) + T_k(\phi,\, F_2).$$

Now let $n > m$, say $n = m + k$ for some $k > 0$. Then $T_n(\phi, F) \leq T_m(\phi, F) + T_k(\phi, F_2)$, whence we readily get $H_n(\phi, F) \leq H_m(\phi, F) + H_k(\phi, F_2)$. Since m is fixed, dividing by n and letting $n \to \infty$ (so $k \to \infty$ as well), we deduce that $H(\phi, F) \leq H(\phi, F_2) \leq \mathrm{ent}(\phi\restriction_H)$, as desired. ■

As a consequence of Lemma 1.4 we obtain a main property of the entropy function.

Proposition 1.6. *Let the Abelian group G be the direct limit of ϕ-invariant subgroups $\{G_i : i \in I\}$, where $\phi \in \mathrm{End}(G)$. Then*

$$\mathrm{ent}(\phi) = \sup \mathrm{ent}(\phi\restriction_{G_i}). \tag{1}$$

Moreover, if $\mathrm{ent}(\phi) < \infty$*, then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_{G_i})$ *for some $i \in I$.*

The proof follows immediately from the definition, from Lemma 1.4 and from the fact that, for every finite subgroup F of G, the ϕ-trajectory of F is contained in some of the subgroups G_i.

This property is useful when the torsion group G is presented as the union of its fully invariant subgroups $G[n!]$, so that (1) has the form

$$\mathrm{ent}(\phi) = \sup_n \mathrm{ent}(\phi\restriction_{G[n!]}). \tag{2}$$

The next result allows us to get rid of finite ϕ-invariant subgroups.

Proposition 1.7. *Let G be an Abelian group, $\phi \in \mathrm{End}(G)$ and K a ϕ-invariant finite subgroup of G. Then* $\mathrm{ent}(\phi) = \mathrm{ent}(\bar{\phi})$*, where $\bar{\phi}$ is the induced endomorphism of G/K.*

Proof. It suffices to prove that $H(\phi, F) = H(\bar{\phi}, F_1)$ for each finite subgroup F of G, where $F_1 = \pi(F)$ and $\pi : G \to G/K$ is the canonical surjection. An easy computation show that, for each $n \geq 1$,

$$T_n(\bar{\phi}, F_1) \cong T_n(\phi, F)/(T_n(\phi, F) \cap K)$$

hence

$$H_n(\phi, F) = \log |T_n(\phi, F) \cap K| + H_n(\bar{\phi}, F_1).$$

Since K is finite, $\log |T_n(\phi, F) \cap K|/n$ tends to 0 when $n \to \infty$. Consequently $H(\phi, F) = H(\bar{\phi}, F_1)$, as required. ∎

1.2 Examples and elementary properties.

We start with some examples of endomorphisms with zero, positive, and infinite algebraic entropy, respectively.

Example 1.8. Given an arbitrary group G, the endomorphism induced by the multiplication by an integer n has zero algebraic entropy, as $nH \leq H$ for every subgroup H of G; a similar result holds for multiplication by a p-adic integer π when G is a p-group. ∎

Example 1.9. (a) Let K be a finite group and $G = \bigoplus_{i \geq 1} K_i$, where $K_i \cong K$ for all i. Let $\sigma_K : G \to G$ be the classical *Bernoulli shift endomorphism*, defined by setting $\sigma(k_1, k_2, \ldots) = (0, k_1, k_2, \ldots)$, $(k_i \in K_i)$. We claim that

$$\mathrm{ent}(\sigma_K) = \log|K|.$$

Indeed, if $F = K_1$, then for each $n \geq 1$ we have $H_n(\sigma_K, F) = \log|\bigoplus_{i \leq n} K_i| = \log|K|^n = n \cdot \log|K|$, consequently $H(\sigma_K, F) = \log|K|$, thus $\mathrm{ent}(\sigma_K) \geq \log|K|$. On the other hand, one readily sees that $T(\sigma_K, K_1) = G$, hence $H(\sigma_K, K_1) = \log|K|$ coincides with $\mathrm{ent}(\sigma_K)$.

(b) The above equality makes sense also in the case when K is an infinite torsion group if one adopts the usual convention that $\log|K| = \infty$.

(c) One can easily verify that the *left* shift $\overleftarrow{\sigma}_K$ defined by $\overleftarrow{\sigma}_K\,(k_1, k_2, \ldots) = (k_2, k_3, \ldots)$ has always zero entropy, irrespective of the size of K, since all the trajectories of $\overleftarrow{\sigma}_K$ are finite. ∎

Example 1.10. Let B be the *standard basic p-group*, i.e., $B = \bigoplus_{n \geq 1} \langle b_n \rangle$, where $\langle b_n \rangle = \mathbb{Z}/p^n\mathbb{Z}$ for all n and let $\sigma : B \to B$ be the endomorphism defined by setting $\sigma(b_n) = pb_{n+1}$ for all n. We claim that

$$\mathrm{ent}(\sigma) = \infty.$$

Fix a positive integer r; for each $n \geq 1$ an easy calculation shows that $H_n(\sigma, \langle b_r \rangle) = \log p^{rn} = n \cdot \log p^r$ and so $H(\sigma, \langle b_r \rangle) = \log p^r$. Hence it follows that $\mathrm{ent}(\sigma) \geq \sup_r \log p^r = \infty$. ∎

We recall now some elementary properties of the algebraic entropy proved by Weiss [W] and we add to the list a further obvious property (namely (g)); note that Weiss's notation for $\mathrm{ent}(\phi)$ was $\mathrm{h}(\phi)$. The proofs of these facts are straightforward and are omitted. Let $\phi : G \to G$ and $\psi : G' \to G'$ be endomorphisms of the groups G and G', respectively; then:

(a) If $\phi : G \to G$ and $\psi : G' \to G'$ are conjugate endomorphisms of isomorphic groups G and G' (i.e., there exists an isomorphism $\theta : G \to G'$ such that $\theta \cdot \phi = \psi \cdot \theta$), then $\mathrm{ent}(\phi) = \mathrm{ent}(\psi)$.

(b) For every nonnegative integer k, $\mathrm{ent}(\phi^k) = k \cdot \mathrm{ent}(\phi)$. If ϕ is an automorphism, $\mathrm{ent}(\phi^k) = |k| \cdot \mathrm{ent}(\phi)$ for every integer k.

(c) If $\phi \oplus \psi$ denotes the endomorphism of $G \oplus G'$ which is the direct sum of ϕ and ψ, then $\mathrm{ent}(\phi \oplus \psi) = \mathrm{ent}(\phi) + \mathrm{ent}(\psi)$.

(d) If $t(G)$ denotes the torsion subgroup of G, then $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_{t(G)})$; in particular, $\mathrm{ent}(G) = 0$ if G is torsion-free.

(e) Let $G = \bigoplus_p G_p$ be a torsion group with p-components G_p: then $\mathrm{ent}(\phi) = \sum_p \mathrm{ent}(\phi_p)$, where ϕ_p is the restriction of ϕ to G_p and the summation is taken over all primes p.

(f) If G is a torsion group and H is a ϕ-invariant subgroup of G, then $\mathrm{ent}(\phi) \geq \mathrm{ent}(\bar{\phi})$, where $\bar{\phi} : G/H \to G/H$ is the induced endomorphism.

(g) If H is a ϕ-invariant subgroup of G, then $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_H)$.

Note that a property analogous to (e) remains true also when ϕ is an endomorphism of a torsion group $G = \bigoplus_{n=1}^{\infty} G_n$ such that every subgroup G_n is ϕ-invariant.

A further consequence of (e) is that, as usual, the study of torsion groups may be reduced to that of p-groups; this will be done without further comment in the remainder of the paper.

It is also worth noting that property (f) does not hold if we drop the torsion hypothesis on G, as the next Example shows.

Example 1.11. Let G be any p-group admitting an endomorphism $\phi : G \to G$ of positive entropy. Let

$$0 \to H \to P \to G \to 0$$

be a free presentation of G. Then, by the projectivity of P, ϕ lifts to an endomorphism ψ of P. One easily deduces that H is ψ-invariant in P, and ϕ is induced by ψ. As P is torsion-free, we have $0 = \mathrm{ent}(\psi) = \mathrm{ent}(\psi\restriction_H) < \mathrm{ent}(\phi)$. ■

Property (d) shows that the notion of algebraic entropy of endomorphisms, defined by the function ent, is vacuous for torsion-free groups.

1.3 Preliminary Results.

The following result provides sufficient conditions for ensuring the existence of endomorphisms of infinite entropy.

Theorem 1.12. *Let $G = \bigoplus_{n\geq 1} G_n$ be a countable direct sum of non-zero torsion groups G_n such that there is an embedding $\phi_n : G_n \to G_{n+1}$ for every n. Then there exists an endomorphism ϕ of G such that* $\mathrm{ent}(\phi) = \infty$.

Proof. Let $\sigma : G \to G$ be the Bernoulli shift relative to the embeddings ϕ_n:

$$\sigma(x_1, x_2, \ldots, x_n, \ldots) = (0, \phi_1(x_1), \phi_2(x_2), \ldots, \phi_n(x_n), \ldots),$$

where the $x_n \in G_n$ are almost all zero.

Let $\dot{\bigcup}_{k\geq 1} I_k = \mathbb{N}$ be a partition of $\mathbb{N}$, where, for each $k \geq 1$, $I_k = \{i_{k1} < i_{k2} < \ldots < i_{kn} < \ldots\}$ is an infinite increasing sequence of positive integers. For each $k \geq 1$, set $A_k = \bigoplus_{n\in I_k} G_n$, so that $G = \bigoplus_{k\geq 1} A_k$; note that each A_k has a right shifting endomorphism $\sigma_k : A_k \to A_k$ induced by the embeddings $\psi_{kn} : G_{i_{kn}} \to G_{i_{k,n+1}}$ obtained by composing the maps ϕ_n. Let $\phi : G \to G$ be the endomorphism

$$\phi = (\sigma_k)_k : \bigoplus_k A_k \to \bigoplus_k A_k.$$

For each $k \geq 1$, pick an element $0 \neq x_k \in A_k$; by a similar argument to that used for the shift endomorphisms in Example 1.9 (a), we see that $H(\sigma_k, x_k) = \log|\mathbb{Z}x_k|$. Now define, for each $n \geq 1$, the following finite subgroup of G

$$F_n = \bigoplus_{k\leq n} x_k\mathbb{Z}.$$

Obviously $H(\phi, F_n) = \sum_{k\leq n} H(\sigma_k, x_k\mathbb{Z}) \geq n \cdot \log 2$, and we conclude that

$$\mathrm{ent}(\phi) = \sup_{F\in\mathcal{F}(G)} H(\phi, F) \geq \sup_{n\geq 1} H(\phi, F_n) = \infty.$$

∎

Theorem 1.12 has two immediate consequences.

Corollary 1.13. *Let $A = M \oplus N$ be a group such that its direct summand M is an infinite direct sum of non-zero p-groups. Then there exists an endomorphism of A of infinite algebraic entropy.* ∎

Corollary 1.14. *Let G be a p-group whose divisible part $d(G)$ is isomorphic to an infinite direct sum of quasi-cyclic groups $\mathbb{Z}(p^\infty)$. Then G has an endomorphism of infinite algebraic entropy.* ∎

Our next result is technical but will prove very useful in the sequel.

Proposition 1.15. *Let ϕ be an endomorphism of the p-group G such that $G = T(\phi, F)$ for an $F \in \mathcal{F}(G)$. Then* $\mathrm{ent}(\phi) \leq \log|F|$. *In particular,* $\mathrm{ent}(\phi) < \infty$.

Proof. If G is finite the claim is trivial, so assume G infinite. As $G = \bigcup_n T_n(\phi, F)$, it follows that for every finite subgroup F_0 of G there exists an index n_0 such that $F_0 \leq T_{n_0}(\phi, F)$. Then for every $k \in \mathbb{N}$ we have that $T_k(\phi, F_0) \leq T_{n_0+k-1}(\phi, F)$, hence, as in the proof of Proposition 1.3, we get

$$H(\phi, F_0) = \lim_{k\to\infty} \frac{H_k(\phi, F_0)}{k} \leq \lim_{k\to\infty} \frac{H_{n_0+k-1}(\phi, F)}{n_0+k-1} \cdot \frac{n_0+k-1}{k} = H(\phi, F).$$

But $H(\phi, F) = \log |\frac{\phi^n F}{\phi^n F \cap T_n(\phi,F)}|$ for n large enough, and the result then follows by observing that

$$\log|\frac{\phi^n F}{\phi^n F \cap T_n(\phi, F)}| \leq \log|\phi^n F| \leq \log|F|.$$

■

However the equality $\mathrm{ent}(\phi) = \log |F|$ may fail even when F is cyclic.

Example 1.16. Let $k > 1$ and let $G = \bigoplus_{n\geq 0}\langle a_n\rangle$, where a_0 has order p^k and a_n has order p for all $n \geq 1$. Take $F = \langle a_0\rangle$ and define $\phi(a_n) = a_{n+1}$ for all $n \geq 0$. Arguing as in Example 1.9 (a), we see that $\mathrm{ent}(\phi) = \log p$, while $\log |F| = k \log p$. ■

The next result reduces the investigation of the entropy of endomorphisms of torsion groups to endomorphisms of reduced groups.

Proposition 1.17. *Let G be a torsion group with divisible part $d(G) = D$. If a p-component D_p of D is an infinite direct sum of quasi-cyclic groups for some p, then G has an endomorphism of infinite entropy. Otherwise, given any endomorphism $\phi : G \to G$, $\mathrm{ent}(\phi) = \mathrm{ent}(\bar{\phi})$, where $\bar{\phi} : G/D \to G/D$ is the induced endomorphism.*

Proof. In the first case the conclusion follows from Corollary 1.14. In the latter case, it suffices to prove that, for a suitable finite subgroup F of G, we have $H(\phi, F) = H(\bar{\phi}, F_1)$, where $F_1 = \pi(F)$ and $\pi : G \to G/D$ is the canonical surjection. Arguing as in the proof of Proposition 1.7, we get

$$H_n(\phi, F) = \log |T_n(\phi, F) \cap D| + H_n(\bar{\phi}, F_1),$$

for all $n > 0$. If $mF = 0$, then $T_n(\phi, F) \cap D = T_n(\phi, F) \cap D[m]$; as our hypothesis on D ensures that $D[m]$ is finite, it follows that $\log |T_n(\phi, F) \cap D|/n$ tends to 0 when $n \to \infty$, and therefore $H(\phi, F) = H(\bar{\phi}, F_1)$, as required. ■

Our next result shows that an endomorphism having strictly positive entropy already reveals this property on the socle.

