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The spectral function for Sturm–Liouville problems where the potential is of Wigner–von Neumann type or slowly decaying

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Abstract

We consider the linear, second-order, differential equation

$$y'' + (\lambda - q(x))y = 0 \quad \text{on } [0, \infty) \tag{$*$}$$

with the boundary condition

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \quad \text{for some } \alpha \in [0, \pi). \tag{$**$}$$

We suppose that $q(x)$ is real-valued, continuously differentiable and that $q(x) \to 0$ as $x \to \infty$ with $q \notin L^1[0, \infty)$. Our main object of study is the spectral function $\rho_\alpha(\lambda)$ associated with $(*)$ and $(**)$. We derive a series expansion for this function, valid for $\lambda \geqslant \Lambda_0$ where Λ_0 is computable and establish a Λ_1, also computable, such that $(*)$ and $(**)$ with $\alpha = 0$, have no points of spectral concentration for $\lambda \geqslant \Lambda_1$. We illustrate our results with examples. In particular we consider the case of the Wigner–von Neumann potential.
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1. Introduction

We consider the linear, second-order, differential equation

$$y'' + (\lambda - q(x))y = 0 \quad \text{on } [0, \infty) \tag{1.1}$$

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with the boundary condition

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0 \quad \text{for some } \alpha\in[0,\pi). \tag{1.2}$$

We suppose that $q(x)$ is real-valued, continuously differentiable and that $q(x)\to 0$ as $x\to\infty$ with $q\notin L^1[0,\infty)$. In this case (1.1) is in the limit point case at infinity and the essential spectrum is $[0,\infty)$. Our main object of study is the spectral function, $\rho_\alpha(\lambda)$, associated with (1.1) and (1.2). It is known that if $q\in L^1[0,\infty)$, then the spectrum is purely absolutely continuous on $(0,\infty)$, $\rho'_\alpha(\lambda)$ exists, is continuous in λ and satisfies $\rho'_\alpha(\lambda)>0$ for $\lambda>0$ (see for example [8,15]). In [9] a series representation was given for $\rho'_\alpha(\lambda)$ for $\lambda>\Lambda_0$ where Λ_0 is computable under general conditions which require little more than $q\in L^1[0,\infty)$. In [6] the question of spectral concentration was also considered under the same circumstances. In this case, points of spectral concentration are defined, roughly, as values of $\lambda\in(0,\infty)$ at which $\rho'_\alpha(\lambda)$ has a local maximum. A more precise definition is given in Section 3 below. This question was also considered in [2] where the physical interpretation of such points was discussed. The results of [6] lead to a computable Λ_1 which is such that $\rho_0(\lambda)$ has no points of spectral concentration for $\lambda\geqslant\Lambda_1$.

Our object in the present paper is to investigate whether similar results can be obtained when $q(x)\to 0$ as $x\to\infty$, but $q\notin L^1[0,\infty)$. This case is much less straightforward, since it is no longer true in general that $\rho'_\alpha(\lambda)$ exists, is continuous in λ and satisfies $\rho'_\alpha(\lambda)>0$ for $\lambda>0$. Indeed, examples have been constructed [10] where $q(x)$ decays arbitrarily more slowly than a Coulomb potential, but for which $\rho_\alpha(\lambda)$ is discontinuous on a dense set of eigenvalues in $[0,\infty)$; moreover, if $q\notin L^2[0,\infty)$ then it is known that the absolutely continuous spectrum may be empty [14], in which case $\rho'_\alpha(\lambda)$ does not exist as a finite limit on a dense set of points in $[0,\infty)$. However, under even quite minimal smoothness conditions, there are classes of decaying, but non-integrable, potentials for which the spectrum is purely absolutely continuous on $(0,\infty)$ (see for example [4]) or on (M,∞) for some $M>0$ (see [1]). In such cases, we seek a series representation of $\rho'_0(\lambda)$ and $\Lambda_0,\Lambda_1\in\mathbb{R}^+$, where the series representation is valid for $\lambda>\Lambda_0$ and there are no points of spectral concentration for $\lambda>\Lambda_1$. We develop a general method which builds on the results of [6] and illustrate the method by the examples

(a) $q(x) = (1+x)^{-\alpha}$, $\frac{2}{3}<\alpha\leqslant 1$ for $0\leqslant x<\infty$,
(b) $q(x) = \frac{\sin((1+x)^{1/2})}{(1+x)^{1/2}}$, for $0\leqslant x<\infty$,
(c) $q(x) = \sum_{k=-M}^{M} h_k(x)e^{2ic_kx}$ for $0\leqslant x<\infty$ where q is real-valued, $c_k\in\mathbb{R}$ and $h_k(x)\to 0$ as $x\to\infty$ for $k=-M,\dots,M$. We also impose differentiability conditions on the $h_k(x)$.

Example (a) is amenable to the analysis of [4] where it is shown that $\rho'_0(\lambda)>0$ for all $\lambda>0$ and that an upper bound exists for the points of spectral concentration.

Example (b) is beyond the range of [4] but is amenable to the analysis of [1], from which it may be inferred that the so-called resonance set is empty, and hence that the spectrum is purely absolutely continuous on $(0, \infty)$.

Potentials of type (c) are known as Wigner–von Neumann potentials and have been widely discussed over the years, we mention in particular [1] and the recent results of [11]. They too are beyond the scope of [4].

We work throughout with the special case $\alpha = 0$ of (1.2). Essentially the same methods work for $\alpha \neq 0$, but the analysis is a bit more complicated. Relations between spectral derivatives for different values of α may be found in [3,5], and we mention the recent result in this direction contained in [12].

2. The main results

In [6] the following theorem was proved

Theorem 1. *Let $q \in L^1[0, \infty)$ and suppose that there exists $\Lambda_1 > 0$ such that for $x \geqslant 0$ and $\lambda > \Lambda_1$*

$$\left| \int_x^\infty e^{2i\lambda^{1/2}t} q(t)\, dt \right| \leqslant a(x)\eta(\lambda),$$

where $a(\cdot) \in L^1$ is decreasing, $\eta(\lambda) \to 0$ as $\lambda \to \infty$ and $32\eta(\lambda) \int_x^\infty a(t)\, dt \leqslant 1$. Then for all $\lambda > \Lambda_1$, $\rho_0''(\lambda)$ exists and satisfies

$$\left| \rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}} \right| \leqslant \frac{4}{\pi\sqrt{\lambda}} \eta(\lambda) \int_0^\infty a(t)\, dt$$

so that $\rho_0''(\lambda) > 0$ for $\lambda > \Lambda_1$ and, in particular, Λ_1 is an upper bound for the points of spectral concentration of $\rho_0(\lambda)$.

The proof of the theorem involved the construction of a series representation for $\rho_0'(\lambda)$, which was valid for $0 < \Lambda_0 < \lambda$, where $\Lambda_0 \leqslant \Lambda_1$ and Λ_0, Λ_1 were computable.

In this paper, we use a similar approach to establish the following analogous result for slowly decaying potentials.

