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HOLOMORPHIC BASIS FOR FAMILIES OF SUBSPACES OF A BANACH SPACE

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Abstract. In this article we investigate the connection between a family of complemented subspaces of a Banach space having a holomorphic basis, and being holomorphically complemented.
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1. Introduction

Perhaps the most fundamental of all selection results is the Axiom of choice: given any collection of non-empty sets $\{X_\alpha\}_{\alpha\in A}$ it is possible to "choose" an element x_α from each X_α. The choice of an element is realised by a "choice function". Additional structure on the indexing set A or on the range spaces X_α allows to refine the problem of "selection". The additional structure on the range spaces can be given by assuming they are, for example, rings, spaces of linear operators, or Banach algebras. Additional structure on the domain can be given by assuming A is an open subset of a topological or even a complex Banach space. This assumption allows us to ask for the choice function to be continuous or holomorphic. The problem now changes from set-theoretic to analytic. The solution typically consists of two stages. The first is finding a local solution about each point; the second is "patching" these local solutions to obtain a global one. Until recently, all selection problems have assumed that the domain is finite-dimensional, and somethimes that the range is finite-dimensional as well. Typically, the choice function has values in an operator space and the question has often been considered in the context of invertibility properties of the operators,e.g. ([1, 2, 19]). The recent work of Lempert and Patyi however has allowed to extend such results to infinite-dimensional domains, as in [5, 6, 7].

In this paper, we concentrate on a different, although related (see [7]) problem - the case when our operators are projections. We introduce holomorphic Schauder basis and study the relationship between families of subspaces with such basis and holomorphically complemented families of subspaces. The case when Ω is a domain in a finite dimensional space was studied by Shubin in [15]. Saphar ([14]) and Bart ([2]), on the other hand, considered finite holomorphic bases over a domain in $\mathbb{C}$. Here we consider the non-trivial generalizations to both infinite-dimensional domains and infinite holomorphic bases.

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2. Notation and definitions

Let X and Y be Banach spaces over $\mathbb{C}$, $\mathcal{L}(X,Y)$ will denote the space of continuous linear mappings from the Banach space X into the Banach space Y, $GL(X,Y)$ will denote the set of all invertible linear operators from X to Y. We let $\mathcal{H}(\Omega;X)$ denote the set of all X-valued holomorphic mappings defined on an open subset Ω of a Banach space. We use the standard notation $X' := \mathcal{L}(X,\mathbb{C})$ and $GL(X) := GL(X,X)$.

We remind the reader that a sequence $(x_n)_n$ in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_n$ so that $x = \sum_{n=1}^{\infty} a_n x_n$. A sequence $(x_n)_n$ which is a Schauder basis of its closed linear span is called a *basic sequence*. Two bases, $(x_n)_n$ for a Banach space X and $(y_n)_n$ for a Banach space Y, are *equivalent* if there exists an isomorphism $T : X \to Y$ such that $T(x_n) = y_n$ for all $n \in \mathbb{N}$.

If $(x_n)_n$ is a Schauder basis for the Banach space X, the bounded linear functionals

$$x_n^*\Big(\sum_{n=1}^{\infty} a_n x_n\Big) = a_n$$

for all n, are called the *biorthogonal functionals* associated to this basis. If for every $x^* \in X'$ the norm of $x^*|_{[x_i]_n^\infty}$, the restriction of x^* to the span of $(x_i)_{i=n}^{\infty}$, tends to zero as $n \to \infty$, then $(x_n)_n$ is called a *shrinking basis*. The biorthogonal functionals $(x_n^*)_n$ form a basis of X' if and only if the basis $(x_n)_n$ is shrinking.

When $(x_n)_n$ is a basic sequence we define $(x_n^*)_n$ by using the relation $x_n^*(x_m) = \delta_{nm}$ and extending by linearity and continuity to all of X (Definition 1.f.1, [11]).

We refer to [12] for background information on operators between Banach spaces, to [4, 13] for the theory of holomorphic mappings on Banach spaces and to [3, 11] for information on Schauder bases and basic sequences.

In the reminder of this section we recall the definition of holomorphic Banach vector bundles and of their sub-bundles, and some of their properties.

Definition 2.1. *Let $\pi : \mathcal{E} \to \Omega$ be a surjective holomorphic map of complex Banach manifolds. We assume that the fibre above $z \in \Omega, \mathcal{E}_z := \pi^{-1}(z)$, has been given a Banach space structure whose topology coincides with the topology induced from $\mathcal{E}$. A collection $(U_\alpha, \tau_\alpha)_{\alpha\in\Gamma}$ is called a trivializing cover for π if $(U_\alpha)_{\alpha\in\Gamma}$ is an open cover of Ω and for each $\alpha \in \Gamma$ there is a Banach space X_α such that $\tau_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times X_\alpha$ is a biholomorphic mapping and conditions (1), (2) and (3) below are satisfied.*