Proposition 1.18. *Let G be a p-group and let $G[p]$ be its socle. If $\phi \in \mathrm{End}(G)$ is such that $\mathrm{ent}(\phi) > 0$, then $\mathrm{ent}(\phi\restriction_{G[p]}) > 0$.*

Proof. Without loss of generality we can assume that the group G is p^k-bounded for some k (take a finite subgroup F whose trajectory is infinite and p^k-bounded, instead of G). We argue by induction on k, the case $k = 1$ being obvious. Let $k > 1$ and consider two cases.

(1) If $\mathrm{ent}(\phi\restriction_{pG}) = 0$, then Lemma 1.5 applied to the ϕ-invariant subgroup $G[p]$ of G yields $\mathrm{ent}(\phi\restriction_{G[p]}) = \mathrm{ent}(\phi) > 0$ (the induced endomorphism $\bar{\phi} : G/G[p] \to G/G[p]$ is conjugate to the restrictions $\phi\restriction_{pG}$ via the isomorphism $G/G[p] \cong pG$, so $\mathrm{ent}(\bar{\phi}) = 0$).

(2) If $\mathrm{ent}(\phi\restriction_{pG}) > 0$, applying the inductive hypothesis to the endomorphism $\phi\restriction_{pG}$ of the subgroup pG of exponent p^{k-1}, one gets that $\mathrm{ent}(\phi\restriction_{pG[p]}) > 0$ and so $\mathrm{ent}(\phi\restriction_{G[p]}) > 0$ as well. ■

Here and several times henceforth we will use the notion of *Ulm-Kaplansky invariants* (of finite index) of a p-group G. For each integer $n \geq 0$, the n-th Ulm-Kaplansky invariant $\alpha_n(G)$ of G is the dimension of the $\mathbb{Z}/p\mathbb{Z}$-vector space $p^nG[p]/p^{n+1}G[p]$, where, as usual, we set $p^nG[p] = p^nG \cap G[p]$. We recall a crucial property of the Ulm-Kaplansky invariants: for each $n \geq 0$, G contains a direct summand isomorphic to a direct sum of $\alpha_n(G)$ copies of $\mathbb{Z}/p^n\mathbb{Z}$.

Theorem 1.19. *Let $\phi : G \to G$ be an endomorphism of the reduced p-group G such that $0 < ent(\phi) < \infty$. Then G has an infinite bounded summand.*

Proof. By Proposition 1.18, $\mathrm{ent}(\phi\restriction_{G[p]}) > 0$. Hence there exists an element $x \in G[p]$ with infinite ϕ-trajectory $T(\phi, x)$; it is readily seen that $T(\phi, x) = \bigoplus_n \langle \phi^n(x) \rangle$ and ϕ acts as the Bernoulli shift on it. $T(\phi, x)$ is a valued vector space with the valuation induced by the height function h_G on G.

CLAIM. The heights of the non-zero elements in $T(\phi, x)$ have as upper bound a positive integer N.

Assume, for a contradiction, that for every positive integer k there exists $y \in G$ such that $0 \neq p^k y = z \in T(\phi, x)$. Then $T(\phi, y) = \bigoplus_n \langle \phi^n(y) \rangle$ and $\mathrm{ent}(\phi\restriction_{T(\phi,y)}) = k \log p$. But k was arbitrary, hence we get a contradiction with the hypothesis $\mathrm{ent}(\phi) < \infty$.

Since $T(\phi, x)$ is countable, it is a free valued vector space (see [F2] or [S, Cor. 9.5, p. 43]). By the CLAIM, it has values $\leq N$, so it is a valued direct sum of homogeneous summands of finite value. Since $T(\phi, x)$ is infinite, one of the homogeneous direct summands, say that of value n, has infinite dimension; but this dimension is $\leq \alpha_n(G)$, the n-th Ulm-Kaplansky invariant of G. Therefore G has a bounded infinite summand isomorphic to $\bigoplus_{\alpha_n(G)} \mathbb{Z}/p^{n+1}\mathbb{Z}$, as claimed. ■

From Proposition 1.17 and Theorem 1.19 we obtain the following result which is somewhat surprising, since its analogue does not hold for topological entropy (see [ADS]).

Corollary 1.20. *If $\mathrm{ent}(G) \neq 0$ for a torsion Abelian group G, then G admits endomorphisms of infinite algebraic entropy. In particular, $\mathrm{ent}(G) = \infty$.* ■

2 The R_ϕ-module structure of p-groups.

The discussion in Section 1.2, and specifically the elementary properties (d) and (e), allow us to confine our investigation to the case of p-groups. Thus from now on we will focus our attention to Abelian p-groups, for a fixed prime number p.

Let G be a p-group and let $\phi : G \to G$ be an endomorphism. The algebraic entropy of ϕ is strongly related to certain properties of the subring of $\mathrm{End}(G)$ generated by ϕ.

It is well known (e.g., see [F, Th. 108.3]) that, given a p-group G, the centre of $\mathrm{End}(G)$ is isomorphic either to the ring J_p of the p-adic integers, or to $\mathbb{Z}/p^n\mathbb{Z}$, respectively when G is unbounded or p^n-bounded (i.e., n is the minimal positive integer such that $p^nG = 0$). We denote by R_ϕ the (commutative) subring of $\mathrm{End}(G)$ generated by the single endomorphism ϕ; hence $R_\phi = J_p[\phi]$ if G is unbounded, and $R_\phi = (\mathbb{Z}/p^n\mathbb{Z})[\phi]$, when G is p^n-bounded.

Obviously, G is an R_ϕ-module, and, for a fixed element $x \in G$, the cyclic R_ϕ-submodule $R_\phi x$ generated by x is nothing else than the ϕ-trajectory $T(\phi, x)$ of x with respect to ϕ.

The next result is well known; its proof is as in [FS, I.3.1].

Lemma 2.1. *Let G be a p-group and ϕ an endomorphism of G. The following conditions are equivalent:*

1) *there exists a monic polynomial $g(X) \in J_p[X]$ such that $g(\phi) = 0$;*

2) *the ring R_ϕ is a finitely generated J_p-module;*

3) *there exists a J_p-subalgebra of $\mathrm{End}(G)$ containing ϕ which is a finitely generated J_p-module.*

If G is p^n-bounded, then J_p can be replaced by $\mathbb{Z}/p^n\mathbb{Z}$.

■

If $\phi \in \mathrm{End}(G)$ satisfies one of the equivalent conditions above, we simply say that ϕ is *integral*. The connection with the algebraic entropy is shown by the following

Lemma 2.2. *Let G be a p-group and ϕ an integral endomorphism of G. Then* $\mathrm{ent}(\phi) = 0$.

Proof. If $g(X)$ is a monic J_p-polynomial of degree m such that $g(\phi) = 0$, then the trajectory $T(\phi, x)$ of every element $x \in G$ equals $T_m(\phi, x)$ and hence it is finite. Then Proposition 1.3 ensures that $\mathrm{ent}(\phi) = 0$. ■

We will see in Proposition 5.13 that the condition that the endomorphism ϕ is integral is not necessary in order that $\mathrm{ent}(\phi) = 0$.

We introduce now two new notions. The first one, which characterizes the endomorphisms with zero algebraic entropy, is furnished by the following weaker version of integrality. An endomorphism ϕ of the p-group G is said to be *point-wise integral* if, for every $x \in G$, there exists a monic polynomial $g(X) \in J_p[X]$ (depending on x), such that $g(\phi)(x) = 0$. Obviously every integral endomorphism of G is point-wise integral.

The second notion is the *ϕ-torsion subgroup* of G, denoted by $t_\phi(G)$: it is the subset of G consisting of the elements $x \in G$ such that $R_\phi x$ is finite. It is readily seen that $t_\phi(G)$ is a ϕ-invariant subgroup of G.

Consider the induced endomorphism $\bar{\phi} : G/t_\phi(G) \to G/t_\phi(G)$; in this notation we have

Lemma 2.3. (1) *The $\bar{\phi}$-torsion subgroup of $G/t_\phi(G)$ is zero.*
(2) *$t_\phi(G)$ contains the subgroup $K_\infty = \bigcup_n \mathrm{Ker}(\phi^n)$.*

Proof. (1) The $\bar{\phi}$-trajectory of an element $x + t_\phi(G) \in G/t_\phi(G)$ is finite if and only if the ϕ-trajectory of x is finite.

(2) If $x \in K_\infty$, then $\phi^n(x) = 0$ for some n. Then the ϕ-trajectory of x is finite, hence $x \in t_\phi(G)$. ∎

The converse containment in Lemma 2.3 (2) does not hold, in general, as the trivial example of the identity map of any non-zero group shows.

We collect in the next quite obvious result the different ways in which we can express the fact that the algebraic entropy of ϕ vanishes.

Proposition 2.4. *Let ϕ be an endomorphism of the p-group G. The following conditions are equivalent:*
(1) *ϕ is point-wise integral;*
(2) *the ϕ-trajectory of every $x \in G$ is finite;*
(3) $\mathrm{ent}(\phi) = 0$*;*
(4) *G coincides with its ϕ-torsion subgroup $t_\phi(G)$;*
(5) *for every $x \in G$ there exist $0 < m < n$ such that $\phi^m(x) = \phi^n(x)$.*

Proof. Assume (1) and let $x \in G$. If $g(X) \in J_p[X]$ is a monic polynomial such that $g(\phi) = 0$, and if n is its degree, then $\phi^n(x) \in T_n(\phi, x)$. Hence we have $T(\phi, x) = T_n(\phi, x)$, so (2) follows. The equivalence of (2), (3) and (4) immediately follows from Proposition 1.3. Assume (2). Then two different powers of ϕ must coincide on x, hence (5) holds true. Finally, (5) trivially implies (1). ∎

For convenience, we introduce the following notation:

$$\mathrm{Ent}_0(G) = \{\phi \in \mathrm{End}(G) : \mathrm{ent}(\phi) = 0\}.$$

Lemma 2.5. *Let G be a p-group. If $\phi, \psi \in \mathrm{Ent}_0(G)$ commute, then $\phi\psi \in \mathrm{Ent}_0(G)$ and $a\phi + b\psi \in \mathrm{Ent}_0(G)$ for any $a, b \in J_p$. Consequently, for any polynomial $f(X, Y) \in J_p[X, Y]$, we have* $\mathrm{ent}(f(\phi, \psi)) = 0$.

Proof. Let F be any finite subgroup of G. If ϕ, ψ commute, it is easy to verify that $T(\phi\psi, F) \leq T(\phi, T(\psi, F))$ and $T(a\phi + b\psi, F) \leq T(\phi, F) + T(\psi, F) + T(\phi, T(\psi, F))$, hence they are both finite, since $\mathrm{ent}(\phi) = 0 = \mathrm{ent}(\psi)$. The statement on $f(X, Y)$ follows, since the elements of J_p have zero entropy and are central in $\mathrm{End}(G)$. ∎

Example 2.6. We give an example of $\phi, \psi \in \mathrm{Ent}_0(G)$ such that both $\phi\psi$ and $\phi + \psi$ are not in $\mathrm{Ent}_0(G)$. Clearly, such a pair of endomorphisms cannot commute.

Take $G = \bigoplus_{n\in\mathbb{Z}} K_n$, where all K_n coincide with a fixed torsion group K. Obviously, every permutation of $\mathbb{Z}$ defines an automorphism of G via the change of coordinates assigned by the permutation. In particular, to the translation $n \mapsto n+1$ corresponds the *two-sided Bernoulli shift* $\sigma : G \to G$ with $\mathrm{ent}(\sigma) = \log |K|$. Let $s : \mathbb{Z} \to \mathbb{Z}$ be the symmetry defined by $s(n) = -n$. Clearly, s gives rise to an involution $\psi : G \to G$. Then also $\phi = \sigma\psi$ is an involution of G, so $\mathrm{ent}(\psi) = \mathrm{ent}(\phi) = 0$. On the other hand we have $\phi\psi = (\sigma\psi)\psi = \sigma$, whence $\mathrm{ent}(\phi\psi) = \log |K| > 0$.

To verify that $\phi+\psi \notin \mathrm{Ent}_0(G)$, it is enough to show that $\mathrm{ent}((\phi+\psi)^2) \neq 0$. Let us fix $0 \neq z \in K$; for all $j \in \mathbb{Z}$ we consider the element $g_j = (a_n)_{n\in\mathbb{Z}} \in G$, where $a_j = z$ and $a_n = 0$ for $n \neq j$. A direct check shows that $(\phi + \psi)^2(g_j) = g_{j-1} + 2g_j + g_{j+1}$. It readily follows that the $(\phi + \psi)^2$-trajectories of the g_j are infinite, whence $\mathrm{ent}((\phi + \psi)^2) \neq 0$, as desired. ∎

The fact that the product of two commuting endomorphisms of zero entropy has again zero entropy can be slightly generalized as follows: if $\phi\psi = \psi\phi$ and $\mathrm{ent}(\phi) = 0$, then $\mathrm{ent}(\phi\psi) = \mathrm{ent}(\psi)$.