Theorem 2. *Let $q(x)$ be continuously differentiable and satisfy $q(x) \to 0$ as $x \to \infty$, $q \notin L^1[0, \infty)$. Define*

$$Q(x, \lambda) := q(x) - R' - R^2 - 2i\sqrt{\lambda}R$$

for $\operatorname{Re}\{\lambda\} > 0$, $\operatorname{Im}\{\lambda\} \geqslant 0$, where $R = R(x, \lambda)$ is chosen so that $Q(\cdot, \lambda) \in L^1[0, \infty)$, R' denotes differentiation with respect to x, and Q, R, $\frac{\partial Q}{\partial \lambda}$, $\frac{\partial R}{\partial \lambda}$ are continuous in x and λ. Suppose that there exists $M > 0$ so that

(a) *for $\operatorname{Re}\{\lambda\} \geqslant 0$, $\operatorname{Im}\{\lambda\} \geqslant 0$, $|\lambda| > M$:*

(i) *there exists* $K \in \mathbb{R}$ *so that for* $0 \leqslant x < t$

$$\operatorname{Re}\{2i\lambda^{1/2}(t-x) + 2\int_x^t R(s,\lambda)\,ds\} \leqslant K,$$

(ii) *for* $0 \leqslant x < t$

$$\left|\int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} Q(t,\lambda)\,dt\right| \leqslant a(x)\eta(\lambda),$$

where $a(x), \eta(\lambda)$ *are real valued functions with* $a(\cdot) \in L^1[0,\infty)$ *and decreasing,* $\eta(\lambda) \to 0$ *as* $|\lambda| \to \infty$ *and* $32\eta(\lambda)e^K \int_0^\infty a(t)\,dt \leqslant 1$,

(iii)
$$\left|\frac{\partial}{\partial\lambda}\int_x^t R(s,\lambda)\,ds\right| \leqslant \operatorname{const}(t-x) \quad \text{for } 0 \leqslant x < t < \infty,$$

(b) *for* $\operatorname{Re}(\lambda) > 0$, $\operatorname{Im}(\lambda) = 0$, $\lambda > M$, *there exists a decreasing function* $b(x)$ *such that for* $x \geqslant 0$,

$$e^K \int_x^\infty \left|\frac{\partial Q}{\partial\lambda}\right| + \left|\frac{i}{\sqrt{\lambda}} + 2\frac{\partial R}{\partial\lambda}\right| a(t)\eta(\lambda)\,dt \leqslant \frac{\eta(\lambda)}{\sqrt{\lambda}} b(x).$$

Then $\rho_0''(\lambda)$ *exists for* $\lambda > M$ *and satisfies*

$$\left|\rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}} - \frac{1}{\pi}\frac{\partial}{\partial\lambda}\operatorname{Im}\{R(0,\lambda)\}\right| \leqslant \frac{3}{\pi\sqrt{\lambda}}\eta(\lambda)b(0).$$

We remark that the overall purpose of both theorems is the same: to enable $\Lambda_1 > 0$ to be determined such that $\rho_0''(\lambda)$ exists and satisfies $\rho_0''(\lambda) > 0$ for $\lambda > \Lambda_1$, in which case there are no local maxima of $\rho_0'(\lambda)$ in (Λ_1, ∞) and hence no points of spectral concentration in (Λ_1, ∞). An added complexity in Theorem 2 is that consideration of the size and sign of $\operatorname{Im}\{R(0,\lambda)\}$ is needed to determine Λ_1.

We note that Theorem 2 reduces to a special case of Theorem 1 if $R(x,\lambda) \equiv 0$, $\operatorname{Im}\{\lambda\} = 0$, $K = 0$, $b(x) = \frac{4}{3}\int_x^\infty a(t)\,dt$, since in this case $Q(x,\lambda) = q(x)$ and conditions (a) (i) and (iii) are trivially satisfied. However, this special case is less general than Theorem 1, where the differentiability condition on $q(x)$ is not required.

It follows from the proof of Theorem 2 that if there exists $M < \infty$ satisfying the conditions of the theorem, then there also exists Λ_0 with $0 < \Lambda_0 \leqslant M$ so that $\rho_0'(\lambda)$ exists as a finite limit for $\lambda > \Lambda_0$, and we have the following corollary.

Corollary 1. *Let* q, Q *and* R *be as in Theorem* 2 *and suppose that* $\Lambda_0 > 0$ *exists such that for* $\operatorname{Re}\{\lambda\} > 0$, $\operatorname{Im}\{\lambda\} \geqslant 0$, $|\lambda| > \Lambda_0$, *conditions* (a) (i) *and* (ii) *of Theorem* 2 *are satisfied. Then for* $\lambda > \Lambda_0$, $\rho_0'(\lambda)$ *has an absolutely and uniformly convergent series representation, so that* $\rho_0'(\lambda)$ *exists and hence the spectrum of* (1.1) *with Dirichlet boundary condition at* $x = 0$ *is purely absolutely continuous on* (Λ_0, ∞).

Note that the Corollary and Theorem 2 provide the means to investigate and compute upper bounds Λ_0 and Λ_1, for embedded singular spectrum and points of spectral concentration, respectively. However, if no finite values of Λ_0 and Λ_1 result from application of the theorem and corollary, then it cannot be inferred that the spectrum is not eventually purely absolutely continuous, respectively, eventually free of points of spectral concentration, although this might well be the case. It is possible to obtain an improved estimate of Λ_0 in the corollary by replacing the inequality $32\eta(\lambda)\int_0^\infty a(t)\,dt \leqslant 1$ by $9\eta(\lambda)\int_0^\infty a(t)\,dt \leqslant 1$. The details are given in the proof of Theorem 2, which is contained in the following section.

3. The method

In the present context we define spectral concentration as follows.

Definition. $\lambda_c \in \mathbb{R}$ is said to be a point of spectral concentration of $\rho_\alpha(\lambda)$ if:

(i) $\rho'_\alpha(\lambda)$ exists finitely and is continuous in a neighborhood of λ_c.
(ii) $\rho'_\alpha(\lambda)$ has a local maximum at λ_c.

It follows immediately from the definition that if $\rho''_\alpha(\lambda)$ exists and has one sign for $\lambda > \Lambda_1$, then Λ_1 is an upper bound for points of spectral concentration of $\rho_\alpha(\lambda)$, and the associated spectrum is purely absolutely continuous on (Λ_1, ∞) (cf. [8, Lemma 4]).

Let θ_α and ϕ_α denote the solutions of (1.1) which satisfy the initial conditions

$$\left.\begin{array}{ll}\theta_\alpha(0,\lambda) = \cos\alpha, & \theta'_\alpha(0,\lambda) = \sin\alpha,\\ \phi_\alpha(0,\lambda) = -\sin\alpha, & \phi'_\alpha(0,\lambda) = \cos\alpha,\end{array}\right\}. \tag{3.1}$$

The hypotheses on $q(x)$ ensure that (1.1) is in the limit point case at infinity, so for $\mathrm{Im}(\lambda) > 0$, the solution $\Psi_\alpha(x,\lambda) = \theta_\alpha(x,\lambda) + m_\alpha(\lambda)\phi_\alpha(x,\lambda)$ of (1.1) belongs to $L^2[0,\infty)$ where $m_\alpha(\lambda)$ is the Titchmarsh–Weyl m-function associated with (1.1) and (1.2). Since $\Psi_\alpha(x,\lambda)$ does not vanish for $x \geqslant 0$, $\mathrm{Im}\{\lambda\} > 0$ we may set $v(x,\lambda) := \frac{\Psi'_\alpha(x,\lambda)}{\Psi_\alpha(x,\lambda)}$ and note that $v(x,\lambda)$ is independent of α and satisfies the Riccati equation

$$v' = -\lambda + q - v^2. \tag{3.2}$$

It follows that

$$\frac{\Psi'_\alpha(x,\lambda)}{\Psi_\alpha(x,\lambda)} = v(x,\lambda) = \frac{\theta'_\alpha(x,\lambda) + m_\alpha(\lambda)\phi'_\alpha(x,\lambda)}{\theta_\alpha(x,\lambda) + m_\alpha(\lambda)\phi_\alpha(x,\lambda)} \tag{3.3}$$

and in particular

$$\frac{\Psi'_\alpha(0,\lambda)}{\Psi_\alpha(0,\lambda)} = \frac{\sin\alpha + m_\alpha(\lambda)\cos\alpha}{\cos\alpha - m_\alpha(\lambda)\sin\alpha}.$$

Specializing to the case $\alpha = 0$ we see that

$$m_0(\lambda) = \frac{\Psi'_0(0,\lambda)}{\Psi_0(0,\lambda)} = v(0,\lambda). \tag{3.4}$$

We note that for $x \geqslant 0$, $\mathrm{Im}\{\lambda\} > 0$, the function $v(x,\lambda)$ satisfying (3.3) may be identified with the Dirichlet m-function associated with (1.1) on $[x,\infty)$ (see for example [7,13]).

