



Technological University Dublin
ARROW@TU Dublin

Articles

School of Mathematics

2019

Riemann-Hilbert Problem, Integrability and Reductions

Vladimir Gerdjikov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, vgerdjikov@gmail.com


Rossen Ivanov

Technological University Dublin, rossen.ivanov@tudublin.ie

Aleksander Stefanov

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

Follow this and additional works at: <https://arrow.tudublin.ie/scschmatart>

 Part of the [Algebra Commons](#), [Geometry and Topology Commons](#), [Non-linear Dynamics Commons](#), and the [Partial Differential Equations Commons](#)

Recommended Citation

Gerdjikov, V.S., Ivanov, R.I. Stefanov, A., (2019) Riemann-Hilbert problem, integrability and reductions. *Journal of Geometric Mechanics*, 2019, 11 (2) : 167-185. DOI : 10.3934/jgm.2019009

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact yvonne.desmond@tudublin.ie, arrow.admin@tudublin.ie, brian.widdis@tudublin.ie.



This work is licensed under a [Creative Commons Attribution-Noncommercial-Share Alike 3.0 License](https://creativecommons.org/licenses/by-nc-sa/3.0/)



RIEMANN-HILBERT PROBLEM, INTEGRABILITY AND REDUCTIONS

VLADIMIR S. GERDJKOV^{1,2,5}, ROSSEN I. IVANOV^{3,*}
AND ALEKSANDER A. STEFANOV^{2,4}

¹ Department of Applied Mathematics
National Research Nuclear University MEPhI
31 Kashirskoe Shosse, Moscow 115409, Russian Federation

² Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
8 Acad. G. Bonchev Street, Sofia 1113, Bulgaria

³ School of Mathematical Sciences
Technological University Dublin - City Campus
Kevin Street, Dublin D08 NF82, Ireland

⁴ Faculty of Mathematics and Informatics
Sofia University St. Kliment Ohridsky

⁵ James Burchier Blvd., Sofia 1164, Bulgaria
Institute for Advanced Physical Studies
New Bulgarian University
21 Montevideo Street, Sofia 1618, Bulgaria

ABSTRACT. The present paper is dedicated to integrable models with Mikhailov reduction groups $G_R \simeq \mathbb{D}_h$. Their Lax representation allows us to prove, that their solution is equivalent to solving Riemann-Hilbert problems, whose contours depend on the realization of the G_R -action on the spectral parameter. Two new examples of Nonlinear Evolution Equations (NLEE) with \mathbb{D}_h symmetries are presented.

1. Introduction. One of the standard methods in the study of integrable systems is based on the Lax representation and the inverse scattering method [54, 16, 1, 26]. The Inverse Scattering Problem (ISP) for a given Lax operators requires the knowledge of the so-called Fundamental Analytic Solutions (FAS) [55]. The next step is the transformation of the ISP to an equivalent Riemann-Hilbert problem (RHP) [61, 62], which proved to be very fruitful.

Indeed, already the scalar RHP provides a tool for the implementation of the inverse scattering method and for calculation of the soliton solutions of some very important and interesting equations such as the KdV [54], the Kaup-Boussinesq [39, 32], the Camassa-Holm [11, 36] and the 2-component Camassa-Holm ¹ [37] equations.

The more general problem is a RHP in matrix form

$$\xi^+(x, t, \lambda) = \xi^-(x, t, \lambda)G(x, t, \lambda), \quad \lambda \in \mathbb{R}. \quad (1.1)$$

2010 *Mathematics Subject Classification.* Primary: 35K10, 35Q15; Secondary: 37K15, 35Q55.
Key words and phrases. Inverse scattering, semisimple Lie algebras, solitons.

* Corresponding author: R. I. Ivanov.

¹The Camassa-Holm equation introduced in [10] triggered an enormous flow of publications and many interesting generalizations.

for Lie group-valued functions $\xi^\pm(x, t, \lambda)$, analytic for $\lambda \in \mathbb{C}_\pm$. Such RHP is a tool for solving the ISP for the generalized Zakharov-Shabat system:

$$L_{\text{gZS}}\psi \equiv i\frac{\partial\psi}{\partial x} + (Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0, \quad (1.2)$$

where $J = \text{diag}(a_1, \dots, a_n)$ is a diagonal matrix with different real-valued eigenvalues [55, 56]. Using L_{gZS} Zakharov and Manakov discovered the integrability of the N -wave equations [60], see also [40]. Particular cases of L_{gZS} , where some of the a_j are equal, can be applied to the Manakov model [46] as well as to other vector (multi-component) nonlinear Schrödinger equations (NLS) related to symmetric spaces. Using RHP's (1.1) Zakharov and Shabat introduced their dressing method for calculating the soliton solutions of all these Nonlinear Evolution Equations (NLEE) [61, 62].

Another important method for deriving new NLEE was proposed by Mikhailov [48], where he introduced the group of reductions G_R . The relevant Lax operator is of the form L_{gZS} but now a_j are powers of $\omega = \exp(2\pi i/h)$.

The problem of constructing FAS for generalized Zakharov-Shabat systems with complex-valued a_j was solved by Beals and Coifman [5] for the $sl(N)$ -algebras and was later generalized to any simple Lie algebras [27]. Thus the ISP for Lax operators having \mathbb{Z}_h and \mathbb{D}_h as reduction groups again can be reduced to a RHP, but now its contour in the λ -plane is a set of h lines closing angles π/h (see fig. 1):

$$\xi_k(x, t, \lambda) = \xi_{k-1}(x, t, \lambda)G_k(x, t, \lambda), \quad \lambda \in l_k, \quad k = 0, \dots, h-1. \quad (1.3)$$

Here $\xi_k(x, t, \lambda)$ is an analytic function in the sector $k\pi/h \leq \arg \lambda \leq (k+1)\pi/h$ and $k-1$ is evaluated modulo h . Among the relevant new NLEE with \mathbb{Z}_h and \mathbb{D}_h symmetries related to such RHP we note the 2-dimensional Toda field theories [48], mKdV equations related to the Kac-Moody algebras [63, 47, 65, 53, 15, 64], see also [17, 31, 22, 19, 23, 30, 24, 25] and others.

Already at this level we see that the formulation of the RHP depends substantially on the group of reductions G_R imposed on the Lax pair. Indeed, the contour Γ of the RHP depends on the explicit realization of G_R as a subgroup of the group of conformal transformation Conf on \mathbb{C} . From now on for definiteness we will assume that the reduction group $G_R \simeq \mathbb{D}_h$ is isomorphic to the dihedral group.

The 2-dimensional Toda field theories [48, 49] and the mKdV equations [15] provide paradigmatic examples of NLEE with \mathbb{Z}_h and \mathbb{D}_h symmetries. Their Lax representations in fact are closely related to the Kac-Moody algebras [15]. The spectral properties of their Lax operators [29, 28, 30] however are not completely established and deserve further studies.

The main focus of this paper is on the Lax operators with a reduction group G_R isomorphic to the dihedral group \mathbb{D}_h . We are especially interested in reductions containing the transformation $\lambda \rightarrow \frac{1}{\lambda}$, where λ is the spectral parameter. As a starting point though, we will review the operators with a \mathbb{Z}_h reduction group.

The paper is structured as follows. Section 2 contains some preliminaries. There we formulate Mikhailov's reduction group (Subsection 2.1) as well as the standard formulation of the RHP for Lax pairs with a \mathbb{Z}_h reduction group. In Section 3 we analyze Lax pairs with a \mathbb{D}_h reduction groups. We demonstrate that they are equivalent to Riemann-Hilbert problems, whose contours depend on the realization of the G_R -action on the spectral parameter. Section 4 contains two new examples NLEE with \mathbb{D}_h symmetries. The first one is a generalization of the GI eq. It is in the list of integrable NLS-type equations classified by Mikhailov, Shabat, Yamilov

and Sokolov [50, 51, 52]. The second example is a deformation of the Heisenberg ferromagnet equation.

2. Preliminaries. Let us start with an example for a typical Lax pair, given below. It is polynomial in the spectral parameter λ and its potentials $U(x, t, \lambda)$ and $V(x, t, \lambda)$ are taking values in the simple Lie algebra \mathfrak{g} .

$$\begin{aligned} L\psi &\equiv i\frac{\partial\psi}{\partial x} + U(x, t, \lambda)\psi(x, t, \lambda) = 0, \\ M\psi &\equiv i\frac{\partial\psi}{\partial t} + V(x, t, \lambda)\psi(x, t, \lambda) = 0. \end{aligned} \tag{2.1}$$

Here $U(x, t, \lambda)$ and $V(x, t, \lambda)$ depend polynomially on λ and with proper Mikhailov reduction it can generate various important equations, like the mKdV and NLS-type equations related to \mathfrak{g} , see for example [15, 26, 17, 26, 19, 20, 21, 23, 38].