When G is a p-bounded group, hence a vector space over the field with p elements $F_p = \mathbb{Z}/p\mathbb{Z}$, then the ring R_ϕ is either finite, when ϕ is integral, or it is an Euclidean domain, when ϕ is not integral, being isomorphic to the polynomial ring $F_p[X]$. In the first case $\mathrm{ent}(\phi) = 0$. In the latter case, if G is a finitely generated R_ϕ-module (equivalently, if G is the ϕ-trajectory of a finite subset F), then G is a direct sum of cyclic R_ϕ-modules: $G = \bigoplus_{i\leq n} T(\phi, x_i)$. As every ϕ-trajectory is a ϕ-invariant submodule, the elementary property (c) shows that

$$\mathrm{ent}(\phi) = \sum_{i\leq n} \mathrm{ent}(\phi\restriction_{T(\phi,x_i)}).$$

Hence the computation of $\mathrm{ent}(\phi)$ reduces to the case when G is the ϕ-trajectory of a single element, i.e., a cyclic R_ϕ-module. In this case we have the following result, whose straightforward proof is left to the reader.

Proposition 2.7. *Let G be a p-bounded group and ϕ an endomorphism of G such that $G = T(\phi, x)$ for a suitable element $x \in G$. Then the following conditions are equivalent:*

(1) *G is infinite;*

(2) *$G = \bigoplus_{n\geq 0}\langle \phi^n(x)\rangle$ and ϕ acts as the Bernoulli shift.*

In such a case, $\mathrm{ent}(\phi) = \log p$*; otherwise* $\mathrm{ent}(\phi) = 0$.

∎

The nice behavior of p-bounded groups described above is not inherited by p^n-bounded p-groups, for $n > 1$, as the next example shows.

Example 2.8. There exists a p^2-bounded homogeneous group G with an endomorphism ϕ such that G is a 2-generated indecomposable R_ϕ-module.

Let $R = (\mathbb{Z}/p^2\mathbb{Z})[\phi]$, where ϕ is an indeterminate, and consider the maximal ideal G of R generated by p and ϕ. Regarded to as a p-group, G is p^2-bounded, and the multiplication by ϕ can be identified with a group endomorphism. A direct verification shows that G is not a principal ideal, that is, it is not a cyclic R-module. Moreover G is an indecomposable R-module, since we can directly prove that G is not a direct sum of two nonzero ideals of R. (We will prove this fact, using the Addition Theorem, in the next Example 3.9.) ■

2.1 The Poincaré - Birkhoff Recurrence Theorem.

The notion of *recurrence* is a standard concept in ergodic theory - see for example [Pet]. Here we develop the concept in the context of Abelian p-groups endowed with the p-adic topology: a mapping $\phi : G \to G$ is said to be *recurrent* on a subset $X \subseteq G$ if for all $x \in X$ and all $N > 0$, there is an $n > 0$, depending on x, N, such that $x - \phi^n(x) \in p^N G$ i.e. for all $x \in G$, there exists $n > 0$ such that $\phi^n(x)$ is in any given neighbourhood of x. The mapping is *strongly recurrent* on X if for all $x \in X$, there is an $n > 0$, depending on x, such that $\phi^n(x) = x$; when $X = G$ we simply say *recurrent* or *strongly recurrent*.

Recall that G is Hausdorff in the p-adic topology if and only if it is *separable*, i.e., $p^\omega G = \bigcap_{n>0} p^n G = 0$. For G separable, the notions of recurrence and strong recurrence are significant only when $\phi \in \mathrm{End}(G)$ is a monomorphism, since any nonzero element of the kernel of ϕ makes it impossible for ϕ to be recurrent.

The most basic result in ergodic theory is the so-called *Poincaré Recurrence Theorem* or its topological analogue the *Birkhoff Recurrence Theorem* - see Theorems 2.3.2 and 4.2.2 respectively in [Pet]; these classical results derive recurrence properties for measure-preserving transformations (resp. homeomorphisms), but do not explicitly involve the entropy of the transformation. A surprising, partly analogous result, directly characterizing zero entropy mappings, holds in a strong form for monomorphisms of a p-group, even without assuming that the group G is separable.

Proposition 2.9. *Let G be an Abelian p-group and $\phi \in \mathrm{End}(G)$ a monomorphism. Then ϕ is strongly recurrent if, and only if* $\mathrm{ent}(\phi) = 0$.

Proof. Suppose that $\mathrm{ent}(\phi) = 0$, then the ϕ-trajectory $T(\phi, x)$ of any element $x \in G$ is finite. In particular, there exist $m > n$ such that $\phi^n(x) = \phi^m(x)$. Since ϕ is a monomorphism, we get $\phi^{m-n}(x) = x$; since the element $x \in G$ was arbitrary, we conclude that ϕ is strongly recurrent.

Conversely suppose that ϕ is strongly recurrent. For each $x \in F$, there is an n_x such that $\phi^{n_x}(x) = x$. It follows that ϕ is point-wise integral, hence $\mathrm{ent}(\phi) = 0$, by virtue of Proposition 2.4. ■

Note. It is an easy consequence of Proposition 2.9 that a group with zero entropy is necessarily co-Hopfian (i.e. every monomorphism is an automorphism);

it would be interesting to know if there was any connection between zero entropy and the Hopfian property, where a group is said to be Hopfian if every epimorphism is an automorphism.

Example 2.10. If the entropy of a monomorphism ϕ is not zero, we may not even get recurrence. Consider the group $G = \bigoplus_{n=1}^{\infty} K_n$ where each K_n is isomorphic to a fixed separable p-group K. Then the shift mapping σ sending K_n isomorphically to K_{n+1} is monic and has nonzero entropy (actually infinite when K is infinite). However, for any $x \in K_1 \setminus pK_1$, it is immediate that $x - \sigma^n(x) \notin pG$ and so σ is not recurrent. ■

Of course, it is crucial to show that the notions of recurrence and strong recurrence do not coincide for separable p-groups. We need the following simple lemma:

Lemma 2.11. *Let G be a p-group, ϕ an injective endomorphism of G, and let ψ be the unique extension of ϕ to the torsion completion $\overline{G}$ of G. If ϕ is recurrent, then also ψ is recurrent.*

Proof. We want to show that, for any $x \in \overline{G}$ and $N > 0$, there exists $n > 0$ such that $x - \psi^n(x) \in p^N\overline{G}$. Since G is dense in $\overline{G}$ with respect to the p-adic topology, there exists $y \in G$ such that $y - x \in p^N\overline{G}$. Since ϕ is recurrent, there exists $n > 0$ such that $\phi^n(y) = \psi^n(y) \equiv y$ modulo $p^N G$. Then we get

$$x - \psi^n(x) \equiv y - \psi^n(x) \equiv \psi^n(y) - \psi^n(x) \equiv \psi^n(y - x) \equiv 0 \mod p^N\overline{G},$$

as desired. ■

Proposition 2.12. *There exists a separable p-group $\overline{G}$ which admits recurrent endomorphisms which are not strongly recurrent.*

Proof. We consider the separable group G and its endomorphism ϕ as constructed in the forthcoming Subsection 5.3. Then ϕ is monic and in Proposition 5.13 we show that $\mathrm{ent}(\phi) = 0$, hence ϕ is strongly recurrent, by virtue of Proposition 2.9. By the preceding lemma, the extension ψ of ϕ to the torsion completion $\overline{G}$ of G is recurrent. However, in Proposition 5.14 we prove that $\mathrm{ent}(\psi) = \infty$, hence ψ cannot be strongly recurrent, again by Proposition 2.9. The desired conclusion follows. ■

The case when the p-group G is not separable is irksome from a topological point of view. Nevertheless, in the present circumstances we can get some information even for non-separable groups.

Example 2.13. If the p-group G has non-trivial elements of infinite height, then the full invariance of $p^\omega G$ gives that any endomorphism ϕ is recurrent on $p^\omega G$. An important class of endomorphism is recurrent on precisely $p^\omega G$: they are the locally nilpotent endomorphisms. For suppose that ϕ is locally nilpotent and is recurrent on the element $x \in G$. Then there is an integer k such that

$\phi^k(x) = 0$ and, for any integer N, an integer n such that $x - \phi^n(x) \in p^N G$. If $n \geq k$, then $x \in p^N G$ and, since N was arbitrary $x \in p^\omega G$. If $n < k$, then there is an integer r such that $rn \geq k$. As ϕ is locally nilpotent, the map $1_G - \phi^n$ is locally invertible:

$$(1_G + \phi^n + \phi^{2n} + \cdots + \phi^{(r-1)n})(1_G - \phi^n)(x) = (1_G - \phi^{rn})(x) = x.$$

Since $(1_G - \phi^n)(x) \in p^N G$, it follows by full invariance that $x \in p^N G$ and so $x \in p^\omega G$ as required.

However the converse is not true. If $G = D \oplus H$, where $H = \bigoplus_\omega \mathbb{Z}(p)$ and D is a copy of $\mathbb{Z}(p^\infty)$, then $p^\omega G = D$. Now if $\psi = 1_D \oplus \sigma$, where σ is the Bernoulli shift on H, then it is easy to see that ψ is recurrent on exactly $D = p^\omega G$ but clearly ψ is not locally nilpotent. ∎

The example above gives a simple characterization of divisible groups, a special case of which we record as:

Corollary 2.14. *A p-group G is divisible if, and only if each locally nilpotent endomorphism ϕ of G is recurrent.* ∎

3 The Addition Theorem.

The present section is devoted to the proof of the so-called "Addition Theorem", which relates the entropy of a mapping to the entropy of its restriction to an invariant subgroup and the entropy of the induced map. We also consider some of the consequences of this result.

Theorem 3.1. (Addition Theorem) *Let G be a torsion group, $\phi \in \mathrm{End}(G)$ and H a ϕ-invariant subgroup of G. If $\bar{\phi} : G/H \to G/H$ is the induced endomorphism, we have*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi}). \qquad (*)$$

For brevity, we will say that AT holds for (G, H, ϕ) if the above formula $(*)$ is satisfied.

We remark that, in view of Example 1.11, the restriction to torsion groups is unavoidable.

The proof of this important result will be made via a number of reductions and a series of partial results, before finally proving the result in full generality.

Without loss of generality, we assume that G is a p-group, for a fixed prime number p.

3.1 Preliminary results.

We observe a first important fact: Lemma 1.5 shows that $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_H)$ when $\mathrm{ent}(\bar{\phi}) = 0$; therefore, the formula $(*)$ holds if $\mathrm{ent}(\bar{\phi}) = 0$.

It is worth noting that Example 1.11 shows that the following counterpart of Lemma 1.5 is not true for arbitrary groups: if $\mathrm{ent}(\phi\restriction_H) = 0$, then $\mathrm{ent}(\phi) = \mathrm{ent}(\bar{\phi})$.

The next reduction shows that it suffices to consider only the case when $\mathrm{ent}(\phi)$ is finite.

Lemma 3.2. *In the above notation, if the Addition Theorem holds whenever* $\mathrm{ent}(\phi) < \infty$*, then it also holds when* $\mathrm{ent}(\phi) = \infty$*.*

Proof. We have to show that $\mathrm{ent}(\phi) = \infty$ implies that either $\mathrm{ent}(\phi\restriction_H) = \infty$ or $\mathrm{ent}(\bar{\phi}) = \infty$.

Assume for a contradiction that both $\mathrm{ent}(\phi\restriction_H)$ and $\mathrm{ent}(\bar{\phi})$ are finite. Let F be an arbitrary finite subgroup of G, and set $G_F = T(\phi, F)$, $H_F = H \cap G_F$, (a ϕ-invariant subgroup of G_F) and $\overline{G}_F = (G_F + H)/H$. Then $\mathrm{ent}(\phi\restriction_{G_F})$ is finite, by Proposition 1.15. By our assumption, AT holds for $(G_F, \phi\restriction_{G_F}, H_F)$, and we get

$$\mathrm{ent}(\phi\restriction_{G_F}) = \mathrm{ent}(\phi\restriction H_F) + \mathrm{ent}(\bar{\phi}\restriction_{\overline{G}_F}),$$

since $\bar{\phi}\restriction_{\overline{G}_F}$ corresponds to the endomorphism induced by $\phi\restriction_{G_F}$ on G_F/H_F under the canonical isomorphism $(G_F + H)/H \cong G_F/H_F$. But $\mathrm{ent}(\phi\restriction_{H_F}) \le \mathrm{ent}(\phi\restriction_H)$ and $\mathrm{ent}(\bar{\phi}\restriction_{\overline{G}_F}) \le \mathrm{ent}(\bar{\phi})$ by the elementary property (g). Hence we derive

$$\mathrm{ent}(\phi\restriction_{G_F}) \le \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi}).$$

Since the choice of F was arbitrary, from Lemma 1.4 we can conclude that

$$\mathrm{ent}(\phi) \le \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi}).$$

This proves that $\mathrm{ent}(\phi) < \infty$. The resulting contradiction yields the desired conclusion. ■

In view of the preceding result, we will assume henceforth that all entropies under discussion are finite.

Our next step is to show that the inequality "$\geq$" holds in the formula $(*)$ of the Addition Theorem.

In the next Lemma 3.3 we shall denote an arbitrary ϕ-invariant subgroup of G by K, rather than the more usual H, as we wish to avoid the possibility of confusion with the symbol $H(\phi, -)$ used in the proof.