It is known, see [5,8,15], that when the normal limit $\lim_{\varepsilon\to 0^+} m_0(\mu + i\varepsilon)$ exists, the derivative $\rho'_0(\lambda)$ of the spectral function also exists and satisfies

$$\rho'_0(\mu) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \mathrm{Im}\,\{m_0(\mu + i\varepsilon)\} = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \mathrm{Im}\,\{v(0,\mu + i\varepsilon)\}. \tag{3.5}$$

This relationship suggests that the behavior of the spectral function for large μ can be investigated by analyzing the asymptotic properties of the appropriate solutions of the Riccati equation. We therefore proceed as follows. Guided by knowledge of the asymptotic form of $v(x,\lambda)$ as defined in (3.3) for $\mathrm{Im}\{\lambda\} > 0$ (see for example [6,9,13]), we seek to determine conditions under which a series solution of (3.2) of the form

$$v(x,\lambda) = i\lambda^{1/2} + R(x,\lambda) + \sum_{n=1}^{\infty} v_n(x,\lambda) \tag{3.6}$$

exists and is continuous in λ on the region $\mathrm{Re}\{\lambda\} > 0$, $\mathrm{Im}\{\lambda\} \geqslant 0$, $|\lambda| > \Lambda_0$, where $R(x,\lambda)$ is chosen so that

$$Q(\cdot,\lambda) = q - R' - R^2 - 2i\lambda^{1/2}R \in L^1[0,\infty) \tag{3.7}$$

and $\sum_{n=1}^{\infty} v_n(x,\lambda)$ satisfies

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} v_n(\cdot,\lambda) \in L^1[0,\infty), \\ &\sum_{n=1}^{\infty} v_n(x,\lambda) \to 0 \quad \text{as } x,\ |\lambda| \to \infty. \end{aligned}\right\}. \tag{3.8}$$

Substituting (3.6) into (3.2) and rearranging yields,

$$\sum_{n=1}^{\infty} (v'_n + 2(i\lambda^{1/2} + R)v_n) = Q - v_1^2 - \sum_{n=3}^{\infty}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right) \tag{3.9}$$

which leads, in a similar way to that of [6], to the choices

$$\left.\begin{aligned} &v_1' + (2i\lambda^{1/2} + 2R)v_1 = Q,\\ &v_2' + (2i\lambda^{1/2} + 2R)v_2 = -v_1^2,\\ &v_n' + (2i\lambda^{1/2} + 2R)v_n = -v_{n-1}^2 - 2v_{n-1}\sum_{m=1}^{n-2} v_m,\\ &\quad \text{for } n = 3, 4, 5, \ldots \end{aligned}\right\} \tag{3.10}$$

and

$$\left.\begin{aligned} &v_1(x,\lambda) = -\int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} Q(t,\lambda)\,dt,\\ &v_2(x,\lambda) = \int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} v_1(t,\lambda)^2\,dt,\\ &v_n(x,\lambda) = \int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds}\left(v_{n-1}^2 + 2v_{n-1}\sum_{m=1}^{n-2} v_m\right)dt,\\ &\quad \text{for } n = 3, 4, 5, \ldots \end{aligned}\right\} \tag{3.11}$$

We suppose now the existence of a constant K so that

$$\mathrm{Re}\{2i\lambda^{1/2}(t-x) + 2\int_x^t R(s,\lambda)\,ds\} \leqslant K \quad \text{for } 0 \leqslant x < t,\ \mathrm{Im}\{\lambda\} \geqslant 0. \tag{3.12}$$

We further suppose that

$$\frac{\partial}{\partial\lambda}\int_x^t R(s,\lambda)\,ds \leqslant \mathrm{const}(t-x). \tag{3.13}$$

and that there exist functions $a(x)$ and $\eta(\lambda)$ so that

$$|v_1(x,\lambda)| \leqslant a(x)\eta(\lambda) \tag{3.14}$$

for $0 \leqslant x < \infty$, $\mathrm{Im}\{\lambda\} \geqslant 0$ where $a(\cdot)$ is a decreasing member of $L^1[0,\infty)$ and $\eta(\lambda) \to 0$ as $|\lambda| \to \infty$. We then have

Lemma 1. *If* Λ_0 *is such that for* $|\lambda| > \Lambda_0$, $9\eta(\lambda)e^k \int_0^\infty a(t)\,dt \leqslant 1$, *then for* $0 \leqslant x < \infty$ $|v_n(x,\lambda)| \leqslant \frac{a(x)\eta(\lambda)}{2^{n-1}}$ *for* $0 \leqslant x < \infty$, $|\lambda| > \Lambda_0$.

Proof. This is very similar to the proof of Lemma 2 of [6]. □

It follows from Lemma 1 and (3.10) that $\sum_{n=1}^\infty v_n(x,\lambda)$ and $\sum_{n=1}^\infty v_n'(x,\lambda)$ are uniformly, absolutely convergent for $x \geqslant 0$ and $|\lambda| > \Lambda_0$, $\mathrm{Im}\{\lambda\} \geqslant 0$. Thus, the series of (3.6) does indeed represent a solution of (3.2) on this region. In particular, we note that each of the functions of (3.11) is such that for $\lambda \in \mathbb{R}$, $\lambda > \Lambda_0$, $\lim_{\varepsilon\to 0^+} v_j(x, \lambda + i\varepsilon)$ exists. It follows then from the uniformity of the convergence that the analogous limit exists for the function $v(x,\lambda)$ in (3.6), which represents a solution of (3.2), and hence from (3.5), that $\rho_0'(\lambda)$ exists and

$$\rho_0'(\lambda) = \frac{1}{\pi}\,\mathrm{Im}\left\{ i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(0,\lambda) + R(0,\lambda) \right\} \tag{3.15}$$

is a continuous function of λ on (Λ_0, ∞). From known results relating $\rho_0'(\lambda)$ to spectral properties (see e.g. [8]), we may infer that, since the essential spectrum is $[0, \infty)$ and $\rho_0'(\lambda)$ exists as a finite limit on (Λ_0, ∞), the spectrum is purely absolutely continuous on (Λ_0, ∞), as stated in Corollary 1. A consequence of (3.15) is that we may discuss points of spectral concentration of $\rho_0(\lambda)$ directly, that is, we seek conditions under which $\omega_n(x,\lambda) := \frac{\partial}{\partial\lambda} v_n(x,\lambda)$, $n = 1, 2, 3, \ldots$ exists for $\lambda \in \mathbb{R}$, $\lambda > \Lambda_0$ and which ensure that the series $\sum_{n=1}^{\infty} \omega_n(x,\lambda)$ converges uniformly and absolutely. In order to derive bounds for the terms of this series it is helpful to construct a sequence of differential equations satisfied by the ω_n functions for $n \geqslant 2$. This approach requires the equality of the mixed second-order partial derivatives of v_n. This equality follows if we show that $\frac{\partial v_n}{\partial\lambda}, \frac{\partial v_n}{\partial x}, \frac{\partial^2 v_n}{\partial\lambda\partial x}$ are continuous in x and λ. For $n \geqslant 2$ the continuity of those partial derivatives may be proved by induction on n. The continuity of $\frac{\partial v_n}{\partial x}$ follows from (3.10) and that of $\frac{\partial^2 v_n}{\partial\lambda dx}$ from (3.10) and the induction hypothesis. It is sufficient for the continuity of $\frac{\partial v_n}{\partial\lambda}$ to show that $tv_k^2 \in L^1$ for $k = 2, 3, \ldots$, which may be proved as in [6, Proof of Lemma 3], and also to require the fact that $\frac{\partial}{\partial\lambda} \int_x^t R(s,\lambda) \leqslant \mathrm{const}(t-x)$ which is part of the hypothesis of Theorem 2. We derive a bound for ω_1 separately. Differentiation of the first equality of (3.11) gives

$$\begin{aligned} \frac{\partial v_1}{\partial\lambda} = &- \int_x^\infty \left(i\lambda^{-1/2}(t-x) + 2\int_x^t \frac{\partial R}{\partial\lambda}(s,\lambda)\,ds \right) e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} Q(t,\lambda)\,dt \\ &- \int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} \frac{\partial}{\partial\lambda} Q(t,\lambda)\,dt \\ =:& \, I_1 + I_2. \end{aligned}$$

