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## Essentially Indecomposable Modules Over a Complete Discrete Valuation Ring.

B. GOLDSMITH (\*)

### 1. Introduction.

Torsion-free modules over a complete discrete valuation ring  $R$  are markedly different from abelian groups or modules over an incomplete discrete valuation ring in that the only indecomposable modules which exist have rank 1 and so are isomorphic to  $R$  itself or the field,  $Q$ , of fractions of  $R$  ([7], p. 45). In this paper we investigate how close a reduced torsion-free  $R$ -module of infinite rank can come to being indecomposable. In particular we establish in § 4 the existence of essentially indecomposable modules with basic submodules of countable rank. The results here bear a strong resemblance to results on  $p$ -groups ([2], [5], [9] and [10]). Notation follows the standard works of Fuchs [3], [4] while set-theoretic concepts, which are kept to a minimum, may be found in Jech [6].

### 2. Maximal pure submodules.

Let  $R$  denote a complete discrete valuation ring of cardinality  $\nu$  with unique prime  $p$ . For an infinite cardinal  $\lambda$  we let  $S_\lambda$  (or just  $S$  if no ambiguity is possible) denote a free  $R$ -module of rank  $\lambda$ . Clearly  $S_\lambda$  is not complete in the  $p$ -adic topology and we denote its completion by  $\hat{S}_\lambda$  (or just  $\hat{S}$ ).

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DEFINITION. A  $R$ -module  $X$  is said to be a maximal pure submodule of the complete  $R$ -module  $\hat{S}$  if  $X$  is a pure submodule of  $\hat{S}$  containing  $S$  and  $\hat{S}/X \cong Q$ , the field of fractions of  $R$ . We remark that if  $X$  is a maximal pure submodule of  $\hat{S}$  then for any  $x \in \hat{S} \setminus X$ , we have  $\hat{S} = \langle X, x \rangle_*$ .

LEMMA 2.1. If  $G$  and  $K$  are pure submodules of  $\hat{S}$  containing  $S$  then  $G \cong K$  if and only if there is an automorphism  $\theta$  of  $\hat{S}$  with  $G\theta = K$ .

PROOF. The sufficiency is clear, we establish necessity. Let  $\varphi$  be an isomorphism from  $G$  onto  $K$ . Then  $\varphi$  extends uniquely to an endomorphism  $\hat{\varphi}$  of  $\hat{S}$ . Similarly if  $\psi$  is the inverse of  $\varphi$ , it extends to an endomorphism  $\hat{\psi}$  of  $\hat{S}$ . However since  $G$  and  $K$  are dense subsets of the Hausdorff space  $\hat{S}$ , it follows easily that  $\hat{\varphi}\hat{\psi}$  and  $\hat{\psi}\hat{\varphi}$  act as the identity on  $\hat{S}$ . Thus  $\theta = \hat{\varphi}$  is the required automorphism.

Before examining the endomorphism rings of maximal pure submodules of  $\hat{S}$ , we introduce the concept of an inessential endomorphism (cf. [5]). Let  $X$  be a pure submodule of  $\hat{S}$  containing  $S$ , then, as we noted in the proof of Lemma 2.1, any endomorphism  $\varphi$  of  $X$  has a unique extension  $\hat{\varphi}$  to an endomorphism of  $\hat{S}$ . We define an endomorphism of  $X$  to be inessential if its unique extension to  $\hat{S}$  maps  $\hat{S}$  into  $X$ . It is easily seen that the difference of two inessential endomorphisms is inessential while the composition of two endomorphisms is inessential when either factor is inessential. Thus the inessential endomorphisms of  $X$  form a two-sided ideal  $I(X)$  in the endomorphism ring  $E(X)$  of  $X$ .

THEOREM 2.2. For any infinite cardinal  $\lambda$ , there exists an  $R$ -module  $G$ , with basic submodule of rank  $\lambda$ , such that  $E(G)$  is the ring split extension of  $R$  by  $I(G)$ ,

$$E(G) = R \oplus I(G).$$

PROOF. Let  $S$  be a free  $R$ -module of rank  $\lambda$  and let  $G$  be any maximal pure submodule of  $\hat{S}$ . Clearly  $S$  is basic in  $G$ . We may identify  $E(G)$  as a subring of  $E(\hat{S})$  by identifying each endomorphism  $\varphi$  in  $E(G)$  with its unique extension  $\hat{\varphi}$  in  $E(\hat{S})$ . With this identification  $I(G)$  is a left ideal of  $E(\hat{S})$ .