2.1. Mikhailov's reduction group. The reduction group G_R is a finite group which preserves the Lax representation (2.1), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: i) $G_R \subset \text{Aut } \mathfrak{g}$ and ii) $G_R \subset \text{Conf } \mathbb{C}$, i.e. as conformal mappings of the complex λ -plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [48]:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \tag{2.2}$$

where $C_k \in \text{Aut } \mathfrak{g}$ and $\Gamma_k(\lambda) \in \text{Conf } \mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k . Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

More specifically the automorphisms C_k , $k = 1, \dots, 4$ listed above lead to the following reductions for the matrix-valued functions of the Lax representation:

$$\begin{aligned} 1) \quad & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & C_1(V^\dagger(\kappa_1(\lambda))) &= V(\lambda), \\ 2) \quad & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & C_2(V^T(\kappa_2(\lambda))) &= -V(\lambda), \\ 3) \quad & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & C_3(V^*(\kappa_1(\lambda))) &= -V(\lambda), \\ 4) \quad & C_4(U(\kappa_2(\lambda))) = U(\lambda), & C_4(V(\kappa_2(\lambda))) &= V(\lambda), \end{aligned} \tag{2.3}$$

reductions of type 1) and 2) in (2.3) involve combinations with outer automorphisms of \mathfrak{g} .

Thus the explicit form of the Lax representation and consequently of the NLEE, depends on: (a) the choice of the reduction group G_r ; (b) the choice of its representation as a subgroup of the conformal group acting on λ ; and (c) the choice of the representation of G_R as a subgroup of $\text{Aut } \mathfrak{g}$. It is important also that both representations of G_R are faithful.

2.2. Lax pairs with \mathbb{Z}_h symmetries. Single point orbits. The cyclic group \mathbb{Z}_h has one generating element satisfying the generating relation:

$$s^h = \mathbb{1}. \tag{2.4}$$

The group has h elements: $\mathbb{1}, s^k, k = 1, \dots, h-1$. Typically the action of \mathbb{Z}_h is

$$s(\lambda) = \lambda\omega, \tag{2.5}$$

where $\omega = \exp(2\pi i/h)$. Next, following [48, 15] we choose the Lax pair to be polynomial in λ . Furthermore, we assume that

$$\begin{aligned} U(x, t, \lambda) &= Q(x, t) - \lambda J, \\ V(x, t, \lambda) &= \sum_{k=0}^{s-1} \lambda^k V_k(x, t) - \lambda^s K, \end{aligned} \quad (2.6)$$

where λ is the spectral parameter and $U(x, t, \lambda)$ and $V(x, t, \lambda)$ take values in the simple Lie algebra \mathfrak{g} . In other words we are using a degenerate orbit of the \mathbb{Z}_h group consisting of the only singular point $\{\infty\}$ of the Lax pair.

Typically in what follows by h we will denote the Coxeter number of \mathfrak{g} and the corresponding inner automorphism C such that $C^h = \mathbb{1}$ will be the Coxeter automorphism (either belonging to the Weyl group or an equivalent to it automorphism from the Cartan subgroup).

The Coxeter automorphism introduces grading in the algebra \mathfrak{g} [33], namely:

$$\mathfrak{g} = \bigoplus_{s=0}^{h-1} \mathfrak{g}^{(s)}, \quad [\mathfrak{g}^{(s)}, \mathfrak{g}^{(p)}] \in \mathfrak{g}^{(s+p)}, \quad (2.7)$$

with the properties

$$X \in \mathfrak{g}^{(s)} \Leftrightarrow CXC^{-1} = \omega^s X, \quad Y \in \mathfrak{g}^{(p)} \Leftrightarrow CYC^{-1} = \omega^p Y, \quad (2.8)$$

i.e. $[X, Y] \in \mathfrak{g}^{(p+s)}$ and $p+s$ is understood modulo h . In particular, the Lax operator L in (2.6) will possess \mathbb{Z}_h symmetry if $Q(x, t) \in \mathfrak{g}^{(0)}$ and $J \in \mathfrak{g}^{(1)}$.

The spectral properties of such Lax operators are well known, see [27, 30] and the references therein. The continuous spectrum of L fills up the set of rays

$$\Gamma = \bigoplus_{s=0}^{2h-1} l_s, \quad l_s \equiv \left\{ \lambda \mid \arg \lambda = \frac{s\pi}{h}, k = 0, \dots, 2h-1 \right\}. \quad (2.9)$$

These rays split the complex plane \mathbb{C} into $2h$ sectors $\Omega_k : k\pi/h \leq \arg \lambda \leq (k+1)\pi/h$, $k = 0, \dots, 2h-1$, see figure 1.

Let Δ be the root system of \mathfrak{g} . As usual, it splits into two parts $\Delta = \Delta_+ \cup \Delta_-$, where Δ_+ denotes the set of the positive roots, and Δ_- the set of the negative roots. Then to each ray we can relate a subset of roots

$$\delta_k \equiv \{\alpha \in \Delta, \quad \operatorname{Im} \lambda \alpha(J) = 0, \quad \lambda \in l_k\}. \quad (2.10)$$

Each of the subsets also can be split into positive and negative roots by

$$\begin{aligned} \delta_k^+ &\equiv \{\alpha \in \delta_k, \quad \operatorname{Im} \lambda \alpha(J) > 0, \quad \lambda \in e^{i0} l_k\}, \\ \delta_k^- &\equiv \{\alpha \in \delta_k, \quad \operatorname{Im} \lambda \alpha(J) < 0, \quad \lambda \in e^{-i0} l_k\}. \end{aligned} \quad (2.11)$$

Note, that this ordering is specific for each ray l_k .

With each of the sectors Ω_k one can relate a FAS of L $\chi_k(x, t, \lambda)$. Next we can introduce the limits of the FAS for $x \rightarrow \pm\infty$ which will play the role of the minimal sets of scattering data of L , see [27, 30]:

$$\lim_{x \rightarrow -\infty} e^{-i\lambda J x} \chi_k(x, t, \lambda) = S_k^\pm, \quad \lim_{x \rightarrow \infty} e^{-i\lambda J x} \chi_k(x, t, \lambda) = T_k^\mp D_k^\pm, \quad \lambda \in e^{\pm i0} l_k, \quad (2.12)$$

where

$$\begin{aligned} S_k^\pm(\lambda, t) &= \exp\left(\sum_{\alpha \in \delta_k^\pm} s_\alpha^\pm(\lambda, t) E_{\pm\alpha}\right), & D_k^\pm(\lambda) &= \exp\left(\sum_{\alpha \in \delta_k^\pm} d_\alpha^\pm(\lambda) H_\alpha\right), \\ T_k^\pm(\lambda, t) &= \exp\left(\sum_{\alpha \in \delta_k^\pm} \tau_\alpha^\pm(\lambda, t) E_{\pm\alpha}\right), \end{aligned} \quad (2.13)$$

As minimal sets of scattering data one can use each of the sets of functions: (i) $S_0^\pm(\lambda, t)$ and $S_1^\pm(\lambda, t)$ for $\lambda \in l_0$ and $\lambda \in l_1$ respectively; or (ii) $T_0^\pm(\lambda, t)$ and $T_1^\pm(\lambda, t)$ for $\lambda \in l_0$ and $\lambda \in l_1$ respectively, [27, 30]. Note that all other sets of functions $S_k^\pm(\lambda, t)$ (resp. $T_k^\pm(\lambda, t)$) are recovered from $S_0^\pm(\lambda, t)$ and $S_1^\pm(\lambda, t)$ (resp. $T_0^\pm(\lambda, t)$ and $T_1^\pm(\lambda, t)$) by the symmetry conditions:

$$S_{2s+k}^\pm(\lambda\omega^s, t) = C^s S_k^\pm(\lambda, t) C^{-s}, \quad T_{2s+k}^\pm(\lambda\omega^s, t) = C^s T_k^\pm(\lambda, t) C^{-s}, \quad (2.14)$$

with $k = 0, 1$.

From eqs. (2.12) it follows, that the FAS satisfy:

$$\chi_k(x, t, \lambda) = \chi_{k-1}(x, t, \lambda) G_k^{(0)}(t, \lambda), \quad G_k^{(0)}(t, \lambda) = S_k^{-, -1}(t, \lambda) S_k^+(t, \lambda), \quad (2.15)$$

for $k = 0, \dots, 2h - 1$. In order to derive the proper RHP we need to introduce:

$$\xi_k(x, t, \lambda) = \chi_k(x, t, \lambda) e^{iJ\lambda x}. \quad (2.16)$$

Then the relations (2.15) can be viewed as a RHP on the contour Γ allowing canonical normalization:

$$\begin{aligned} \xi_k(x, t, \lambda) &= \xi_{k-1}(x, t, \lambda) G_k(x, t, \lambda), \\ \lim_{\lambda \rightarrow \infty} \xi_k(x, t, \lambda) &= \mathbf{1}. \end{aligned} \quad (2.17)$$

The x -dependence of the sewing function is given by

$$G_k(x, t, \lambda) = e^{i\lambda Jx} G_k^{(0)}(t, \lambda) e^{-i\lambda Jx}, \quad \lambda \in l_k. \quad (2.18)$$

Thus, starting from the Lax operator and its FAS we derived canonically normalized RHP.