I_1 may be rewritten as

$$\begin{aligned} &- \int_x^\infty \left(i\lambda^{-1/2} + 2\frac{\partial R}{\partial\lambda}(s,\lambda) \right) e^{2i\lambda^{1/2}(s-x)+2\int_x^s R(\tau,\lambda)\,d\tau} \\ &\times \int_s^\infty e^{2i\lambda^{1/2}(t-s)+2\int_s^t R(\tau,\lambda)\,d\tau} Q(t,\lambda)\,dt\,ds \end{aligned}$$

and

$$|I_1| \leqslant e^K \int_x^\infty \left| i\lambda^{-1/2} + 2\frac{\partial R(s,\lambda)}{\partial\lambda} \right| a(s)\eta(\lambda)\,ds.$$

It may readily be seen that

$$|I_2| \leqslant e^K \int_x^\infty \left|\frac{\partial Q(t,\lambda)}{\partial \lambda}\right| dt,$$

whence, by condition (b) of the hypothesis of Theorem 2

$$|\omega_1(x,\lambda)| \leqslant 2\lambda^{-1/2}\eta(\lambda)b(x).$$

For $n \geqslant 2$ we differentiate (3.10) with respect to λ and, using the equality of the mixed second-order derivatives, see that

$$\left.\begin{aligned} \omega_2' + [2i\lambda^{1/2} + 2R]\omega_2 &= -2v_1\omega_1 - \left(i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right)v_2, \\ \omega_n' + [2i\lambda^{1/2} + 2R]\omega_n &= -2v_{n-1}\omega_{n-1} - 2\omega_{n-1}\sum_{m=1}^{n-2} v_m \\ &\quad -2v_{n-1}\sum_{m=1}^{n-2}\omega_m - \left(i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right)v_n. \end{aligned}\right\} \qquad (3.16)$$

It follows from (3.16) that

$$\left.\begin{aligned} \omega_2(x,\lambda) &= \int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R\,ds}\left(2v_1\omega_1 + \left(i\lambda^{-1/2} + \frac{2\partial R}{\partial \lambda}\right)v_2\right)dt, \\ \omega_n(x,\lambda) &= \int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R\,ds}\left(2v_{n-1}\sum_{m=1}^{n-2}\omega_m\right. \\ &\quad \left. +2\omega_{n-1}\sum_{m=1}^{n-1} v_m + \left(i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right)v_n\right)dt \end{aligned}\right\} \qquad (3.17)$$

for $n = 3, 4, \ldots$.

Lemma 2. *If M is such that for all $\lambda > M$, $32\eta(\lambda)e^K \int_0^\infty a(t)\,dt \leqslant 1$ then for $\lambda > M$*

$$|\omega_j(x,\lambda)| \leqslant \frac{\lambda^{-1/2}\eta(\lambda)b(x)}{2^{j-2}} \quad \textit{for } 0 \leqslant x < \infty \textit{ and } j = 1, 2, 3, \ldots.$$

Proof. The result has already been shown for $j = 1$. Consider now the case $j = 2$.

$$\begin{aligned} |\omega_2(x,\lambda)| &\leqslant e^K \int_x^\infty 2|v_1|\,|\omega_1| + \left|i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right||v_2|\,dt \\ &\leqslant e^K \int_x^\infty 4a(t)b(t)\lambda^{-1/2}\eta(\lambda)^2\,dt + \frac{e^K}{2}\int_x^\infty \left|i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right| a(t)\eta(\lambda)\,dt \\ &\leqslant \lambda^{-1/2}\eta(\lambda)b(x)\left\{4e^K\eta(\lambda)\int_0^\infty a(t)\,dt + \frac{1}{2}\right\} \end{aligned}$$

and the result follows.

Suppose the result was true up to, and including, $n-1$. Then

$$
\begin{aligned}
|\omega_n(x,\lambda)| &\leqslant e^K \int_x^\infty 2|v_{n-1}| \sum_{m=1}^{n-2} |\omega_m| + 2|\omega_{n-1}| \sum_{m=1}^{n-1} |v_m|\, dt \\
&\quad + e^K \int_x^\infty \left| i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda} \right| |v_n|\, dt \\
&\leqslant e^K \lambda^{-1/2} \eta(\lambda)^2 \int_x^\infty \frac{2a(t)b(t)}{2^{n-2}} \left(1 + \sum_{m=2}^\infty \frac{1}{2^{m-2}}\right) + \frac{2a(t)b(t)}{2^{n-3}} \sum_{m=1}^\infty \frac{1}{2^{m-1}}\, dt \\
&\quad + \frac{e^K \eta(\lambda)}{2^{n-1}} \int_x^\infty \left| i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda} \right| a(t)\, dt \\
&\leqslant \frac{\lambda^{-1/2}\eta(\lambda) b(x)}{2^{n-2}} \left\{ 14 e^K \eta(\lambda) \int_0^\infty a(t)\, dt + \frac{1}{2} \right\}
\end{aligned}
$$

as required. □

We are now able to complete the proof of Theorem 2. It follows from Lemma 2 that, provided λ is sufficiently large, the series $\sum_{n=1}^\infty v_n(x,\lambda)$ may be differentiated term by term to give

$$
\frac{\partial}{\partial \lambda} \sum_{n=1}^\infty v_n(x,\lambda) = \sum_{n=1}^\infty \omega_n(x,\lambda).
$$

Thus from (3.15) and Lemma 2 we have

$$
\begin{aligned}
\rho_0'(\lambda) &= \frac{1}{\pi} \left\{ \lambda^{1/2} + \mathrm{Im}\{R(0,\lambda)\} + \sum_{n=1}^\infty \mathrm{Im}\{v_n(0,\lambda)\} \right\} \quad \text{for } \lambda \geqslant \Lambda_0, \\
\rho_0''(\lambda) &= \frac{1}{\pi} \left\{ \frac{1}{2}\lambda^{-1/2} + \frac{\partial}{\partial \lambda} \mathrm{Im}\{R(0,\lambda)\} + \sum_{n=1}^\infty \mathrm{Im}\{\omega_n(0,\lambda)\} \right\} \quad \text{for } \lambda \geqslant M \qquad (3.18)
\end{aligned}
$$

so

$$
\begin{aligned}
\left| \rho_0''(\lambda) - \frac{1}{2\pi\lambda^{1/2}} - \frac{1}{\pi}\frac{\partial}{\partial \lambda} \mathrm{Im}\{R(0,\lambda)\} \right| &\leqslant \frac{1}{\pi} \sum_{n=1}^\infty |\omega_n(0,\lambda)| \\
&\leqslant \frac{\lambda^{-1/2}}{\pi} \eta(\lambda) b(0) \left(\sum_{n=2}^\infty 2^{-n+2} + 1 \right) \\
&\leqslant \frac{3\lambda^{-1/2}}{\pi} \eta(\lambda) b(0) \quad \text{for } \lambda > M. \qquad (3.19)
\end{aligned}
$$

Depending on the behaviour of $\mathrm{Im}\{R(0,\lambda)\}$ for large λ, it is often possible to use (3.19) to compute an upper bound, Λ_1, for points of spectral concentration of $\rho_0(\lambda)$. In the next sections we illustrate the method with examples.