Pick  $x \in \hat{S} \setminus G$ . Then for arbitrary  $\varphi$  in  $E(G)$  we must have  $q(x\hat{\varphi}) = tx + g$ , some  $q, t \in R, g \in G$ . Since every element of  $R$  is a product of a power of  $p$  and a unit, there is no loss in generality in supposing

$q = p^r, t = p^s$ . We consider two cases:

(i)  $r \leq s$ .

In this case  $p^r(x\phi - p^{s-r}x) = g$ . The purity of  $G$  in  $\hat{S}$  implies that  $x(\phi - p^{s-r}1)$  belongs to  $G$ . Since  $G$  is invariant under  $\phi - p^{s-r}1$  and  $\langle G, x \rangle_* = \hat{S}$ , it is clear that  $\hat{S}(\phi - p^{s-r}1)$  is contained in  $G$ . Thus  $\phi - p^{s-r}1 \in I(G)$  and so  $E(G) = R + I(G)$ .

(ii)  $r > s$ .

We show that this case cannot arise. As before we can show that  $x(p^{r-s}\phi - 1) \in G$  and deduce that  $p^{r-s}\phi - 1 \in I(G)$ . Suppose  $p^{r-s}\phi - 1 = \theta$ , where  $\theta \in I(G)$ . Since  $r > s$ ,  $p^{r-s}\phi$  belongs to the Jacobson radical of  $E(\hat{S})$  (see [8]) and this forces  $\theta$  to be a unit in  $E(\hat{S})$ . However since  $G$  is certainly not a homomorphic image of  $\hat{S}$ ,  $I(G)$  is a proper left ideal of  $E(\hat{S})$  which contains a unit-contradiction. So case (ii) does not arise.

Since  $G$  is pure in  $\hat{S}$ ,  $R \cap I(G) = 0$  and *quâ* modules,  $E(G) = R \oplus \oplus I(G)$ . However this is clearly a ring split extension also and we have established the result.

### 3. Essentially-rigid systems of $R$ -modules.

As we noted in the introduction indecomposable  $R$ -modules have rank 1 whereas indecomposable abelian groups of arbitrary large rank exist [11]. One useful tool in the investigation of indecomposable abelian groups was the concept of a rigid system of groups (see [4], § 88). In this section we define and explore an analogous concept for  $R$ -modules.

We extend the concept of inessential to homomorphisms between different reduced torsion-free  $R$ -modules  $X_i, X_j$  by defining  $I_i(X_j) = \{\varphi \in \text{Hom}(X_i, X_j) \mid \hat{X}_i \phi \leq X_j\}$  where  $\phi$  denotes the unique extension of  $\varphi$  to a map  $\hat{X}_i \rightarrow \hat{X}_j$ .

DEFINITION. A family  $\{X_j\}$  ( $j \in J$ ) of reduced torsion-free  $R$ -modules is said to be essentially rigid if

$$\text{Hom}(X_i, X_j) = \begin{cases} R \oplus I(X_i) & \text{if } i = j \\ I_i(X_j) & \text{if } i \neq j, \end{cases}$$

for all  $i, j \in J$ .

Suppose throughout this section that  $R$  is a complete discrete valuation ring of cardinality  $\nu$  and  $\lambda$  is an infinite cardinal satisfying  $\mu = \lambda^{\aleph_0} = 2^\lambda$  and  $\nu < \mu$ . For an infinite cardinal  $\sigma$ , let  $\sigma^+$  denote the successor of  $\sigma$ . We can now state the main result of this section.

**THEOREM 3.1.** If  $\lambda$  is an infinite cardinal with the property that  $\mu = \lambda^{\aleph_0} = 2^\lambda$  and  $R$  is a complete discrete valuation ring of cardinality  $\nu < \mu$ , then there exists an essentially-rigid family of  $R$ -modules having  $\mu^+$  members.

**REMARK.** (i) By assuming G.C.H. we may, of course replace  $\mu^+$  by  $2^\mu$ .

(ii) Cardinals of the form  $\lambda$  do exist for values of  $\lambda$

other than  $\lambda = \aleph_0$  e.g. assuming G.C.H., any cardinal of cofinality  $\aleph_0$  will do.