(1) *$\tau_{\alpha,z} := \tau_\alpha|_{\mathcal{E}_z}$ is a linear isomorphism from $\mathcal{E}_z$ onto X_α, modulo identifying $\{z\} \times X_\alpha$ and X_α, for each $z \in U_\alpha$.*
(2) *$\pi = \pi_\alpha \circ \tau_\alpha$, where π_α is the canonical projection from $U_\alpha \times X_\alpha$ onto U_α. Conditions (1) and (2) imply that $\rho_{\alpha\beta} := \tau_\alpha \circ \tau_\beta^{-1}|_{U_\alpha \cap U_\beta}$ has the form $\rho_{\alpha\beta}(z,x) = (z, g_{\alpha\beta}(z)x)$ where $g_{\alpha\beta}(z) \in \mathcal{L}(X_\beta, X_\alpha)$ whenever $\alpha, \beta \in \Gamma$ and $z \in U_\alpha \cap U_\beta \neq \emptyset$.*
(3) *If $\alpha, \beta \in \Gamma$ and $U_\alpha \cap U_\beta \neq \emptyset$ then the map $z \mapsto g_{\alpha\beta}(z)$ from $U_\alpha \cap U_\beta$ into $\mathcal{L}(X_\beta, X_\alpha)$ is holomorphic.*

Two trivializing covers are said to be *equivalent* if their union is also a trivializing cover.

Definition 2.2. *A holomorphic Banach vector bundle is a triple* $(\mathcal{E}, \pi, \Omega)$*, where* $\pi : \mathcal{E} \longrightarrow \Omega$ *is a surjective holomorphic map of complex Banach manifolds, together with a class of equivalent trivializing covers.*

The bundle structure is specified by any one trivializing cover. If Ω is connected, then all the X_U are isomorphic to a common Banach space X, called the *fibre type* of the bundle. We call $\mathcal{E}$ the *bundle space*, π the *projection* of the bundle, Ω the *base* of the bundle, τ_α a *trivialization* of $\pi^{-1}(U_\alpha)$ and $g_{\alpha\beta}$ a *transition map.* If X is a Banach space and Ω is a complex manifold, the triple $(\Omega \times X, \pi, \Omega)$, where π is the canonical projection from $\Omega \times X$ onto Ω, together with the trivializing covers equivalent to the trivializing cover $\{(\Omega, \pi)\}$, is called the *trivial bundle.* For convenience, we often write $\mathcal{E}$ in place of $(\mathcal{E}, \pi, \Omega)$.

A *holomorphic section* of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ is a holomorphic mapping $f : \Omega \longrightarrow \mathcal{E}$ such that $\pi \circ f = \mathbf{1}_\Omega$. The set of all holomorphic sections is denoted by $\mathcal{H}(\Omega; \mathcal{E})$. When $(\mathcal{E}, \pi, \Omega)$ is the trivial bundle $(\Omega \times X, \pi, X)$ we write $\mathcal{H}(\Omega; X)$ in place of $\mathcal{H}(\Omega; \mathcal{E})$. Under the restriction maps, the collections $\mathcal{H}(U; \mathcal{E})$, $U \subset \Omega$ open, make up a sheaf $\mathcal{O}^{\mathcal{E}}$ over Ω.

An *endomorphism* of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ is a holomorphic mapping $f : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ f = \pi$, $f_z := f|_{\mathcal{E}_z}$ is a continuous linear mapping for all $z \in \Omega$, and the mapping

$$z \in U \longrightarrow \tau_z \circ f_z \circ \tau_z^{-1} \in \mathcal{L}(X)$$

is holomorphic for any trivialising map $\tau : \pi^{-1}(U) \to U \times X$. We denote by $\mathcal{M}(\mathcal{E})$ the set of all endomorphisms of $\mathcal{E}$. If $f_z^2 = f_z$ for all $z \in \Omega$ we call f a *projection.*

A *sub-bundle* of $(\mathcal{E}, \pi, \Omega)$ is a bundle $(\mathcal{E}', \pi', \Omega)$ where $\mathcal{E}'$ is a subset of $\mathcal{E}$, $\pi' = \pi|_{\mathcal{E}'}$, $\mathcal{E}'_z$ is a closed subspace of $\mathcal{E}_z$ for all $z \in \Omega$ and the following condition holds:

for each z in Ω there exists an open neighbourhood U of z in Ω, a subspace Y_U of X_U and trivializations $\tau : \pi^{-1}(U) \longrightarrow U \times X_U$ and $\sigma : (\pi')^{-1}(U) \longrightarrow U \times Y_U$ such that

$$\tau_z \circ (\sigma^{-1})_z = Id_{U \times Y_U}.$$

A sub-bundle $(\mathcal{E}', \pi', \Omega)$ is *direct* if its fibres are complemented subspaces of the corresponding fibres of $(\mathcal{E}, \pi, \Omega)$. We say that there is a *projection* from $(\mathcal{E}, \pi, \Omega)$ onto $(\mathcal{E}', \pi', \Omega)$ if there is an endomorphism of $\mathcal{E}$ which on each fibre is a continuous projection onto the corresponding fibre of $\mathcal{E}'$.