4. Application to slowly decaying potentials

Example 1. Let $q(x) := (1+x)^{-\alpha}$, $\frac{2}{3}<\alpha\leqslant 1$, $0\leqslant x<\infty$ and set $R(x,\lambda) := R_1 + R_2 + R_3$, where

$$R_1(x,\lambda) := \frac{-iq(x)}{2\sqrt{\lambda}}, \quad R_2(x,\lambda) := \frac{q'(x)}{4\lambda}, \quad R_3(x,\lambda) := \frac{-iq(x)^2}{8\lambda^{3/2}}.$$

Then,

$$\begin{aligned} Q(x,\lambda) &= q - R' - R^2 - 2i\lambda^{1/2}R \\ &= \frac{-q''}{4\lambda} - \frac{(q')^2}{16\lambda^2} + \frac{q^4}{64\lambda^3} + \frac{q^3}{8\lambda^2} + \frac{iq^2q'}{16\lambda^{5/2}} + \frac{iqq'}{2\lambda^{3/2}} \end{aligned} \tag{4.1}$$

so that $\int_x^\infty |Q(t,\lambda)|\,dt = O((1+x)^{(-3\alpha+1)}) \in L^1[0,\infty)$.

It is straightforward to show that

$$\mathrm{Re}\{2i\lambda^{1/2}(t-x) + 2\int_x^t R(s,\lambda)\,ds\} \leqslant 0$$

for $0\leqslant x<t$, $\mathrm{Im}(\lambda)\geqslant 0$ where we take the branch of the square root with $\lambda^{1/2} = \alpha + i\beta$, $\alpha,\beta\geqslant 0$, so that condition (a)(i) of the hypothesis of Theorem 2 is satisfied with $K=0$. We now have from (3.11) and (4.1) that

$$|v_1(x,\lambda)| \leqslant e^0 \int_x^\infty |Q(t,\lambda)|\,dt \leqslant \frac{3}{4|\lambda|}(1+x)^{1-3\alpha},$$

where we have supposed, in order to improve our final estimate for Λ_1, that $\mathrm{Re}\{\lambda\}\geqslant 9$ and $\mathrm{Im}\{\lambda\}\geqslant 0$. Then we set

$$a(x) := (1+x)^{1-3\alpha}, \quad 0\leqslant x<\infty,$$

$$\eta(\lambda) := \tfrac{3}{4}|\lambda|^{-1}, \quad \mathrm{Re}(\lambda)\geqslant 9.$$

It may readily be seen that the condition $32\eta(\lambda)e^K \int_0^\infty a(t)\,dt \leqslant 1$ is satisfied provided that $|\lambda| \geqslant 24(3\alpha-2)^{-1}$, from which it follows that condition (a)(ii) of the hypothesis holds for the same values of λ. Condition (a)(iii) also holds for these values of λ, since $R(x,\lambda)$ is bounded in x and λ for $x\geqslant 0$ and $|\lambda|\geqslant$const.>0.

We note that for $\frac{2}{3}<\alpha\leqslant 1$, Λ_0 is at least 24 so for the remaining estimates we suppose that λ is real and greater than 24.

To determine $b(x)$ satisfying the hypothesis of Theorem 2 we note first that

$$\left|\frac{\partial R}{\partial\lambda}\right| = \left|\frac{iq}{4\lambda^{3/2}} - \frac{q'}{4\lambda^2} + \frac{3iq^2}{16\lambda^{5/2}}\right| \leqslant \frac{(1+x)^{-\alpha}}{\lambda^{3/2}}\left[\frac{1}{4} + \frac{1}{4\lambda^{1/2}} + \frac{3}{16\lambda}\right] \leqslant \frac{5(1+x)^{-\alpha}}{16\lambda^{3/2}}$$

$$\left|\frac{\partial Q}{\partial\lambda}\right| = \left|\frac{q''}{4\lambda^2} + \frac{(q')^2}{8\lambda^3} - \frac{3q^4}{64\lambda^4} - \frac{q^3}{4\lambda^3} - \frac{5iq^2q'}{32\lambda^{7/2}} - \frac{3iqq'}{4\lambda^{5/2}}\right| \leqslant \frac{9}{16\lambda^2(1+x)^{3\alpha}}$$

and

$$\left|\frac{i}{\lambda^{1/2}} + 2\frac{\partial R}{\partial\lambda}\right| a(x) \leqslant \frac{17}{16}\lambda^{-1/2}(1+x)^{-3\alpha+1}.$$

It follows that

$$e^0\int_x^\infty \left|\frac{\partial Q}{\partial\lambda}\right| + \left|\frac{i}{\lambda^{1/2}} + 2\frac{\partial R}{\partial\lambda}\right| a(t)\eta(\lambda)\,dt \leqslant \int_x^\infty \frac{9(1+t)^{-3\alpha}}{16\lambda^2} + \frac{51(1+t)^{1-3\alpha}}{64\lambda^{3/2}}\,dt$$
$$\leqslant \frac{11}{8}\left(\frac{1}{3\alpha-2}\right)\lambda^{-3/2}(1+x)^{-3\alpha+2} = \lambda^{-1/2}\frac{3}{4\lambda}\frac{11}{6}\frac{1}{(3\alpha-2)}(1+x)^{-3\alpha+2}.$$

Hence, we may choose

$$b(x) := \frac{11}{6}\frac{1}{(3\alpha-2)}(1+x)^{-3\alpha+2}.$$

We have now shown that the hypothesis of Theorem 2 is satisfied with $R(x,\lambda)$ as chosen above and $M = 24(3\alpha-2)^{-1}$. Hence for $\lambda > M$, $\rho_0''(\lambda)$ exists and satisfies (3.19), so that

$$\left|\rho_0''(\lambda) - \frac{1}{2\pi\lambda^{1/2}} - \frac{1}{\pi}\left\{\frac{1}{4}\lambda^{-3/2} + \frac{3}{16}\lambda^{-5/2}\right\}\right|$$
$$\leqslant \frac{1}{\pi}\frac{3}{\lambda^{1/2}}\frac{3}{4\lambda}\frac{11}{6(3\alpha-2)} = \frac{33}{\pi 8(3\alpha-2)\lambda^{3/2}}.$$

It follows that $\rho_0''(\lambda) > 0$ if $\lambda > \max\left\{M, \frac{33}{4(3\alpha-2)}\right\} = \frac{24}{3\alpha-2}$ and so there are no points of spectral concentration for $\lambda > \Lambda_1$, with $\Lambda_1 = M = 24(3\alpha-2)^{-1}$.