**LEMMA 3.2.** Let  $V$  be a vector space of dimension  $\alpha$ , an infinite cardinal, over a field. Let  $\{W_i\}$  ( $i < \beta$ ) be a family of subspaces of  $V$  indexed by the cardinal  $\beta < \alpha$ , such that  $\dim W_i = \alpha$  for all  $i < \beta$ . Then there exist  $\alpha^+$  subspaces  $\{U\}$  of  $V$  such that each subspace  $U$  is of codimension 1 in  $V$  and no subspace  $W_i$  is contained in any subspace  $U$ .

**PROOF.** By a result of Beaumont and Pierce ([1], Lemma 5.2) there exists at least one such subspace,  $U_0$  say. Suppose that the subspaces  $\{U_i\}$  ( $i < \zeta$ ) have been constructed and  $\zeta < \alpha^+$ . Then the set of subspaces consisting of the given  $W_i$  together with the constructed subspaces  $U_i$  constitutes a family of at most  $\alpha$  subspaces each of dimension  $\alpha$ . Applying Beaumont and Pierce's result to this family yields another subspace of codimension 1. Call this subspace  $U_\zeta$ . The result follows easily by transfinite induction.

**LEMMA 3.3.** If  $S$  is free of rank  $\lambda$  then there exist  $\mu^+$  maximal pure submodules of  $\hat{S}$  with the property that none of them contains a submodule isomorphic to  $\hat{S}$ .

**PROOF.** Since  $S$  is free of rank  $\lambda$ ,  $|\hat{S}| = \max(\lambda^{\aleph_0}, \nu^{\aleph_0})$ . But  $\nu < \mu$  implies that  $\nu^{\aleph_0} < \mu^{\aleph_0} = (\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0} = \mu$ . So  $\hat{S}/S$  is a  $Q$ -vector space of dimension  $\mu = \lambda^{\aleph_0}$ . Now let  $\{W_k\}$  ( $k \in K$ ) be the collection of submodules of  $\hat{S}$  which are isomorphic to  $\hat{S}$ . Each of these submodules is determined by an endomorphism of  $\hat{S}$ . However each endomorphism

of  $\hat{S}$  is completely determined by its action on  $S$  which has rank  $\lambda$ , so  $|E(\hat{S})| = \mu^\lambda = (2^\lambda)^\lambda = 2^{\lambda^2} = \mu$ . Hence  $\{W_k\}$  ( $k \in K$ ) is a family of at most  $\mu$  submodules of  $\hat{S}$ .

Let  $\bar{W}_k = \langle W_k + S \rangle_*/S$ . Then  $\{\bar{W}_k\}$  ( $k \in K$ ) is a family of at most  $\mu$  subspaces of the  $Q$ -vector space  $\hat{S}/S$  which has dimension  $\mu$ . Since  $\bar{W}_k \cong (Q \otimes_R (W_k + S))/Q \otimes_R S$ , it follows that each  $\bar{W}_k$  has dimension  $\mu$ . By Lemma 3.2 we can find  $\mu^+$  subspaces  $U$  such that no  $\bar{W}_k$  is contained in any  $U$  and, moreover, each  $U$  has codimension 1 in  $\hat{S}/S$ . If  $G$  is a submodule of  $\hat{S}$  with  $G/S = U$ , then  $G$  is a maximal pure submodule of  $\hat{S}$  and clearly no  $W_k$  is contained in any  $G$ . Thus we have constructed the required family of  $\mu^+$  maximal pure submodules.

Let  $\mathfrak{G}_\lambda$  denote the collection of all maximal pure submodules of  $\hat{S}_\lambda$  which do not contain an isomorphic copy of  $\hat{S}_\lambda$ .

LEMMA 3.4. If  $\{G_\alpha\}$  ( $\alpha < \beta$ ) is a subset of  $\mathfrak{G}_\lambda$  and  $|\beta| \leq \mu$ , then there exist  $\mu^+$  submodules  $G$  in  $\mathfrak{G}_\lambda$  such that  $\text{Hom}(G_\alpha, G) = I_\alpha(G)$  for all  $\alpha < \beta$ .

