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ZAKHAROV-SHABAT SYSTEM WITH CONSTANT BOUNDARY CONDITIONS. REFLECTIONLESS POTENTIALS AND END POINT SINGULARITIES

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Abstract.   We consider scalar defocusing nonlinear Schrödinger equation with constant boundary conditions. We aim here to provide a self contained pedagogical exposition of the most important facts regarding integrability of that classical evolution equation. It comprises the following topics: direct and inverse scattering problem and the dressing method.

1. Introduction

The integrability of the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0 \tag{1}$$

where $q : \mathbb{R}^2 \to \mathbb{C}$ is an infinitely smooth function and subscripts mean partial differentiations, was discovered in the pioneer papers by Zakharov and Shabat [35, 36]. Historically, it was the second nonlinear evolution equations (NLEE) solved by means of inverse scattering method after Gardner, Greene, Kruskal and Muira solved the Korteweg-de Vries equation [10, 11, 25] and proved its complete integrability. NLS has numerous applications in physics and mathematics. In nonlinear optics it models quasi-monochromatic wave packets propagating in nonlinear media [1] while in plasma physics it describes Langmuir waves in plasma [9, 19]. Another application of NLS is in fluid mechanics [31] where it appears in the context of deep water gravity waves. NLS also occurs in classical differential geometry of curves moving in three dimensional Euclidean space [26].

Despite the fact that NLS and its multicomponent counterparts have been thoroughly studied [1, 7, 8, 14, 17, 22, 29, 34] they are still an attractive subject to study [6, 15, 18, 28, 32] due to recently established applications of multicomponent NLS in Bose-Einstein condensation, see for example [16, 20, 30] and references therein. There is also increasing interest in derivation and study of nontrivial background solutions to NLS [2, 4, 5, 18]. The latter are claimed to model extreme

physical phenomena like rogue (freak) waves [3,27]. All this motivated the authors of the present paper to summarize the most important results on scalar NLS with constant boundary conditions.

In the following we shall deal with the repulsive (or defocusing) NLS equation [36]

$$\mathrm{i}q_t + q_{xx} - 2\left(|q|^2 - \rho^2\right)q = 0, \qquad \rho \in \mathbb{R}_+ \tag{2}$$

where the extra linear term introduced above ensures that the x-asymptotics of q

$$\lim_{x \to \pm\infty} q(x,t) = \rho\mathrm{e}^{\mathrm{i}\theta_\pm}, \qquad \theta_\pm \in [0, 2\pi) \tag{3}$$

are time independent. The purpose of that paper is to give an accessible summary of results on inverse scattering method applied to the NLS equation (2) with boundary conditions (3). In doing this the we shall try, following [13, 14, 29], to provide a self-contained pedagogical exposition.

Our paper is organized as follows. In Section 2 we introduce basic notions and facts needed for our further considerations. We outline the construction of the Jost solutions and of the fundamental analytic solutions (FAS) of L. In Section 3 we derive the singularities of the Jost solutions and the FAS in the vicinity of the points of the continuous spectrum of L. Sections 4 and Section 5 are dedicated to two methods for integration of a given integrable NLEE. These are the Gelfand-Levitan-Marchenko integral equations and the Zakharov-Shabat's dressing technique. Taking into account the form of Lax pair and the boundary conditions for $q(x,t)$ (3) one can fit both of these methods to effectively generate special solutions, dark solitons in particular. In Section 6 we introduce the kernel of the resolvent of L expressed in terms of the FAS and derive its singularities at the end points of the spectrum.

2. Preliminaries. Zakharov-Shabat Spectral Problem with Constant Boundary Conditions

In this section we are going to discuss some basic properties of the Zakharov-Shabat (AKNS) auxiliary linear system

$$\mathrm{i}\partial_x\Psi(x,t,\lambda) + (Q(x,t) - \lambda\sigma_3)\Psi(x,t,\lambda) = 0 \tag{4}$$

where $\lambda \in \mathbb{C}$ is spectral parameter and $\sigma_3 = \mathrm{diag}\,(1,-1)$ is one of Pauli matrices. The potential

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -q^*(x,t) & 0 \end{pmatrix} \tag{5}$$

where $*$ stands for complex conjugation, is assumed to obey constant boundary conditions

$$\lim_{x \to \pm\infty} Q(x,t) = Q_\pm = \begin{pmatrix} 0 & q_\pm \\ -q_\pm^* & 0 \end{pmatrix}, \qquad q_\pm = \rho e^{i\theta_\pm}, \qquad \theta_\pm \in [0, 2\pi). \quad (6)$$

Matrix-valued function Ψ is viewed as an arbitrary fundamental solution to (4), i.e. its columns are given by two linearly independent solutions to the Zakharov-Shabat linear system. It is our aim here to present the inverse scattering method (ISM) for (4). In doing this we are going to formulate direct scattering problem for Zakharov-Shabat systems with constant boundary conditions and analyze spectral properties of scattering operator

$$L(\lambda) = i\partial_x + Q(x,t) - \lambda\sigma_3 \quad (7)$$

introduced in (4). Our considerations shall mostly follow the ideas and the notation used in [12, 13, 17].

The linear system (4) with boundary conditions (6) underlies the inverse scattering method application to the defocusing NLS equation (2). It is equivalent to zero curvature condition $[L(\lambda), A(\lambda)] = 0$ for $L(\lambda)$ given by (7) and the second operator being in the form

$$A(\lambda)\Psi(x,t,\lambda) = i\partial_t\Psi + (V_0 + 2\lambda Q - 2\lambda^2\sigma_3)\Psi(x,t,\lambda) = \Psi(x,t,\lambda)f(\lambda)$$
$$V_0 = \left(\rho^2 - |q|^2\right)\sigma_3 + \frac{i}{2}[\sigma_3, Q_x]. \quad (8)$$

The function $f(\lambda)$ will be calculated below; it determines the dispersion law of the NLS.

Due to intrinsic U(1) symmetry of equation (2) we can set one of the phases in the asymptotic values of Q to be zero. Thus we shall fix $\theta_- = 0$ from now on and denote the other phase simply by θ in order to simplify our notation.

The space of allowed potentials $\mathcal{M}_{\rho,\theta}$, i.e. the set of all smooth matrix-valued functions of the form (5) satisfying boundary condition (6) for ρ and θ being fixed, **is not** a linear space. However, the difference of any two allowed potentials $Q_1(x,t)$ and $Q_2(x,t) \in \mathcal{M}_{\rho,\theta}$ is such that $Q_1(x,t) - Q_2(x,t)$ vanishes as $x \to \pm\infty$. Thus we can obtain the whole space $\mathcal{M}_{\rho,\theta}$ by adding up to a fixed allowed potential any potential satisfying vanishing boundary conditions.

The form of the potential implies that it obeys the following symmetry condition

$$\sigma_3 Q^\dagger(x,t)\sigma_3 = Q(x,t) \quad (9)$$

where $\dagger$ stands for Hermitian conjugation. Following the concepts by Mikhailov [23, 24] one can interpret (9) in terms of a finite reduction group group acting on

the set of solutions $\{\Psi(x, t, \lambda)\}$ to the Zakharov-Shabat system. In our case the reduction group is $\mathbb{Z}_2$ acting on the fundamental solutions in the following way:

$$\sigma_3 \hat{\Psi}^\dagger(x, t, \lambda^*)\sigma_3 = \Psi(x, t, \lambda), \tag{10}$$

where $\Psi(x, t, \lambda)$ ia any of the Jost solutions of (4) and 'hat' means the inverse of a matrix:

$$\hat{\Psi}(x, t, \lambda) = [\Psi(x, t, \lambda)]^{-1}.$$

It proves to be convenient to introduce a gauge transform denoted by $\varphi_\pm(\lambda)$ to diagonalize the asymptotic values of $Q(x, t) - \lambda\sigma_3$ when $x \to \pm\infty$, i.e. we have

$$\varphi_\pm(\lambda)(Q_\pm - \lambda\sigma_3)\hat{\varphi}_\pm(\lambda) = -j(\lambda)\sigma_3, \qquad j(\lambda) = \sqrt{\lambda^2 - \rho^2}. \tag{11}$$

where $\varphi_\pm$ are given by

$$\varphi_\pm(\lambda) = \frac{1}{\sqrt{2j(\lambda)(\lambda + j(\lambda))}} \begin{pmatrix} \lambda + j(\lambda) & -q_\pm \\ -q_\pm^* & \lambda + j(\lambda) \end{pmatrix}. \tag{12}$$