Example 2. Let $q(x) := \frac{\sin((1+x)^{1/2})}{(1+x)^{1/2}}$, $0 \leqslant x < \infty$ and for $\mathrm{Re}\{\lambda\} > 0$ let $R(x,\lambda) = R_1 + R_2 + R_3$ be defined in terms of q as in Example 1. Then as before

$$Q(x,\lambda) = -\frac{q''}{4\lambda} - \frac{(q')^2}{16\lambda^2} + \frac{q^4}{64\lambda^3} + \frac{q^3}{8\lambda^2} + \frac{iq^2q'}{16\lambda^{5/2}} + \frac{iqq'}{2\lambda^{3/2}}$$

so that, in this case,

$$\int_x^\infty |Q(t,\lambda)|\,dt = O((1+x)^{-1/2}) \notin L^1[0,\infty).$$

However, we may assert that for $\mathrm{Re}\{\lambda\} \geqslant 25$

$$\mathrm{Re}\left\{2i\lambda^{1/2}(t-x) + 2\int_x^t R(s,\lambda)\,ds\right\} \leqslant \frac{4}{|\lambda|^{1/2}} + \frac{1}{|\lambda|} \leqslant \frac{21}{25} \quad \text{for } 0 \leqslant x < t < \infty$$

since for all x, t

$$\left|\int_x^t R_1(s,\lambda)\,ds\right| = \left|\frac{i}{\lambda^{1/2}}\{\cos((1+s)^{1/2})\}\Big|_x^t\right| \leqslant 2|\lambda|^{1/2},.$$

$$\left|\int_x^t R_2(s,\lambda)\,ds\right| = \left|\frac{\sin((1+s)^{1/2})}{4\lambda(1+s)^{1/2}}\Big|_x^t\right| \leqslant \frac{1}{2}|\lambda|^{-1}$$

and

$$\mathrm{Re}\left\{2i\lambda^{1/2}(t-x) + 2\int_x^t R_3(s,\lambda)\,ds\right\} \leqslant 0 \quad \text{for } \mathrm{Re}\{\lambda\} > 0,\ \mathrm{Im}\{\lambda\} \geqslant 0.$$

Thus, condition (a)(i) of the hypothesis of Theorem 2 is satisfied with $M = 25$, $K = \frac{21}{25}$. To satisfy condition (a)(ii), we need to identify suitable functions $a(x)$, $\eta(\lambda)$ such that $|v_1(x,\lambda)| \leqslant a(x)\eta(\lambda)$. Using (3.11) we first integrate by parts twice to give

$$\begin{aligned} v_1(x,\lambda) &= -\int_x^\infty e^{2i\lambda^{1/2}(t-x)+2\int_x^t R(s,\lambda)\,ds} Q(t,\lambda)\,dt \\ &= \frac{Q(x,\lambda)}{2i\lambda^{1/2}} + \left[\frac{e^{2i\lambda^{1/2}(t-x)}}{(2i\lambda^{1/2})^2}(2RQ+Q')e^{2\int_x^t R(s,\lambda)\,ds}\Big|_x^\infty\right] \\ &\quad - \int_x^\infty \frac{e^{2i\lambda^{1/2}(t-x)}}{(2i\lambda^{1/2})^2}\{2R'Q + 4RQ' + Q'' + 4R^2Q\}e^{2\int_x^t R(s,\lambda)\,ds}\,dt \\ &=: \frac{Q(x,\lambda)}{2i\lambda^{1/2}} - \frac{2R(x,\lambda)Q(x,\lambda) + Q'(x,\lambda)}{(2i\lambda^{1/2})^2} + I(x,\lambda), \end{aligned} \tag{4.2}$$

where, for $\mathrm{Re}\{\lambda\} \geqslant 25$

$$|I(x,\lambda)| \leqslant \frac{e^{\frac{21}{25}}}{4|\lambda|} \int_x^\infty |Q'' + 2R'Q + 4RQ' + 4R^2Q|\, dt \tag{4.3}$$

and

$$\left.\begin{aligned}
|R(x,\lambda)| &\leqslant \frac{3}{5|\lambda|^{1/2}(1+x)^{1/2}}, \quad |R'(x,\lambda)| \leqslant \frac{3}{5|\lambda|^{1/2}(1+x)},\\
|Q(x,\lambda)| &\leqslant \frac{3}{5|\lambda|(1+x)^{3/2}}, \quad |Q'(x,\lambda)| \leqslant \frac{7}{5|\lambda|(1+x)^2},\\
|Q''(x,\lambda)| &\leqslant \frac{26}{5|\lambda|(1+x)^{5/2}}.
\end{aligned}\right\} \tag{4.4}$$

Substitution of (4.4) into (4.3) gives

$$|I(x,\lambda)| \leqslant \frac{e^{\frac{21}{25}}}{4|\lambda|} \int_x^\infty \frac{1}{4}(1+t)^{-5/2}\, dt \leqslant \frac{1}{10}|\lambda|^{-1}(1+x)^{-3/2}$$

and hence from (4.2) we have

$$|v_1(x,\lambda)| \leqslant \tfrac{1}{5}|\lambda|^{-1}(1+x)^{-3/2}$$

so we may choose

$$a(x) := (1+x)^{-3/2}, \quad \eta(\lambda) := \tfrac{1}{5}|\lambda|^{-1}.$$

It is now straightforward to check that the inequality

$$32\eta(\lambda)e^{\frac{21}{25}} \int_0^\infty a(t)\, dt \leqslant 1$$

holds for $\mathrm{Re}\{\lambda\} \geqslant 30$ and, proceeding as in Example 1, we see that condition (a)(iii) of the hypothesis is also satisfied for such λ.

To determine $b(x)$ satisfying part (b) of the hypothesis of Theorem 2 we note that in this case with λ real and at least 25,

$$\left|\frac{\partial R}{\partial \lambda}\right| \leqslant \frac{1}{3}\lambda^{-3/2}(1+x)^{-1/2},$$
$$\left|\frac{\partial Q}{\partial \lambda}\right| \leqslant \frac{1}{2}\lambda^{-2}(1+x)^{-3/2}$$

and $\left|\dfrac{i}{\lambda^{1/2}}+2\dfrac{\partial R}{\partial\lambda}\right|a(x)\leqslant\dfrac{16}{15}\lambda^{-1/2}(1+x)^{-3/2}$ so that

$$\begin{aligned}
&e^{\frac{21}{25}}\int_x^\infty \left|\frac{\partial Q}{\partial\lambda}\right|+\left|\frac{i}{\lambda^{1/2}}+2\frac{\partial R}{\partial\lambda}\right|a(t)\eta(\lambda)\,dt\\
&\leqslant\left(\frac{3}{2}\lambda^{-2}+\frac{16}{25}\lambda^{-3/2}\right)\int_x^\infty (1+t)^{-3/2}\,dt\\
&\leqslant\frac{1}{\lambda^{1/2}}\frac{1}{5\lambda}\frac{47}{5}(1+x)^{-1/2}.
\end{aligned}$$

Hence, we may choose

$$b(x)=\frac{47}{5(1+x)^{1/2}}$$

and to satisfy all the hypotheses of Theorem 2 we take $M=\max\{25,30\}=30$. We now have that for $\lambda>30$, $\rho_0''(\lambda)$ exists and satisfies (3.19), so that

$$\begin{aligned}
&\left|\rho_0''(\lambda)-\frac{1}{2\pi}\lambda^{-1/2}-\frac{1}{\pi}\left\{\frac{\sin(1)}{4\lambda^{3/2}}+\frac{3\sin^2(1)}{16\lambda^{5/2}}\right\}\right|\\
&\leqslant\frac{1}{\pi}\frac{3}{\lambda^{1/2}}\frac{1}{5\lambda}\frac{47}{5}=\frac{141}{25\pi\lambda^{3/2}}.
\end{aligned}$$

Hence $\rho_0''(\lambda)>0$ if $\lambda>30$ and $\dfrac{1}{2\pi}\lambda^{-1/2}>\dfrac{141}{25\pi\lambda^{3/2}}$, that is if $\lambda>\max\{12,30\}=30$. Thus, we may take $\Lambda_1=M=30$ to be an upper bound for points of spectral concentration.