Lemma 3.3. *Let G be a p-group, $\phi \in \mathrm{End}(G)$, K a ϕ-invariant subgroup of G, and let $\bar{\phi} \in \mathrm{End}(G/K)$ be induced by ϕ. Then we have*

$$\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_K) + \mathrm{ent}(\bar{\phi}).$$

Proof. Since $\mathrm{ent}(\bar{\phi})$ and $\mathrm{ent}(\phi\restriction_K)$ are assumed to be finite, there exist finite subgroups F' of G/K and F_0 of K such that $\mathrm{ent}(\bar{\phi}) = H(\bar{\phi}, F')$ and $\mathrm{ent}(\phi\restriction_K) = H(\phi\restriction_K, F_0)$. Then $F' = (F_1 + K)/K$ for a finite subgroup F_1 of G. Let $F =$

$F_0 + F_1$. Note that $(F + K)/K = F'$, whence $H(\bar{\phi}, (F + K)/K) = \mathrm{ent}(\bar{\phi})$. Moreover $H(\phi\restriction_K, F_0) = H(\phi\restriction_K, F \cap K)$, since $H(\phi\restriction_K, F_0)$ is the maximum and $F_0 \leq F \cap K$. Since $\mathrm{ent}(\phi) \geq H(\phi, F)$, our conclusion will follow, once we show that

$$H(\phi, F) \geq H(\bar{\phi}, (F + K)/K) + H(\phi\restriction_K, F \cap K).$$

For each $n > 0$ consider the exact sequence

$$0 \to T_n(\phi, F) \cap K \to T_n(\phi, F) \to (T_n(\phi, F) + K)/K \to 0$$

which, since $(T_n(\phi, F) + K)/K = T_n(\bar{\phi}, (F + K)/K)$, gives

$$|T_n(\phi, F)| = |T_n(\bar{\phi}, (F + K)/K)| \cdot |T_n(\phi, F) \cap K|.$$

Note that $T_n(\phi\restriction_K, F \cap K)$ is a subgroup of $T_n(\phi, F) \cap K$ and so, taking logs, dividing by n and passing to the limit we get the desired inequality. ■

Using the above inequality we get the following crucial result, which implies a weak form of the Addition Theorem, valid for cyclic R_ϕ-modules. The final step in our proof will be to deduce the full Addition Theorem from this result.

Theorem 3.4. *Let G be an infinite p-group such that $G = T(\phi, x)$ for suitable $\phi \in \mathrm{End}(G)$ and $x \in G$. Then $\mathrm{ent}(\phi) = k \cdot \log p$, where k is the largest positive integer such that the Ulm-Kaplansky invariant $\alpha_{k-1}(G)$ is infinite.*

In particular, if $\bar{\phi} : G/pG \to G/pG$ is the induced map, we have

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_{pG}) + \mathrm{ent}(\bar{\phi}) = \mathrm{ent}(\phi\restriction_{pG}) + \log p. \qquad (\dagger)$$

Proof. If p^n is the order of x, then G is p^n-bounded and hence is a direct sum of cyclic groups of orders $\leq p^n$. We induct on n. If $n = 1$, then $\alpha_0(G)$ is infinite and the proof follows from Proposition 2.7. So assume $n > 1$ and that claim is true for $n - 1$. Consider the exact sequence

$$0 \to p^{n-1}G \to G \to G/p^{n-1}G \to 0$$

where $p^{n-1}G$ is a F_p-vector space of dimension $\alpha_{n-1}(G)$. If $k < n$, $\alpha_{n-1}(G)$ is finite, and so $p^{n-1}G$ is also finite. From Proposition 1.7 we get $\mathrm{ent}(\phi) = \mathrm{ent}(\bar{\phi})$, where $\bar{\phi} : G/p^{n-1}G \to G/p^{n-1}G$ is the induced endomorphism. But now $G/p^{n-1}G = T(\bar{\phi}, x + p^{n-1}G)$, and its Ulm-Kaplansky invariants satisfy the equalities $\alpha_i(G/p^{n-1}G) = \alpha_i(G)$ for all $i \leq n - 2$, hence the claim follows by induction. So it remains only to examine the case when $k = n$; we know that

$$\mathrm{ent}(\phi) \geq \mathrm{ent}(\bar{\phi}) + \mathrm{ent}(\phi\restriction_{p^{n-1}G}).$$

The inductive hypothesis implies that $\mathrm{ent}(\bar{\phi}) = (n - 1)\log p$, since $G/p^{n-1}G$ is a cyclic $R_{\bar{\phi}}$-module and $\alpha_{n-2}(G/p^{n-1}G)$ is infinite since $k = n$. Moreover, Proposition 2.7 gives $\mathrm{ent}(\phi\restriction_{p^{n-1}G}) = \log p$, and therefore we get

$$\mathrm{ent}(\phi) \geq n \log p.$$

The converse inequality follows from Proposition 1.15 and so we have established the main result.

The formula (†) follows easily since $pG = T(\phi, px)$ and $G/pG = T(\bar{\phi}, \bar{x})$ (where $\bar{x} = x + pG$) are both infinite groups (except in the trivial case when $pG = 0$). ∎

3.2 Proof of the Addition Theorem.

The following lemma is a main tool to prove the Addition Theorem.

Lemma 3.5. *Let G be a p-group, ϕ an endomorphism, H a ϕ-invariant subgroup of G, and $\bar{\phi} : G/H \to G/H$ the induced endomorphism. If $\mathrm{ent}(\bar{\phi}) > 0$, then there exists an element $x \in G$ such that:*

(a) $px \in H$;

(b) the trajectory $T(\bar{\phi}, x + H)$ is an infinite subgroup of G/H (so $x \notin H$).

In such case, the trajectory $L = T(\phi, x)$ in G satisfies $L \cap H = pL$.

Proof. By Lemma 1.18, the hypothesis $\mathrm{ent}(\bar{\phi}) > 0$ implies that the restriction of $\bar{\phi}$ to the socle $(G/H)[p]$ of G/H still has positive entropy. Therefore there exists an element $\bar{x} \in (G/H)[p]$ such that $T(\bar{\phi}, \bar{x})$ is an infinite subgroup of G/H. If $\bar{x} = x + H$, then the element $x \in G$ obviously satisfies (a) and (b).

In order to prove that L satisfies $L \cap H = pL$, first note that $pL \le L \cap H$, by (a). Let $z \in T_n(\phi, x) \cap H$. Then $z = k_0 x + k_1 \phi(x) + \ldots + k_{n-1}\phi^{n-1}(x)$ for some $k_0, k_1, \ldots, k_{n-1} \in \mathbb{Z}$. If now p does not divide some k_i, then, since $px \in H$, we conclude that $\bar{\phi} \in \mathrm{End}(G/H)$ is point-wise integral at $\bar{x}$: this would lead to the finiteness of the trajectory $T(\bar{\phi}, \bar{x})$ (see Proposition 2.4), contrary to (b). Thus, necessarily, p divides all coefficients k_i, whence $z \in pL$. We conclude that $L \cap H = pL$. ∎

Remark 3.6. In the notation of the preceding lemma, if H is a direct summand of G, then x can be chosen in the socle of G. Indeed, let $G = H \oplus K$, and choose an element x satisfying the requirements of Lemma 3.5. Then x has the form $x = h + y$, where $h \in H$ and $y \in K[p]$ (since $px \in H$). Now $T(\phi, y)$ cannot be finite modulo H, since $T(\phi, x) \le T(\phi, h) + T(\phi, y) \le H + T(\phi, y)$. Hence y lies in the socle of G and satisfies the requirements of Lemma 3.5.

The next lemma is the final step needed for the proof of the Addition Theorem. It reveals a property of the entropy that does not involve quotient groups. The full symmetry between the subgroups H and K is noteworthy (cf the statement of the Addition Theorem). Note also that, as a consequence of the Addition Theorem, it is possible to show that equality actually holds in the formula (∗∗).

Lemma 3.7. *Let G be a torsion Abelian group and let $\phi \in \mathrm{End}(G)$. If H, K are ϕ-invariant subgroups of G such that $G = H + K$, then*

$$\mathrm{ent}(\phi) \le \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\phi\restriction_K) - \mathrm{ent}(\phi\restriction_{H\cap K}). \qquad (**)$$

Proof. Define $f : G \oplus G \to G$ by the evaluation map $f(x, y) = x + y$ and let $\Phi : H \oplus K \to H \oplus K$ be the restriction of the map $\phi \oplus \phi : G \oplus G \to G \oplus G$ to $H \oplus K$. Finally, let $f' = f\upharpoonright_{H\oplus K}$ and $D = \{(x, -x) : x \in H \cap K\}$. Then $f'(H \oplus K) = G$ and $\ker f' = D$. Denoting by $i : D \hookrightarrow H \oplus K$ the inclusion map, we get the following commutative diagram:

$$\begin{array}{ccccc} H \cap K \cong D & \xrightarrow{\ i\ } & H \oplus K & \xrightarrow{\ f'\ } & G \\ {\scriptstyle \phi\upharpoonright_{H\cap K}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle \phi} \\ H \cap K \cong D & \xrightarrow{\ i\ } & H \oplus K & \xrightarrow{\ f'\ } & G \end{array}$$

The subgroup D of $H \oplus K$ is Φ-invariant and the induced endomorphism $\bar{\Phi} : (H\oplus K)/D \to (H\oplus K)/D \cong G$ can be identified with $\phi : G \to G$. Moreover, $\Phi\upharpoonright_D$ can be identified with $\phi\upharpoonright_{H\cap K}$. Applying Lemma 3.3 to the triple $(H \oplus K, \Phi, D)$ we get $\mathrm{ent}(\Phi) \geq \mathrm{ent}(\phi) + \mathrm{ent}(\Phi\upharpoonright_D)$. Since $\mathrm{ent}(\Phi) = \mathrm{ent}(\phi\upharpoonright_H) + \mathrm{ent}(\phi\upharpoonright_K)$ and $\mathrm{ent}(\Phi\upharpoonright_D) = \mathrm{ent}(\phi\upharpoonright_{H\cap K})$, we conclude that $\mathrm{ent}(\phi\upharpoonright_H) + \mathrm{ent}(\phi\upharpoonright_K) \geq \mathrm{ent}(\phi) + \mathrm{ent}(\phi\upharpoonright_{H\cap K})$. ■

Now we are in the position to complete the proof of the Addition Theorem.

Proof of the Addition Theorem. Recall that we may assume that $\mathrm{ent}(\phi) < \infty$. Since the values of the entropy have the form $m \log p$, for a suitable non-negative integer m, we will argue by induction on $m = \mathrm{ent}(\bar{\phi})(\log p)^{-1}$. Lemma 1.5 takes care of the case $m = 0$. Thus we can assume $m > 0$ (equivalently, $\mathrm{ent}(\bar{\phi}) > 0$). Pick a ϕ-invariant subgroup $L = T(\phi, x)$ of G as in Lemma 3.5. Then $L \cap H = pL$. Let $H_1 = H + L$; then H_1 is also ϕ-invariant and we may consider the $\bar{\phi}$-invariant subgroup $\bar{L} = H_1/H \cong L/pL$ of G/H. Since $\bar{L}$ is an infinite F_p-vector space, in view of Proposition 2.7 we can identify $\bar{\phi}\upharpoonright_{\bar{L}}$ with the Bernoulli shift, and so we get $\mathrm{ent}(\bar{\phi}\upharpoonright_{\bar{L}}) = \log p$. Applying the formula (†) of Theorem 3.4 to L we get $\mathrm{ent}(\phi\upharpoonright_L) = \mathrm{ent}(\phi\upharpoonright_{pL}) + \log p$, so that

$$\mathrm{ent}(\phi\upharpoonright_L) - \mathrm{ent}(\phi\upharpoonright_{pL}) = \log p. \tag{1}$$

Let ψ be the endomorphism of G/H_1 induced by ϕ; note that, since $G/H_1 \cong (G/H)/(H_1/H)$, we may also regard at ψ as being induced by $\bar{\phi}$. Let us apply Lemma 3.3 to the triple $(G/H, \bar{\phi}, \bar{L})$; since $\mathrm{ent}(\bar{\phi}\upharpoonright_{\bar{L}}) > 0$ we get $\mathrm{ent}(\psi) < \mathrm{ent}(\bar{\phi})$.

Hence our inductive hypothesis implies that AT holds for both (G, ϕ, H_1) and $(G/H, \bar{\phi}, H_1/H)$. Thus we get

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi\upharpoonright_{H_1}) + \mathrm{ent}(\psi)\ ;\ \mathrm{ent}(\bar{\phi}) = \mathrm{ent}(\bar{\phi}\upharpoonright_{\bar{L}}) + \mathrm{ent}(\psi). \tag{2}$$

Subtracting the equalities in (2) and recalling that $\mathrm{ent}(\bar{\phi}\upharpoonright_{\bar{L}}) = \log p$, we obtain

$$\mathrm{ent}(\phi) - \mathrm{ent}(\bar{\phi}) = \mathrm{ent}(\phi\upharpoonright_{H_1}) - \log p. \tag{3}$$

To compute $\mathrm{ent}(\phi\upharpoonright_{H_1})$ we use Lemma 3.7 to get

$$\mathrm{ent}(\phi\upharpoonright_{H_1}) \leq \mathrm{ent}(\phi\upharpoonright_H) + \mathrm{ent}(\phi\upharpoonright_L) - \mathrm{ent}(\phi\upharpoonright_{pL}). \tag{4}$$

From (1), (3) and (4) we readily obtain

$$\mathrm{ent}(\phi) \leq \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi}).$$

Since the converse inequality holds by Lemma 3.3, the desired conclusion follows. ■

3.3 Consequences of the Addition Theorem

In this section we prove some consequences of the Addition Theorem.

Firstly we show that the inequality (∗∗) in Lemma 3.7 is in fact an equality.

Corollary 3.8. *Let G be a torsion Abelian group and let $\phi \in \mathrm{End}(G)$. If H, K are ϕ-invariant subgroups of G such that $G = H + K$, then*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\phi\restriction_K) - \mathrm{ent}(\phi\restriction_{H\cap K}). \qquad (***)$$

Proof. We adopt the notation of Lemma 3.7. In its proof, an application of the Addition Theorem to the triple $(H \oplus K, \Phi, D)$ yields

$$\mathrm{ent}(\Phi) = \mathrm{ent}(\phi) + \mathrm{ent}(\Phi\restriction_D),$$

which equals $(***)$. ■

Example 3.9. Here we reconsider the Example 2.8. Making use of the entropy of ϕ, we give an indirect proof of the fact that G is an indecomposable 2-generated R-module.