Thus the spectral parameter λ lives in a two-sheet Riemann surface $\mathcal{S} \equiv \mathcal{S}_+ \cup \mathcal{S}_-$ associated with $j(\lambda)$. To construct $\mathcal{S}$ one cuts the complex plane from $-\infty$ to $-\rho$ and from ρ to ∞ along real axis. The sheets $\mathcal{S}_+$ and $\mathcal{S}_-$ are determined by

$$\mathcal{S}_+ : \operatorname{Im} j(\lambda) > 0, \qquad \mathcal{S}_- : \operatorname{Im} j(\lambda) < 0. \tag{13}$$

The transform $\varphi_\pm(\lambda)$ allows one to define Jost solutions through the equality below

$$\lim_{x \to \infty} \psi(x, t, \lambda)\hat{E}_+(x, \lambda) = \mathbb{1}, \qquad \lim_{x \to -\infty} \phi(x, t, \lambda)\hat{E}_-(x, \lambda) = \mathbb{1} \tag{14}$$

where

$$E_\pm(x, \lambda) = \hat{\varphi}_\pm(\lambda)e^{-ij(\lambda)\sigma_3 x} \tag{15}$$

are solutions to the equation

$$i\partial_x E_\pm + (Q_\pm - \lambda\sigma_3)E_\pm = 0.$$

It straightforwardly follows from (15) that $\psi(x, t, \lambda)$ and $\phi(x, t, \lambda)$ are unimodular matrices. The transition matrix

$$T(t, \lambda) = \hat{\psi}(x, t, \lambda)\phi(x, t, \lambda) \tag{16}$$

between the Jost solutions is called scattering matrix. As a result of the reduction (10) we deduce that the scattering matrix obeys the symmetry

$$\sigma_3 \hat{T}^\dagger(t, \lambda^*)\sigma_3 = T(t, \lambda). \tag{17}$$

From the compatibility of the linear problems (4) and (8) we find that the time evolution of T is determined by

$$T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t}T(0,\lambda)\mathrm{e}^{-\mathrm{i}f(\lambda)t}. \tag{18}$$

where the dispersion law $f(\lambda)$ for the NLS is given by:

$$f(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) = -2\lambda j(\lambda)\sigma_3. \tag{19}$$

Further on in text the variable t will not be essential so we shall omit it.

Let us now discuss the spectral properties of the scattering operator. Generally speaking the spectrum of $L(\lambda)$ consists of a continuous and a discrete part. The operator (7) is equivalent to a self-adjoint eigenvalue problem:

$$\mathcal{L}\Psi \equiv \mathrm{i}\sigma_3\frac{\partial\Psi}{\partial x} + \sigma_3 Q(x)\Psi(x,\lambda) = \lambda\Psi(x,\lambda), \tag{20}$$

and therefore its spectrum must be on the real axis. The continuous part of its spectrum is determined by equality $\mathrm{Im}\, j(\lambda) = 0$, i.e. it coincides with set $\mathbb{R}_\rho = (-\infty, -\rho) \cup (\rho, \infty)$. The discrete eigenvalues of $L(\lambda)$ are simple and must belong to $(-\rho, \rho)$. The Jost solutions and the scattering matrix are defined for $\lambda \in \mathbb{R}_\rho$ only. To see this one needs to introduce the auxiliary functions

$$\begin{aligned}
\xi_+(x,\lambda) &= \varphi_+(\lambda)\psi(x,\lambda)\hat{E}_+(x,\lambda)\hat\varphi_+(\lambda) \\
\xi_-(x,\lambda) &= \varphi_-(\lambda)\phi(x,\lambda)\hat{E}_-(x,\lambda)\hat\varphi_-(\lambda)
\end{aligned} \tag{21}$$

which satisfy differential equation

$$\mathrm{i}\partial_x\xi_\pm + \tilde{Q}_+(x,\lambda)\xi_\pm - j(\lambda)[\sigma_3,\xi_\pm] = 0 \tag{22}$$

where

$$\tilde{Q}_\pm(x,\lambda) = \varphi_\pm(\lambda)(Q(x) - Q_\pm)\hat\varphi_\pm(\lambda).$$

Equivalently, $\xi_\pm$ can be viewed as solutions to the Volterra type integral equation

$$\xi_\pm(x,\lambda) = \mathbb{1} + \mathrm{i}\int_{\pm\infty}^x \mathrm{e}^{\mathrm{i}j(\lambda)\sigma_3(y-x)}\tilde{Q}_\pm(y,\lambda)\xi_\pm(y,\lambda)\mathrm{e}^{-\mathrm{i}j(\lambda)\sigma_3(y-x)}\mathrm{d}y. \tag{23}$$

Let us consider now the function ξ_+. It is easily from the integral equation that the second column of ξ_+ is analytic on $\mathcal{S}_+$ hence the second column of $\psi(x,\lambda)$ is analytic on $\mathcal{S}_+$. Similarly, the first column of $\phi(x,\lambda)$ are analytic on $\mathcal{S}_+$ while the second one as well as the first column of $\psi(x,\lambda)$ are analytic on $\mathcal{S}_-$. That is why we will denote the Jost solutions by:

$$\psi(x,\lambda) = ||\psi^-(x,\lambda), \psi^+(x,\lambda)||, \qquad \phi(x,\lambda) = ||\phi^+(x,\lambda), \phi^-(x,\lambda)|| \tag{24}$$

where the superscript $+$ (resp. $-$) refers to the analyticity properties of the corresponding column on the sheet $\mathcal{S}_+$ (resp. on $\mathcal{S}_-$).

As a result of the above considerations one can construct another pair of solutions χ^+ and χ^- which are analytic in $\mathcal{S}_+$ and $\mathcal{S}_-$ respectively. This is done using the formulae below

$$
\begin{aligned}
\chi^+(x,\lambda) &\equiv ||\phi^+(x,\lambda), \psi^+(x,\lambda)|| = \phi(x,\lambda)S^+(\lambda) = \psi(x,\lambda)T^-(\lambda) \\
\chi^-(x,\lambda) &\equiv ||\psi^-(x,\lambda), \phi^-(x,\lambda)|| = \phi(x,\lambda)S^-(\lambda) = \psi(x,\lambda)T^+(\lambda)
\end{aligned}
\tag{25}
$$

where

$$
S^+(\lambda) = \begin{pmatrix} 1 & -b^*(\lambda) \\ 0 & a(\lambda) \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} a(\lambda) & 0 \\ b(\lambda) & 1 \end{pmatrix}
\tag{26}
$$

$$
S^-(\lambda) = \begin{pmatrix} a^*(\lambda) & 0 \\ -b(\lambda) & 1 \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} 1 & b^*(\lambda) \\ 0 & a^*(\lambda) \end{pmatrix}.
\tag{27}
$$

The triangular matrices $S^\pm(\lambda)$ and $T^\pm(\lambda)$ are LU-decomposition of the scattering matrix $T(\lambda)$:

$$
T(\lambda) = \begin{pmatrix} a(\lambda) & b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix} = T^-(\lambda)\hat{S}^+(\lambda) = T^+(\lambda)\hat{S}^-(\lambda).
\tag{28}
$$

Due to reduction (10) it is seen that χ^+ and χ^- satisfy relation

$$
\sigma_3 \left[\hat{\chi}^+(x,\lambda^*)\right]^\dagger \sigma_3 = \chi^-(x,\lambda).
\tag{29}
$$

The fundamental analytic solutions satisfy the following interrelation

$$
\chi^-(x,\lambda) = \chi^+(x,\lambda)G(\lambda), \qquad \lambda \in \mathbb{R}_\rho.
\tag{30}
$$

This is a manifestation of the fact that the fundamental analytic solutions satisfy local Riemann-Hilbert problem [17, 29, 34].