5. The Wigner von–Neumann Case

We suppose now that

$$q(x)=\sum_{k=-M}^{M}h_k(x)e^{2ic_kx},\tag{5.1}$$

where $c_k\in\mathbb{R}$, $c_k\neq0$, $h_k(x)\to0$ as $x\to\infty$, $h_k(\cdot)\in C^L[0,\infty)$ for $k=-M,\dots,M$, $h_k^{(L+1)}\in AC[0,\infty)$ and there exists $p(x)$ with

$$|h_k^{(j)}(x)|\leqslant p(x)^{j+1}\quad\text{for }0\leqslant x<\infty,\quad j=0,\dots,L+1,\tag{5.2}$$

where

$$xp(x)^{L+2}\in L^1[0,\infty).\tag{5.3}$$

We set $R(x,\lambda) \coloneqq \sum_{n=1}^{L+1} R_n(x,\lambda)$ and

$$\begin{aligned} Q(x,\lambda) &= q - \sum_{n=1}^{L+1} R_n' - 2i\lambda^{1/2} \sum_{n=1}^{L+1} R_n - R_1^2 - \sum_{n=2}^{L+1} \left(R_n^2 + 2R_n \sum_{l=1}^{n-1} R_l \right) \\ &= (q - R_1' - 2i\lambda^{1/2} R_1) - (R_2' + 2i\lambda^{1/2} R_2 + R_1^2) \\ &\quad - \sum_{k=3}^{l+1} \left\{ R_k' + 2i\lambda^{1/2} R_k + \left(R_k^2 + 2R_{k-1} \sum_{l=1}^{k-2} R_l \right) \right\} \\ &\quad - R_{L+1}^2 - 2R_{L+1} \sum_{l=1}^{L} R_l. \end{aligned} \tag{5.4}$$

We choose the $\{R_l\}$ so that

$$\left. \begin{array}{c} q - R_1' - 2i\lambda^{1/2} R_1 = E_1, \\ R_2' + 2i\lambda^{1/2} R_2 + R_1^2 = E_2, \\ R_k' + 2i\lambda^{1/2} R_k + \left(R_{k-1}^2 + 2R_{k-1} \sum_{l=1}^{k-2} R_l \right) = E_k, \\ \text{for } k = 3, \dots, L+1, \end{array} \right\} \tag{5.5}$$

where

$$xE_j(x,\lambda) \in L^1[0,\infty) \quad \text{for } j = 1, \dots, L+1.$$

From (5.5) we may take

$$\begin{aligned} R_1(x,\lambda) &= -e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} q(t)\,dt + e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} E_1(t,\lambda)\,dt, \\ R_2(x,\lambda) &= e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} R_1(t,\lambda)^2\,dt - e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} E_2(t,\lambda)\,dt, \end{aligned}$$

$$\begin{aligned} R_k(x,\lambda) = {} & e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} \left\{ R_{k-1}^2 + 2R_{k-1} \sum_{l=1}^{k-2} R_l \right\} dt \\ & - e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} E_k(t,\lambda)\,dt \end{aligned}$$

for $k = 3, \dots, L+1$.

We set $c_* \coloneqq \max_{k=-M,\dots,M} |c_k|$ and suppose that λ is such that $|\lambda| - 2^L c_* > 0$. From (5.1)

$$e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i\lambda^{1/2}t} q(t)\,dt = \sum_{k=-M}^{M} e^{-2i\lambda^{1/2}x} \int_x^\infty e^{2i(\lambda^{1/2}+c_k)t} h_k(t)\,dt. \tag{5.6}$$

Now, by integration by parts

$$e^{-2i\lambda^{1/2}x}\int_x^\infty e^{2i(\lambda^{1/2}+c_k)t}h_k^{(j)}(t)\,dt$$
$$= -\frac{e^{2ic_kx}h_k^{(j)}(x)}{2i(\lambda^{1/2}+c_k)} - \frac{e^{2i\lambda^{1/2}x}}{2i(\lambda^{1/2}+c_k)}\int_x^\infty e^{2i(\lambda^{1/2}+c_k)t}h_k^{(j+1)}(t)\,dt$$

so from (5.6)

$$-e^{-2i\lambda^{1/2}x}\int_x^\infty e^{2i\lambda^{1/2}t}q(t)\,dt = \sum_{k=-M}^{M}\left\{e^{2ic_kx}\sum_{l=0}^{L}\frac{(-1)^l h_k^{(l)}(x)}{[2i(\lambda^{1/2}+c_k)]^{L+1}} + \frac{(-1)^L e^{-2i\lambda^{1/2}x}}{[2i(\lambda^{1/2}+c_k)]^{L+1}}\int_x^\infty e^{2i(\lambda^{1/2}+c_k)t}k_k^{(L+1)}(t)\,dt\right\}$$

and we set

$$E_1(t,\lambda) := (-1)^{L+1}\sum_{k=-M}^{M}\frac{e^{2ic_kt}h_k^{(L+1)}(t)}{(2i[\lambda^{1/2}+c_k])^{(L+1)}}.$$

We then have

$$R_1(x,\lambda) = \sum_{k=-M}^{M}e^{2ic_kx}\sum_{l=0}^{L}\frac{(-1)^l h_k^{(l)}(x)}{[2i(\lambda^{1/2}+c_k)]^{l+1}} \tag{5.7}$$

and note from (5.2) and (5.3) that

$$|E_1(t,\lambda)| \leqslant \frac{\text{const}\, p(t)^{L+2}}{(|\lambda|^{1/2}-c_*)^{L+1}}.$$

Consider now $R_2(x,\lambda)$. From (5.7), $R_1^2(t,\lambda)$ is a sum of terms of the form

$$\frac{h_{k_1}^{(m_1)}(t)h_{k_2}^{(m_2)}(t)e^{2i(c_{k_1}+c_{k_2})t}}{(2i[\lambda^{1/2}+c_{k_1}])^{m_1+1}(2i[\lambda^{1/2}+c_{k_2}])^{m_2+1}}, \tag{5.8}$$

where $0\leqslant m_1, m_2\leqslant L$, $-M\leqslant k_1, k_2\leqslant M$. It follows from (5.8) that

$$R_2(x,\lambda) = e^{2i\lambda^{1/2}x}\sum_{m_1,m_2,k_1,k_2}e^{2i(\lambda^{1/2}+c_{k_1}+c_{k_2})t}\times\frac{h_{k_1}^{(m_1)}(t)h_{k_2}^{(m_2)}(t)\,dt}{(2i[\lambda^{1/2}+C_{k_1}])^{m_1+1}(2i[\lambda^{1/2}+C_{k_2}])^{m_2+1}}$$
$$- e^{2i\lambda^{1/2}x}\int_x^\infty e^{2i\lambda^{1/2}t}E_2(t,\lambda)\,dt, \tag{5.9}$$

where E_2 is to be chosen. If $m_1+m_2\geqslant L$ then the terms of the sum in (5.9) are bounded above by $\frac{\text{const}\, p(t)^{L+2}}{(|\lambda|^{1/2}-c_*)^{L+2}}$ and may be absorbed into $E_2(t,\lambda)$. If $m_1+m_2=$

$l \leqslant L-1$, then such terms are differentiable and successive integrations by parts give rise to a sum of integrated terms like

$$\frac{e^{2i(c_{k_1}+c_{k_2})x} h_{k_1}^{(n_1)}(x) h_{k_2}^{(n_2)}(x)}{(2i[\lambda^{1/2}+c_{k_1}+c_{k_2}])^p (2i[\lambda^{1/2}+c_{k_1}])^{m_1+1} (2i[\lambda^{1/2}+c_{k_2}])^{m_2+1}}, \tag{5.10}$$

where $n_1+n_2 \geqslant m_1+m_2$, $p \geqslant 0$. Each integration by parts increases p and n_1+n_2 by 1. Eventually we reach a point where $n_1+n_2 = L-1$ and $m_1+m_2+p = L$, where the integrated terms are less than or equal to $\frac{\text{const}\, p(t)^{L+2}}{(|\lambda|^{1/2}-2c_*)^{L+2}}$ and may be absorbed into E_2. Thus, $E_2(x,\lambda)$ and $R_2(x,\lambda)$ consist of sums of terms like (5.10) and

$$|E_2(x,\lambda)| \leqslant \frac{\text{const}\, p(x)^{L+2}}{(|\lambda|^{1/2}-2c_*)^{L+2}}.$$