PROOF. The proof is similar to that of Lemma 3.3. Suppose  $\{W_{\alpha_i}\}$  denotes the set of endomorphic images of  $G_\alpha$  which have rank  $\mu$ . Then for each  $\alpha$ , the set  $\{W_{\alpha_i}\}$  is of cardinality at most  $\mu$ . Since  $|\beta| \leq \mu$ , the union of all such collections is a set of at most  $\mu$  submodules of  $\hat{S}$ . Call this set  $\mathcal{W}$ . Now let  $\mathcal{U}$  denote the set of endomorphic images of  $\hat{S}$  which are isomorphic to  $\hat{S}$ . Then  $\mathcal{W} \cup \mathcal{U}$  is a collection of at most  $\mu$  submodules of  $\hat{S}$ , say  $\mathcal{W} \cup \mathcal{U} = \{X_i\}$  ( $i < \mu$ ). Note that each  $X_i$  has rank  $\mu$ . Set  $\bar{X}_i = \langle X_i + S \rangle_*/S$ ; then  $\{\bar{X}_i\}$  ( $i < \mu$ ) is a collection of  $\mu$  subspaces of the  $Q$ -vector space  $\hat{S}/S$  and each  $\bar{X}_i$  has dimension  $\mu$ . By Lemma 3.2 there exist  $\mu^+$  maximal subspaces  $U$  such that no  $\bar{X}_i$  is contained in a  $U$ . Choose a maximal pure submodule  $G$  such that  $G/S = U$ . Clearly  $G \in \mathfrak{G}_\lambda$ .

Now consider  $\text{Hom}(G_\alpha, G)$  for any  $\alpha$ . If  $\varphi: G_\alpha \rightarrow G$  is not inessential then  $\text{Ker } \varphi$  is contained in  $G_\alpha$  which forces  $\text{Ker } \varphi$  to have rank less than  $\mu$ . But then  $\text{Im } \varphi \cong G_\alpha/\text{Ker } \varphi$  is an endomorphic image of  $G_\alpha$  of rank  $\mu$  and is contained in  $G$ -contradiction. So we conclude that  $\text{Hom}(G_\alpha, G) = I_\alpha(G)$  for each  $\alpha$ .

LEMMA 3.5. Given any maximal pure submodule  $G$  of  $\hat{S}_\lambda$ , there are at most  $\mu$  maximal pure submodules  $G_i$  of  $\hat{S}_\lambda$  for which  $\text{Hom}(G_i, G) \neq I_i(G)$ .

PROOF. Suppose there exists a family  $\{G_i\}$  ( $i \in J$ ) of more than  $\mu$  submodules. For each  $i \in J$ , pick a homomorphism  $\varphi_i: G_i \rightarrow G$ . Then  $\{\hat{\varphi}_i\}$  ( $i \in J$ ) is a family of more than  $\mu$  endomorphisms of  $\hat{S}_\lambda$ . Since  $|E(\hat{S}_\lambda)| = \mu$ , we must have  $\hat{\varphi}_i = \hat{\varphi}_j$  for some  $i \neq j \in J$ . But then

$$\hat{S}\hat{\varphi}_i = (G_i + G_j)\hat{\varphi}_i \leq G_i\hat{\varphi}_i + G_j\hat{\varphi}_j = G.$$

Thus  $\varphi_i$  is inessential-contradiction. This establishes the lemma.

PROOF OF THEOREM 3.1. Choose  $G_0$  to be any member of  $\mathfrak{G}_\lambda$ . Suppose the essentially-rigid family  $\{G_\alpha\}$  ( $\alpha < \beta$ ) has been constructed for  $\beta < \mu^+$ . By Lemma 3.4 there exist  $\mu^+$  maximal pure submodules  $G$  such that  $\text{Hom}(G_\alpha, G) = I_\alpha(G)$ . However for each  $\alpha < \beta$ , there are, by Lemma 3.5, at most  $\mu$  of these submodules  $G$  for which  $\text{Hom}(G, G_\alpha) \neq I(G_\alpha)$ . Then deleting all such submodules  $G$  deletes at most  $\mu$  submodules from the original collection since  $\beta \leq \mu$ . So there exists  $G \in \mathfrak{G}_\lambda$  with  $\text{Hom}(G, G_\alpha) = I(G_\alpha)$  and  $\text{Hom}(G_\alpha, G) = I_\alpha(G)$  for all  $\alpha < \beta$ . Set  $G_\beta = G$ . Then the family  $\{G_\alpha\}$  ( $\alpha \leq \beta$ ) is essentially-rigid. The proof is completed by transfinite induction.