It also follows from (25)–(27) that

$$
\det\chi^+(x,\lambda) = a(\lambda).
\tag{31}
$$

Therefore $a(\lambda)$ is an analytic function on the whole sheet $\mathcal{S}^+$. In what follows we will need to know the structure of χ^+ and χ^- and their inverse in the vicinity of discrete eigenvalues $\{\lambda_j\}_{j=1}^N$ of the operator $L(\lambda)$. The discrete eigenvalues are simple zeroes of $a(\lambda)$, i.e. in vicinity of λ_j we have the following Taylor expansion

$$
a(\lambda) = (\lambda - \lambda_j)\left(\dot{a}_j + \frac{1}{2}\ddot{a}_j(\lambda - \lambda_j) + \cdots\right)
\tag{32}
$$

where dot stands for differentiation with respect to λ. Due to (31) at any point λ_j the columns of $\chi^+(x, \lambda_j)$ are proportional to each other, i.e. there exists $b_j \in \mathbb{C}$ such that

$$\phi_j^+(x) = b_j \psi_j^+(x), \qquad \phi_j^-(x) = \frac{1}{b_j^*} \psi_j^-(x), \tag{33}$$

where $\phi_j^\pm(x) = \phi^\pm(x, \lambda_j$ and $\psi_j^\pm(x) = \psi^\pm(x, \lambda_j$. Thus we find:

$$\begin{aligned} \chi_j^+(x, \lambda_j) &= \psi_j^+(x)(b_j, 1) = \phi_j^+(x)(1, 1/b_j) \\ \chi_j^-(x, \lambda_j) &= \psi_j^-(x)(1, -b_j^*) = \phi_j^-(x)(-1/b_j^*, 1). \end{aligned} \tag{34}$$

In what follows we will need similar formulae also for the inverse of $\chi^\pm(x, \lambda)$ for $\lambda \simeq \lambda_j$. It is easy to see that

$$\hat{\chi}^+(x, \lambda) = \frac{1}{a(\lambda)} \begin{pmatrix} \tilde{\psi}^+ \\ -\tilde{\phi}^+ \end{pmatrix}, \qquad \hat{\chi}^-(x, \lambda) = \frac{1}{a^*(\lambda)} \begin{pmatrix} \tilde{\phi}^- \\ -\tilde{\psi}^- \end{pmatrix} \tag{35}$$

where the operation 'tilde' applied to the vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ maps it onto the row $(y_2, -y_1)$. Thus we obtain:

$$\begin{aligned} \hat{\chi}^+(x, \lambda) &\underset{\lambda \to \lambda_j}{\simeq} \begin{pmatrix} 1 \\ -b_j \end{pmatrix} \frac{\tilde{\psi}_j^+}{(\lambda - \lambda_j)\dot{a}_j} \simeq \begin{pmatrix} -1/b_j \\ 1 \end{pmatrix} \frac{\tilde{\phi}_j^+}{(\lambda - \lambda_j)\dot{a}_j} \\ \hat{\chi}^-(x, \lambda) &\underset{\lambda \to \lambda_j}{\simeq} \begin{pmatrix} 1 \\ -1/b_j^* \end{pmatrix} \frac{\tilde{\psi}_j^-}{(\lambda - \lambda_j)\dot{a}_j^*} \simeq -\begin{pmatrix} b_j^* \\ 1 \end{pmatrix} \frac{\tilde{\phi}_j^-}{(\lambda - \lambda_j)\dot{a}_j^*} \end{aligned} \tag{36}$$

Given the potential $Q(x)$ one can obtain the Jost solutions uniquely by solving the integral equations (23). The Jost solutions in turn determine uniquely the scattering matrix $T(\lambda)$. $Q(x)$ contains one independent complex-valued function $q(x)$ of x. Thus it is natural to expect that only one of the coefficients of $T(\lambda)$ for $\lambda \in \mathbb{R}_\rho$, will be independent.

At the same time the matrix elements of $T(\lambda)$ (28) are determined by the complex-valued functions $a(\lambda)$ and $b(\lambda)$ and their complex conjugate, satisfying the condition $|a|^2 - |b|^2 = 1$. It is important that $a(\lambda)$ (resp. $a^*(\lambda)$) are analytic functions of λ for $\lambda \in \mathcal{S}_+$ (resp. $\lambda \in \mathcal{S}_-$). This fact allows one to determine $a(\lambda)$ using its values on the cuts $\mathbb{R}_\rho$ and the set of its zeroes λ_j. We assume that $a(\lambda)$ has a finite number of zeroes λ_j; it is well known that $a(\lambda)$ may have only simple zeroes [13, 14, 29]. Skipping the details we introduce two equivalent minimal sets of scattering data:

$$\mathcal{T}_1 \equiv \mathcal{T}_{1,c} \cup \mathcal{T}_{1,d}, \qquad \mathcal{T}_2 \equiv \mathcal{T}_{2,c} \cup \mathcal{T}_{2,d}, \tag{37}$$

where

$$\begin{aligned}
\mathcal{T}_{1,\mathrm{c}} &\equiv \{\rho(\lambda), \quad \lambda \in \mathbb{R}_\rho\}, & \mathcal{T}_{1,\mathrm{d}} &\equiv \{C_j, \lambda_j\}_{j=1}^N \\
\mathcal{T}_{2,\mathrm{c}} &\equiv \{\tau(\lambda), \quad \lambda \in \mathbb{R}_\rho\}, & \mathcal{T}_{2,\mathrm{d}} &\equiv \{M_j, \lambda_j\}_{j=1}^N.
\end{aligned} \tag{38}$$

The reflection coefficients $\rho(\lambda)$ and $\tau(\lambda)$ and the coefficients C_j and M_j are given by:

$$\rho(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \qquad \tau(\lambda) = -\frac{b^*(\lambda)}{a(\lambda)}, \qquad C_j = \frac{b_j}{\dot{a}_j}, \qquad M_j = \frac{1}{b_j \dot{a}_j}. \tag{39}$$

Now we need to compute the asymptotics of the FAS solutions for large values of $|\lambda|$. The results are summarized in the tables below:

	$\operatorname{Im}\lambda > 0, \ \lambda \in \mathcal{S}_+$	$\operatorname{Im}\lambda < 0, \ \lambda \in \mathcal{S}_+$	
$\chi^+(x,\lambda)\mathrm{e}^{\mathrm{i}j(\lambda)\sigma_3 x}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$-\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta_+} \\ \mathrm{e}^{-\mathrm{i}\theta_-} & 0 \end{pmatrix}$	(40)
$a^+(\lambda)$	1	$\mathrm{e}^{\mathrm{i}(\theta_+ - \theta_-)}$	

	$\operatorname{Im}\lambda > 0, \ \lambda \in \mathcal{S}_-$	$\operatorname{Im}\lambda < 0, \ \lambda \in \mathcal{S}_-$	
$\chi^-(x,\lambda)\mathrm{e}^{\mathrm{i}j(\lambda)\sigma_3 x}$	$-\mathrm{i}\begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta_-} \\ \mathrm{e}^{-\mathrm{i}\theta_+} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	(41)
$a^-(\lambda)$	$\mathrm{e}^{-\mathrm{i}(\theta_+ - \theta_-)}$	$1.$	

The functions $b^\pm(\lambda)$ in general do not admit analytic continuations, however if one considers special functional classes one can argue that [17, 29]

$$\lim_{|\lambda|\to\infty} b(\lambda) = 0.$$

3. The end points of the spectrum

The Lax operator L with constant boundary condition is one of the basic examples of ordinary differential operator whose continuous spectrum multiplicity varies. Obviously the continuous spectrum on the rays $\mathbb{R}_\rho \equiv (-\infty, -\rho) \cup (\rho, \infty)$ has multiplicity 2, while on the lacuna $(-\rho, \rho)$ it has vanishing multiplicity.

In the limit $\lambda \to \lambda\varepsilon\rho$ the matrices $\varphi_\pm$ become singular. So in order to introduce the Jost solutions we will need regularized definitions, see below. Besides, we have to take into account that the eigenfunctions of $L_{\varepsilon\rho,\mathrm{as}}$ are given by:

$$L_{\pm\mathrm{as}}|_{\lambda=\varepsilon\rho} E_{\varepsilon\rho} = 0, \qquad E_{\varepsilon\rho} = \begin{pmatrix} 1 & \mathrm{i}\rho x \mathrm{e}^{\mathrm{i}\theta_\pm} \\ \varepsilon\rho\mathrm{e}^{-\mathrm{i}\theta_\pm} & \mathrm{i}\rho\varepsilon x + 1 \end{pmatrix}, \tag{42}$$

where $\varepsilon = \pm 1$. The inverse of $E_{\varepsilon\rho}$ is

$$\hat{E}_{\varepsilon\rho} = \begin{pmatrix} i\rho\varepsilon x + 1 & -\rho x e^{i\theta_\pm} \\ -\varepsilon\rho e^{i\theta_\pm} & 1 \end{pmatrix}. \tag{43}$$

Thus the Jost solutions for $\lambda = \varepsilon\rho$ may be defined by:

$$\lim_{\lambda\to\varepsilon\rho}\lim_{x\to\infty} \hat{E}_{\varepsilon\rho}\psi(x,\varepsilon\rho) = \mathbb{1}, \qquad \lim_{\lambda\to\varepsilon\rho}\lim_{x\to-\infty} \hat{E}_{\varepsilon\rho}\phi(x,\varepsilon\rho) = \mathbb{1}. \tag{44}$$

This definition shows that the Jost solutions and the FAS may develop singularities at the end points of the spectrum. This definition is valid for the generic case, because the Lax operator L will have two linearly independent eigenfunctions also at the end points $\lambda = \varepsilon\rho$. Along with the generic case we will consider also the possibility of virtual eigenvalues at the end points. In general both possibilities have been analyzed in [29]. We will present slightly different approach which is gauge covariant.