An important fact, proved by Zakharov and Shabat [61, 62] is, that from the RHP (2.16) one can derive the Lax operator. Indeed, following Zakharov and Shabat [61, 62] it became possible to develop the dressing method for explicit calculation of the soliton solutions. Later results by Dickey, Gelfand [14] and Mark Adler [2] established the fact that the solution of the RHP in each of the sectors Ω_k for $\lambda \gg 1$ can be represented as an asymptotic series in the form [18]:

$$\xi_k(x, t, \lambda) = \exp(\mathcal{Q}^+(x, t, \lambda)), \quad \mathcal{Q}^+(x, t, \lambda) = \sum_{s=1}^{\infty} Q_s(x, t) \lambda^{-s}, \quad \lambda \gg 1. \quad (2.19)$$

This representation is compatible with the canonical normalization of RHP and allows to parameterise the Lax pair by:

$$U^+(x, t, \lambda) = -(\lambda \xi_k J \xi_k^{-1})_+, \quad V^+(x, t, \lambda) = -(\lambda^3 \xi_k J \xi_k^{-1})_+, \quad (2.20)$$

where the subscript $+$ means that only terms with positive powers of λ in the corresponding expansion in powers of λ are retained. In [18] it was demonstrated

that the first few coefficients of $\mathcal{Q}(x, t, \lambda)$ were sufficient to parameterize U and V of a polynomial in λ Lax pair. Indeed, for a Lax operator linear in λ we have

$$U^+(x, t, \lambda) = -(\lambda \xi_k J \xi_k^{-1})_+ = -\lambda J + [J, Q_1(x, t)]. \quad (2.21)$$

The formulae (2.19) and (2.20) allow one to calculate effectively compatible Lax pairs when both L and M are quadratic in λ or polynomials in λ of any order greater than 2, see [18]. One can also impose on $\mathcal{Q}^+(x, t, \lambda)$ reductions of \mathbb{Z}_h -type; the resulting Lax pair will naturally possess \mathbb{Z}_h as reduction group.

3. Lax pairs with \mathbb{D}_h symmetries.

3.1. \mathbb{D}_h symmetries: General construction. The dihedral group \mathbb{D}_h has two generating elements satisfying the generating relations:

$$r^2 = s^h = \mathbb{1}, \quad sr s^{-1} = s^{-1}. \quad (3.1)$$

The group has $2h$ elements: $\{s^k, r s^k, k = 1, \dots, h\}$ and allows several inequivalent realization on the complex λ -plane. Some of them are:

$$\begin{aligned} \text{(i)} \quad s(\lambda) &= \lambda \omega, & r(\lambda) &= \epsilon \lambda^*, & \text{(ii)} \quad s(\lambda) &= \lambda \omega, & r(\lambda) &= \frac{\epsilon}{\lambda^*}, \\ \text{(iii)} \quad s(\lambda) &= \lambda \omega, & r(\lambda) &= \epsilon \lambda, & \text{(iv)} \quad s(\lambda) &= \lambda \omega, & r(\lambda) &= \frac{\epsilon}{\lambda}, \end{aligned} \quad (3.2)$$

where $\omega = \exp(2\pi i/h)$ and $\epsilon = \pm 1$. In Section 4 we will present two examples of new NLEE whose Lax representation conform with case (iv) of (3.2).

Detailed analysis of the realizations of \mathbb{D}_h as reduction groups of Lax pairs has been performed by Mikhailov, Lombardo and Bury [43, 44, 42, 45, 8, 9]. It depends substantially not only on the realization of $r(\lambda)$, but also on the choice of the orbit of \mathbb{D}_h . Here we will limit ourselves to the two-point orbit containing the points $\{0, \infty\}$.

We will also need a pair of automorphisms C_s and C_r such that

$$C_s^h = \mathbb{1}, \quad C_r^2 = \mathbb{1}, \quad C_s C_r C_s^{-1} = C_r^{-1}. \quad (3.3)$$

The realizations (i) and (iii) of \mathbb{D}_h were used by Mikhailov to derive the 2-dimensional Toda field theories [48] and by Drinfeld and Sokolov for obtaining the mKdV equations related to the Kac-Moody algebras. The potentials of the relevant Lax representations take values in the simple Lie algebras \mathfrak{g} of rank r , which is graded using the Coxeter automorphism of order h . Each of the above mentioned NLEE is a system of r equations for r real-valued functions. The fact that the corresponding inverse scattering problem is equivalent to a RHP was noted by Mikhailov [48]. It has the form (2.16) and the contour is the union of h rays, see fig. 1. The additional \mathbb{Z}_2 generating element r in the group does not affect the contour of the RHP because the FAS of the Lax operators are again analytic in the sectors Ω_k , see fig. 1 where the case with $h = 3$ is shown.

The situation changes if we consider the cases (ii) and (iv) in (3.2). Then the Lax pair becomes polynomial in λ and λ^{-1} , see [8, 43]. Now we must use two asymptotic expansions of $\xi_k(x, t, \lambda)$: one for $|\lambda| \gg 1$ and another valid for $|\lambda| \ll 1$:

$$\xi_k^+(x, t, \lambda) = \exp(\mathcal{Q}^+(x, t, \lambda)), \quad \mathcal{Q}^+(x, t, \lambda) = \sum_{s=1}^{\infty} Q_s(x, t) \lambda^{-s}, \quad \lambda \gg 1,$$

$$\xi_{k+2h}^-(x, t, \lambda) = \exp(\mathcal{Q}^-(x, t, \epsilon\lambda)), \quad \mathcal{Q}^-(x, t, \lambda) = \sum_{s=1}^{\infty} C_r(Q)_s(x, t)(\epsilon\lambda)^s, \quad \lambda \ll 1. \quad (3.4)$$

Obviously the second asymptotic expansion in (3.4) is obtained from the first one by the action of the reduction group. In other words the solution of the RHP in each of the sectors Ω_{k+h} for $\lambda \ll 1$ can be obtained from $\xi_k^+(x, \lambda)$ by applying the automorphism $C_r : \xi_k^-(x, \lambda) = C_r(\xi_k^+)(x, \epsilon\lambda^{-1})$.

Now equations (2.20) allow us to derive the parts of $U(x, t, \lambda)$ and $V(x, t, \lambda)$ that are polynomial in λ . Analogously, using the formulae

$$U^-(x, t, \lambda) = -(\lambda^{-1}\xi_k^- J \xi_k^{-,-1})_-, \quad V^-(x, t, \lambda) = -(\lambda^{-1}\xi_k^- J \xi_k^{-,-1})_-, \quad (3.5)$$

we derive the parts of $U(x, t, \lambda)$ and $V(x, t, \lambda)$ that are polynomial in λ^{-1} . The subscript “-” in eq. (3.5) means that only the non-positive powers of the expansion of the corresponding expressions in powers of λ is retained. The first few coefficients of $\mathcal{Q}(x, t, \lambda)$ are sufficient to parameterize U^- and V^- for the non-polynomial in λ part of the Lax pairs:

$$\begin{aligned} U^-(x, t, \lambda) &= -(\epsilon\lambda^{-1}\xi_k^- C_r(J)\xi_k^{-,-1})_- = -\frac{1}{\epsilon\lambda}C_r(J) + [C_r(J), C_r(Q)_1(x, t)], \\ V^-(x, t, \lambda) &= -(\epsilon\lambda^{-1}\xi_k^- C_r(K)\xi_k^{-,-1})_- = -\frac{1}{\epsilon\lambda}C_r(K) + [C_r(K), C_r(Q)_1(x, t)]. \end{aligned} \quad (3.6)$$

Taking the average of these two procedures allows one to obtain a Lax representation, invariant with respect to this representation of the dihedral group.

Important examples whose Lax pairs are with \mathbb{Z}_3 -symmetry are the Degasperis-Procesi equation ² [12], the Tzitzeica equation [3, 4] and the Kaup-Kupershmidt equation [41].

Indeed, let us assume that the Lax pair L and M (2.1) already possesses \mathbb{Z}_h symmetry with one point orbit. Using it we can construct new Lax pair \tilde{L} and \tilde{M} with \mathbb{D}_h symmetry with two-point orbit as follows:

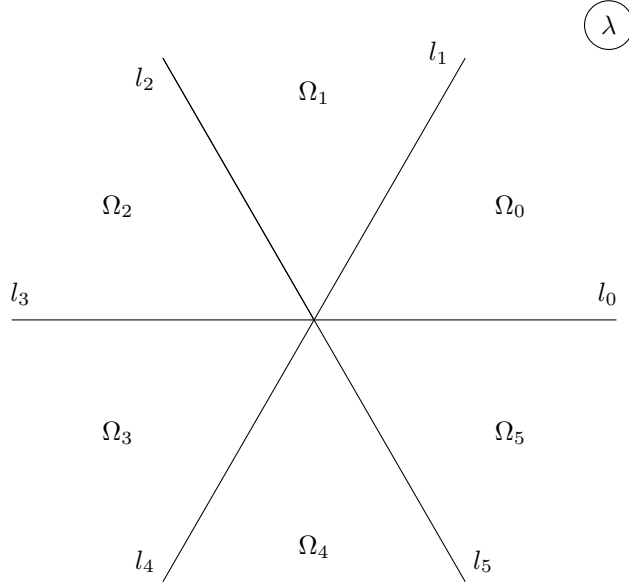
$$\begin{aligned} \tilde{L}\psi &\equiv i\frac{\partial\psi}{\partial x} + (U(x, t, \lambda) + C_r(U(x, t, \epsilon/\lambda)))\psi(x, t, \lambda) = 0, \\ \tilde{M}\psi &\equiv i\frac{\partial\psi}{\partial x} + (V(x, t, \lambda) + C_r(V(x, t, \epsilon/\lambda)))\psi(x, t, \lambda) = 0, \end{aligned} \quad (3.7)$$

Now, depending on the specific choice of C_r the FAS of \tilde{L} (3.7) may have substantially different analytic properties. The contour of the RHP consists of $2h$ -rays closing angles π/h intersected by the unit circle S^1 . As a result FAS can be defined in each of the $4h$ sectors and the formulation of the RHP changes into:

$$\begin{aligned} \xi_k^+(x, t, \lambda) &= \xi_{k-1}^+(x, t, \lambda)G_k^+(x, t, \lambda), & \lambda \in l_k^+, & \quad k = 0, \dots, 2h-1 \\ \xi_k^-(x, t, \lambda) &= \xi_{k-1}^-(x, t, \lambda)G_k^-(x, t, \lambda), & \lambda \in l_k^-, & \quad k = h, \dots, 2h-1 \\ \xi_{k+2h}^{(0)}(x, t, \lambda) &= \xi_k^{(0)}(x, t, \lambda)G_k^{(0)}(x, t, \lambda), & \lambda \in a_k, & \quad k = 0, \dots, 2h-1, \end{aligned} \quad (3.8)$$

where by l_k^+ (resp. l_k^-) is the the part of the ray l_k with $|\lambda| > 1$ (resp. $|\lambda| < 1$) and a_k are the arcs $k\pi/h \leq \arg \lambda \leq (k+1)\pi/h$ of the unit circle.