#### 4. Essentially-indecomposable modules.

In this section we show that a slightly stronger result than Theorem 3.1 can be deduced and apply this new result to the construction of essentially indecomposable modules.

We shall use the term basic rank of a homomorphism to denote the rank of a basic submodule of the image of the homomorphism.

DEFINITION. If  $S$  is a free  $R$ -module of infinite rank  $\lambda$  and  $X$  is a pure submodule of  $\hat{S}$  containing  $S$ , then we define

$$I_\lambda(X) = \{\varphi \in E(X) \mid \hat{S}\hat{\varphi} \leq X \text{ and } \hat{\varphi} \text{ has basic rank } < \lambda\}$$

Clearly  $I_\lambda(X)$  is an ideal in  $E(X)$ .

THEOREM 4.1. If  $R$  is a complete discrete valuation ring of cardinality  $\nu$  and  $\lambda$  is an infinite cardinal such that  $\mu = \lambda^{\aleph_0} = 2^\lambda$  and  $\nu < \mu$ , then there exists a family of  $\mu^+$   $R$ -modules  $\{G_j\}$  ( $j \in J$ ) such that

- (i) for each  $j \in J$ ,  $E(G_j) = R \oplus I_\lambda(G_j)$ ;
- (ii) for distinct  $j, k \in J$ , every homomorphism  $G_j \rightarrow G_k$  is inessential and has basic rank less than  $\lambda$

PROOF. This stronger result comes by observing in the proof of Theorem 3.1 that all of the modules constructed actually belong to  $\mathfrak{G}_\lambda$ . Since the image of an inessential homomorphism is complete, it must be the completion of a free module of rank less than  $\lambda$ . This gives the desired result.

Recall that  $E_0(G)$  denotes the ideal of  $E(G)$  consisting of all endomorphisms of finite rank.

COROLLARY 4.2 (G.C.H.). If  $R$  is a complete discrete valuation ring of cardinality  $2^{\aleph_0}$ , then there exists a family of  $2^{2^{\aleph_0}}$   $R$  modules  $\{G_j\}$  ( $j \in J$ ) each with basic submodules of rank  $\aleph_0$  such that

- (i) for each  $j \in J$ ,  $E(G_j) = R \oplus E_0(G_j)$ ;
- (ii) for distinct  $j, k \in J$ , every homomorphism  $G_j \rightarrow G_k$  has finite rank.

PROOF. Since  $2^{\aleph_0} \leq (\aleph_0)^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ , we see that  $\lambda = \aleph_0$  satisfies the cardinality requirements of Theorem 4.1. However if a homomorphism from  $G_j$  has finite basic rank then it clearly also has finite rank. The result now follows from Theorem 4.1 and G.C.H.

DEFINITION. If  $\lambda$  is an infinite cardinal we say that a reduced torsion-free  $R$ -module  $G$  is  $\lambda$ -essentially indecomposable if in any decomposition  $G = A \oplus B$ , one of  $A, B$  is the completion of a free module of rank less than  $\lambda$ .

In the case  $\lambda = \aleph_0$  we are requiring that in any direct decomposition one of the summands is complete of finite rank. A module with this property is said to be essentially indecomposable (cf. essentially indecomposable  $p$ -groups, [9], § 15).

The existence of  $\lambda$ -essentially indecomposable modules follows rather easily from Theorem 4.1 in the case  $\lambda^{\aleph_0} = 2^\lambda$ . For if  $G$  is one of the modules constructed in Theorem 4.1 and we have a decomposition  $G = A \oplus B$  with associated projections  $\pi_1$  and  $\pi_2$ , then one of  $\pi_1, \pi_2$  belongs to  $I_\lambda(G)$  since the quotient  $E(G)/I_\lambda(G)$  is a domain. If  $\pi_1 \in I_\lambda(G)$  then clearly  $A$  is the completion of a free  $R$ -module of rank less than  $\lambda$ . In particular if  $\lambda = \aleph_0$  we have established the existence of  $2^{2^{\aleph_0}}$  essentially indecomposable  $R$ -modules for any complete discrete valuation ring  $R$  of cardinality  $2^{\aleph_0}$ .