Recall that $R = (\mathbb{Z}/p^2\mathbb{Z})[\phi]$ and G is the maximal ideal of R generated by p and ϕ. We can write $G = \bigoplus_{n\geq 0}\langle b_n\rangle$, where $\langle b_0\rangle \cong \mathbb{Z}/p\mathbb{Z}$, and $\langle b_n\rangle \cong \mathbb{Z}/p^2\mathbb{Z}$ for all $n \geq 1$. Then $\phi \in \mathrm{End}(G)$ acts as follows:

$$\phi(b_0) = pb_1\ ,\ \phi(b_n) = b_{n+1} \quad (n \geq 1).$$

Since ϕ acts as the Bernoulli shift on $\bigoplus_{n\geq 1}\langle b_n\rangle \cong G/\langle b_0\rangle$, and $\langle b_0\rangle$ is finite, by Proposition 1.7 we get

$$\mathrm{ent}(\phi) = 2 \cdot \log p.$$

We want to show that G is indecomposable as an R-module. Assume, for a contradiction, that $G = H_1 \oplus H_2$, with the H_i non-zero R-submodules. Since no element in G has finite ϕ-trajectory, each H_i must contain an infinite cyclic R-module, so $\mathrm{ent}(\phi\restriction_{H_i}) \geq \log p$. Moreover, at least one of the Ulm-Kaplansky invariants $\alpha_1(H_j)$ is infinite. Then, using Theorem 3.4, we see that the corresponding $\mathrm{ent}(\phi\restriction_{H_j})$ is at least $2\cdot\log p$, whence $\mathrm{ent}(\phi) \geq 3\cdot\log p$, a contradiction.

Using the same group G, we now give an example of computation using the formula $(***)$.

We can write $G = \phi R + (\phi - p)R$. It is easy to check that $\phi G = \phi R \cap (\phi - p)R$. Since ϕ induces the Bernoulli shift on ϕR, we have

$$\mathrm{ent}(\phi\restriction_{\phi R}) = 2 \cdot \log p.$$

In a similar way we prove that $\mathrm{ent}(\phi\restriction_{(\phi-p)R}) = 2\cdot\log p$. From the formula $(***)$ we get

$$\mathrm{ent}(\phi\restriction_{\phi R\cap(\phi-p)R}) = \mathrm{ent}(\phi\restriction_{\phi R}) + \mathrm{ent}(\phi\restriction_{(\phi-p)R}) - \mathrm{ent}(\phi) = 2\cdot\log p,$$

in accordance with the fact that $\phi R\cap(\phi-p)R = \phi G$ is isomorphic to G as an R-module. ∎

Using the Addition Theorem we get further information on the values of the entropy of endomorphisms of bounded groups.

Corollary 3.10. *Let $\phi : G \to G$ be an endomorphism of the p^n-bounded group G. Consider, for $1 \leq i \leq n$, the endomorphism $\phi_i : p^{i-1}G[p]/p^iG[p] \to p^{i-1}G[p]/p^iG[p]$ induced by ϕ. Then*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi_1) + 2\,\mathrm{ent}(\phi_2) + 3\,\mathrm{ent}(\phi_3) + ... + n\,\mathrm{ent}(\phi_n).$$

In particular, if G is a direct sum of copies of $\mathbb{Z}/p^n\mathbb{Z}$, then $\mathrm{ent}(\phi) = nk\log p$, *for a suitable integer $k \geq 0$.*

Proof. By induction on n. If $n = 1$ the claim is trivial. Assume $n > 1$ and set $H = G[p^{n-1}]$. Then $p^iG[p] = p^iH[p]$ for $i \leq n-2$, and $p^{n-1}G[p] > p^{n-1}H[p] = 0$. Let $\psi : H \to H$ be the restriction of ϕ to H, and, for $1 \leq i \leq n-1$, let $\psi_i : p^{i-1}H[p]/p^iH[p] \to p^{i-1}H[p]/p^iH[p]$ be the map induced by ψ. Then the inductive hypothesis, applied to H, yields

$$\mathrm{ent}(\psi) = \mathrm{ent}(\psi_1) + 2\,\mathrm{ent}(\psi_2) + 3\,\mathrm{ent}(\psi_3) + ... + (n-1)\,\mathrm{ent}(\psi_{n-1}).$$

By the Addition Theorem, since $\phi_n = \phi\restriction_{p^{n-1}G}$, we get

$$\mathrm{ent}(\phi) = \mathrm{ent}(\psi) + \mathrm{ent}(\phi_n).$$

Now, for $i \leq n-2$, we have $\phi_i = \psi_i$. Hence, to conclude, it is enough to prove that

$$\mathrm{ent}(\psi_{n-1}) = \mathrm{ent}(\phi_{n-1}) + \mathrm{ent}(\phi_n).$$

This equality follows from the Addition Theorem applied to the exact sequence

$$0 \to p^{n-1}G \to p^{n-2}G[p] = p^{n-2}H[p] \to p^{n-2}G[p]/p^{n-1}G \to 0$$

and the three endomorphisms: $\phi_n : p^{n-1}G \to p^{n-1}G$, $\psi_{n-1}; p^{n-2}H[p] \to p^{n-2}H[p]$ and $\phi_{n-1} : p^{n-2}G[p]/p^{n-1}G[p] \to p^{n-2}G[p]/p^{n-1}G[p]$.

The final assertion is clear, since $\mathrm{ent}(\phi)$ is always equal to $\log p$ times a nonnegative integer. ∎

From the preceding corollary and Theorem 3.4 we derive

Corollary 3.11. *Let G be an infinite p-group such that $G = T(\phi, x)$ for suitable $\phi \in \mathrm{End}(G)$ and $x \in G$, and let k be the largest integer such that the Ulm-Kaplansky invariant α_{k-1} is infinite. Then, in the above notation, we have*

(a) $\mathrm{ent}(\phi_k) = \log p$, $\mathrm{ent}(\phi) = k\,\mathrm{ent}(\phi_k)$, and $\mathrm{ent}(\phi_i) = 0$ for every $i \neq k$;

(b) for every $i \neq k$ and every $z \in p^{i-1}G[p]$, there exists a monic polynomial $g(X)$ in $J_p[X]$ such that $g(\phi)(z) \in p^iG[p]$.

Proof. (a) From Theorem 3.4 we know that $\mathrm{ent}(\phi) = k \log p$, where k is the largest positive integer such that $\alpha_{k-1}(G) = \aleph_0$. Then $p^{k-1}G/p^kG$ is a cyclic $R_{\bar{\phi}}$-module, where $\bar{\phi}$ is the endomorphism of $p^{k-1}G/p^kG$ induced by ϕ. Since $V = p^{k-1}G[p]/p^kG[p] \cong (p^{k-1}G[p] + p^kG))/p^kG$ is an $R_{\bar{\phi}}$-submodule and $\bar{\phi}$ restricted to V coincides with ϕ_k, we have that V is also a cyclic R_{ϕ_k}-module; this fact shows that ϕ_k acts as the Bernoulli shift on V, hence $\mathrm{ent}(\phi_k) = \log p$ by Proposition 2.7. Therefore we deduce that $\mathrm{ent}(\phi) = k\,\mathrm{ent}(\phi_k)$ and, from Corollary 3.10, we get $\mathrm{ent}(\phi_i) = 0$ for all $i \neq k$.

(b) is an immediate consequence of Proposition 2.4 and the fact that $\mathrm{ent}(\phi_i) = 0$ for $i \neq k$. ∎

Assuming that a group G is a finitely generated R_ϕ-module, for a suitable $\phi \in \mathrm{End}(G)$, we get a connection between the vanishing of $\mathrm{ent}(\phi_i)$, where the endomorphisms ϕ_i are defined as in Corollary 3.10, and the finiteness of the Ulm-Kaplansky invariant $\alpha_{i-1}(G)$.

Corollary 3.12. *Let ϕ be an endomorphism of the p-group G and assume that G is a finitely generated R_ϕ-module. Then, in the notation of Corollary 3.10, $\mathrm{ent}(\phi_i) = 0$ if and only if $\alpha_{i-1}(G)$ is finite.*

Proof. The sufficiency is obvious; so, let us assume that $\mathrm{ent}(\phi_i) = 0$. In the same way as in the proof of Corollary 3.11, one can show that $p^{i-1}G[p]/p^iG[p]$ is a finitely generated R_{ϕ_i}-module; this implies that it is a finite sum of finite subspaces, hence its dimension is finite, as desired. ∎

We have a subgroup canonically related to an endomorphism ϕ, namely the largest ϕ-invariant subgroup of G where ϕ has zero entropy. This subgroup coincides with the ϕ-torsion subgroup $t_\phi(G)$ of G.

Proposition 3.13. *For an Abelian group G and an endomorphism ϕ of G, the ϕ-torsion subgroup $t_\phi(G)$ is the largest ϕ-invariant subgroup of G such that $\mathrm{ent}(\phi\restriction_{t_\phi(G)}) = 0$. Moreover, $\mathrm{ent}(\phi) = \mathrm{ent}(\bar{\phi})$, where $\bar{\phi} : G/t_\phi(G)(\phi) \to G/t_\phi(G)$ is the induced endomorphism.*

Proof. We may regard at $t_\phi(G)$ as the sum of all finite ϕ-invariant subgroups of G; thus the first assertion follows. In view of Lemma 3.3, to prove the second assertion it suffices to show that $\mathrm{ent}(\phi) \leq \mathrm{ent}(\bar{\phi})$. This follows from the Addition Theorem. ∎

3.4 An alternative proof using Topological Entropy

In the preceding discussion we gave a direct and purely algebraic proof of the Addition Theorem. In the present subsection we sketch an alternative and indirect proof, which makes use of the Addition Theorem for the topological entropy, proved by Bowen [B, Theorem 19], and of the main result in Weiss's paper [W]. Note that this proof also needs the preceding Lemma 3.3.

We remark that for an Abelian group G and an endomorphism $\phi : G \to G$ of G there exists a countable ϕ-invariant subgroup S_ϕ such that $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_{S_\phi})$. Moreover, if $\mathrm{ent}(\phi) < \infty$, then S_ϕ can be chosen of the form $T(\phi, F)$ for some finite subgroup F of G and $\mathrm{ent}(\bar{\phi}) = 0$, where $\bar{\phi} : G/S_\phi \to G/S_\phi$ is the induced endomorphism.

In fact, every finite subgroup F of G is contained in the countable ϕ-invariant subgroup $G_F = T(\phi, F)$. Moreover, from Lemma 1.4 we know that

$$\mathrm{ent}(\phi) = \sup_{F \in \mathcal{F}(\mathcal{G})} \mathrm{ent}(\phi\restriction_{G_F}).$$

Either in case of finite or infinite entropy, the above sup may be attained by a sequence $\{F_n\}_n$ contained in $\mathcal{F}(\mathcal{G})$. Then the countable ϕ-invariant subgroup S_ϕ generated by all G_{F_n} does the job, since $\mathrm{ent}(\phi\restriction_{S_\phi}) \geq \mathrm{ent}(\phi\restriction_{G_{F_n}})$ for every n, so $\mathrm{ent}(\phi\restriction_{S_\phi}) \geq \sup_n \mathrm{ent}(\phi\restriction_{G_{F_n}}) = \mathrm{ent}(\phi)$.

When $\mathrm{ent}(\phi) < \infty$, Lemma 1.4 shows that $\mathrm{ent}(\phi) = H(F_0, \phi)$ for a suitable finite subgroup F_0 of G, hence we can take the F_n constantly equal to F_0 and $S_\phi = T(\phi, F)$. Finally, $\mathrm{ent}(\bar{\phi}) = 0$ follows from Lemma 3.3.

We observe that the subgroup S_ϕ is not uniquely determined by ϕ, but if some other countable ϕ-invariant subgroup $S'_\phi \leq G$ has the same property (i.e., $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{S'_\phi})$), then the subgroup $S_\phi \cap S'_\phi$ keeps the same property. It is tempting to call these subgroups of G *ϕ-large*.

The family $\mathcal{B}_\phi$ of all ϕ-large subgroups of G is a filter-base if $\mathrm{ent}(\phi)$ is finite. This gives rise to a group topology τ_ϕ on G such that ϕ is τ_ϕ-continuous.

Lemma 3.14. *If the Addition Theorem holds for all countable groups and their endomorphisms, then it holds in the general case.*

Proof. Assume the Addition Theorem holds for all countable groups. By Lemma 3.3 we need to prove

$$\mathrm{ent}(\phi) \leq \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi})$$

for an arbitrary Abelian group G, an endomorphism $\phi : G \to G$ of G and ϕ-invariant subgroup H of G. If G' is a countable ϕ-invariant subgroup, then $H' = G' \cap H$ is a ϕ-invariant subgroup of G', hence by our hypothesis applied to $\phi' = \phi\restriction_{G'}$, $\mathrm{ent}(\phi') \leq \mathrm{ent}(\phi'\restriction_{H'}) + \mathrm{ent}(\bar{\phi}')$, where $\bar{\phi}'$ is the endomorphism induced by ϕ' on G'/H'. But $\bar{\phi}' = \bar{\phi}\restriction_{G'/H'}$ as well when we consider $G'/H' \cong (G' + H)/H$ as a subgroup of G/H. Hence, $\mathrm{ent}(\bar{\phi}') \leq \mathrm{ent}(\bar{\phi})$. Obviously, $\mathrm{ent}(\phi\restriction_{H'}) \leq \mathrm{ent}(\phi\restriction_H)$. Therefore, $\mathrm{ent}(\phi\restriction_{G'}) \leq \mathrm{ent}(\phi\restriction_H) + \mathrm{ent}(\bar{\phi})$ for every countable ϕ-invariant subgroup G' of G. By the above discussion we get the desired conclusion. ∎

We can now give the following

Alternative proof of the Addition Theorem. The preceding lemma shows that we may confine ourselves to the case when G is countable.

Now we can make use of Bowen's Addition theorem for the entropy of continuous endomorphisms of compact metrizable topological groups. To be able to use this theorem we apply Weiss's theorem transferring the properties of topological entropy of continuous endomorphisms of compact totally disconnected Abelian groups to those of the algebraic entropy of endomorphisms of discrete torsion Abelian groups. Since the duals of compact metrizable Abelian groups are countable discrete groups, we get the result for G countable, as desired. ∎

4 Endomorphisms of infinite algebraic entropy.

In this section we investigate the class of p-groups which admit endomorphisms of infinite entropy.