²The scalar Lax pair is derived in [13].

FIGURE 1. Contour of a RHP with \mathbb{Z}_3 symmetry

The x dependence of the sewing functions $G_k^{(b)}(x, t, \lambda)$, $b = 0, \pm$ is defined by:

$$\begin{aligned} i \frac{\partial G_k^\pm}{\partial x} - [\mathcal{J}(\lambda), G_k^\pm(x, t, \lambda)] &= 0, & \lambda \in l_k^\pm, & \quad k = 0, \dots, h-1, \\ i \frac{\partial G_k^{(0)}}{\partial x} - [\mathcal{J}(\lambda), G_k^{(0)}(x, t, \lambda)] &= 0, & \lambda \in a_k, & \quad k = 0, \dots, h-1, \end{aligned} \quad (3.9)$$

where $\mathcal{J}(\lambda)$ is invariant with respect to the reduction group G_R . Similarly the t dependence will be given by:

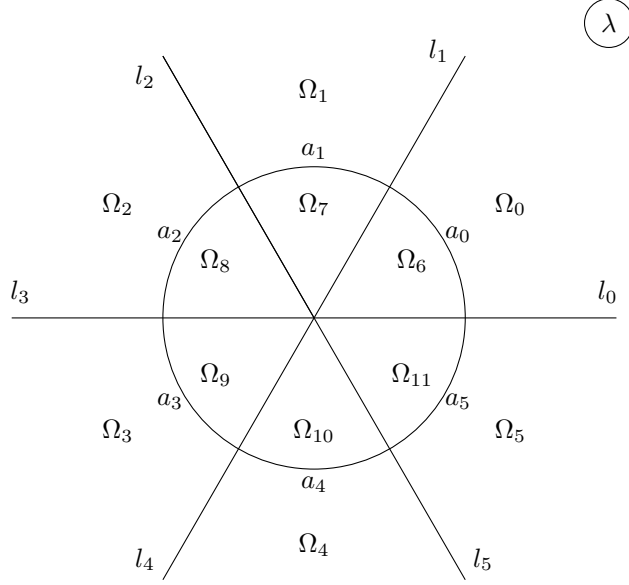
$$\begin{aligned} i \frac{\partial G_k^\pm}{\partial t} - [\mathcal{K}(\lambda), G_k^\pm(x, t, \lambda)] &= 0, & \lambda \in l_k^\pm, & \quad k = 0, \dots, h-1, \\ i \frac{\partial G_k^{(0)}}{\partial t} - [\mathcal{K}(\lambda), G_k^{(0)}(x, t, \lambda)] &= 0, & \lambda \in a_k, & \quad k = 0, \dots, h-1, \end{aligned} \quad (3.10)$$

where $\mathcal{K}(\lambda)$ must also be invariant with respect to the reduction group G_R . The explicit form of $\mathcal{J}(\lambda)$ and $\mathcal{K}(\lambda)$ depends on the chosen model. The contours of the RHP for the case $h = 3$ are given in fig. 2.

3.2. Equivalence of the RHP to a Lax representation. One of the famous results of Zakharov and Shabat was the proof of the equivalence between the RHP and the Lax representation [61, 62].

Our aim in this subsection is to outline the analog of Zakharov–Shabat theorem for the case of a RHP with a \mathbb{D}_h reduction group. Following the ideas of [61, 62] we introduce the functions

$$\begin{aligned} g_k^b(x, t) &= i \frac{\partial \xi_k^b}{\partial x} (\xi_k^b)^{-1} - \xi_k^b \mathcal{J}(\lambda) (\xi_k^b)^{-1}, \\ h_k^b(x, t) &= i \frac{\partial \xi_k^b}{\partial t} (\xi_k^b)^{-1} - \xi_k^b \mathcal{K}(\lambda) (\xi_k^b)^{-1}, \end{aligned} \quad (3.11)$$

FIGURE 2. Contour of the RHP \mathbb{D}_3 symmetry

with $b = 0, \pm$. Then it is not difficult to show that

$$g_k^b(x, t, \lambda) = g_{k-1}^b(x, t, \lambda), \quad h_k^b(x, t, \lambda) = h_{k-1}^b(x, t, \lambda), \quad (3.12)$$

for all $k = 0, \dots, h-1$. The proof uses the conditions that both $\mathcal{J}(\lambda)$ and $\mathcal{K}(\lambda)$ are invariant under the action of the group \mathbb{D}_h , and eqs. (3.10) hold. The functions $g_k^b(x, t, \lambda)$ and $h_k^b(x, t, \lambda)$ satisfy the appropriate reduction conditions:

$$\begin{aligned} C_s(g_k^b(x, t, \lambda\omega)) &= g_{k+1}^b(x, t, \lambda), & C_r(g_k^b(x, t, \lambda)) &= g_{k+2h}^b\left(x, t, \frac{\epsilon}{\lambda}\right), \\ C_s(h_k^b(x, t, \lambda\omega)) &= h_{k+1}^b(x, t, \lambda), & C_r(h_k^b(x, t, \lambda)) &= h_{k+2h}^b\left(x, t, \frac{\epsilon}{\lambda^*}\right), \end{aligned} \quad (3.13)$$

This means that $g_k^b(x, t, \lambda)$ and $h_k^b(x, t, \lambda)$ take values in the automorphic Lie algebra $\mathcal{A}(\mathfrak{g})$ introduced by Mikhailov and Lombardo [43]. In our case (two-point orbit of \mathbb{D}_h) it will be enough to introduce basis in the algebra $\mathcal{A}(\mathfrak{g})$ which is λ -dependent and is the analog of Cartan-Weyl basis:

$$\mathcal{E}_\alpha^{(k)}(\lambda) = \lambda^k E_\alpha + C_r(\lambda^{-k} E_\alpha), \quad \mathcal{H}_p^{(k)}(\lambda) = \lambda^k H_p + C_r(\lambda^{-k} H_p). \quad (3.14)$$

By $C_r(\lambda^{-k} E_\alpha)$ (resp. $C_r(\lambda^{-k} H_p)$) we mean the action of the reduction group element both on λ and the Weyl generator E_α (resp. the Cartan generator H_p). Obviously, also the potentials of the Lax operators will be taking values in $\mathcal{A}(\mathfrak{g})$. One can also prove the following

Corollary 1. *Let us consider the function $U(x, t, \lambda) \in \mathcal{A}(\mathfrak{g})$ such that for $\lambda \rightarrow \infty$*

$$\lim_{\lambda \rightarrow \infty} U(x, t, \lambda) \simeq \sum_{k=1}^{N_0} \lambda^k \left(\sum_{\alpha} U_{k,\alpha}(x, t) E_\alpha + \sum_{p=1}^r U_p(x, t) H_p \right). \quad (3.15)$$

Then

$$U(x, t, \lambda) = \sum_{k=1}^{N_0} \left(\sum_{\alpha} U_{k,\alpha}(x, t) \mathcal{E}_{\alpha}^{(k)}(\lambda) + \sum_{p=1}^r U_p(x, t) \mathcal{H}_p^{(k)}(\lambda) \right). \quad (3.16)$$

Thus we conclude that the asymptotics of $U(x, t, \lambda)$ for $\lambda \rightarrow \infty$ determines it uniquely as a function taking values in $\mathcal{A}(\mathfrak{g})$. The same holds true also for the functions $g_k^b(x, t, \lambda)$ and $h_k^b(x, t, \lambda)$.

Remark 1. The asymptotic of $g_k^b(x, t, \lambda)$ and $h_k^b(x, t, \lambda)$ for $\lambda \rightarrow \infty$ determine them uniquely. Indeed, we just need to combine this asymptotic with the reduction conditions (3.13).