In [7] Dineen and the second author proved the following proposition:

Proposition 2.3. *Let* Ω *be a pseudo-convex open subset of a Banach space with an unconditional basis and* $(\mathcal{E}, \pi, \Omega)$ *be a holomorphic Banach vector bundle over* Ω*. If* $(\mathcal{F}, \pi', \Omega)$ *is sub-bundle of the holomorphic vector bundle* $(\mathcal{E}, \pi, \Omega)$*, then* $(\mathcal{F}, \pi', \Omega)$ *is a direct sub-bundle if and only if there exists a holomorphic projection* $p \in \mathcal{M}(\mathcal{E})$ *such that* $p(\mathcal{E}) = \mathcal{F}$*.*

This result relied upon the following important theorem of Lempert ([9, 10]):

Theorem 2.4. *Let* Z *be a Banach space with a Schauder basis,* $\Omega \subset Z$ *pseudo-convex open,* $\mathcal{E} \to \Omega$ *a holomorphic Banach vector bundle. If plurisubharmonic domination holds in every pseudo-convex open subset of* Ω*, then the sheaf coh† omogy groups* $H^q(\Omega, \mathcal{O}^{\mathcal{E}})$ *vanish for all* $q \geq 1$*.*

Theorem 2.4 implies the solvability of the *additive Cousin problem.*
We will not go into details about plurisubharmonic domination - let us just say that in his recent paper [18] Patyi showed that plurisubharmonic domination holds on a pseudo-convex open set Ω of a space with a Schauder (not necessarily unconditional) basis, and on a convex set Ω in a separable space. For the rest of this article we will assume the former case, i.e. that Ω is a pseudo-convex open subset of a space with a Schauder basis, but it is worth remembering that the same results will hold when Ω is a convex open subset of a separable space.

Let $\{M(z)\}_{z\in\Omega}$ be a family of complemented subspaces of E. If there exists $P \in \mathcal{H}(\Omega, \mathcal{L}(E))$ such that $P(z)$ is a projection mapping of E onto $M(z)$ for all $z \in \Omega$, we will call $\{M(z)\}_{z\in\Omega}$ a *holomorphically complemented family* of subspaces of E. This definition, together with Patyi's result, allows us to re-state and generalize Proposition 2.3 in the following form:

Proposition 2.5. *Let Ω be a pseudo-convex open subset of a Banach space with a Schauder basis and E be a Banach space. Suppose $\{M(z)\}_{z\in\Omega}$ is a family of complemented subspaces of E. The following are equivalent:*

(1) $\{M(z)\}_{z\in\Omega}$ *a holomorphically complemented family of subspaces of E.*
(2) $(z \in \Omega, M(z))$ *is a direct holomorphic sub-bundle of $\Omega \times E$.*

A *multiplicative Cousin data* for $(U_\alpha)_{\alpha\in\Gamma}$, an open covering of Ω, is a collection of functions $(f_{\alpha\beta})_{\alpha,\beta\in\Gamma} \subset \mathcal{H}(U_{\alpha\beta}, \mathcal{GL}(E))$ on $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, satisfying

$$f_{\alpha\beta} \circ f_{\beta\alpha} = 1$$

on $U_{\alpha\beta}$, and

$$f_{\alpha\beta} \circ f_{\beta\gamma} \circ f_{\gamma\alpha} = 1$$

on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ whenever $U_{\alpha\beta\gamma} \neq \varnothing$.
The *multiplicative Cousin problem* consists in finding a collection of holomorphic mappings $(f_\alpha)_{\alpha\in\Gamma} \subset \mathcal{H}(U_\alpha), \mathcal{GL}(E))$ such that

$$f_\alpha|_{U_{\alpha\beta}} \circ f_\beta^{-1}|_{U_{\alpha\beta}} = f_{\alpha\beta}$$

whenever $U_{\alpha\beta} \neq \varnothing$.
The following Theorem ([17, 16]) shows the multiplicative Cousin problem is solvable on certain domains:

Theorem 2.6. *Let Z be a Banach space with a Schauder basis, $\Omega \subset Z$ be pseudo-convex and open. If plurisubharmonic domination holds in every pseudo-convex open subset of Ω, then for any Banach space E any multiplicative Cousin problem for $\mathcal{O}^{GL(E)}$ is solvable over Ω as soon as it is continuously solvable.*

In particular, since for contractible (i.e. homotopically equivalent to a point) set Ω the bundle $\mathcal{O}^{GL(E)}$ is continuously trivial, under the constraints of Theorem 2.6 it will be holomorphically trivial.

3. Holomorphic bases

Definition 3.1. *Let E and X be Banach spaces and let Ω be an open subspace of X. The sequence $\big(x_n\big)_{n\in\mathbb{N}}$ with $x_n \in \mathcal{H}(\Omega, E)$ for all $n \in \mathbb{N}$ is said to form a* holomorphic basis *(resp.* holomorphic basic sequence*) for E if the following two conditions are satisfied:*