A major role in the structure of the endomorphism rings of Abelian p-groups is played by small endomorphisms, introduced by Pierce in his seminal paper [Pi]. Recall that an endomorphism ϕ of the p-group G is *small* if, given an arbitrary positive integer k, there exists an integer $n \geq 0$ such that $\phi(p^n G[p^k]) = 0$. An obvious example of small endomorphism is furnished by the bounded endomorphism, i.e., those $\phi \in \mathrm{End}(G)$ such that $p^n \phi(G) = 0$ for some n. Small homomorphisms between two p-groups A and B are defined similarly, and form a group denoted by $\mathrm{Hom}_s(A, B)$.

Pierce proved that the small endomorphisms of G form a two-sided ideal $E_s(G)$ of the ring $\mathrm{End}(G)$, and that $\mathrm{End}(G)$ is an extension of $E_s(G)$ by a ring which is a torsion-free complete module over the ring J_p of the p-adic integers. If G is unbounded, then the center Z_G of $\mathrm{End}(G)$ is isomorphic to the ring J_p (acting as local multiplications by their partial sums) and intersects $E_s(G)$ trivially. Furthermore, $\mathrm{End}(G)$ contains, as a J_p-direct summand, the sub-J_p-algebra

$$J_p \cdot 1 \oplus E_s(G)$$

generated by $E_s(G)$.

Recall that we always have the following J_p-module decomposition

$$\mathrm{End}(G) = A \oplus E_s(G),$$

where A is the completion of a free J_p-module containing the center J_p (see [Pi, Theorem 7.5]). This result, however, says nothing on the ring structure of the factor ring $\mathrm{End}(G)/E_s(G)$, which is indeed the crucial point with respect to the behavior of the entropy.

Firstly we look at some necessary conditions for the non-existence of endomorphisms of infinite entropy.

Recall that a p-group G is called *semi-standard* (see [C1, p. 287]) if its Ulm-Kaplansky invariants of finite index are all finite. A semi-standard group is *essentially finitely indecomposable* (e.f.i. for short) if it does not admit a direct summand that is an infinite direct sum of cyclic subgroups.

The next result shows that to be semi-standard and e.f.i. is a necessary condition for a p-group to have no endomorphisms of infinite entropy.

Proposition 4.1. *A reduced p-group G with no endomorphisms of infinite algebraic entropy is semi-standard, essentially finitely indecomposable and has zero entropy. Consequently such a group has cardinality not exceeding $2^{\aleph_0}$ and, if it is bounded, then it is finite.*

Proof. It is well known that $\alpha_n(G) \geq \aleph_0$ implies that G has a summand isomorphic to an infinite direct sum of cyclic groups $\mathbb{Z}(p^{n+1})$, so G is semi-standard and essentially finitely indecomposable, in view of Corollary 1.13. Moreover, from Theorem 1.19 it follows that G has zero entropy. The last assertion follows from an observation of Kulikov (see [F, Corollary 34.4]): if G is semi-standard, then a basic subgroup B is countable and so $|G| \leq |B|^{\aleph_0} = 2^{\aleph_0}$. ∎

In view of the above proposition, a natural question is to ask, conversely, whether a semi-standard e.f.i. group necessarily has entropy zero.

This is not the case, as shown by Theorem 4.4 below, where the standard group G we will exhibit is even essentially indecomposable, that is, whenever $G = G_1 \oplus G_2$ one of the G_i must be finite.

The proof of the following lemma is essentially contained in that of Proposition 5.1 of [C1]. We prove it here for the sake of completeness.

Lemma 4.2. *Let G be a semi-standard group such that $End(G) = A \oplus E_s(G)$, where A is a J_p-algebra without nontrivial idempotents. Then G is essentially indecomposable.*

Proof. We may assume that G is unbounded, since it is semi-standard. Let us assume, for a contradiction, that $G = G_1 \oplus G_2$, where both G_1 and G_2 are infinite. This implies that the G_i are both unbounded. Let π_i be the projection onto G_i $(i = 1, 2)$. Then the π_i are not small endomorphisms, since the identity of an unbounded group cannot be small. Thus $\pi_1 \notin E_s(G)$, whence $\pi_1 = f + \theta$, where $f \in A$, $\theta \in E_s(G)$, and $f \neq 0$. We get $\pi_1 = f + \theta = \pi_1^2 = f^2 + \theta'$, with $\theta' \in E_s(G)$; this yields $f = f^2$, whence $f = 1$, since A has only trivial idempotents and $f \neq 0$. But then $\pi_2 = 1 - \pi_1 = -\theta$ is small, a contradiction. ∎

We recall an important result proved by Corner [C1, Theorem 2.1].

Theorem 4.3. *(Corner) Let $\bar{B}$ be a torsion-complete p-group with an unbounded basic subgroup B of cardinal $\leq 2^{\aleph_0}$, and let Φ be a separable closed subring of* $\operatorname{End}(\bar{B})$ *that leaves B invariant and satisfies the condition*

(C) if $\phi \in \Phi$ and $\phi(p^n\bar{B}[p]) = 0$ for some n, then $\phi \in p\Phi$.

Then there exits a family G_ρ $(\rho \in P)$ of $2^{2^{\aleph_0}}$ pure subgroups of $\bar{B}$ containing B such that

(a) for each $\rho \in P$, $\operatorname{End}(G_\rho) = \Phi \oplus E_s(G_\rho)$.

(b) for distinct $\rho, \sigma \in P$, every homomorphism $G_\rho \to G_\sigma$ is small.

Corner's theorem allows us to prove the following

Theorem 4.4. *There exists a standard essentially indecomposable group G which admits an endomorphism of infinite entropy.*

Proof. Let $B = \bigoplus_{n>0} B_n$ be the standard group, where $B_n = \langle b_n \rangle$ is cyclic of order p^n, for all $n > 0$, and let $\bar{B}$ be its torsion completion. We denote by σ the endomorphism of B determined by the assignments $b_n \mapsto pb_{n+1}$; it has a unique extension to $\bar{B}$, which we continue to denote by σ. Now consider the subring $J_p[\sigma] = R_\sigma$ of $\mathrm{End}(\bar{B})$ generated by $1, \sigma$. We have already seen in Section 2 that R_σ is isomorphic to the ring of polynomials with coefficients in J_p; denote by Φ its p-adic completion, still contained in $\mathrm{End}(\bar{B})$. We remark that, since $\Phi = R_\sigma + p\Phi$, it easily follows that $p\Phi$ is a prime ideal of Φ and that Φ is an integral domain. We show that the hypotheses of the above Corner's theorem are satisfied in our situation. Clearly Φ leaves B invariant, since $\sigma(B) \subseteq B$ and $\Phi = R_\sigma + p^n\Phi$ for all $n > 0$. It remains to show that the J_p-algebra Φ satisfies the Crawley's condition (C) of the preceding statement.

It suffices to show that, whenever $\phi \notin p\Phi$, we have $\phi(p^n B[p]) \neq 0$ for all $n \geq 0$. Since $\Phi = R_\sigma + p\Phi$, we can write $\phi = f + p\psi$, where $f \in J_p[\sigma]$ and all the coefficients of f are not divisible by p. Then we have $\phi(p^n B[p]) = f(p^n B[p])$. For all $n > 0$ we have $0 \neq p^n b_{n+1} \in p^n B[p]$. Let now $a_k\sigma^k$ be the monomial of f of highest degree, where $a_k \in J_p$; then $a_k\sigma^k(p^n b_{n+1}) = a_k p^{n+k} b_{n+k+1}$ is a nonzero element of B_{n+k+1} (recall that p does not divide a_k). Since for each other monomial $a_i\sigma^i$ of f, with $i < k$, we have $a_i\sigma^i(p^n b_{n+1}) \in B_{n+i+1}$, it readily follows that $f(p^n b_{n+1}) \neq 0$. We have proved that (C) is satisfied.

Thus we have seen that we are in a position to apply the preceding theorem of Corner. Then there exists a group G, pure in $\bar{B}$ and containing B, such that $\mathrm{End}(G) = \Phi \oplus E_s(G)$. Then G is trivially semi-standard and, of course, σ is an endomorphism of G of infinite entropy, since it has infinite entropy when restricted to B. Finally G is essentially indecomposable by virtue of the preceding lemma, since Φ is an integral domain. ■

We now show that unbounded groups belonging to some important classes of p-groups admit endomorphisms of infinite algebraic entropy. Of course, by Proposition 4.1, the next result is significant only when the groups are semi-standard.

Theorem 4.5. *The reduced unbounded p-groups belonging to any of the following classes of p-groups, admit endomorphisms of infinite algebraic entropy:*

(1) *totally projective groups;*
(2) *$p^{\omega+1}$-projective groups;*
(3) *torsion-complete groups.*

Proof. (1) By Corollary 1.13, it suffices to show that an unbounded reduced totally projective group G has a direct summand which is an infinite direct sum of cyclic groups. Assume first that the length $l(G)$ of G is at most ω_1 (the first uncountable ordinal). Then G is a direct sum of countable groups, by a well known result of Nunke (see [F, Theorem 82.4]), and at least one summand is unbounded, hence infinite. As countably infinite reduced p-groups decompose into the direct sum of infinitely many nontrivial groups (see [F, Proposition 77.3]), the claim easily follows. Assume now that $\tau = l(G) > \omega_1$. By [F, Theorem 83.6], the Ulm invariants $f_\sigma(G)$ $(\sigma < \tau)$ give rise to a τ-admissible function; this means that $\tau = \sup\{\sigma + 1 : f_\sigma(G) \neq 0\}$ and, for all σ such that $\sigma + \omega < \tau$, the following inequality holds

$$\sum_{n<\omega} f_{\sigma+n}(G) \geq \sum_{\rho \geq \sigma+\omega} f_\rho(G).$$

In particular, we have that

$$\sum_{n<\omega} f_n(G) \geq \sum_{\rho \geq \omega} f_\rho(G).$$

But $\tau > \omega_1$ implies that $\sum_{\rho \geq \omega} f_\rho(G) \geq \aleph_1$, hence at least one invariant $f_n(G)$, for some $n < \omega$, must be infinite (actually, infinitely many are $\geq \aleph_1$). It follows that G has a summand which is an infinite direct sum of cyclic groups, as desired.

(2) The results on $p^{\omega+1}$-projective groups we invoke here may be found in the paper by Fuchs and Irwin [FI]. A $p^{\omega+1}$-projective group G decomposes as $G = A \oplus T$, where A is $p^{\omega+1}$-projective separable and T is totally projective. By Corollary 1.13 and point (1), it is enough to consider the case of $G = A$, and in this case G has a direct summand which is an unbounded direct sum of cyclics, by [FI, Corollary 2], so we are done again by Corollary 1.13.

(3) A basic subgroup of an unbounded torsion-complete group G is unbounded, hence it admits an endomorphism ϕ of infinite algebraic entropy. Now ϕ extends, by continuity and the torsion-completeness of G, to an endomorphism of G, which necessarily also has infinite algebraic entropy. ■

5 Groups with zero algebraic entropy.

5.1 Small endomorphisms and entropy.

In this Section we focus our attention on p-groups with zero algebraic entropy.

The next results show the relevant role played by small endomorphisms and by the subring $J_p \cdot 1 \oplus E_s(G)$ in our investigation.

Lemma 5.1. *Let G be a p-group with the first e Ulm-Kaplansky invariants $\alpha_0(G), \dots, \alpha_{e-1}(G)$ finite, and ϕ an endomorphism of G. Then, for each $x \in G[p^e]$ and for each $N \in \mathbb{N}$, there exists a monic polynomial $f(\phi)$ in ϕ, with integer coefficients of degree $\leq N \cdot \sum_{0 \leq i \leq e-1} \alpha_i(G)$, such that $f(\phi)(x) \in p^N G$.*

Proof. By induction on N. So, let us assume first that $N = 1$.

If $x \in pG$ the claim is trivial. So let $x \notin pG$ and let B be a basic subgroup of G. Denote by $\pi : G \to B/pB$ the composition of canonical surjection $G \to G/pG$

followed by the canonical isomorphism $G/pG \cong B/pB$. Let $B = \oplus_{n\geq 1} B_n$, where B_n is isomorphic to $\alpha_{n-1}(G)$ copies of $\mathbb{Z}(p^n)$:

$$B_n = \bigoplus_{1\leq i\leq \alpha_{n-1}(G)} \langle b_{ni}\rangle.$$

Then $B/pB = \bigoplus_{n,i}\langle b_{ni} + pB\rangle$. We claim that $\pi(x) \in \bigoplus_{n\leq e}\langle b_{ni} + pB\rangle$.

Since $G = B + pG$, $x = \sum k_{ni}b_{ni} + pw$ where the sum is finite and $w \in G$. So $0 = p^e x$ implies that $\sum k_{ni}p^e b_{ni} = -p^{e+1}w \in p^{e+1}G \cap B = p^{e+1}B$, consequently, for all $n > e$, p divides the coefficients k_{ni}. Therefore $\pi(x) = \sum_{n\leq e} k_{ni}b_{ni} + pB$, as claimed.

It follows that, setting $k = \sum_{0\leq i\leq e-1}\alpha_i(G)$, the elements

$$\pi(x), \pi(\phi(x)), \pi(\phi^2(x)), \ldots, \pi(\phi^k(x))$$

are linearly dependent, since $\bigoplus_{n\leq e}\langle b_{ni} + pB\rangle$ has dimension k over the field with p elements. Thus for some $h \leq k$ we have

$$r_0\pi(x) + r_1\pi(\phi(x)) + \ldots + r_{h-1}\pi(\phi^{h-1}(x)) + \pi(\phi^h(x)) = 0$$

for certain $r_i \in \{0, 1, \ldots, p-1\}$. This implies that $\pi(r_0x + r_1\phi(x) + \ldots + \phi^h(x)) = 0$, hence $r_0x + r_1\phi(x) + \ldots + \phi^h(x) = (r_0 + r_1\phi + \ldots + \phi^h)(x) \in pG$, as desired.