4. Examples.

4.1. **Generalization of the GI equation** [20, 21]. Consider the Lax pair

$$\begin{aligned} L\psi &\equiv i \frac{\partial \psi}{\partial x} + U(x, t, \lambda) \psi(x, t, \lambda) = 0, \\ M\psi &\equiv i \frac{\partial \psi}{\partial t} + V(x, t, \lambda) \psi(x, t, \lambda) = 0, \end{aligned} \quad (4.1)$$

with

$$\begin{aligned} U(x, t, \lambda) &= \left(Q_0 + \lambda Q_1 - \lambda^2 J + \frac{1}{\lambda} \tilde{Q}_1 - \frac{1}{\lambda^2} \tilde{J} \right), \\ V(x, t, \lambda) &= \left(V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 - \lambda^4 K \right. \\ &\quad \left. + \frac{1}{\lambda} \tilde{V}_1 + \frac{1}{\lambda^2} \tilde{V}_2 + \frac{1}{\lambda^3} \tilde{V}_3 - \frac{1}{\lambda^4} \tilde{K} \right), \end{aligned} \quad (4.2)$$

where by ‘‘tilde’’ we mean

$$\tilde{X} = -BX^T B^{-1}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.3)$$

Here V_i, Q_i, J, K take values in the Lie algebra $\mathfrak{sl}(2)$. V_i, Q_i are functions of x and t and J and K are constant matrices. We will impose two reductions (types one and four from (2.3)). Their effect on the potential of the Lax operator is given by

$$\begin{aligned} 1) \quad & U^\dagger(x, t, \lambda^*) = U(x, t, \lambda), \\ 2) \quad & \tilde{U} \left(x, t, \frac{1}{\lambda} \right) = U(x, t, \lambda). \end{aligned} \quad (4.4)$$

The same holds for $V(x, t, \lambda)$.

The compatibility condition $[L, M] = 0$ leads to the following set of recursion relations:

$$\begin{aligned} \lambda^6 : \quad & [J, K] = 0, \\ \lambda^5 : \quad & [J, V_3] = [K, Q_2], \\ \lambda^4 : \quad & [J, V_2] = [K, Q_0] + [Q_1, V_3], \\ \lambda^3 : \quad & [J, V_1] = i \frac{\partial}{\partial x} V_3 + [Q_1, V_2] + [Q_0, V_3] - [\tilde{Q}_1, K], \\ \lambda^2 : \quad & [J, V_0] = i \frac{\partial}{\partial x} V_2 + [Q_0, V_2] + [Q_1, V_1] + [\tilde{Q}_1, V_3] \end{aligned}$$

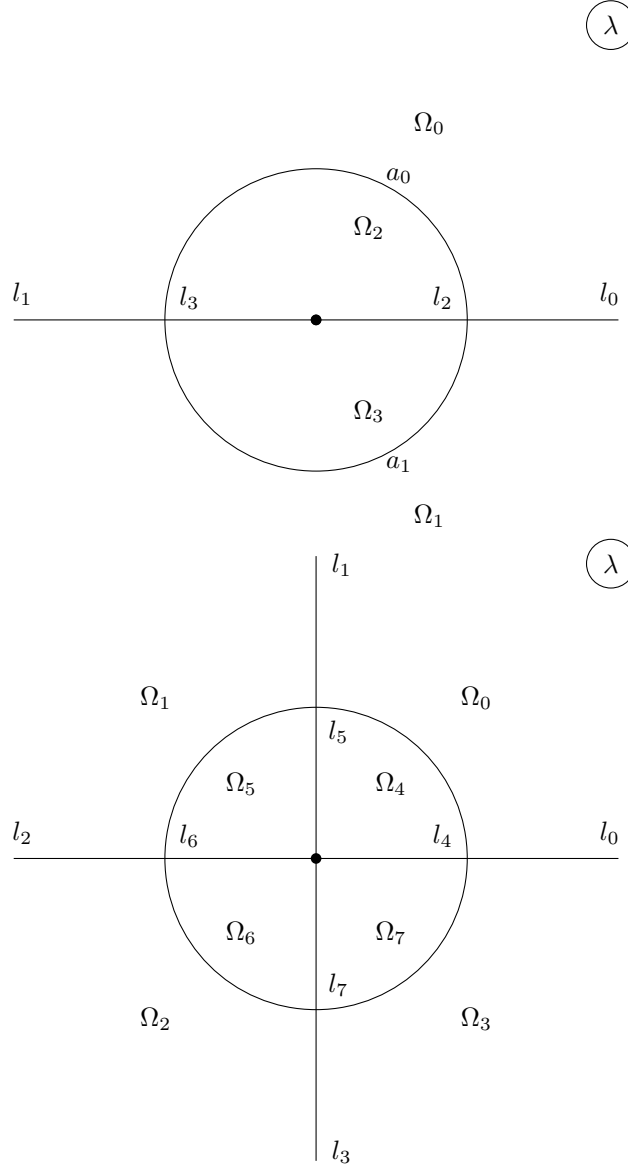


FIGURE 3. Contour of the RHP for \mathbb{D}_2 symmetry (upper panel) and for \mathbb{D}_4 symmetry (lower panel)

$$\begin{aligned}
 & + [\tilde{J}, K], \\
 \lambda^1 : & \quad i \frac{\partial}{\partial t} Q_1 = i \frac{\partial}{\partial x} V_1 + [Q_0, V_1] + [Q_1, V_0] + [\tilde{Q}_1, V_2] \\
 & \quad - [J, \tilde{V}_1] - [\tilde{J}, V_3], \\
 \lambda^0 : & \quad i \frac{\partial}{\partial t} Q_0 = i \frac{\partial}{\partial x} V_0 + [Q_0, V_0] + [Q_1, \tilde{V}_1] + [\tilde{Q}_1, V_1] \\
 & \quad - [J, \tilde{V}_2] - [\tilde{J}, V_2].
 \end{aligned} \tag{4.5}$$

We will impose two additional restrictions. The first is that the corresponding RHP should have canonical normalization (this means that (2.20) holds). The second will be an additional reduction. Let $\xi(x, t, \lambda)$ be a FAS. Then

$$\xi(x, t, -\lambda) = \xi^{-1}(x, t, \lambda). \quad (4.6)$$

This, together with (2.20), the reductions (4.4), and the recursion relations (4.5) leads to the following form for the coefficients of $U(x, t, \lambda)$ and $V(x, t, \lambda)$

$$\begin{aligned} Q_0 = V_2 &= \begin{pmatrix} -2qp & 0 \\ 0 & 2qp \end{pmatrix}, & V_3 &= \begin{pmatrix} 0 & 2q \\ -2p & 0 \end{pmatrix}, \\ V_1 &= \begin{pmatrix} 0 & i\partial_x q + 2q \\ i\partial_x p - 2p & 0 \end{pmatrix}, & & \\ V_0 &= \text{diag}\left(2q^2 p^2 + iq\partial_x p - ip\partial_x q - 4qp, -2q^2 p^2 - iq\partial_x p + ip\partial_x q + 4qp\right), \end{aligned} \quad (4.7)$$

where $p = q^*$. The λ^1 terms in (4.5) give the following equation

$$i\frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial^2 q}{\partial x^2} + 2iq^2\frac{\partial q^*}{\partial x} + 4q|q|^4 - 8q|q|^2 + 4q = 0. \quad (4.8)$$

This is GI equation with an additional cubic nonlinearity (and a linear term). The λ -independent term vanishes, provided that q is a solution of (4.8).

This equation has appeared in the list of integrable NLS-type equations classified by [50, 51]. Indeed, consider the equations (1.1) in [51]:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + f(u, v, u_x, v_x), \quad -\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + g(u, v, u_x, v_x), \quad (4.9)$$

with $v = u^*$, $g = f^*$ and fix up the function f according to eq. (1.10) in [51], i.e.:

$$f(u, v, u_x, v_x) = 2auvv_x + bu^2v_x + \frac{b(a-b)}{2}u^3v^2 + cu^2v. \quad (4.10)$$

Let us now put

$$\tau = \frac{i}{2}t, \quad u(x, \tau) = q(x, t)e^{-4it}, \quad a = 0, \quad b = 4i, \quad c = -16. \quad (4.11)$$

Then one can easily check that the first equation in (4.9) coincides with (4.8), while the second equation in (4.9) is obtained from (4.8) by complex conjugation. Similar arguments show one that: i) eq. (4.8) is equivalent to eq. (4.3.15) in [52] and, ii) eq. (24) in [57] is a vector generalization to (4.8).