Skipping the details we collect the formulae, giving the asymptotics of the Jost solutions, the FAS and the scattering matrix for $\lambda \to \varepsilon\rho$.

3.1. Generic case – asymptotics

The Jost solutions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ develop singularities at the end points of the spectrum, which are consequence of the singularity of $E_\pm(x,\lambda)$. Indeed, for $\lambda \to \varepsilon\rho$ we have:

$$E_\pm(x,\lambda) = \sqrt{\frac{\rho}{j(\lambda)}} \left(E_{\varepsilon,0} + \frac{j(\lambda)}{2\rho} E_{\varepsilon,1} + \mathcal{O}(j^2) \right) e^{-ij(\lambda)x\sigma_3},$$

$$E_{\varepsilon,0} = \begin{pmatrix} 1 & -e^{i\theta_\pm} \\ -e^{-i\theta_\pm} & 1 \end{pmatrix}, \qquad E_{\varepsilon,1} = \begin{pmatrix} 1 & e^{i\theta_\pm} \\ e^{-i\theta_\pm} & 1 \end{pmatrix}, \tag{45}$$

In the vicinity of end points $\lambda \simeq \varepsilon\rho$ the Jost solutions become:

$$\psi^\pm(x,\lambda) = \sqrt{\frac{\rho}{2j(\lambda)}} \left(\psi^\pm_{\varepsilon\rho,0}(x) + j(\lambda)\psi^\pm_{\varepsilon\rho,1}(x) + \mathcal{O}(j^2) \right),$$

$$\phi^\pm(x,\lambda) = \sqrt{\frac{\rho}{2j(\lambda)}} \left(\phi^\pm_{\varepsilon\rho,0}(x) + j(\lambda)\phi^\pm_{\varepsilon\rho,1}(x) + \mathcal{O}(j^2) \right), \tag{46}$$

$$a_{\varepsilon\rho} = \frac{a_{\varepsilon\rho,0}}{j(\lambda)} + a_{\varepsilon\rho,1} + \mathcal{O}(j), \qquad b_{\varepsilon\rho} = -\varepsilon\frac{ib_{\varepsilon\rho,0}}{j(\lambda)} + b_{\varepsilon\rho,1} + \mathcal{O}(j),$$

$$\psi^-_{\varepsilon\rho}(x) = \varepsilon i \psi^+_{\varepsilon\rho}(x), \qquad \phi^+_{\varepsilon\rho}(x) = \varepsilon i \phi^-_{\varepsilon\rho}(x), \tag{47}$$

i.e.
$$\lim_{\lambda \to \pm\rho} \frac{a_{\varepsilon\rho}}{b_{\varepsilon\rho}} = \varepsilon \mathrm{i}.$$

Similar relations can be derived also for the asymptotic values of the FAS, namely:

$$
\begin{aligned}
\chi_{\varepsilon\rho}^+(x,\lambda) &\equiv ||\phi_{\varepsilon\rho}^+(x), \psi_{\varepsilon\rho}^+(x)|| \\
&= \sqrt{\frac{\rho}{2j(\lambda)}} \left(\chi_{\varepsilon\rho,0}^+(x) + j(\lambda)\chi_{\varepsilon\rho,1}^+(x) + \mathcal{O}(j^2) \right), \\
\chi_{\varepsilon\rho}^-(x,\lambda) &\equiv ||\psi_{\varepsilon\rho}^-(x), \phi_{\varepsilon\rho}^-(x)|| \\
&= \sqrt{\frac{\rho}{2j(\lambda)}} \left(\chi_{\varepsilon\rho,0}^-(x) + j(\lambda)\chi_{\varepsilon\rho,1}^-(x) + \mathcal{O}(j^2) \right).
\end{aligned}
\tag{48}
$$

as well as for their inverse:

$$
\hat{\chi}_{\varepsilon\rho}^+(x,\lambda) = \sqrt{\frac{2j(\lambda)}{\rho}} \left(\hat{\chi}_{\varepsilon\rho,0}^+(x) - j(\lambda)\hat{\chi}_{\varepsilon\rho,0}^+(x)\chi_{\varepsilon\rho,1}^+(x)\hat{\chi}_{\varepsilon\rho,0}^+(x) + \mathcal{O}(j^2) \right),
$$

$$
\hat{\chi}_{\varepsilon\rho}^-(x,\lambda) = \sqrt{\frac{2j(\lambda)}{\rho}} \left(\hat{\chi}_{\varepsilon\rho,0}^-(x) - j(\lambda)\hat{\chi}_{\varepsilon\rho,0}^-(x)\hat{\chi}_{\varepsilon\rho,1}^-(x)\hat{\chi}_{\varepsilon\rho,0}^-(x) + \mathcal{O}(j^2) \right).
\tag{49}
$$

where $\hat{\chi}_{\varepsilon\rho,0}^\pm(x) = (\chi_{\varepsilon\rho,0}^-(x))^{-1}$.

3.2. Virtual eigenvalue – asymptotics

In this case the Jost solutions become degenerate for $\lambda = \varepsilon\rho$, i.e. the matrices $\psi_{\varepsilon\rho,0}(x,\lambda)$ are $\phi_{\varepsilon\rho,0}(x,\lambda)$ become degenerate

$$
\psi_{\varepsilon\rho,0}(x) = \sqrt{\frac{\rho}{2j(\lambda)}}\psi_{\varepsilon\rho,0}^-(x)(1,\varepsilon\mathrm{i}), \quad \phi_{\varepsilon\rho,0}(x) = \sqrt{\frac{\rho}{2j(\lambda)}}\phi_{\varepsilon\rho,0}^+(x,\lambda)(1,\varepsilon\mathrm{i}),
\tag{50}
$$

Then both $a(\lambda)$ and $b(\lambda)$ are regular for $\lambda \to \pm\rho$:

$$
\lim_{\lambda \to \varepsilon\rho} a_{\varepsilon\rho} = a_{\varepsilon\rho,1}, \qquad \lim_{\lambda \to \varepsilon\rho} b_{\varepsilon\rho} = b_{\varepsilon\rho,1}.
\tag{51}
$$

Similarly we can analyze the behavior of the FAS in the vicinity of the end points of the spectrum. The results are:

$$
\begin{aligned}
\chi_{\varepsilon\rho}^+(x) &= \sqrt{\frac{\rho}{2j(\lambda)}}\psi_{\varepsilon\rho}^+(x)(b_{\varepsilon\rho},1) = \sqrt{\frac{\rho}{2j(\lambda)}}\phi_{\varepsilon\rho}^+(x)(1,1/b_{\varepsilon\rho}) \\
\chi_{\varepsilon\rho}^-(x,\lambda_j) &= \sqrt{\frac{\rho}{2j(\lambda)}}\psi_{\varepsilon\rho}^-(x)(1,-b_{\varepsilon\rho}^*) = \sqrt{\frac{\rho}{2j(\lambda)}}\phi_{\varepsilon\rho}^-(x)(-1/b_{\varepsilon\rho}^*,1).
\end{aligned}
\tag{52}
$$

and

$$
\hat{\chi}^{+}(x,\lambda) \underset{\lambda \to \varepsilon \rho}{\simeq} \sqrt{\frac{\rho}{2j(\lambda)}} \begin{pmatrix} 1 \\ -b_{\varepsilon\rho} \end{pmatrix} \tilde{\psi}_{\varepsilon\rho}^{+} \simeq \sqrt{\frac{\rho}{2j(\lambda)}} \begin{pmatrix} -1/b_{\varepsilon\rho} \\ 1 \end{pmatrix} \tilde{\phi}_{\varepsilon\rho}^{+}
$$
$$
\hat{\chi}^{-}(x,\lambda) \underset{\lambda \to \varepsilon \rho}{\simeq} -\sqrt{\frac{\rho}{2j(\lambda)}} \begin{pmatrix} 1/b_{\varepsilon\rho}^{*} \\ 1 \end{pmatrix} \tilde{\psi}_{\varepsilon\rho}^{-} \simeq \sqrt{\frac{\rho}{2j(\lambda)}} \begin{pmatrix} 1 \\ b_{\varepsilon\rho} \end{pmatrix} \tilde{\phi}_{\varepsilon\rho}^{-}.
$$

$$(53)$$

We will use this in Section 6 below for analyzing the singularities of the resolvent at the end points of the spectrum.