We conclude this section by constructing an essentially indecomposable module which is not a maximal pure submodule of a complete module.

PROPOSITION 4.3. If  $R$  is a complete discrete valuation ring of cardinality  $2^{\aleph_0}$  and  $S$  is a free  $R$ -module of countably infinite rank, then there exists a pure submodule  $H$  of  $\hat{S}$  containing  $S$  with  $\hat{S}/H \cong \cong Q \oplus Q$  and such that  $E(H) = R \oplus E_0(H)$ .

PROOF. Choose distinct maximal pure submodules  $G$  and  $G_1$  belonging to the family constructed in Corollary 4.2. Set  $H = G \cap G_1$ . Clearly  $S \leq H \leq \hat{S}$  and both inclusions are pure. Also  $\hat{S}/H \cong Q \oplus Q$ . Let  $\hat{S} = \langle H, x, y \rangle_*$  where  $G = \langle H, x \rangle_*$  and  $G_1 = \langle H, y \rangle_*$ . Let  $\varphi$  be any endomorphism of  $H$ . Then as in the proof of Theorem 2.2 we may write

$$q(x\hat{\varphi}) = h_0 + \alpha x + \beta y$$

$$q(y\hat{\varphi}) = h_1 + \gamma x + \delta y$$

where  $q, \alpha, \beta, \gamma, \delta \in R$  and  $h_0, h_1 \in H$ .

Now  $x(q\hat{\varphi} - \alpha 1) \in G_1$  and so  $q\hat{\varphi} - \alpha 1$  maps  $G$  into  $G_1$ . From the properties of  $G$  and  $G_1$  we conclude that  $q\hat{\varphi} - \alpha 1$  is an inessential endomorphism. It follows as in the proof of Theorem 2.2, that  $\hat{\varphi}|_G$  belongs to  $R \oplus I(G_1)$ . Hence we can write  $\varphi = r + \theta$  where  $r \in R$ ,  $\theta \in I(G_1)$ . But then  $\theta \in E(H) \cap I(G_1) = E(H) \cap E_0(G) = E_0(H)$ . So  $\varphi \in R \oplus E_0(H)$  and  $E(H) \leq R \oplus E_0(H)$ . The reverse inequality is clearly true so  $E(H) = R \oplus E_0(H)$ .

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#### REFERENCES

- [1] R. A. BEAUMONT - R. S. PIERCE, *Some invariants of  $p$ -groups*, Michigan Math. J., **11** (1964), pp. 137-149.
- [2] A. L. S. CORNER, *On endomorphism rings of primary abelian groups*, Quart. J. Math. (Oxford) (2), **20** (1969), pp. 277-296.

- [3] L. FUCHS, *Infinite Abelian Groups* - Vol. I, Academic Press, New York and London, 1970.
- [4] L. FUCHS, *Infinite Abelian Groups* - Vol. II, Academic Press, New York and London, 1973.
- [5] B. Goldsmith, *Essentially-rigid families of abelian  $p$ -groups*, J. London Math. Soc., (2), **18** (1978), pp. 70-74.
- [6] T. J. JECH, *Lectures in Set Theory, Lecture Notes in Mathematics* - Vol. 217, Springer-Verlag, Berlin 1971.
- [7] I. KAPLANSKY, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954.
- [8] W. LIEBERT, *Characterisation of the endomorphism rings of divisible torsion modules and reduced complete torsion-free modules over complete discrete valuation rings*, Pacific J. Math., **37** (1971), pp. 141-170.
- [9] R. S. PIERCE, *Homomorphisms of primary abelian groups*, Topics in Abelian Groups, pp. 215-310, Scott, Foresman & Co., Chicago, 1963.
- [10] S. SHELAH, *Existence of rigid-like families of abelian  $p$ -groups*, Model Theory and Algebra, Lecture Notes in Mathematics Vol. 498, pp. 384-402, Springer-Verlag, Berlin, 1975.
- [11] S. SHELAH, *Infinite Abelian Groups, Whitehead problem and some constructions*, Israel J. Math., **18** (1974), pp. 243-256.

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