Assume now that $N > 1$. By the inductive hypothesis, there exists a monic polynomial $g(\phi)$ of degree $\leq (N-1)\cdot\sum_{0\leq i\leq e-1}\alpha_i(G)$ with coefficients in $\mathbb{Z}_p$ such that $g(\phi)(x) = p^{N-1}y$ for some $y \in G$. If $y \in pG$ we are done, otherwise, by the case $N = 1$, there exists a monic polynomial $g'(\phi)$ of degree $\leq \sum_{0\leq i\leq e-1}\alpha_i(G)$ such that $g'(\phi)(y) = py'$ for some $y' \in G$. Thus we have:

$$p^N y' = p^{N-1}g'(\phi)(y) = g'(\phi)(p^{N-1}y) = g'(\phi)(g(\phi)(x)) = (g'(\phi)g(\phi))(x)$$

where the product $g'(\phi)g(\phi)$ is still a monic polynomial in ϕ, of degree $\leq N\cdot\sum_{0\leq i\leq e-1}\alpha_i(G)$. ■

One can easily derive from Lemma 5.1 the following theorem on small endomorphisms; recall that, by Proposition 4.1, a necessary condition for a p-group G to have zero algebraic entropy is that G is semi-standard.

Theorem 5.2. *Let G be a semi-standard p-group. If ϕ is a small endomorphism, then ϕ is point-wise integral, so* $\mathrm{ent}(\phi) = 0$.

Proof. Let F be a finite subgroup of G. Fix an element $x \in F$; if $x \in G[p^n]$, choose $N \in \mathbb{N}$ such that $\phi(p^N G[p^n]) = 0$. By Lemma 5.1, there exists a monic polynomial $f(\phi)$ such that $f(\phi)(x) = z \in p^N G$. If K is the degree of f, then $\phi^K(x) = z + g(\phi)(x)$, where the polynomial g has degree $< K$. Therefore $\phi^{K+1}(x) = \phi(z) + \phi(g(\phi)(x))$, where $\phi(z) = 0$, as $z \in p^N G[p^n]$. Since the polynomial $\phi(g(\phi))$ has degree $\leq K$, we deduce that $\phi^{K+1}(x) \in \sum_{i\leq K}\phi^i(x)\mathbb{Z}$. Therefore, ϕ is point-wise integral, and so $\mathrm{ent}(\phi) = 0$. ■

Obviously, if $\phi \in \mathrm{End}(G)$ is the sum of a p-adic integer and a point-wise integral endomorphism, then ϕ is point-wise integral, too, hence $\mathrm{ent}(\phi) = 0$. Therefore, from Theorem 5.2 we get

Corollary 5.3. *Let G be a semi-standard p-group. Then every endomorphism in the subring $J_p \cdot 1 \oplus E_s(G)$ of $\mathrm{End}(G)$ has zero entropy.* ∎

From Corollary 5.3 we obtain plenty of reduced unbounded p-groups with zero entropy.

We can get examples as follows. Recall that, given a p-group G, $\bar{G}$ denotes the torsion part of its p-adic completion, and that a quasi-complete p-group G is a p-group such that the closure of a pure subgroup is again pure; it is characterized by the property that every subsocle S of G is the socle of a pure subgroup of G containing a pre-assigned pure subgroup H of G such that $H[p]$ is contained in S (see [F, 74] and [S, p. 49]). These groups have been studied by Hill and Megibben in [HM] and [M]. Torsion complete groups are quasi-complete, and quasi complete groups which are not torsion complete are called *proper*. Megibben proved in [M, Theorem 3.7] that a proper quasi-complete group G such that $\bar{G}/G \cong \mathbb{Z}(p^\infty)$ satisfies $\mathrm{End}(G) = J_p \cdot 1 \oplus E_s(G)$. An example of a semi-standard group of this form was first constructed by Pierce [Pi]. Corollary 5.3 shows that $\mathrm{ent}(G) = 0$.

However, using Corner's results in [C1] we obtain much more.

Theorem 5.4. *There exists a family of $2^{2^{\aleph_0}}$ semi-standard p-groups G of length ω with zero algebraic entropy and only small homomorphisms between the different members of the family.*

Proof. The desired family of groups may be obtained from a realization theorem proved by Corner [C1, Theorem 1.1] (or also from the slightly stronger Theorem 2.1 [C1] recorded above). ∎

Looking at Corollary 5.3, one could ask whether the condition $\mathrm{End}(G) = J_p \cdot 1 \oplus E_s(G)$ is not only sufficient, but also necessary, in order that $\mathrm{ent}(G) = 0$, at least for separable p-groups. This is not the case, as we will see in the next subsection.

In this section and in the preceding two, we restricted our attention to torsion groups. This is a significant limitation, since a mixed group can have zero algebraic entropy while its torsion subgroup admits endomorphisms of infinite algebraic entropy. The obvious explanation of this phenomenon, illustrated in the next Example 5.5, is that very few endomorphisms of the torsion subgroup can be extended to the whole group. It is easy to see that when all endomorphisms of the torsion subgroup extend to the whole group, the entropies are equal; this is the situation when the torsion subgroup splits or when one takes the cotorsion hull of a torsion group.

Example 5.5. There exists a non-splitting mixed group G with $\mathrm{ent}(G) = 0$, such that $t(G)$ admits endomorphisms of infinite algebraic entropy.

Let $B = \bigoplus_{n\geq 1} \mathbb{Z}(p^n)$ and $\hat{B}$ its p-adic completion. We have seen in Example 1.10 that B admits an endomorphism of infinite algebraic entropy. By [CG], there exists a pure J_p-submodule G of $\hat{B}$ such that $t(G) = B$ and $\mathrm{End}(G) = Z_G \oplus E_b(G)$, where Z_G is the center of $\mathrm{End}(G)$ consisting of the multiplications by the p-adic integers, and $E_b(G)$ denotes the two-sided ideal of $\mathrm{End}(G)$ consisting of the bounded endomorphisms. As bounded endomorphisms are trivially small when restricted to the torsion subgroup, Corollary 5.3 implies that $\mathrm{ent}(G) = 0$.■

5.2 Further results relating zero entropy and integrality.

In order to get some stronger results we need first the following definition and theorem:

$$Cl(E_s) = \{\phi \in \mathrm{End}(G) : \bar{\phi} = \phi + E_s(G) \text{ is integral over } J_p \text{ in } \mathrm{End}(G)/E_s(G)\}.$$

In general, $Cl(E_s)$ is closed neither under sums nor under products, although $Cl(E_s)$ is closed under sums and products of commuting endomorphisms. Furthermore, $Cl(E_s)$ contains $J_p \cdot 1 \oplus E_s(G)$.

Recall that we denote by $\mathrm{Ent}_0(G)$ the set of all endomorphisms of G with zero entropy: by Proposition 2.4 above, these are precisely the point-wise integral ones.

Theorem 5.6. *Let G be an unbounded reduced semi-standard p-group, then $\mathrm{Ent}_0(G)$ contains $Cl(E_s)$.*

Proof. Let $\phi \in \mathrm{End}(G)$ be an endomorphism such that $\bar{\phi} = \phi + E_s(G)$ is an element integral over J_p of the factor algebra $\mathrm{End}(G)/E_s(G)$. We have to prove that $\mathrm{ent}(\phi) = 0$. By hypothesis we have that $\bar{\phi}^n + r_{n-1}\bar{\phi}^{n-1} + \ldots + r_1\bar{\phi} + r_0\bar{1} = \bar{0}$ for suitable $n \geq 1$ and $r_i \in J_p$, equivalently:

$$\phi^n + r_{n-1}\phi^{n-1} + \ldots + r_1\phi + r_0 = \theta$$

where $\theta \in E_s(G)$ is a small endomorphism. Now, Theorem 5.2 ensures that θ is point-wise integral. A monic polynomial in θ gives rise to a monic polynomial in ϕ, and so ϕ is also point-wise integral, and hence $\mathrm{ent}(\phi) = 0$, by Proposition 2.4. ■

From the above theorem we immediately derive an important corollary, which is a main tool in the discussion that follows.

Corollary 5.7. *Let G be an unbounded reduced semi-standard p-group G such that the J_p-algebra $\mathrm{End}(G)/E_s(G)$ is integral over J_p. Then $\mathrm{ent}(G) = 0$.* ■

Note that the hypothesis of Corollary 5.7 is always satisfied when $\mathrm{End}(G)/E_s(G)$ has finite rank, since then it is a finitely generated free J_p-module.

The following two results show the existence of plenty of groups G such that $\mathrm{ent}(G) = 0$ and $\mathrm{End}(G) \neq J_p \cdot 1 \oplus E_s(G)$,

Corollary 5.8. *Let $G = G_1 \oplus G_2 \oplus \ldots \oplus G_n$ be a finite direct sum of semi-standard p-groups such that $\mathrm{Hom}(G_i, G_j) = \mathrm{Hom}_s(G_i, G_j)$ for $i \neq j$, and $\mathrm{End}(G_i)/E_s(G_i)$ is integral over J_p for all i. Then $\mathrm{ent}(G) = 0$.*

Proof. $A = \mathrm{End}(G)/E_s(G)$ is a block diagonal matrix $\mathrm{Diag}(A_1, \ldots, A_n)$, where each $A_i \cong \mathrm{End}(G_i)/E_s(G_i)$ is integral over J_p for each i. Clearly A is also integral over J_p, and the result follows from Corollary 5.7. ∎

Corollary 5.9. *Let G be an unbounded reduced semi-standard p-group G such that the J_p-algebra $\mathrm{End}(G)/E_s(G)$ is integral over J_p. Then $\mathrm{ent}(G^n) = 0$ for all n.*

Proof. $\mathrm{End}(G^n)/E_s(G^n)$ is the J_p-algebra $M_n(A)$ of the $n \times n$ matrices with entries in $A = \mathrm{End}(G)/E_s(G)$. Clearly $M_n(A)$ is also integral over J_p, hence the conclusion comes from Corollary 5.7. ∎

It follows from Theorem 5.6 that $\mathrm{Ent}_0(G)$ contains $Cl(E_s)$. However the containment may be strict: in the next subsection we will give an example of an endomorphism in $\mathrm{Ent}_0(G)$ that is not in $Cl(E_s)$.

Recall that Corner proved in [C1, Theorem 4.1] the following

Theorem 5.10. *(Corner) Let A be a J_p-algebra which is the completion of a free J_p-module of countable rank. If A satisfies the following condition:*

(*) there exists a descending sequence of right ideals $A \geq A_1 \geq A_2 \geq \ldots \geq A_n \geq \ldots$ such that A_i/A_{i+1} is a free J_p-module of finite rank for each i and $pA = \bigcap_i (pA + A_i)$,

then there exists a separable semi-standard p-group G such that $\mathrm{End}(G) = A \oplus E_s(G)$. ∎

Using Corollary 5.7 and the above theorem we get

Corollary 5.11. *If A is an integral J_p-algebra satisfying the hypotheses of Corner's Theorem 5.10, then there exists a separable p-group G such that $\mathrm{End}(G) = A \oplus E_s(G)$, and, consequently, $\mathrm{ent}(G) = 0$.* ∎

We want to show that the preceding corollary is useful in situations other than the simple case when the J_p-algebra has finite rank. Accordingly we give an example of a J_p-algebra of infinite rank satisfying the hypotheses of Corollary 5.11.

Example 5.12. Recall the notion of *Nagata's idealization*, first introduced in the classical book [N]. Let R be a ring and B an R-module. The idealization D of B, denoted by

$$D = R(+)B$$

is additively the direct sum of R and B, but with multiplication defined by

$$(r, b)(r', b') = (rr', rb' + r'b).$$

These operations make D a ring with identity element $(1, 0)$, containing R. Note that the definition of the multiplication yields $B^2 = 0$.

Our aim is to construct a J_p-algebra A of infinite rank satisfying the hypotheses of Corollary 5.11. Namely, we look for a J_p-algebra A such that

(i) as a J_p-module, A is the completion of a free J_p-module of countable rank;

(ii) every element of A is integral over J_p;

(iii) the technical condition (*) from Corner's Theorem 5.10 holds true.

Now consider the free J_p-module $B = \bigoplus_{i \geq 1} B_i$, where $B_i \cong J_p$ for all $i \geq 1$. Let $\widehat{B}$ be its completion in the p-adic topology and form the idealization

$$A = J_p(+)\widehat{B}.$$

We show that A is the required example.

Of course, A is the completion of a free J_p-algebra of countable rank, thus (i) holds.

To see that (ii) holds, first observe that any element $b \in \widehat{B}$ is integral, since $b^2 = 0$. Now an arbitrary element $\eta \in A$ has the form $\eta = r + b$, where $r \in J_p$ and $b \in \widehat{B}$. Therefore we have $(\eta - r)^2 = 0$, which implies that η is also integral.

Finally, let us define the ideals A_i satisfying (iii). We must have $A_0 = A$. For $i \geq 1$, we define A_i to be the completion of $\bigoplus_{j \geq i} B_j$ in the p-adic topology. Note that $A_i = B_i \oplus A_{i+1}$, for $i \geq 1$, and $A_0/A_1 = A/B \cong J_p$, and therefore all the quotients are free of rank one, as desired. It remains to show that if $\eta \in \bigcap_{i<\omega}(A_i + pA)$ then $\eta \in pA$. Since A is a submodule of $J_p \oplus \prod_{i \geq 1} B_i$, we may regard at η as a sequence $\eta = (r_i)_{i<\omega}$. It suffices to show that for every $j \geq 0$, the j-th component r_j of η lies in pJ_p. Pick an index $i > j$; since $\eta \in A_i + pA$, we have $\eta = a_i + pz_i$, for suitable $a_i \in A_i$ and $z_i \in A$. Since $i > j$, the j-th component of a_i is zero, and so the r_j coincides with the j-th component of pz_i, whence $r_j \in pJ_p$, as desired. ∎

It is interesting to observe that the preceding example of a complete torsion-free J_p-algebra A that has infinite rank and is integral over J_p, cannot be improved by requiring that A is also a commutative domain. In fact one can show that a complete J_p-algebra A of infinite rank cannot be integral over J_p, whenever A is a commutative domain. We will not give a proof of this fact here, since it is not relevance for the present discussion.