4. Gelfand-Levitan-Marchenko Equations

An effective method to derive soliton solutions and the corresponding eigenfunctions is based upon the Gelfand-Levitan-Marchenko integral equations (GLM). Here we just briefly outline the basic facts of derivation of GLM [13, 17, 29].

Let us consider the linear problems

$$
L_0 \Psi_0 \equiv i\partial_x \Psi_0 + (Q_0 - \lambda\sigma_3)\Psi_0 = 0 \tag{54}
$$
$$
L\Psi \equiv i\partial_x \Psi + (Q - \lambda\sigma_3)\Psi = 0. \tag{55}
$$

where both potentials are of the form (5) and have equal asymptotic values as $x \to \infty$, i.e.

$$
\lim_{x \to \infty} Q_0(x) = \lim_{x \to \infty} Q(x) = Q_+. \tag{56}
$$

The Jost solutions of (54) and (55) by definition obey the equalities

$$
\lim_{x \to \infty} \psi_0(x,\lambda)\hat{E}_+(x,\lambda) = \mathbb{1} \qquad \lim_{x \to \infty} \psi(x,\lambda)\hat{E}_+(x,\lambda) = \mathbb{1}. \tag{57}
$$

We assume now that Q_0 is known and we shall refer to it as bare (or seed) potential, while the other one, to be found, will be called dressed potential

Remark 1 *It is not possible for that the bare potential and the dressed one share the same asymptotic values both at $x \to -\infty$ and $x \to \infty$. For the derivation of GLM it suffices to have consistency of just one of the asymptotics, say as $x \to \infty$.*

The dressed Jost solution can be expressed from the bare one through the integral transformation:

$$
\psi(x,\lambda) = \psi_0(x,\lambda) + \int_{\infty}^{x} dy\, \Gamma_+(x,y)\psi_0(y,\lambda),
$$
$$
\phi(x,\lambda) = \phi_0(x,\lambda) + \int_{-\infty}^{x} dy\, \Gamma_-(x,y)\phi_0(y,\lambda),
$$

$$(58)$$

where the integral kernels $\Gamma_\pm$ satisfy

$$\lim_{y\to\infty} \Gamma_+(x,y) = 0, \qquad \lim_{y\to-\infty} \Gamma_-(x,y) = 0. \qquad (59)$$

In order for transformation (58) to be consistent the kernel must satisfy certain differential constraints. Indeed, after substituting (58) into (55) and taking into account that the bare Jost solution fulfils (54) one obtains the following relations

$$\mathrm{i}\frac{\partial\Gamma_\pm(x,y)}{\partial x} + \mathrm{i}\sigma_3\frac{\partial\Gamma_\pm(x,y)}{\partial y}\sigma_3 + Q(x)\Gamma_\pm(x,y) - \sigma_3\Gamma_\pm(x,y)\sigma_3 Q_0(y) = 0$$
$$Q(x) - Q_0(x) + \mathrm{i}(\Gamma_\pm(x,x) - \sigma_3\Gamma_\pm(x,x)\sigma_3) = 0. \qquad (60)$$

Obviously the solutions of (60) will be parametrized by the scattering data of both operators L and L.

Let us develop this idea first considering the Jost solution $\psi(x,\lambda)$ and the transformation operator with kernel $\Gamma_+(x,y)$. This transformation operator maps the Jost solution $\psi_0(x,\lambda)$ of the 'naked' operator L_0 into the Jost solution $\psi(x,\lambda)$ of the 'dressed' operator L. Each of these operators has its own scattering data (38): reflection coefficients $\rho_0(\lambda)$ and $\rho(\lambda)$ and sets of discrete eigenvalues

$$\mathcal{D}_0 \equiv \{\lambda_{0j}\colon a_0(\lambda_{0j}) = 0\}_{j=1}^{N_0}, \qquad \mathcal{D} \equiv \{\lambda_j\colon a(\lambda_j) = 0\}_{j=1}^{N}.$$

Note that some of the eigenvalues of L_0 may coincide with the eigenvalues of L.

The generic solution of (60) can be presented in the form:

$$\Gamma_+(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}_\rho} \mathrm{d}\lambda \left(c_+(\lambda)\psi^+(x,\lambda)\hat{\psi}_0^+(y,\lambda) - c_+^*(\lambda)\psi^-(x,\lambda)\hat{\psi}_0^-(y,\lambda)\right)\sigma_3$$
$$- \sum_{\lambda_j\in\mathcal{D}} c_j\psi_j^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j\in\mathcal{D}_0} c_{0j}\psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3, \qquad (61)$$

where $\psi_{0j}^+(x) = \psi_0^+(x,\lambda_j)$ and $\psi_j^+(x) = \psi^+(x,\lambda_j)$. The coefficients c_j, c_{0j} and the function $c(\lambda)$ can be expressed in terms of the scattering data, e.g. $c(\lambda) = \rho(\lambda) - \rho_0(\lambda)$.

By $\psi_0^\pm(x,\lambda)$ and $\psi^\pm(x,\lambda)$ we mean the corresponding columns of Jost solutions (see (24)) to (54) and (55) respectively, while $\tilde{\psi}_j(x)$ and $\tilde{\psi}_{0,j}(x)$ are row vectors of inverse Jost solutions (see (35). Obviously they satisfy:

$$\mathrm{i}\partial_x\psi_0^\pm(x) + (Q_0(x) - \lambda\sigma_3)\,\psi_0^\pm(x,\lambda) = 0$$
$$\mathrm{i}\partial_x\psi^\pm(x) + (Q(x) - \lambda\sigma_3)\,\psi^\pm(x,\lambda) = 0, \qquad (62)$$

and

$$i\partial_x \tilde{\psi}_0^\pm(x) - \tilde{\psi}_0^\pm(x, \lambda)\left(Q_0(x) - \lambda\sigma_3\right) = 0$$
$$i\partial_x \tilde{\psi}^\pm(x) - \tilde{\psi}^\pm(x, \lambda)\left(Q(x) - \lambda\sigma_3\right) = 0, \tag{63}$$

respectively. Therefore $\psi_{0j}^+(x)$, $\psi_j^+(x)$ and $\tilde{\psi}_{0j}^+(x)$, $\tilde{\psi}_j^+(x)$ will be solutions to (62) and (63) with $\lambda = \lambda_{0j}$ and $\lambda = \lambda_j$ respectively.

Taking into account that from eq. (58) there follows

$$\psi^\pm(x, \lambda) = \psi_0^\pm(x, \lambda) + \int_\infty^x dy\ \Gamma_+(x, y)\psi_0^\pm(y, \lambda).,$$
$$\psi_j^\pm(x) = \psi_{0j}^\pm(x) + \int_\infty^x dy\ \Gamma_+(x, y)\psi_{0j}^\pm(y)., \tag{64}$$

we can easily rewrite eq. (61) as the well GLM equation:

$$\Gamma_+(x, y) + F_+(x, y) + \int_\infty^x dz\ \Gamma_+(x, z)F_+(z, y) = 0, \tag{65}$$

where

$$F_+(x, y) = \frac{-1}{2\pi}\int_{\mathbb{R}_\rho} d\lambda\ \left(c(\lambda)\psi_0^+(x, \lambda)\hat{\tilde{\psi}}_0^+(y, \lambda) - c^*(\lambda)\psi_0^-(x, \lambda)\hat{\tilde{\psi}}_0^-(y, \lambda)\right)\sigma_3$$
$$+ \sum_{\lambda_j \in \mathcal{D}} c_j\psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3 - \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}\psi_{0j}^+(x)\tilde{\psi}_{0,j}^+(y)\sigma_3, \tag{66}$$

This is the most general form of the GLM equation which relates two Lax operators L and L_0 with generic choice for their spectral data. If one knows the Jost solutions of the 'naked' operator L_0 then solving the GLM eq. one can construct the Jost solutions of the 'dressed' operator L. However, the Jost solutions of L_0 for generic spectral data (i.e., non-vanishing $\rho_0(\lambda)$) can not be evaluated explicitly. There is however, a special class of potential – the so-called reflectionless potentials, that can be constructed explicitly by solving the GLM eq.