(1) $\big(x_n(z)\big)_{n\in\mathbb{N}}$ *is a Schauder basis (resp. basic sequence) for* E *for every* $z \in \Omega$*;*
(2) *for every* $z_0 \in \Omega$ *there exist a neighbourhood* V_0 *of* z_0 *and continuous mappings* $l_0 : V_0 \to \mathbb{R}^+$ *and* $L_0 : V_0 \to \mathbb{R}^+$ *such that*

$$l_0(z)\Big\|\sum_{n=1}^{N} a_n x_n(z_0)\Big\| \le \Big\|\sum_{n=1}^{N} a_n x_n(z)\Big\| \le L_0(z)\Big\|\sum_{n=1}^{N} a_n x_n(z_0)\Big\|$$

for all $\big(a_n\big)_{n\in\mathbb{N}}$*, all* $N \in \mathbb{N}$*, and all* $z \in V_0$*.*

If $\big(x_n\big)_{n\in\mathbb{N}}$ is a holomorphic basis (basic sequence) than the bases (resp. basic sequences) $\big(x_n(z_0)\big)_{n\in\mathbb{N}}$ and $\big(x_n(z)\big)_{n\in\mathbb{N}}$ are equivalent for all $z \in V_0$. Note that if a Banach space has a Schauder basis, then it has infinitely many non-equivalent bases (see [11]), thus allowing us to ask whether we can 'select' in such a way that we obtain a holomorphic basis over Ω.

Example 3.2.

Let E be a subspace of a Banach space F. Suppose E a Schauder basis $(e_n)_n$ and let Ω be an open subset of E. Let $f \in \mathcal{H}(\Omega, E)$ is such that for each z in Ω the derivative of f at z, $\hat{d}f(z)$, is an invertiable linear mapping from E into E. For each n in $\mathbb{N}$ let $x_n(z) = \hat{d}f(z)e_n$. Then for each z in Ω we have that $(x_n(z))_n$ is a Schauder basis for E. Let z_0 be a point of Ω. Since the function $\hat{d}f\colon z \to \hat{d}f(z)$ is holomorphic we can choose a neighbourhood V_0 of z_0 so that $\|\hat{d}f(z) - \hat{d}f(z_0)\| < \dfrac{1}{\|\hat{d}f(z_0)^{-1}\|}$ for all z in V_0. Then for z in V_0 we have

$$\hat{d}f(z) = \hat{d}f(z_0) + (\hat{d}f(z) - \hat{d}f(z_0)) = \hat{d}f(z_0)\Big(I + \hat{d}f(z_0)^{-1}(\hat{d}f(z) - \hat{d}f(z_0))\Big).$$

Hence for for each z in V_0, each sequence of complex numbers $(a_n)_n$ and each $N \in \mathbb{N}$ we have that

$$l_0(z)\left\|\sum_{n=1}^{N} a_n x_n^*(z_0)\right\| \le \left\|\sum_{n=1}^{N} a_n x_n^*(z)\right\| \le L_0(z)\left\|\sum_{n=1}^{N} a_n x_n^*(z_0)\right\|$$

where

$$l_0(z) = \frac{1}{\|(I + \hat{d}f(z_0)^{-1}(\hat{d}f(z) - \hat{d}f(z_0)))^{-1}\|}$$

and

$$L_0(z) = \left\|I + \hat{d}f(z_0)^{-1}\Big(\hat{d}f(z) - \hat{d}f(z_0)\Big)\right\|.$$

Thus we have that $(x_n(z))_n$ is a holomorphic basis for E over Ω. Regarding x_n as a holomorphic function from Ω into F we get that $(x_n(z))_n$ is a holomorphic basic sequence for F over Ω.

In particular, if $f\colon \Omega \to f(\Omega)$ is bi-holomorphic then $(x_n(z))_n$ is a holomorphic basic sequence for F over Ω.

We will need the following lemma, proven in [6]:

Lemma 3.3. *If* P *and* P' *are projections in* $\mathcal{L}(X)$ *and* $\|P - P'\| < 1$ *then* $(\mathbf{1}_X - P + P') \in GL(X)$ *and* $(\mathbf{1}_X - P + P')(P(X)) = P'(X)$*. In particular,* $P(X) \simeq P'(X)$*.*

Proposition 3.4. *Let Ω be a contractible pseudo-convex open subset of a Banach space with a Schauder basis and E be a Banach space. Suppose $\{M(z)\}_{z\in\Omega}$ is a holomorphically complemented family of subspaces of E and that for some $z_0 \in \Omega$ the space $M(z_0)$ has a Schauder basis. Then $\{M(z)\}_{z\in\Omega}$ has a holomorphic basis.*

Proof. Let $P \in \mathcal{H}(\Omega, \mathcal{L}(E))$ be a projection onto $\{M(z)\}_{z\in\Omega}$ and z_α be fixed in Ω. Let V_α be a neighbourhood of z_α such that $\|P(z_\alpha) - P(z)\| < 1$ when $z \in V_\alpha$. By Lemma 3.3, $(\mathbf{1}_E - P(z_\alpha) + P(z)) \in GL(E)$ and $(\mathbf{1}_E - P(z_\alpha) + P(z))(M(z_\alpha)) = M(z)$ for all $z \in V_\alpha$. In this way we obtain an open cover $\Gamma := \{V_\alpha\}_{z_\alpha\in\Omega}$ for Ω. Let $Q_\alpha(z) := \mathbf{1}_E - P(z_\alpha) + P(z)$. As Ω is connected and open it is path-connected, hence there exists a continuous path in Ω connecting z_0 and z_α. This path is a compact set, hence it can be covered by a finite number of sets $\{V_i\}_{i=0}^k \subset \Gamma$. For simplicity of notation let V_0 be the set in Γ corresponding to z_0, $V_\alpha = V_k$, and $V_i \cap V_{i+1} \neq \varnothing$ for $i = 0 \ldots, k-1$. Let $(x_n^0)_{n=1}^\infty$ denote the Schauder basis of $M(z_0)$. If $\bar{z}_0 \in V_0 \cap V_1$, then