4.2. Ferromagnet type equation. Let us denote by a tilde a given algebra automorphism. Let us take the Lax pair in the form

$$\begin{aligned} \left(\frac{\partial}{\partial x} + S_0 + \zeta S(x, t) + \frac{a}{\zeta}\tilde{S}\right)\psi(x, t, \zeta) &= 0, & S_0 &= \tilde{S}_0, \\ \left(\frac{\partial}{\partial t} + M_0 + \zeta W(x, t) + \frac{a}{\zeta}\tilde{W} + \zeta^2 S(x, t) + \frac{a^2}{\zeta^2}\tilde{S}\right)\psi(x, t, \zeta) &= 0, & & \\ M_0 &= \tilde{M}_0. \end{aligned} \quad (4.12)$$

where $S \in \mathfrak{g}$, a is a constant and ζ is a spectral parameter³. Moreover, we assume that $S^2 = \mathbb{1}$ and thus S defines an involution. Furthermore, a symmetric space is

³In this example the letters S , X , a , α etc are not related to any quantities from the previous (sub)sections.

defined as

$$\mathfrak{k} = \{X, \quad SXS = -X\}.$$

The equations arising from the Lax pair are

$$\begin{aligned} -S_x + [S, W] + [S_0, S] &= 0, \\ -\tilde{S}_x + [\tilde{S}, \tilde{W}] + [S_0, \tilde{S}] &= 0 \\ S_t - W_x + [S, M_0] + [S_0, W] + a[\tilde{S}, S] &= 0, \\ \tilde{S}_t - \tilde{W}_x + [\tilde{S}, M_0] + [S_0, \tilde{W}] + a[S, \tilde{S}] &= 0 \\ S_{0,t} - M_{0,x} + [S_0, M_0] + a[\tilde{S}, W] + a[S, \tilde{W}] &= 0. \end{aligned} \tag{4.13}$$

Then it is easy to spot that S_x, S_t and W belong to the symmetric space \mathfrak{k} . Moreover,

$$\frac{1}{4}\text{ad}_S^2 = \mathbb{1} \quad \text{on} \quad \mathfrak{k}.$$

From the first two equations

$$W(x, t) = S_0 + \frac{1}{4}[S, S_x], \quad \tilde{W}(x, t) = S_0 + \frac{1}{4}[\tilde{S}, \tilde{S}_x].$$

We can take for simplicity $S_0 = 0$, then

$$M_{0,x} = \frac{a}{4}[\tilde{S}, [S, S_x]] + \frac{a}{4}[S, [\tilde{S}, \tilde{S}_x]] \tag{4.14}$$

and one can write formally

$$M_0 = \frac{a}{4} \int_{-\infty}^x \left([\tilde{S}, [S, S_x]] + [S, [\tilde{S}, \tilde{S}_x]] \right) dx'.$$

So it seems that M_0 is in general nonlocal. The evolution equation becomes

$$S_t - \frac{1}{4}[S, S_{xx}] + [S, M_0] + a[\tilde{S}, S] = 0.$$

In the particular case when $\tilde{S} = -S^T$ and $\lim_{|x| \rightarrow \infty} S = J$ is real and diagonal with $J^2 = \mathbb{1}$, we clearly have

$$\lim_{|x| \rightarrow \infty} \tilde{S} = -J$$

and the asymptotic Lax operator satisfies

$$\left(\frac{\partial}{\partial x} + \left(\zeta - \frac{a}{\zeta} \right) J \right) \psi_a(x, t, \zeta) = 0$$

with

$$\psi_a(x, t, \zeta) = \exp \left(- \left(\zeta - \frac{a}{\zeta} \right) Jx \right). \tag{4.15}$$

In order to compare the spectrum with the spectrum of the Heisenberg ferromagnet model as it appears in most textbooks [16, 26] we introduce a spectral parameter $\lambda = \zeta/i = -i\zeta$. Then

$$\zeta - \frac{a}{\zeta} = -i \left(\lambda + \frac{a}{\lambda} \right)$$

and writing $\lambda = |\lambda|e^{i\gamma}$ we obtain

$$\zeta - \frac{a}{\zeta} = -i \left(|\lambda|e^{i\gamma} - \frac{a}{|\lambda|}e^{-i\gamma} \right) = \left(|\lambda| - \frac{a}{|\lambda|} \right) \sin \gamma - i \left(|\lambda| + \frac{a}{|\lambda|} \right) \cos \gamma.$$

Thus the continuous spectrum is where the exponent in (4.15) is oscillatory, that is $\sin \gamma = 0$, comprising the horizontal axis of the λ -plane and the set $|\lambda| = \frac{a}{|\lambda|}$ or

$|\lambda|^2 = a$ which is the circle of radius \sqrt{a} in the case when $a > 0$, as shown on fig. 3, the upper panel.

Let us now move to the $S \in su(2)$ case. We take the following parametrization:

$$\begin{aligned} S &= \begin{pmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{pmatrix}, & S &= S^\dagger \\ \tilde{S} &= -US^TU^{-1}. \end{aligned} \quad (4.16)$$

Since the above transformation is an involution, then

$$S = U(US^TU^{-1})^TU^{-1} = UU^{-T}SU^TU^{-1}.$$

Therefore $U^TU^{-1} = \pm \mathbb{1}$. The most general form of U is

$$U = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ -\bar{\mathbf{b}} & \bar{\mathbf{a}} \end{pmatrix}, \quad |\mathbf{a}|^2 + |\mathbf{b}|^2 = 1.$$

The case $U^TU^{-1} = -\mathbb{1}$ or $U^T = -U$ has only one nontrivial representative,

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and does not produce a new nonlinear evolution equation. Therefore, parameterising

$$\mathbf{a} = e^{i\alpha} \cos \theta, \quad \mathbf{b} = e^{i\beta} \sin \theta$$

with $U^T = U$ we have the most general form of U compatible with the involution, which necessitates $b = i \sin \theta$ and

$$U = \begin{pmatrix} e^{i\alpha} \cos \theta & i \sin \theta \\ i \sin \theta & e^{-i\alpha} \cos \theta \end{pmatrix}, \quad (4.17)$$

where α and θ are constant real parameters. In the special case when $\sin \theta = 0$, $\cos \theta = 1$ we have a diagonal transformation matrix

$$U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.$$

Introducing a vector form notation

$$\underline{S} = (S_1, S_2, S_3)^T, \quad S_1^2 + S_2^2 + S_3^2 = 1$$

from (4.16), (4.17) we obtain

$$\tilde{\underline{S}} = A \cdot \underline{S}$$

where the matrix A is explicitly given by

$$A = \begin{pmatrix} -\sin^2 \theta - \cos^2 \theta \cos 2\alpha & \cos^2 \theta \sin 2\alpha & -\sin 2\theta \sin \alpha \\ \cos^2 \theta \sin 2\alpha & \cos^2 \theta \cos 2\alpha - \sin^2 \theta & -\sin 2\theta \cos \alpha \\ -\sin 2\theta \sin \alpha & -\sin 2\theta \cos \alpha & -\cos 2\theta \end{pmatrix}. \quad (4.18)$$

The matrix A has the following properties,

$$A^2 = \mathbb{1}, \quad A = A^T, \quad A^T = A^{-1}.$$

Moreover, the diagonalization of A is

$$A = V \text{diag}(1, -1, -1) V^{-1}, \quad V^{-1} = V^T.$$

Next, we use the correspondence between the commutator of $su(2)$ matrices and the cross-product of 3-vectors:

$$[X, Y] = (2i)\mathcal{M}(\underline{X} \times \underline{Y})$$

where

$$\mathcal{M}(\underline{X}) = \begin{pmatrix} X_3 & X_1 - iX_2 \\ X_1 + iX_2 & -X_3 \end{pmatrix} \equiv X.$$

From (4.14)

$$\begin{aligned} \partial_x M_0 &= a \frac{(2i)^2}{4} \mathcal{M} \left(\tilde{\underline{S}} \times (\underline{S} \times \underline{S}_x) + \underline{S} \times (\tilde{\underline{S}} \times \tilde{\underline{S}}_x) \right) \\ &= -a \mathcal{M} \left((\tilde{\underline{S}} \cdot \underline{S}_x) \underline{S} + (\underline{S} \cdot \tilde{\underline{S}}_x) \tilde{\underline{S}} - (\underline{S} \cdot \tilde{\underline{S}}) (\underline{S} + \tilde{\underline{S}})_x \right) \\ &= -a (F_x (S + \tilde{S}) - 2F (S + \tilde{S})_x) \end{aligned} \quad (4.19)$$

where

$$F = \frac{1}{2} A_{ij} S_i S_j \quad (4.20)$$

is a scalar (summation is assumed). Indeed, since A is symmetric,

$$\tilde{\underline{S}} \cdot \underline{S}_x = A_{ij} S_j S_{i,x} = \frac{1}{2} (A_{ij} S_j S_i)_x$$

and similarly for $(\underline{S} \cdot \tilde{\underline{S}}_x)$.

In a vector form

$$\partial_x \underline{M}_0 = -a (F_x (\underline{S} + \tilde{\underline{S}}) - 2F (\underline{S} + \tilde{\underline{S}})_x)$$

Next, we represent the vectors $\underline{S} = VV^{-1}\underline{S}$ and $\tilde{\underline{S}} = A\underline{S} = V\Lambda V^{-1}\underline{S}$ where $\Lambda = \text{diag}(1, -1, -1)$, hence

$$\underline{S} + \tilde{\underline{S}} = V \text{diag}(2, 0, 0) V^{-1} \underline{S}.$$

Introducing the orthogonal transformation $\underline{\Sigma} = V^{-1}\underline{S}$ with

$$\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 1$$

and using the fact that

$$2F = \underline{S} \cdot (A\underline{S}) = \underline{\Sigma} \cdot (\Lambda \underline{\Sigma}) = \Sigma_1^2 - \Sigma_2^2 - \Sigma_3^2 = -1 + 2\Sigma_1^2$$

we further obtain

$$\begin{aligned} \underline{M}_{0,x} &= -aV \left(F_x \begin{pmatrix} 2\Sigma_1 \\ 0 \\ 0 \end{pmatrix} - 2F \begin{pmatrix} 2\Sigma_1 \\ 0 \\ 0 \end{pmatrix}_x \right) \\ &= -2aV \cdot \begin{pmatrix} F_x \Sigma_1 - 2F \Sigma_{1,x} \\ 0 \\ 0 \end{pmatrix} = -2aV \cdot \begin{pmatrix} \Sigma_{1,x} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (4.21)$$

Thus M_0 in this case is local. Moreover,

$$\underline{M}_0 = -2aV \cdot \begin{pmatrix} \Sigma_1 \\ 0 \\ 0 \end{pmatrix} = -aV(\mathbb{1} + \Lambda) \cdot \underline{\Sigma} = -aV(\mathbb{1} + \Lambda)V^{-1}\underline{S} = -a(\underline{S} + \tilde{\underline{S}}). \quad (4.22)$$

Hence $M_0 = -a(S + \tilde{S})$ and the equation becomes

$$S_t - \frac{1}{4}[S, S_{xx}] + 2a[\tilde{S}, S] = 0$$

or with a proper redefinition of the time variable (by a constant of $2i$), in a vector form

$$\underline{S}_t = \frac{1}{4} \underline{S} \times \underline{S}_{xx} + 2a \underline{S} \times \tilde{\underline{S}} = \frac{1}{4} \underline{S} \times \underline{S}_{xx} + 2a \underline{S} \times (A \cdot \underline{S}).$$

The matrix A is given in (4.18). This equation is very similar to the integrable Landau-Lifshitz model [16], in which the matrix A however is diagonal and to the integrable Landau-Lifshitz-type model derived by A. Borovik in [7] where the matrix A is a projector of rank 1. The Landau-Lifshitz type models describe, in general, the dynamics of an anisotropic ferromagnetic medium.