Indeed, let us now assume that the $Q_0(x, t) = Q_+$. Then $\psi_0(x, \lambda) = E_+(x, \lambda)$ and the kernel F is degenerate, i.e. we have $c(\lambda) = c_0(\lambda) = 0$, $N_0 = 0$ and $N = 1$. In this special case GLM equations reduces to a set of linear algebraic equations and one can construct the solution explicitly. For $N = 1$ equations (61) and (64) lead to the following result for the dressed solutions

$$\psi_1^+(x) = \left(1 - c_1\int_\infty^x \left(\tilde{\psi}_{0,1}^+(y)|\psi_{0,1}^+(y)\right)\right)^{-1}\psi_{0,1}^+(x). \tag{67}$$

Taking into account the explicit form of $E_+(x, \lambda)$ (see equalities (12) and (15)) one can perform the integration above to obtain

$$\psi_1^+(x) = \frac{1}{1 + V_0(x)} \psi_{0,1}^+(x), \qquad V_0(x) = \frac{c_1 \rho e^{-2\sqrt{\rho^2 - \lambda_1^2}x} e^{i\theta}}{2i(\rho^2 - \lambda_1^2)}. \qquad (68)$$

After substituting (68) into (61) for the transformation operator kernel we get

$$\Gamma_+(x, y) = \frac{\rho \sin \theta_1 V_0(x) e^{\sqrt{\rho^2 - \lambda_1^2}(x-y)}}{1 + V_0(x)} \begin{pmatrix} 1 & e^{i(\theta - \theta_1)} \\ e^{-i(\theta - \theta_1)} & 1 \end{pmatrix}. \qquad (69)$$

Thus we have all information needed to find the dressed potential. Making use of relation (60) one derives the following result

$$\begin{aligned} Q(x) &= Q_+ + ic_1\sigma_3 \left[\psi_1^+(x)\tilde{\psi}_{0,1}^+(x), \sigma_3 \right] \\ &= \frac{\rho}{1 + V_0(x)} \begin{pmatrix} 0 & e^{i\theta} \left(1 + e^{-2i\theta_1} V_0(x)\right) \\ -e^{-i\theta} \left(1 + e^{2i\theta_1} V_0(x)\right) & 0 \end{pmatrix}. \end{aligned} \qquad (70)$$

We remind that $\lambda_1 + i\sqrt{\rho^2 - \lambda_1^2} = \rho e^{i\theta_1}$ and one can pick up the discrete eigenvalue λ_1 in such a way that $\theta = 2\theta_1$ is fulfilled.

Now we shall construct the 1-soliton eigenfunctions. In order to do so we substitute (69) into (58) and perform the integration required. The result reads

$$\psi(x, \lambda) = \begin{pmatrix} A - f_1 \frac{V_0}{1 + V_0} & e^{i\theta} \left(B + f_2 \frac{V_0}{1 + V_0} e^{-i\theta_1}\right) \\ e^{-i\theta} \left(B - f_1 \frac{V_0}{1 + V_0} e^{i\theta_1}\right) & A + f_2 \frac{V_0}{1 + V_0} \end{pmatrix} e^{-ij(\lambda)\sigma_3 x} \qquad (71)$$

where

$$\begin{aligned} f_1 &= \frac{\sqrt{\rho^2 - \lambda_1^2}(A + Be^{-i\theta_1})}{ij(\lambda) + \sqrt{\rho^2 - \lambda_1^2}}, & A(\lambda) &= \sqrt{\frac{\lambda + j(\lambda)}{2j(\lambda)}} \\ f_2 &= \frac{\sqrt{\rho^2 - \lambda_1^2}(A + Be^{i\theta_1})}{ij(\lambda) - \sqrt{\rho^2 - \lambda_1^2}}, & B(\lambda) &= \sqrt{\frac{\lambda - j(\lambda)}{2j(\lambda)}}. \end{aligned} \qquad (72)$$

After an elementary transformation of expressions above the dressed Jost solution are rewritten as follows:

$$\psi(x, \lambda) = \left\{ \mathbb{1} + \frac{i\sqrt{\rho^2 - \lambda_1^2}V_0(x)}{(\lambda - \lambda_1)(1 + V_0(x))} \begin{pmatrix} 1 & -e^{i\theta/2} \\ e^{-i\theta/2} & -1 \end{pmatrix} \right\} E_+(x, \lambda). \qquad (73)$$

Similarly, one can use the second transformation operator $\Gamma_-(x,y)$ in eq. (58) connecting the other pair of Jost solutions $\phi(x,\lambda)$ and $\phi_0(x,\lambda)$, and apply the same considerations as before. The dual' GLM eq. is

$$\Gamma_-(x,y) + F_-(x,y) + \int_{-\infty}^{x} \mathrm{d}z\, \Gamma_-(x,z)F_-(z,y) = 0. \tag{74}$$

If we choose a generic kernel

$$F_-(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}_\rho} \mathrm{d}\lambda \, \left(c_-(\lambda)\phi_0^+(x,\lambda)\hat{\phi}_0^+(y,\lambda) - c_-^*(\lambda)\phi_0^-(x,\lambda)\hat{\phi}_0^-(y,\lambda) \right) \sigma_3$$
$$- \sum_{\lambda_j \in \mathcal{D}} c_j^- \phi_{0j}^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}^- \phi_{0j}^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3, \tag{75}$$

then its generic solution of (60) can be presented in the form:

$$\Gamma_-(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}_\rho} \mathrm{d}\lambda \, \left(c_-(\lambda)\phi^+(x,\lambda)\hat{\phi}_0^+(y,\lambda) - c_-^*(\lambda)\phi^-(x,\lambda)\hat{\phi}_0^-(y,\lambda) \right) \sigma_3$$
$$- \sum_{\lambda_j \in \mathcal{D}} c_j^- \phi_j^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3 + \sum_{\lambda_j \in \mathcal{D}_0} c_{0j}^- \phi_j^+(x)\tilde{\phi}_{0,j}^+(y)\sigma_3, \tag{76}$$

where $\phi_{0j}^+(x) = \phi_0^+(x,\lambda_j)$ and $\phi_j^+(x) = \phi^+(x,\lambda_j)$. The coefficients c_j^-, c_{0j}^- and the function $c_-(\lambda)$ again can be expressed in terms of the scattering data, e.g. $c_-(\lambda) = \tau(\lambda) - \tau_0(\lambda)$.

If we take the simplest nontrivial kernel with $N = 1$ and $Q_0 = Q_-$ then the final result for the dressed potential reads

$$Q(x) = Q_+ + \mathrm{i}c_1\sigma_3 \left[\phi_1^+(x)\tilde{\phi}_{0,1}^+(x), \sigma_3 \right]$$
$$= \frac{\rho}{1 + \tilde{V}_0(x)} \begin{pmatrix} 0 & 1 + \mathrm{e}^{\mathrm{i}\theta}\tilde{V}_0(x) \\ -\left(1 + \mathrm{e}^{-\mathrm{i}\theta}\tilde{V}_0(x)\right) & 0 \end{pmatrix} \tag{77}$$

where

$$\tilde{V}_0(x) = \frac{c_1\rho\mathrm{e}^{2\sqrt{\rho^2-\lambda_1^2}\,x}}{2\mathrm{i}(\rho^2 - \lambda_1^2)}. \tag{78}$$

The corresponding Jost solution $\phi(x,\lambda)$ is given by

$$\phi(x,\lambda) = \left\{ \mathbb{1} - \frac{\mathrm{i}\sqrt{\rho^2 - \lambda_1^2}\,\tilde{V}_0(x)}{(\lambda - \lambda_1)(1 + \tilde{V}_0(x))} \begin{pmatrix} 1 & -\mathrm{e}^{\mathrm{i}\theta/2} \\ \mathrm{e}^{-\mathrm{i}\theta/2} & -1 \end{pmatrix} \right\} E_-(x,\lambda). \tag{79}$$

Let us now calculate the dressed scattering matrix:

$$T(\lambda) = \lim_{x\to-\infty} \psi(x,\lambda)\phi(x,\lambda) = \mathrm{diag}\,(a(\lambda), 1/a(\lambda)) \tag{80}$$

where

$$a(\lambda) = \frac{A - Be^{i\theta_1}}{A - Be^{-i\theta_1}} = \frac{\lambda + j - \lambda_1 - ij_1}{\lambda + j - \lambda_1 + ij_1}. \tag{81}$$

The dressed fundamental analytic solutions are constructed through equalities:

$$\chi^+(x, \lambda)e^{ij\sigma_3 x} = u(x, \lambda) \begin{pmatrix} A & Be^{i\theta} \\ B & A \end{pmatrix} \tag{82}$$

where the dressing factor reads:

$$u(x, \lambda) = \left\{ \mathbb{1} + \frac{f_2(A - Be^{i\theta_1})}{(1 + V_0(x))} \begin{pmatrix} 1 \\ e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} 1, & V_0 e^{i(\theta - \theta_1)} \end{pmatrix} \right\} e^{-ij\sigma_3 x}. \tag{83}$$

5. Dressing Method

The dressing method is another indirect method to solve a nonlinear evolution equation, i.e. it allows one to find a particular solution from a known one. For this to be done one uses substantially the existence of Lax representation and the connection between ISM and Riemann-Hilbert problem [17, 21, 34, 37].