$$x_n^0(\bar{z}_0) = (\mathbf{1}_E - P(z_0) + P(\bar{z}_0))(x_n^0) = Q_0(z)(x_n^0)$$

is a basis for $M(\bar{z}_0)$. Since $\bar{z}_0 \in V_1$, the mapping $Q_1^{-1}(\bar{z}_0) = (\mathbf{1}_E - P(z_1) + P(\bar{z}_0))^{-1}$ is well defined, and

$$x_n^1(z_1) = Q_1^{-1}(\bar{z}_0)(x_n^0(\bar{z}_0))$$

is a basis for $M(z_1)$. Moreover,

$$x_n^1(z) = Q_1(z)x_n^1(z_1) = Q_1(z)Q_1^{-1}(\bar{z}_0)(x_n^0(\bar{z}_0))$$

will form holomorphic bases for $\{M(z)\}_{z\in V_1}$.
Next we choose $\bar{z}_1 \in V_1 \cap V_2$, and by repeating the steps above we obtain

$$x_n^2(z) = Q_2(z)Q_2^{-1}(\bar{z}_1)(x_n^1(\bar{z}_1)),$$

a holomorphic basis for $\{M(z)\}_{z\in V_2}$. Let at each step $A_i := Q_i^{-1}(\bar{z}_i)Q_{i-1}(\bar{z}_i)$ where $i = 1, \ldots, k$. After a finite number of steps we will get

$$x_n^k := A_k \ldots A_1(x_n^0),$$

and

$$x_n^k(z) := Q_k(z)A_k \ldots A_1(x_n^0)$$

is a holomorphic basis for $\{M(z)\}_{z\in V_k}$. Clearly the bases $(x_i(z))_{z\in V_i}$ are equivalent for all $i = 0 \ldots, k$.
Suppose V_α and V_β belong to Γ and $V_\alpha \cap V_\beta \neq \varnothing$. Let $z \in V_\alpha \cap V_\beta$, then as before we can construct $A_\alpha \in GL(E)$ such that $x_n^\alpha(z) = Q_\alpha(z)A_\alpha(x_n^0)$ and $A_\beta \in GL(E)$ such that $x_n^\beta(z) = Q_\beta(z)A_\beta(x_n^0)$. The mapping $A_\beta A_\alpha^{-1}$ is a linear isomorphism mapping x_n^α onto x_n^β for all n. Then

$$x_n^\beta(z) = Q_\beta(z)A_\beta A_\alpha^{-1}\big(Q_\alpha(z)\big)^{-1}(x_n^\alpha(z))$$

for all n. Let $T_{\beta\alpha}(z) := Q_\beta(z)A_\beta A_\alpha^{-1}\big(Q_\alpha(z)\big)^{-1}$, then $T_{\beta\alpha}$ is holomorphic on $V_\alpha \cap V_\beta$ and $T_{\beta\alpha}(z) \in GL(M(z))$ for every $z \in V_\alpha \cap V_\beta$. Let $V_\alpha \cap V_\beta \cap V_\gamma \neq \varnothing$, then if $z \in V_\alpha \cap V_\beta \cap V_\gamma$ we have

$$T_{\alpha\beta}(z) \circ T_{\beta\gamma}(z) \circ T_{\gamma\alpha}(z)(x_n^\alpha(z)) = x_n^\alpha(z)$$

for all n. Hence $T_{\alpha\beta}(z) \circ T_{\beta\gamma}(z) \circ T_{\gamma\alpha}(z) = \mathbf{1}_{M(z)}$, so we can consider a holomorphic vector bundle $\mathcal{S}$ with base Ω, open cover $\{V_\alpha\}_{\alpha\in\Gamma}$, fibre $M(z)$ and transition mappings $\{T_{\alpha\beta}\}$. Clearly the mappings $\{T_{\alpha\beta}\}$ form a (multiplicative) Cousin data for $\{V_\alpha\}_{\alpha\in\Gamma}$. By Theorem 2.6 the multiplicative Cousin problem is solvable over Ω.