The Hamiltonian of the model is

$$H = \frac{1}{2} \int \left(\frac{1}{4} \underline{S}_x^2 - 4aF(\underline{S}) \right) dx$$

where $F(\underline{S})$ is the quadratic form (4.20). The Lie-Poisson bracket is

$$\{F_1, F_2\}(S) = - \int \left\langle S, \left[\frac{\delta F_1}{\delta S}, \frac{\delta F_2}{\delta S} \right] \right\rangle dx = - \int \left\langle \left[S, \frac{\delta F_1}{\delta S} \right], \frac{\delta F_2}{\delta S} \right\rangle dx$$

where $\langle \cdot, \cdot \rangle$ is the pairing in the Lie algebra. Writing the Lie-Poisson bracket in a vector form where the pairing is the usual Euclidean scalar product

$$\{F_1, F_2\}(S) = - \int \left\langle \underline{S} \times \frac{\delta F_1}{\delta \underline{S}}, \frac{\delta F_2}{\delta \underline{S}} \right\rangle dx,$$

(see the details in [34, 35]) gives

$$\underline{S}_t = \{\underline{S}, H\} = \underline{S} \times \frac{\delta H}{\delta \underline{S}}.$$

Generalizations of ferromagnet-type models related to Lie-algebras and symmetric spaces are studied in [59, 58, 22].

5. Discussions and conclusions. The scope of the present paper is limited to the \mathbb{Z}_h and \mathbb{D}_h reduction groups. Of course there are quite a few examples of NLEE related to the tetrahedral \mathbb{T} , octahedral \mathbb{O} and even to the icosahedral \mathbb{Y} reduction groups, see [8] and the references therein. The deeper studies of these NLEE should be based on the relevant automorphic Lie algebras $\mathcal{A}(\mathfrak{g})$ [43, 44, 45, 42].

The fact that the generalized GI eq. (4.8) is integrable has been known for long time now [50, 51]. The new facts about it are the Lax representation which possesses \mathbb{D}_4 symmetry. The solution of the inverse scattering problem (4.1), (4.2) requires construction of its fundamental analytic solutions (FAS). Skipping the technical details we remark here, that these FAS satisfy an equivalent RHP with nontrivial contour splitting the complex λ -plane into eight domains, see lower panel of fig. 3. The calculation of the soliton solutions of (4.8) can be done via the Zakharov-Shabat dressing method.

We point out also several open problems. The list of examples of NLEE having \mathbb{D}_h as group of reductions can be naturally extended by considering more general Lax pairs than (4.1), (4.2) related to symmetric spaces of higher rank. This could lead to Lax pairs for some of the vector NLEE whose integrability was proposed in [57]. It is rather natural to expect that these NLEE possess hierarchies of Hamiltonian structures, whose phase spaces are co-adjoint orbits of $\mathcal{A}(\mathfrak{g})$ passing through conveniently chosen element $\mathcal{H}_p^{(k)}(\lambda)$.

More detailed studies of the mapping between the potential of L and its scattering data would require the study of the Wronskian relations. While for \mathbb{Z}_h and \mathbb{D}_h reduction groups such construction seems to be rather straightforward, for \mathbb{T} , \mathbb{O} and \mathbb{Y} this would require additional efforts.

Acknowledgments. R.I. is thankful to the organizers of the conference *New Trends in Applied Geometric Mechanics – Celebrating Darryl Holm’s 70th birthday* at IC-MAT (Madrid, Spain) July 3–7, 2017 for their kind hospitality. One of us, V.S.G. is grateful to Prof. N. A. Kudrayshov and to New Bulgarian University for their support. R. I. I. and V. S. G. are grateful to Professors D. D. Holm and A. V. Mikhailov for many stimulating discussions. V. S. G and A. A. S. acknowledge funding from the NTS-Russia grant 02/101 from 23.10.2017.

REFERENCES

- [1] M. J. Ablowitz, B. Prinari and A. D. Trubach, *Discrete and Continuous Nonlinear Schrödinger Systems*, Cambridge University press, London Mathematical Society Lecture Note Series, 2004.
- [2] M. Adler, On a trace functional for pseudo-differential operators and the symplectic structure of the Korteweg-deVries Equation, *Invent. Math.*, **50** (1979), 219–248.
- [3] N. C. Babalic, R. Constantinescu and V. Gerdjikov, On the solutions of a family of Tzitzeica equations, *J. Geom. Symm. Physics*, **37** (2015), 1–24.
- [4] N. C. Babalic, R. Constantinescu and V. S. Gerdjikov, On Tzitzeica equation and spectral properties of related Lax operators, *Balkan Journal of Geometry and Its Applications*, **19** (2014), 11–22.
- [5] R. Beals and R. Coifman, Inverse Scattering and Evolution Equations, *Commun. Pure Appl. Math.*, **38** (1985), 29–42.
- [6] G. Berkeley, A. V. Mikhailov and P. Xenitidis, Darboux transformations with tetrahedral reduction group and related integrable systems, *Journal of Mathematical Physics*, **57** (2016), 092701, 15pp, [arXiv:1603.03289](https://arxiv.org/abs/1603.03289)
- [7] A. Borovik, N -soliton solutions of the Landau-Lifshitz equation, *Pis'ma Zh. Eksp. Teor. Fiz.*, **28** (1978), 629–632.
- [8] R. T. Bury, *Automorphic Lie Algebras, Corresponding Integrable Systems and Their Soliton Solutions*, PhD thesis, University of Leeds, 2010.
- [9] R. T. Bury, A. V. Mikhailov and J. P. Wang, Wave fronts and cascades of soliton interactions in the periodic two dimensional Volterra system, *Phys. D*, **347** (2017), 21–41, [arXiv:1603.03106v1](https://arxiv.org/abs/1603.03106v1) [nlin.SI]
- [10] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664; [arXiv:patt-sol/9305002](https://arxiv.org/abs/patt-sol/9305002)
- [11] A. Constantin, V. S. Gerdjikov and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inv. Problems*, **22** (2006), 2197–2207; [arXiv:nlin/0603019v2](https://arxiv.org/abs/nlin/0603019v2) [nlin.SI].
- [12] A. Constantin, R. Ivanov and J. Lenells, Inverse scattering transform for the Degasperis-Procesi equation, *Nonlinearity*, **23** (2010), 2559–2575.
- [13] A. Degasperis, D. Holm and A. Hone, A new integrable equation with peakon solutions, *Theor. Math. Phys.*, **133** (2002), 1461–1472.
- [14] L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, World scientific, 2003.
- [15] V. Drinfel'd and V. V. Sokolov, Lie algebras and equations of Korteweg - de Vries type, *Sov. J. Math.*, **22** (1995), 25–86.
- [16] L. D. Faddeev and L. A. Takhtadjan, *Hamiltonian Methods in the Theory of Solitons*, Springer Verlag, Berlin, 1987.
- [17] V. S. Gerdjikov, Algebraic and analytic aspects of N -wave type equations, *Contemporary Mathematics*, **301** (2002), 35–68.
- [18] V. S. Gerdjikov, Riemann-Hilbert Problems with canonical normalization and families of commuting operators, *Pliska Stud. Math. Bulgar.*, **21** (2012), 201–216; [arXiv:1204.2928v1](https://arxiv.org/abs/1204.2928v1) [nlin.SI].
- [19] V. S. Gerdjikov, Derivative nonlinear Schrödinger equations with \mathbb{Z}_N and \mathbb{D}_N -reductions, *Romanian Journal of Physics*, **58** (2013), 573–582.
- [20] V. S. Gerdjikov, M. I. Ivanov, The quadratic bundle of general form and the nonlinear evolution equations. I. Expansions over the “squared” solutions are generalized Fourier transforms, *Bulgarian J. Phys.*, **10** (1983), 13–26 (In Russian).
- [21] V. S. Gerdjikov and M. I. Ivanov, The quadratic bundle of general form and the nonlinear evolution equations. II. Hierarchies of Hamiltonian structures, *Bulgarian J. Phys.*, **10** (1983), 130–143 (In Russian).