Let us consider once again the auxiliary linear problems (54) and (55). We shall denote by g the dressing transform $\Psi_0 \to \Psi$. By comparing (54) and (55) we see that g satisfies:

$$i\partial_x g + Qg - gQ_0 - \lambda[\sigma_3, g] = 0. \tag{84}$$

We are going to use a dressing factor in the form:

$$g(x, \lambda) = \mathbb{1} + \frac{A(x)}{\lambda - \lambda_1}, \qquad \lambda_1 \in (-\rho, \rho). \tag{85}$$

Due to reduction (10) the dressing factor obeys the following symmetry condition:

$$\sigma_3 \hat{g}^\dagger(x, \lambda^*)\sigma_3 = g(x, \lambda). \tag{86}$$

Hence the inverse factor $\hat{g}$ reads

$$\hat{g}(x, \lambda) = \mathbb{1} + \frac{\sigma_3 A^\dagger(x)\sigma_3}{\lambda - \lambda_1}. \tag{87}$$

Let us evaluate the limit $|\lambda| \to \infty$ in equation (84). As a result we obtain a relation between bare potential Q_0 and dressed one Q, namely

$$Q = Q_0 + [\sigma_3, A]. \tag{88}$$

Thus we can find Q if we only know the residue A. As it turns out the latter can be expressed in terms of a fundamental solution of the bare linear problem (54). In order to find A we consider the identity $g\hat{g} = \mathbb{1}$ which gives rise to the following algebraic relations:

$$A\sigma_3 A^\dagger = 0 \tag{89}$$

$$A + \sigma_3 A^\dagger \sigma_3 = 0. \tag{90}$$

It follows from (89) that $A(x)$ is a degenerate matrix so there exist two vectors $X(x)$ and $F(x)$ such that $A = XF^T$. Substituting this decomposition into (89) we obtain

$$F^T \sigma_3 F^* = 0. \tag{91}$$

From (89), (90) and (91) it follows that

$$X = -\mathrm{i}\frac{\sigma_3 F^*}{\alpha} \tag{92}$$

for some real function α to be determined further on. F and α can be found if one considers equation (84). We shall skip all technical details here and give the final result. Both F and α are expressed through a solution Ψ_0 to the bare linear problem in the following way

$$F^T(x) = F_0^T \hat{\Psi}_0(x, \lambda_1) \tag{93}$$

$$\alpha(x) = \mathrm{i}F_0^T \hat{\Psi}_0(x, \lambda_1)\partial_\lambda|_{\lambda=\lambda_1}\Psi_0(x, \lambda)C_0(\lambda_1)\sigma_3 F_0^* + \alpha_0 \tag{94}$$

where the 2-vector[1] F_0 as well as $\alpha_0 \in \mathbb{R}$ are constants of integration. The constant matrix $C_0(\lambda_1)$ appears in the reduction condition (10), namely

$$C_0(\lambda) = \hat{\Psi}_0(x, \lambda)\sigma_3 \hat{\Psi}_0^\dagger(x, \lambda^*)\sigma_3 \tag{95}$$

The final step in our considerations consists in recovering the time evolution in all quantities. To achieve this one can use the following rule

$$F_0^T \to F_0^T \mathrm{e}^{-\mathrm{i}f(\lambda_1)t}$$
$$\alpha_0 \to \alpha_0 - F_0^T \frac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda_1)C_0(\lambda_1)\sigma_3 F_0^* t. \tag{96}$$

Let us recall that $f(\lambda)$ is the dispersion law of the NLEE under consideration. Formulae (96) are derived by analyzing the equation

$$\mathrm{i}g_t + Vg - gV^{(0)} = 0 \tag{97}$$

---

[1] The vector F_0 is usually called polarization vector.

where $V^{(0)}$ and V are involved in the Lax operators

$$
\begin{aligned}
A_0(\lambda) &= \mathrm{i}\partial_t + V^{(0)}(x,t,\lambda) \\
A(\lambda) &= \mathrm{i}\partial_t + V(x,t,\lambda)
\end{aligned}
$$

respectively. Let us illustrate this general scheme in the following example.

Example 2 *Let us consider the case when the bare solution is constant, i.e. $q_0 = \rho$. Then the corresponding fundamental solution is given by:*

$$
\Psi_0(x,\lambda) = \hat{\varphi}_0(\lambda)\mathrm{e}^{-\mathrm{i}j(\lambda)\sigma_3 x} \tag{98}
$$

where $\varphi_0(\lambda)$ is given by eq. (12). For our purposes it suffices to pick up the polarization vector F_0 as follows $F_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the 2-vector F and the scalar function α acquire the form

$$
\begin{aligned}
F(x) &= \frac{\mathrm{e}^{-\mathrm{i}j(\lambda_1)x}}{\sqrt{2j(\lambda_1)(\lambda_1 + j(\lambda_1))}} \begin{pmatrix} -\rho \\ \lambda_1 + j(\lambda_1)) \end{pmatrix} \\
\alpha(x) &= -\frac{\rho\mathrm{e}^{-2\mathrm{i}j(\lambda_1)x}}{2j^2(\lambda_1)} + \alpha_0.
\end{aligned} \tag{99}
$$

Finally substitution of all these results into (88) leads to

$$
\begin{aligned}
q(x) &= \rho\,\frac{1 + \mathrm{e}^{2\mathrm{i}\theta_1}\mathrm{e}^{2\sqrt{\rho^2-\lambda_1^2}(x-x_0)}}{1 + \mathrm{e}^{2\sqrt{\rho^2-\lambda_1^2}(x-x_0)}} \\
\tan\theta_1 &= \frac{\sqrt{\rho^2-\lambda_1^2}}{\lambda_1}, \qquad x_0 = \frac{1}{2\sqrt{\rho^2-\lambda_1^2}}\ln\frac{2(\rho^2-\lambda_1^2)\alpha_0}{\rho}
\end{aligned} \tag{100}
$$

for the reflectionless potential. It coincides with the result obtained in the previous section, see (70). In order to recover the t-dependence in (100) we have to replace x with $x + 2\lambda_1 t$. Thus we have

$$
q(x,t) = \rho\,\frac{1 + \mathrm{e}^{2\mathrm{i}\theta_1}\mathrm{e}^{2\sqrt{\rho^2-\lambda_1^2}(x+2\lambda_1 t-x_0)}}{1 + \mathrm{e}^{2\sqrt{\rho^2-\lambda_1^2}(x+2\lambda_1 t-x_0)}}. \tag{101}
$$

This is the well-known dark soliton for the defocusing NLS [13, 29]. It is immediately seen that asymptotic value of the dark soliton as $x \to -\infty$ coincide with that of vacuum solution while the other asymptotic differs, i.e. Q and Q_0 belong to different phase spaces, $\mathcal{M}_{2\theta_1}$ and $\mathcal{M}_0$ respectively. This means that in general

the dressing procedure **does not** *respect the constant boundary conditions. This motivates one to introduce a more general space*

$$\mathcal{M}_\rho = \bigcup_\theta \mathcal{M}_{\rho,\theta}$$

which is dressing invariant. The the dressing procedure describe above will map $\mathcal{M}_{\rho,\theta}$ into $\mathcal{M}_{\rho,\theta+2\theta_1}$. $\square$

In order to find more complicated solutions one can pursue either of the following two ways: apply the discussed procedure to the solution already dressed and thus generate a sequence of solutions; or use a multiple poles dressing factor in the form

$$g(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^N \frac{A_k(x,t)}{\lambda - \lambda_k}, \qquad \lambda_k \in \mathbb{R}. \tag{102}$$

In the latter case the dressed solution can be obtained through the following relation

$$Q = Q_0 + \sum_{k=1}^N [\sigma_3, A_k]. \tag{103}$$

As before in order to find residues of the dressing factor we consider the algebraic relations:

$$A_k \sigma_3 A_k^\dagger = 0 \tag{104}$$

$$A_k \sigma_3 \Omega_k^\dagger \sigma_3 + \sigma_3 A_k^\dagger \sigma_3 \Omega_k = 0 \tag{105}$$

where

$$\Omega_k(x,t) = \mathbb{1} + \sum_{j \neq k} \frac{A_j(x,t)}{\lambda_k - \lambda_j}.$$