The solution $\left(x_n^\alpha\right)_{n\in N}$ gives us the desired holomorphic basis. Indeed, part (1) of Definition 3.1 is clearly satisfied. To show that part (2) of Definition 3.1 is satisfied, take a fixed z_α in Ω, on the neighbourhood V_α the bounded and invertible linear operator $Q_\alpha(z)$ maps $(x_n(z_\alpha))_n$ to $(x_n(z))_n$. Hence for each z in V_α, each sequence of complex numbers $(a_n)_n$ and each N in $\mathbb{N}$ we have that

$$\frac{1}{\|Q_\alpha(z)^{-1}\|}\left\|\sum_{n=1}^{N} a_n x_n(z_\alpha)\right\| \leq \left\|\sum_{n=1}^{N} a_n x_n(z)\right\| \leq \|Q_\alpha(z)\|\left\|\sum_{n=1}^{N} a_n x_n(z_\alpha)\right\|$$

on V_α. □

4. Applications and properties of holomorphic bases

The following proposition is a partial converse to Proposition 3.4:

Proposition 4.1. *Let Ω be a pseudo-convex open subset of a Banach space with a basis and E be a Banach space. Suppose $\left(x_n\right)_{n\in\mathbb{N}}$ is a holomorphic basic sequence such that the closed linear span of $\left(x_n(z)\right)_{n\in\mathbb{N}}$, $M(z)$, is a complemented subspace of E for every $z\in\Omega$. Then $\{M(z)\}_{z\in\Omega}$ is holomorphically complemented in E.*

Proof. Suppose $z_0\in\Omega$ is fixed, and let P_0 denote a continuous projection from E onto $M(z_0)$. Let $x\in E$. By part (2) of Definition 3.1 there exist a neighbourhood V_0 of z_0 and a continuous mapping $L_0 : V_0 \to \mathbb{R}^+$ such that

$$\Big\|\sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z)\Big\| \leq L_0(z)\Big\|\sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z_0)\Big\|$$

for all $z\in V_0$. Hence

$$\begin{aligned}
&\Big\|x - P_0x + \sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z)\Big\| \\
\leq\ & \|1-P_0\|\|x\| + \Big\|\sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z)\Big\| \\
\leq\ & \|1-P_0\|\|x\| + L_0(z)\Big\|\sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z_0)\Big\| \\
=\ & \|1-P_0\|\|x\| + L_0(z)\|P_0\|\|x\|.
\end{aligned}$$

Thus the mapping defined by

$$A(z)x = x - P_0x + \sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z),$$

is continuous on V_0. For each $k\in\mathbb{N}$, the function $\sum_{n=1}^{k}[x_n^*(z_0)(P_0x)]x_n(z_0)$ is holomorphic. Since $\sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z_0)$ converges uniformly on V_0, it is the uniform limit of a series of holomorphic functions, hence $A\in\mathcal{H}(\Omega,\mathcal{L}(E))$. Moreover,

$$A(z_0)x = x - P_0x + \sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x)]x_n(z_0)] = x,$$

i.e. $A(z_0) = I_E$. Hence there exists a neighbourhood of z_0 such that $A \in \mathcal{H}(V_0, GL(E))$. Without loss of generality we will assume this neighbourhood is V_0. Clearly,

$$A(z)x_n(z_0) = \sum_{n=1}^{\infty}[x_n^*(z_0)(P_0x_n(z_0))]x_n(z) = x_n(z).$$

The mapping $Q(z) = A(z)P_0A^{-1}(z)$ is a projection onto M_z and $Q \in \mathcal{H}(V_0, \mathcal{L}(E))$. The mappings

$$V_0 \times E \to V_0 \times E, (z_0, x) \to (z_0, A^{-1}(z)x)$$

provide a trivialization so that $V_0 \times M_z$ is a direct holomorphic sub-bundle of the trivial bundle $V_0 \times E$. By Proposition 2.5, the family of subspaces $\{M(z)\}_{z\in\Omega}$ is holomorphically complemented in E. □

Lemma 4.2. *Let Ω be an open subset of a Banach space X, E and F be Banach spaces. Suppose that $f \in \mathcal{H}(\Omega, \mathcal{L}(E, F))$ is holomorphic. Then f^t given by $f^t(z) = f(z)^t$ for z in Ω, is holomorphic.*

Proof. Take z_0 in Ω. Then we can find a neighbourhood V_0 of z_0 and $M > 0$ such that $\|f(z)\| < M$ for z in V_0. Then we have that $\|f^t(z)\| = \|f(z)^t\| = \|f(z)\| < M$ for all z in V_0 and thus we have that f^t is locally bounded. Given x in E and φ in F' we have that

$$\langle \varphi \otimes x, f^t(z)\rangle = \left(f^t(z)\varphi\right)(x) = \varphi(f(z)x).$$

Hence the function $z \to \langle \varphi \otimes x, f^t(z)\rangle$ if holomorphic for all x in E and φ in F'. As $\{\varphi \otimes x : x \in E, \varphi \in F'\}$ is a separating subset for $\mathcal{L}(E, F)$, Theorem 3 of [8] implies that f^t is holomorphic. □

Proposition 4.3. *Let Ω be a connected pseudo-convex open subset of a Banach space with a basis and E be a Banach space with holomorphic basic sequence $(x_n)_n$ on Ω such that the closed linear span of $(x_n(z))_n$, M_z, is a complemented subspace of E for every z in Ω. Then the associated biorthogonal functionals x_n^* belong to $\mathcal{H}(\Omega, E')$ for all n. Moreover, if there is z_s in Ω such that $(x_n(z_s))_n$ is a shrinking basis for M_{z_s}, then $(x_n^*)_n$ is a holomorphic basis sequence on Ω.*