- [22] V. S. Gerdjikov, G. G. Grahovski, A. V. Mikhailov and T. I. Valchev, On soliton interactions for the hierarchy of a generalised Heisenberg ferromagnetic model on $SU(3)/S(U(1) \times U(2))$ symmetric space, *Journal of Geometry and Symmetry in Physics*, **25** (2012), 23–55.
- [23] V. S. Gerdjikov, D. M. Mladenov, A. A. Stefanov and S. K. Varbev, MKdV-type of equations related to $B_2^{(1)}$ and $A_4^{(2)}$, in *Nonlinear Mathematical Physics and Natural Hazards*, (eds: Boyka Aneva, Mihaela Kouteva-Guentcheva), Springer Proceedings in Physics, **163** (2015), 59–69. ISBN: 978-3-319-14327-9 (Print) 978-3-319-14328-6 (Online).
- [24] V. S. Gerdjikov, D. M. Mladenov, A. A. Stefanov and S. K. Varbev, Soliton equations related to the affine Kac-Moody algebra $D_4^{(1)}$. *Eur. Phys. J. Plus*, **130** (2015), 106–123; [arXiv:1412.2383v1](#) [nlin.SI].
- [25] V. S. Gerdjikov, D. M. Mladenov, A. A. Stefanov and S. K. Varbev, On mKdV equations related to the affine Kac-Moody algebra $A_5^{(2)}$, *J. Geom. Sym. Phys.*, **39** (2015), 17–31, [arXiv:1512.01475](#) nlin.SI.
- [26] V. Gerdjikov, G. Vilasi and A. Yanovski, *Integrable Hamiltonian Hierarchies*. Spectral and Geometric Methods, Lecture Notes in Physics, **748**, Springer, Berlin - Heidelberg, 2008.
- [27] V. S. Gerdjikov and A. B. Yanovski, Completeness of the eigenfunctions for the Caudrey–Beals–Coifman system, *J. Math. Phys.*, **35** (1994), 3687–3725.
- [28] V. S. Gerdjikov and A. B. Yanovski, Riemann-Hilbert Problems, families of commuting operators and soliton equations, *Journal of Physics: Conference Series*, **482** (2014), 012017.
- [29] V. S. Gerdjikov and A. B. Yanovski, On soliton equations with \mathbb{Z}_h and \mathbb{D}_h reductions: conservation laws and generating operators, *J. Geom. Symmetry Phys.*, **31** (2013), 57–92.
- [30] V. S. Gerdjikov and A. B. Yanovski, CBC systems with Mikhailov reductions by Coxeter automorphism. I. Spectral theory of the recursion operators, *Studies in Applied Mathematics*, **134** (2015), 145–180.
- [31] M. Gürses, A. Karasu and V. V. Sokolov, On construction of recursion operators from Lax representation, *Journal of Mathematical Physics*, **40** (1999), 6473–6490.
- [32] J. Haberland and T. Lyons, Solitons of shallow-water models from energy-dependent spectral problems, *Eur. Phys. J. Plus*, **133** (2018), 16, [arXiv:1705.04989](#) [math-ph]
- [33] S. Helgasson, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York-London, 1978.
- [34] D. D. Holm, *Geometric Mechanics Part I: Dynamics and Symmetry*, Imperial College Press: London, 2011.
- [35] D. D. Holm, *Geometric Mechanics Part II: Rotating, Translating and Rolling*, Imperial College Press: London, 2011.
- [36] D. Holm and R. Ivanov, Smooth and peaked solitons of the CH equation, *J. Phys. A: Math. Theor.*, **43** (2010), 434003 (18pp).
- [37] D. Holm and R. Ivanov, Two-component CH system: Inverse scattering, peakons and geometry, *Inverse Problems*, **27** (2011), 045013, 19pp, [arXiv:1009.5374v1](#) [nlin.SI]
- [38] R. Ivanov, On the dressing method for the generalised Zakharov-Shabat system, *Nuclear Physics B*, **694** (2004), 509–524; math-ph/0402031.
- [39] R. Ivanov and T. Lyons, Integrable models for shallow water with energy dependent spectral problems, *Journal of Nonlinear Mathematical Physics*, **19** (2012), 1240008 (17 pages).
- [40] D. J. Kaup, The three-wave interaction - a nondispersive phenomenon, *Stud. Appl. Math.*, **55** (1976), 9–44.
- [41] D. J. Kaup, On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, *Stud. Appl. Math.*, **62** (1980), 189–216.
- [42] V. Knibbeler, S. Lombardo and J. A. Sanders, Higher-Dimensional Automorphic Lie Algebras, *Found. Comput. Math.*, **17** (2017), 987–1035.
- [43] S. Lombardo and A. V. Mikhailov, Reductions of integrable equations: Dihedral group, *J. Phys. A*, **37** (2004), 7727–7742; [arXiv:nlin/0404013](#) [nlin.SI]
- [44] S. Lombardo, A.V. Mikhailov, Reduction groups and Automorphic Lie Algebras, *Communication in Mathematical Physics*, **258** (2005), 179–202.
- [45] S. Lombardo and J. Sanders, On the classification of automorphic Lie algebras, *Communications in Mathematical Physics*, **299** (2010), 793–824; [arXiv:0912.1697](#) [math.RA].
- [46] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys. JETP*, **38** (1974), 248.

- [47] S. V. Manakov and V. E. Zakharov, Soliton theory, in: *Physics Review*, Vol. I, I. M. Khalatnikov (ed.), London, (1979), 133–190.
- [48] A. V. Mikhailov, The reduction problem and the inverse scattering problem, *Physica D*, **3** (1981), 73–117.
- [49] A. V. Mikhailov, M. A. Olshanetsky, and A. M. Perelomov, Two-dimensional generalized Toda lattice, *Comm. Math. Phys.*, **79** (1981), 473–488.
- [50] A. V. Mikhailov, A. B. Shabat and R. I. Yamilov, The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems, *Russian Math. Surveys*, **42** (1987), 1–63.
- [51] A. V. Mikhailov, A. B. Shabat and R. I. Yamilov, Extension of the module of invertible transformations. classification of integrable systems, *Commun. Math. Phys.*, **115** (1988), 1–19.
- [52] A. V. Mikhailov, A. B. Shabat and V. V. Sokolov, The symmetry approach to classification of integrable equations, In: *Zakharov V.E. (eds) What Is Integrability?*, 115–184, Springer Series in Nonlinear Dynamics. Springer, Berlin, 1991.
- [53] A. V. Mikhailov and V. E. Zakharov, On the integrability of classical spinor models in two-dimensional space–time, *Commun. Math. Phys.*, **74** (1980), 21–40.
- [54] S. Novikov, S. Manakov, L. Pitaevskii and V. Zakharov, *Theory of Solitons: The Inverse Scattering Method*, Plenum, Consultants Bureau, New York, 1984.
- [55] A. B. Shabat, Inverse scattering problem for a system of differential equations, *Funkts. Anal. Prilozhen.*, **9** (1975), 75–78; English translation: *Funct. Anal. Appl.*, **9** (1975), 244–247.
- [56] A. B. Shabat, A one-dimensional scattering theory. I, *Differ. Uravn.*, **8** (1972), 164–178 (In Russian).
- [57] V. V. Sokolov and T. Wolf, Classification of integrable polynomial vector evolution equations, *J. Phys. A: Math. Gen.*, **34** (2001), 11139–11148.
- [58] T. I. Valchev, On certain reductions of integrable equations on symmetric spaces, *AIP Conference Proceedings*, **1340** (2011), 154–164.
- [59] A. B. Yanovski and T. I. Valchev, Pseudo-Hermitian reduction of a generalized Heisenberg ferromagnet equation. I. Auxiliary system and fundamental properties; *Journal of Nonlinear Mathematical Physics*, **25** (2018), 324–350.
- [60] V. E. Zakharov and S. V. Manakov, On the theory of resonance interactions of wave packets in nonlinear media, *Zh. Exp. Teor. Fiz.*, **69** (1975), 1654–1673.
- [61] V. E. Zakharov and A. B. Shabat, A scheme for integrating nonlinear evolution equations of mathematical physics by the inverse scattering problem. I, *Funkts. Anal. Prilozhen.*, **8** (1974), 43–53; English translation: *Funct. Anal. Appl.*, **8** (1974), 226–235.
- [62] V. E. Zakharov and A. B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering II., *Funkts. Anal. Prilozhen.*, **13** (1979), 13–22; English translation: *Funct. Anal. Appl.*, **13** (1979), 166–174.
- [63] V. E. Zakharov, Exact solutions of the problem of parametric interaction of wave packets, *Dokl. Akad. Nauk SSSR*, **228** (1976), 1314–1316.
- [64] V. E. Zakharov, The inverse scattering method, In: *Solitons, R. K. Bullough and P. J. Caudrey (editors)*, Springer-Verlag, Berlin, (1980), 243–286.
- [65] V. E. Zakharov, Integrable systems in multidimensional spaces, in *Mathematical Problems in Theoretical Physics*, Lecture Notes in Phys., **153**, Springer-Verlag, Berlin, (1982), 190–216.

Received April 2018; revised March 2019.

E-mail address: vgerdjikov@gmail.com

E-mail address: rossen.ivanov@dit.ie

E-mail address: aleksander.a.stefanov@gmail.com