Relation (104) means that each residue $A_k(x,t)$ is a degenerate matrix hence there exists couples of vectors X_k and F_k, $k = 1, \ldots, N$ such that $A_k = X_k F_k^T$. Due to (104) the components of F_k are not independent but satisfy the relations

$$F_k^T \sigma_3 F_k^* = 0. \tag{106}$$

Relation (105) can be reduced to:

$$\Omega_k \sigma_3 F_k^* = i\alpha_k X_k \tag{107}$$

for some matrices $\alpha_k(x)$, yet to be determined. The system (107) can be viewed as a linear system for the vectors X_k

$$\sigma_3 F_k^* = \sum_{j=1}^N \mathcal{B}_{kj} X_j, \tag{108}$$

where

$$\mathcal{B}_{kk} = \mathrm{i}\alpha_k, \qquad \mathcal{B}_{kj} = \frac{F_j^T \sigma_3 F_k^*}{\lambda_j - \lambda_k}, \quad k \neq j.$$

This allows us to express X_k in terms of F_k and α_k Similarly to the one-pole case the F_k and α_k are expressed through a seed solution as follows:

$$F_k^T(x) = F_{k,0}^T \hat{\Psi}_0(x, \lambda_k) \tag{109}$$

$$\alpha_k(x) = \mathrm{i}F_k^T(x)\partial_\lambda \Psi_0(x, \lambda_k)C_0(\lambda_k)\sigma_3 F_{k,0}^* + \alpha_{k,0} \tag{110}$$

where $F_{k,0}$ and $\alpha_{k,0}$ are integration constants. Finally in order to recover the time dependence one uses the following formulae:

$$F_{k,0}^T \rightarrow F_{k,0}^T \mathrm{e}^{-\mathrm{i}f(\lambda_k)t} \tag{111}$$

$$\alpha_{k,0} \rightarrow \alpha_{k,0} - F_{k,0}^T \frac{\mathrm{d}f}{\mathrm{d}\lambda}(\lambda_k)C_0(\lambda_k)\sigma_3 F_{k,0}^* t. \tag{112}$$

6. FAS and the Resolvent of L

The FAS play important role in analyzing the spectral properties of the Lax operator. Here we will demonstrate that the FAS can be used to construct the kernel of the resolvent of L.

Let us now show how the resolvent $R(\lambda)$ can be expressed through the FAS of $L(\lambda)$. Indeed, let us write down $R(\lambda)$ in the form:

$$R(\lambda)f(x) = \int_{-\infty}^{\infty} R(x, y, \lambda)f(y), \tag{113}$$

where the kernel $R(x, y, \lambda)$ of the resolvent is given by:

$$R^{\pm}(x, y, \lambda) = \frac{1}{\mathrm{i}}\chi^{\pm}(x, \lambda)\Theta^{\pm}(x - y)\hat{\chi}^{\pm}(y, \lambda). \tag{114}$$

Here

$$\Theta^+(x - y) = \mathrm{diag}\left(-\theta(y - x), \theta(x - y)\right)$$
$$\Theta^-(x - y) = \mathrm{diag}\left(\theta(x - y), -\theta(y - x)\right). \tag{115}$$

Theorem 3 *Let $Q(x) \in \mathcal{M}_\theta$ and let $-\rho < \lambda_j < \rho$ be the simple zeroes of the $a(\lambda)$. Then*

1. *$R^{\pm}(x, y, \lambda)$ is an analytic function of λ for $\lambda \in \mathcal{S}_{\pm}$ having pole singularities at λ_j;*

2. $R^{\pm}(x, y, \lambda)$ is a kernel of a bounded integral operator for $\lambda \in \mathcal{S}_{\pm}$;

3. $R(x, y, \lambda)$ is uniformly bounded function for $\lambda \in \mathbb{R}_{\rho}$ and provides a kernel of an unbounded integral operator;

4. $R^{\pm}(x, y, \lambda)$ satisfy the equation:

$$L(\lambda)R^{\pm}(x, y, \lambda) = \mathbb{1}\delta(x - y). \tag{116}$$

5. If $Q(x)$ is such that for $\lambda \to \varepsilon\rho$ the FAS have generic behavior then the kernel of resolvent is regular for $\lambda \to \varepsilon\rho$;

6. If $Q(x)$ is such that for $\lambda \to \varepsilon\rho$ $a(\lambda)$ and $b(\lambda)$ remain finite then $R^{\pm}(x, y, \lambda)$ behaves like $1/j(\lambda)$ for $\lambda \to \varepsilon\rho$.

Proof:

1. is obvious from the fact that $\chi^{\pm}(x, \lambda)$ are the FAS of $L(\lambda)$;

2. Assume that $\lambda \in \mathcal{S}_{+}$ and consider the asymptotic behavior of $R^{+}(x, y, \lambda)$ for $x, y \to \infty$. Equation (115) can be rewritten as

$$R^{+}(x, y, \lambda) = \tfrac{1}{i}X^{+}(x, \lambda)e^{-ij(\lambda)\sigma_3 x}\Theta^{+}(x - y)e^{ij(\lambda)\sigma_3 y}\hat{X}^{+}(y, \lambda) \tag{117}$$

where $X^{+}(x, \lambda) = \chi^{+}(x, \lambda)e^{ij(\lambda)\sigma_3 x}$. Note that due to eqs. (25)–(27) the functions $X^{+}(x, \lambda)$ are bounded for $\lambda \in \mathcal{S}_{+}$ where $\mathrm{Im}\, j(\lambda) > 0$. But for $\mathrm{Im}\, j(\lambda) > 0$ both exponential factors in (117) fall off exponentially for $x, y \to \infty$. All other possibilities are treated analogously.

3. For $\lambda \in \mathbb{R}_{\rho}$ the arguments of 2) can not be applied because the exponents in the right hand side of (117) $\mathrm{Im}\, j(\lambda) = 0$ only oscillate. Thus we conclude that $R^{\pm}(x, y, \lambda)$ for $\lambda \in \mathbb{R}_{\rho}$ is only a bounded function and thus the corresponding operator $R(\lambda)$ is an unbounded integral operator.

4. The proof of eq. (116) follows from the fact that $L(\lambda)\chi^{+}(x, \lambda) = 0$ and

$$\partial_x\Theta^{\pm}(x - y) = \mp\mathbb{1}\delta(x - y). \tag{118}$$

5. From eqs. (35), (47) and (48) there follows that

$$\lim_{\lambda \to \varepsilon\rho} R^{+}(x, y, \lambda) = \frac{1}{2ia_{\varepsilon\rho,0}}\chi_{\varepsilon\rho,0}^{+}(x)\Theta^{+}(x - y)\tilde{\chi}_{\varepsilon\rho,0}^{+}(y), \tag{119}$$

where $\tilde{\chi}_{\varepsilon\rho,0}^{+}(y)$ is a non-degenerate constant matrix. The limit $\lim_{\lambda \to \varepsilon\rho} R^{-}(x, y, \lambda)$ is treated analogously.

6. In the case of virtual eigenvalue we use eqs. (35), (50) and (51). Now the result is

$$\lim_{\lambda \to \varepsilon\rho} R^+(x, y, \lambda) \simeq \frac{1}{2\mathrm{i}j(\lambda)} \psi^+_{\varepsilon\rho,0}(x) \tilde{\phi}^+_{\varepsilon\rho,0}(y) + \mathcal{O}(1)$$

$$\lim_{\lambda \to \varepsilon\rho} R^-(x, y, \lambda) \simeq \frac{1}{2\mathrm{i}j(\lambda)} \phi^+_{\varepsilon\rho,0}(x) \tilde{\psi}^+_{\varepsilon\rho,0}(y) + \mathcal{O}(1).$$

(120)

∎

It is well known that applying the contour integration method on the kernel of the resolvent one can prove the completeness relation for the Jost solutions [13, 22]. From the above theorem it is obvious that, if the potential $Q(x)$ satisfies the virtual eigenvalue condition, then there will be additional terms in this relations.

7. Conclusion

We have outlined the construction of the Jost solutions and the FAS for the Zakharov-Shabat system L with constant boundary conditions. We also calculated their singularities at the end points $\pm\rho$ of the continuous spectrum of L. We also demonstrated the derivation of the reflectionless potentials of L and the dark soliton solutions for the relevant NLS equation (2) using first the GLM approach and second – the dressing Zakharov-Shabat method. Finally we constructed the kernel of the resolvent of L and proved that in the regular case $R^\pm(x, y, \lambda)$ are regular for $\lambda \to \varepsilon\rho$, while in the virtual soliton case $R^\pm(x, y, \lambda)$ develop pole singularities for $\lambda \to \varepsilon\rho$. The explicit form of the resolvent can be used to derive the completeness relation for the Jost solutions. Our result shows that in the virtual soliton case this relation will contain additional term corresponding to a discrete eigenvalue at $\varepsilon\rho$.

The results can be used also to derive the generalized Fourier transform, i.e. the expansions over the 'squared solutions' of L. This will be published elsewhere.

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