Proof. Fix z_0 in Ω and let P_0 be a continuous projection from E onto M_{z_0}. For x in E let

$$A(z)x = x - P_0x + \sum_{n=1}^{\infty} x_n^*(z_0)[(P_0x)]x_n(z).$$

Then as shown in Proposition 4.1 A is bounded, continuous and invertible on some neighbourhood V_0 of z_0. Moreover, we have that $A(z)x_n(z_0) = x_n(z)$ for all n in $\mathbb{N}$. Let $B(z) = (A(z)^{-1})^t$ for z in V_0. It follows from Lemma 4.2 that B is analytic on V_0. Also, if z belongs to V_0 then for all n, m in $\mathbb{N}$

$$\begin{aligned}\langle x_n(z), B(z)x_m^*(z_0)\rangle =&\langle A(z)x_n(z_0), (A(z)^{-1})^t x_m^*(z_0)\rangle\\ =&\langle A(z)^{-1}A(z)x_n(z_0), x_m^*(z_0)\rangle\\ =&\langle x_n(z_0), x_m^*(z_0)\rangle\\ =&\delta_{nm}\end{aligned}$$

proving that $x_m^*(z) = B(z)x_m^*(z_0)$. It follows that x_m^* is holomorphic on a neighbourhood of z_0, and hence on Ω.

Suppose there is z_s in Ω such that $(x_n(z_s))_n$ is a shrinking basis for M_{z_s}, and let P_s be a continuous projection from E onto M_{z_s}. We have already shown that each of the biorthogonal functionals x_n^* belongs to $\mathcal{H}(\Omega, E')$ and there is a neighbourhood V_s of z_s so that for each z in V_s,

$$A(z)x = x - P_s x + \sum_{n=1}^{\infty} x_n^*(z_s)[(P_s x)]x_n(z)$$

is bounded and invertible linear operator which maps $x_n(z_s)$ to $x_n(z)$ and hence M_{z_s} onto M_z. In addition we have that $B(z) := (A(z)^{-1})^t$ maps $x_n^*(z_s)$ to $x_n^*(z)$ for each z in V_s. Let z belong to V_s and take x^* in M_z'. Then $B(z)^{-1}x^*$ belongs to M_{z_s}'. As $(x_n^*(z_s))_n$ is a basis for M_{z_s}', we can find a sequence of complex numbers $(a_n)_n$ such that $B(z)^{-1}x^* = \sum_{n=1}^{\infty} a_n x_n^*(z_s)$. Applying $B(z)$ we get that $x^* = \sum_{n=1}^{\infty} a_n B(z) x_n^*(z_s) = \sum_{n=1}^{\infty} a_n x_n^*(z)$ and thus $(x_n^*(z))_n$ is a basis for M_z'. Moreover, for each z in V_s, each sequence of complex numbers $(a_n)_n$ and each N in $\mathbb{N}$ we have that

$$\frac{1}{\|B(z)^{-1}\|}\left\|\sum_{n=1}^{N} a_n x_n^*(z_s)\right\| \leq \left\|\sum_{n=1}^{N} a_n x_n^*(z)\right\| \leq \|B(z)\| \left\|\sum_{n=1}^{N} a_n x_n^*(z_s)\right\|$$

showing that part (2) of Definition 3.1 is satisfied on V_s. We now repeat the above procedure with each z in V_s to get a neighbourhood V_z of z such that for each w in V_z we have an invertible continuous linear operator $B_z(w)$ on E' which maps M_z' onto M_w', in the process mapping $x_n^*(z)$ to $x_n^*(w)$ for each n in $\mathbb{N}$. As in the above it follows that $(x_n^*(w))_n$ is a holomorphic basis for M_w with

$$\frac{1}{\|B_z(w)^{-1}\|}\left\|\sum_{n=1}^{N} a_n x_n^*(z)\right\| \leq \left\|\sum_{n=1}^{N} a_n x_n^*(w)\right\| \leq \|B_z(w)\| \left\|\sum_{n=1}^{N} a_n x_n^*(z)\right\|$$

for all sequence of complex numbers $(a_n)_n$ and each N in $\mathbb{N}$. Using the same method as in the proof of Proposition 3.4, we will eventually reach each point of Ω. Moreover, as the sequence $(x_n^*(z))_n$ is the dual of the sequence $(x_n(z)_n$, it is uniquely determined and we have that $(x_n^*)_n$ is a holomorphic basis sequence on Ω. □

As an application of holomorphic bases, we will use them to show the existence of holomorphic generalized inverses. To remind the reader: if $T \in \mathcal{L}(X, Y)$ and there exists $S \in \mathcal{L}(Y, X)$ such that $TST = T$ and $STS = S$, we call S a *generalized inverse* for T.