5.3 Example of non-integrality and failure of the Completion Theorem.

We give an example of an endomorphism with zero entropy which is not integral over J_p modulo the small endomorphisms (hence the converse of Corollary 5.7 does not hold).

The interest of this example is three-fold, since it also leads to a counterexample of the so-called "Completion Theorem" for our setting, and to examples of recurrent endomorphisms which are not strongly recurrent, as required in the proof of Proposition 2.12.

We define a suitable semi-standard group. For every $n > 0$ and $1 \leq i \leq n$, let $\langle b_{ni} \rangle$ be a cyclic group isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. For $n > 0$ let

$$B_n = \bigoplus_{i=1}^{n} \langle b_{ni} \rangle.$$

and set $G = \bigoplus_{n>0} B_n$.

For each $n > 0$, consider the endomorphism $\phi_n : B_n \to B_n$ that extends the assignments

$$b_{ni} \mapsto b_{n,i+1},\ (i < n)\ ;\quad b_{nn} \mapsto b_{n1}.$$

Then we get the endomorphism $\phi : G \to G$, where $\phi = (\phi_n)_{n>0}$.

Our aim is to show that ϕ provides the required example.

Proposition 5.13. *In the above notation we have:*

(i) ϕ has zero entropy;

(ii) ϕ is not integral over J_p modulo the small endomorphisms (in particular ϕ is not small).

Proof. (i) Pick any $z \in G$. Then $z \in B_1 \oplus \cdots \oplus B_k$, for a suitable $k > 0$, and $\phi(z) = \phi_1(z) + \cdots + \phi_k(z) \in B_1 \oplus \cdots \oplus B_k$. If follows at once that z has finite trajectory with respect to ϕ.

(ii) Assume, for a contradiction, that there exist $a_0, a_1, \ldots, a_{m-1} \in J_p$ such that

$$\phi^m + a_{m-1}\phi^{m-1} + \cdots + a_1\phi + a_0 = \vartheta,$$

where ϑ is a small endomorphism. Then, by the definition of smallness, there exist $N > 0$ such that $\vartheta(p^N G[p]) = 0$. Let us choose $k > N + m$. For convenience, for $1 \leq i \leq k$ we set $z_i = p^{k-1} b_{ki} \in B_k$; then, by construction, $0 \neq z_i \in p^N G[p]$, hence, in particular, $\vartheta(z_i) = 0$. Now, since $k > m$, we have

$$\phi^m(z_1) = z_{m+1};\quad a_j\phi^j(z_1) = a_j z_{j+1},$$

for $j = 0, \ldots, m-1$.

It follows that $\vartheta(z_1) = a_0z_1 + a_1z_2 + \cdots + a_{m-1}z_m + z_{m+1}$, and this element in nonzero, since the b_{ki} are independent and $z_{m+1} \neq 0$.

We have thus reached the required contradiction, and our conclusion follows. ■

Note that the endomorphism ϕ is not integral, but it is point-wise integral, since $\mathrm{ent}(\phi) = 0$.

Now denote by $\bar{G}$ the torsion completion of G, and by ψ the endomorphism of $\bar{G}$ which uniquely extends ϕ.

Proposition 5.14. *In the above notation, $\psi \in End(\bar{G})$ has infinite entropy. In particular, the entropy of an endomorphism need not be preserved when we pass to the extensions to torsion completions.*

Proof. For any fixed $k \geq 0$ we define an element $\eta = (e_n)_{n>0} \in \bar{G}$, where $e_n \in B_n$ is chosen as follows: $e_n = p^{n-k}b_{n1}$ if $n \geq k$, and $e_n = 0$ otherwise. Let $F = \mathbb{Z}\eta$; note that η has order p^k. Take any $m > 0$; it is easy to verify that $\psi^m F \cap (F + \psi F + \cdots + \psi^{m-1}F) = 0$. In fact, as soon as $n > m$, the n-th component of $\psi^m(\eta)$ is $0 \neq p^{n-k}b_{n,m+1} \in \langle b_{n,m+1}\rangle$, while the n-th component of any element of $T_n = F + \psi F + \cdots + \psi^{m-1}F$ never lies in $\langle b_{n,m+1}\rangle$, when $n > m$.

We conclude that

$$\alpha_{m+1} = |\psi^m F/(T_n \cap \psi^m F)| = |\psi^m F| = p^k.$$

Therefore an application of Proposition 1.3 (ii) yields

$$H(\psi, F) = \log(p^k).$$

In particular, we get $\mathrm{ent}(\psi) \geq k \log p$. Since $k > 0$ was arbitrary, we conclude that $\mathrm{ent}(\psi) = \infty$, as desired. ■

The preceding proposition says that a "Completion Theorem" is not valid for algebraic entropy.

We also remark that $\bar{G}$ is separable and its endomorphism ψ is recurrent, but not strongly recurrent (cf. Proposition 2.12).

5.4 p-groups of length $> \omega$.

A generalization of small endomorphisms, useful for non-separable p-groups, was introduced by Corner [C2], who called an endomorphism $\phi : G \to G$ *thin* if, for every positive integer k there exists an integer $n \geq 0$ such that $\phi(p^n G[p^k]) \subseteq p^\omega G$. Trivially, small endomorphisms are thin, and the converse is true for separable p-groups. The thin endomorphisms form a two-sided ideal of $\mathrm{End}(G)$ as well, usually denoted by $E_\theta(G)$, which also intersects the centre Z_G trivially.

We now consider the existence of p-groups of length strictly larger than ω with zero algebraic entropy. We start with two technical lemmas on thin endomorphisms.

Lemma 5.15. *If a p-group G satisfies* $\mathrm{End}(G/p^\omega G) = J_p \oplus E_s(G/p^\omega G)$*, then* $\mathrm{End}(G) = J_p \oplus E_\theta(G)$.

Proof. Let $\phi : G \to G$ be an endomorphism. Then ϕ induces an endomorphism $\bar{\phi} : G/p^\omega G \to G/p^\omega G$. By hypothesis, there exists a $\pi \in J_p$ such that $\bar{\phi} - \pi$ is small. Since $\bar{\phi} - \pi$ is induced by $\phi - \pi$, from [C2, Lemma 7.1] we get that $\phi - \pi$ is thin, hence $\phi \in J_p \oplus E_\theta(G)$. ■

Lemma 5.16. *Let G be a semi-standard p-group of length $\lambda < \omega^2$. Then every thin endomorphism of G has zero entropy, hence, in particular, it is point-wise integral.*

Proof. By hypothesis there exists $n > 0$ such that $p^{\omega n}G = 0$. Since ϕ is thin, it follows from Lemma 7.2 of [C2] that ϕ^n is small, hence $\mathrm{ent}(\phi^n) = 0$, which is equivalent to $\mathrm{ent}(\phi) = 0$. ∎

An immediate consequence of the two preceding lemmas is

Corollary 5.17. *Let G be a semi-standard p-group of length $\lambda < \omega^2$, such that* $\mathrm{End}(G/p^\omega G) = J_p \oplus E_s(G/p^\omega G)$*. Then* $\mathrm{ent}(G) = 0$.

Proof. From Lemma 5.15 we have that $\mathrm{End}(G) = J_p \oplus E_\theta(G)$. From Lemma 5.16 we derive that every endomorphism of G has zero algebraic entropy. ∎

Corollary 5.17 shows that, for semi-standard p-groups G of length $< \omega^2$, what is relevant in order to get zero algebraic entropy, is the quotient group $G/p^\omega G$, and it does not matter what the subgroup $p^\omega G$ is. Furthermore, note that, under the hypotheses of Corollary 5.17, no endomorphism of positive algebraic entropy of $p^\omega G$ can be extended to an endomorphism ϕ of G, since $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_{p^\omega G})$.

The next result provides plenty of p-groups of arbitrary length $\lambda < \omega^2$ with zero algebraic entropy. Recall that the same limitation to groups of length $< \omega^2$ was assumed also in [C2, Theorem 10.2].

Theorem 5.18. *Given an ordinal $\lambda < \omega^2$, there exists a family of $2^{2^{\aleph_0}}$ p-groups, each of length λ and with zero entropy, such that there are only thin homomorphisms between the different members of the family.*

Proof. By Corollary 5.17, it is enough to prove the existence of a p-group X of length λ such that $X/p^\omega X \cong G$, where G is a semi-standard p-group considered in Theorem 5.4. Since we have a family of $2^{2^{\aleph_0}}$ p-groups G of this form with only small homomorphisms between the different members of the family, we have that the corresponding family of p-groups X has the desired property, by [C2, Lemma 7.1 (ii)].

Let B be basic in G and T a countable p-group of length λ such that $T/p^\omega T \cong B$. Such a group T exists by Zippin's result (see [F, Corollary 76.2]). Since we have the exact sequence

$$\mathrm{Ext}(G, p^\omega T) \to \mathrm{Ext}(B, p^\omega T) \to 0$$

we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & p^{\omega}T & \to & T & \to & B & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & p^{\omega}T & \to & X & \to & G & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D & = & D & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $D \cong \bigoplus_{2^{\aleph_0}} \mathbb{Z}(p^{\infty})$. Then $p^{\omega}G = 0$ implies $p^{\omega}X \leq p^{\omega}T$, whence $p^{\omega}T = p^{\omega}X \cap T$, so that $p^{\omega}T = p^{\omega}X$. Thus $l(X) = l(T)$ and we are done. ■

6 The uniqueness of the algebraic entropy function.

This final section is inspired by Stojanov's [St] axiomatic characterization of the topological entropy for endomorphisms of compact groups. In the sequel we denote by $\mathcal{T}$ the class of all torsion Abelian groups and by $\mathcal{T}_p$ the class of all Abelian p-groups.

Theorem 6.1. *The algebraic entropy of the endomorphisms of the groups in $\mathcal{T}$ is characterized as the unique collection $h = \{\mathrm{h}_G : G \in \mathcal{T}\}$ of functions $\mathrm{h}_G : \mathrm{End}(G) \to \mathbb{R}^+$ such that:*

(i) the Addition Theorem holds for h;

(ii) h_G is invariant under conjugation for every $G \in \mathcal{T}$;

(iii) $\mathrm{h}_G(\phi^k) = k \cdot \mathrm{h}_G(\phi)$ for every $G \in \mathcal{T}$;

(iv) if $\phi : G \to G$ and G is the direct limit of ϕ-invariant subgroups G_i, $\mathrm{h}_G(\phi) = \sup \mathrm{h}_{G_i}(\phi\restriction_{G_i})$;

(v) (normalization) $\mathrm{h}_G(\sigma_K) = \log |K|$, where $G = \oplus_{\aleph_0} K$, $\sigma_K : G \to G$ is the Bernoulli shift, and K is any non-zero finite group.

Proof. We have to show that $\mathrm{h}_G(\phi) = \mathrm{ent}(\phi)$, for every $G \in \mathcal{T}$ and every $\phi \in \mathrm{End}(G)$. We proceed by steps.

Step 1. If $h = \{\mathrm{h}_G : G \in \mathcal{T}\}$ is a collection of functions with (iii) and (iv), then $h_G(\phi) = 0$ for every point-wise integral $\phi \in \mathrm{End}(G)$.

Indeed, note firstly that (iii) yields $h_F = 0$ for every finite group F. In fact, every $\phi \in \mathrm{End}(F)$ satisfies $\phi^m = \phi^n$ for some $m < n$, so $m\, h_F(\phi) = n\, h_F(\phi)$ and consequently $h_F(\phi) = 0$. It follows that, if $\phi \in \mathrm{End}(G)$ is point-wise integral, then $\mathrm{h}_G(\phi) = 0 = \mathrm{ent}(G)$ by (iv), since G is the direct limit of its finite ϕ-invariant subgroups.

Let us fix now a prime p.

Step 2. Let $\mathcal{T}_{p,1}$ be the class of all p-groups of exponent p; let G be such a group and $\phi : G \to G$ an endomorphism. If ϕ is integral, then it is point-wise integral, so $\mathrm{h}_G(\phi) = 0 = \mathrm{ent}(\phi)$ by Step 1. Assume that ϕ is not integral, so $R_\phi = F_p[\phi]$ is an Euclidean domain. Let $t_\phi(G)$ be the ϕ-torsion subgroup of G; then $\phi\restriction_{t_\phi(G)}$ is point-wise integral on $t_\phi(G)$. Then Step 1 gives $\mathrm{h}_{t_\phi(G)}(\phi\restriction_{t_\phi(G)}) = 0$. By (i), setting $\bar{G} = G/t_\phi(G)$ and $\bar{\phi}$ the induced endomorphism, we get $\mathrm{h}_G(\phi) = \mathrm{h}_{\bar{G}}(\bar{\phi})$. In other words, we can assume that the ϕ-torsion subgroup of G vanishes; so the ϕ-trajectories of all the elements of G are infinite. If G is a finitely generated R_ϕ-module, then it is a direct sum of finitely many, say k, trajectories. Each one is isomorphic to $\bigoplus_{n>0} F_p$, and ϕ acts as the Bernoulli shift on it. By (i), (ii), and (v) we have $\mathrm{h}_G(\phi) = k \cdot \log p = \mathrm{ent}(\phi)$. If G is not a finitely generated R_ϕ-module, then, using (iv), one can easily prove that $\mathrm{h}_G(\phi) = \infty = \mathrm{ent}(\phi)$. This establishes the uniqueness of the entropy function on the class $\mathcal{T}_{p,1}$.

Step 3. Using Step 2, (i) and the induction one can establish the uniqueness of the algebraic entropy function on the class $\mathcal{T}_{p,n}$ of all p-groups of exponent p^n. Then (iv) allows us to extend it to the whole class $\mathcal{T}_p$.

Step 4. The general case for $\mathcal{T}$ easily follows from Step 3, (i) and (iv). ∎

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