Similar considerations apply to $R_3, \dots, R_{L+1}$ and $E_3, \dots, E_{L+1}$. In general, $R_j(x,\lambda)$ is a sum of products of the form

$$\frac{e^{2i(c_{k_1}+\cdots+c_{k_m})} h_{k_1}^{(n_1)}(x) \cdots h_{k_m}^{(n_m)}(x)}{(2i(\lambda^{1/2}+c_{k_1}))^{p_{1,1}} \cdots (2i(\lambda^{1/2}+c_{k_m}))^{p_{1,m}} (2i(\lambda^{1/2}+c_{k_1}+c_{k_2}))^{p_{2,1}} \cdots (2i(\lambda^{1/2}+c_{k_1}+\cdots+c_{k_m}))^{p_{m,n}}} \tag{5.11}$$

and

$$|E_j(x,\lambda)| \leqslant \frac{\text{const}\, p(x)^{L+2}}{(|\lambda|^{1/2}-2^{j-1}c_*)^{L+2}}.$$

We turn now to the requirement that there exists a constant K with

$$2\text{Re}\left\{ i\lambda^{1/2}(t-x) + \int_x^t R(s,\lambda)\, ds \right\} \leqslant K \quad \text{for } 0 \leqslant x < t < \infty.$$

Consider first the contribution to the integral of $R_1(s,\lambda)$. This is a sum of terms of the form

$$\int_x^t e^{2ic_k s} \frac{(-1)^l h_k^{(l)}(s)\, ds}{[2i(\lambda^{1/2}+c_k)]^{l+1}}, \quad \text{where } 0 \leqslant l < L.$$

Upon integration by parts this becomes

$$\frac{(-1)^l}{2ic_k[2i(\lambda^{1/2}+c_k)]^{l+1}} \left\{ e^{2ic_k t} h_k^{(l)}(t) - e^{2ic_k x} h_k^{(l)}(x) - \int_x^t e^{2ic_k s} h_k^{(l+1)}(s)\, ds \right\}.$$

The two integrated terms are bounded by (5.2) and the argument may be repeated with the non-integrated term until we are left with an integral involving $h_k^{(L+1)}(\cdot)$ which belongs to $L^1[0,\infty)$.

A similar argument applies to the $R_j(s,\lambda)$ terms for $j = 2, \dots, L+1$. The only difference is that they now involve products of the $h_k^{(m)}(\cdot)$ terms. This establishes the existence of a real constant K such that condition (a)(i) of the hypothesis of Theorem 2 is satisfied.

From (5.4) and (5.5) we have that

$$Q(x,\lambda) = \sum_{k=1}^{L+1} E_k(x,\lambda) - R_{L+1}(x,\lambda)^2 - 2R_{L+1}(x,\lambda)\sum_{l=1}^{L} R_l(x,\lambda) \qquad (5.12)$$

and from (5.2) and (5.12):

$$|Q(x,\lambda)| \leqslant \frac{cp(x)^{L+2}}{(|\lambda|^{1/2} - 2^L c_*)^{L+2}}. \qquad (5.13)$$

It follows from (5.13) that, in the notation of Theorem 2, we may take

$$\eta(\lambda) := C(|\lambda|^{1/2} - 2^L c_*)^{-(L+2)} \quad \text{and} \quad a(x) := \int_x^\infty p(t)^{L+2}\, dt$$

which have the required properties. We may now infer the following result from Corollary 1 and (3.19); note that Λ_0 is determined by the requirement that $9\eta(\lambda)e^K \int_0^\infty a(t)\, dt \leqslant 1$ for $\lambda > \Lambda_0$ (see remarks following Corollary 1).

Theorem 3. *If $q(x)$ satisfies the conditions of (5.1)–(5.3) there exists $\Lambda_0 > 0$ so that for $\lambda \in \mathbb{R}$ with $\lambda > \Lambda_0$*

$$\begin{aligned}\rho_0'(\lambda) &= \frac{1}{\pi}\{\lambda^{1/2} + \operatorname{Im}(R(0,\lambda)) + v(0,\lambda)\} \\ &= \frac{1}{\pi}\{\lambda^{1/2} + \operatorname{Im}(R(0,\lambda))\} + O\left(|\lambda|^{-\frac{(L+2)}{2}}\right).\end{aligned}$$

We look now at the question of spectral concentration. In order to use Theorem 2 we need to show that condition (a)(iii) is satisfied and that there exists a decreasing function, $b(x)$, with

$$\begin{aligned}&e^K \int_x^\infty \left|\frac{\partial Q}{\partial \lambda}\right| dt + e^K \int_x^\infty \left| i\lambda^{-1/2} + 2\frac{\partial R}{\partial \lambda}\right| a(t)\eta(\lambda)\, dt \\ &\quad = I_1 + I_2 \leqslant \lambda^{-1/2}\eta(\lambda) b(x).\end{aligned} \qquad (5.14)$$

It is clear by direct differentiation and the choice

$$b(x) := \int_x^\infty a(t)\,dt = \int_x^\infty (s-x)p(s)^{L+2}\,ds$$

that I_2 satisfies this condition. Consider now I_1. Again differentiation of the last two terms of (5.12) with respect to λ show that this part of I_1 satisfies (5.14) and we are left with the term

$$e^K \int_x^\infty \left| \sum_{k=1}^{L+1} \frac{\partial}{\partial \lambda} E_k(t,\lambda) \right| dt. \tag{5.15}$$

Each term of the sum of E_k's can be expressed as a sum of constant multiples of terms like (5.11) where $n_1 + \cdots + n_m \geqslant L$. Differentiation with respect to λ yields the required bounds. To see that condition (a)(iii) is satisfied, we note from (5.7), (5.9) and (5.11) that $\frac{\partial}{\partial \lambda} R(x,\lambda)$ is bounded in x on $[0,\infty)$ and in λ on $[C,\infty)$ for any $C>0$. We now have the following result, where Λ_1 is determined by the requirement that $32\eta(\lambda)e^K \int_0^\infty a(t)\,dt \leqslant 1$.

Theorem 4. *There exists Λ_1 such that if q satisfies* (5.1)–(5.3), *$\rho_0''(\lambda)>0$ for $\lambda \in \mathbb{R}$ with $\lambda > \Lambda_1$. In particular there are no points of spectral concentration for $\lambda > \Lambda_1$.*

Proof. By Theorem 2, there exists $M>0$ such that $\rho_0''(\lambda)$ exists and satisfies

$$\left| \rho_0''(\lambda) - \frac{1}{2\pi\sqrt{\lambda}} - \frac{1}{\pi}\mathrm{Im}\{R(0,\lambda)\} \right| \leqslant \frac{3}{\pi\sqrt{\lambda}} \eta(\lambda) b(0)$$

for $\lambda > M$. Since $R(0,\lambda) = O(\lambda^{-1/2})$ and $\eta(\lambda) \to 0$ as $\lambda \to \infty$, there exists $\Lambda_1 \geqslant M$ such that $\rho_0''(\lambda)$ has the required properties for $\lambda > \Lambda_1$. □

Corollary 2. *There exists $M \in \mathbb{R}$ such that $\Lambda_0 < M \leqslant \Lambda_1$ and for which*

(i) *$\rho_0'(\lambda)$ exists and is an absolutely continuous function on (M,∞),*
(ii) *$\rho_0'(\lambda)$ is strictly positive and strictly increasing on (M,∞).*

Note that Corollary 2 is a strengthening for large λ of what is already known in the Wigner–von Neumann case, viz, that the spectrum is purely absolutely continuous on $(0,\infty)$ outside a set of resonances (see [1]).

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