The following definition appears in [7]:

Definition 4.4. *Let $f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y))$, where X and Y are Banach spaces and Ω is an open subset of a Banach space. A mapping $g \in \mathcal{H}(\Omega, \mathcal{L}(Y, X))$ is called a holomorphic generalized inverse for f if, for all $z \in \Omega$, $g(z)$ is a generalized inverse for $f(z)$.*

Also in [7] it is shown that the existence of holomorphic generalized inverse is equivalent to three other conditions - none of which, unfortunately, is easy to check:

Theorem 4.5. *Let Ω be a pseudo-convex open subset of a Banach space with an unconditional basis and X and Y be Banach spaces. Suppose $f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y))$ has a generalized inverse for each $z \in \Omega$. Then the following conditions are equvalent:*

(1) *f has a holomorphic generalized inverse on Ω.*

(2) *There exist holomorphic projections* $P \in \mathcal{H}(\Omega, \mathcal{L}(X))$ *onto* $ker(f(z))$ *and* $Q \in \mathcal{H}(\Omega, \mathcal{L}(Y))$ *onto* $Im(f(z))$.
(3) $\{z \in \Omega : (z, ker\, f(z))\}$ *and* $\{z \in \Omega : (z, Im\, f(z))\}$ *are holomorphic subbundles of* $\Omega \times X$ *and* $\Omega \times Y$ *respectively.*
(4) *For every* $w \in \Omega$ *there exist a neighbourhood* V_w *of* w *and closed subspaces* $X_w \subset X$ *and* $Y_w \subset Y$ *such that for all* $z \in V_w$, $ker\, f(z) \oplus X_w = X$ *and* $Im\, f(z) \oplus Y_w = Y$ *are direct sum decompositions.*

As a straitforward application of Proposition 4.1 we obtain the following:

Proposition 4.6. *Let* Ω *be a pseudo-convex open subset of a Banach space with a basis,* E *and* F *be Banach spaces. Suppose* $T \in \mathcal{H}(\Omega, \mathcal{L}(E, F))$ *has a generalized inverse for each* $z \in \Omega$. *Then if* $\{ker\, T(z)\}_{z\in\Omega}$ *and* $\{Im\, T(z)\}_{z\in\Omega}$ *have holomorphic bases,* T *has a holomorphic generalized inverse.*

Note that if $P \in \mathcal{H}(\Omega, \mathcal{L}(E))$ is a projection then $\mathbf{1}-P$ is a holomorphic projection onto its complement, hence in Proposition 4.6 the condition that $\{ker\, T(z)\}_{z\in\Omega}$ and $\{Im\, T(z)\}_{z\in\Omega}$ have holomorphic bases can be substituted by a condition that their complements can be chosen so that they form families with holomorphic bases.

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References

[1] G.A. Allan, On one-sided inverses in Banach algebras of holomorphic vector-valued functions, J. London Math. Soc., 42, 1967, 463-470.
[2] H. Bart, Holomorphic Relative Inverses of Operator Valued Functions, Math. Ann., 208, 1974, 179-194.
[3] J. Diestel, Sequences and series in Banach spaces, Springer-Verlag, 1984.
[4] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag Monographs in Mathematics, 1999.
[5] S. Dineen, I. Patyi, M.Venkova, Inverses depending holomorphically on a parameter in a Banach space. J. Funct. Anal. 237 (2006), no. 1, 338349.
[6] S. Dineen, M. Venkova, Fredholm-valued holomorphic mappings on a Banach space. Indag. Math. (N.S.) 19 (2008), no. 3, 379389.
[7] S. Dineen, M. Venkova, Holomorphically dependent generalized inverses, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM 103 (2009), no. 1, 19. .
[8] K.-G. Grosse-Erdmann, A weak criterion for vector-valued holomorphy, Math. Proc. Cambridge Philos. Soc., 136, (2), (2004), 399–411.
[9] L. Lempert, The Dolbeault Complex in infinite dimensions II, J. Amer. Math. Soc., 11, 1998, 485-520.
[10] L. Lempert, Vanishing Cohomology for Holomorphic Vector Bundles in a Banach Setting, Asian J. Math., 8, 1, 65-68, 2004.
[11] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I, Springer-Verlag, 1977.
[12] G.J. Murphy, C*-Algebras and Operator Theory, Academic Press, Inc. 1990.
[13] J. Mujica, Complex Analysis in Banach Spaces, North-Holland Mathematical Studies, 120, 1986.
[14] P. Saphar, Contribution á l' étude des applications linéaries dans un espace de Banach, Bull.Soc.Math France, 92, 1964, 363–384.
[15] M.A. Shubin, On Holomorphic Families of Subspaces of a Banach Space, Integral Equations and Operator Theory, 2/3, 1979, 407–420, (Translation from Mat. Issled. (Kishinev) 5, vyp 4 (18),1970, 153–165.)
[16] I. Patyi, On complex Banach submanifolds of a Banach space, Contemph. Math, 435, 2007,343–354.

[17] I. Patyi, On holomorphic Banach vector bundles over Banach spaces, Math. Annalen, 341:2, 2008, 455–482.

[18] I. Patyi, Plurisubharmonic domination in Banach spaces, Advances in Mathematics, 227, 2011, 245–252.

[19] M.G. Zaidenberg, S.G. Krein, P.A. Kuchment, A.A. Pankov, Banach Bundles and Linear Operators, Russian Math. Surveys, 30:5, 1975, 115-175.

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