2002-01-01

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Recommended Citation

R. Ivanov and M.P. Tuite, Rational Generalized Moonshine from Abelian Orbifoldings of the Moonshine Module, Nuclear Physics B 635, n 3, (2002) 435-472

Rational Generalised Moonshine from Abelian Orbifoldings of the Moonshine Module

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Abstract

We consider orbifoldings of the Moonshine Module with respect to the abelian group generated by a pair of commuting Monster group elements with one of prime order $p = 2, 3, 5, 7$ and the other of order pk for $k = 1$ or k prime. We show that constraints arising from meromorphic orbifold conformal field theory allow us to demonstrate that each orbifold partition function with rational coefficients is either constant or is a hauptmodul for an explicitly found modular fixing group of genus zero. We thus confirm in the cases considered the Generalised Moonshine conjectures for all rational modular functions for the Monster centralisers related to the Baby Monster, Fischer, Harada-Norton and Held sporadic simple groups. We also derive non-trivial constraints on the possible Monster conjugacy classes to which the elements of the orbifolding abelian group may belong.

PACS: 11.25.Hf, 02.10.De, 02.20.Bb
Key Words: Conformal Fields, Modular Groups, Moonshine, Orbifolds.
Los Alamos Database Number: math.QA/0106027

1 Introduction

Orbifold constructions [1] in Meromorphic Conformal Field Theory [2],[3] (MCFT) and Vertex Operator Algebras [4],[5],[6] provide the most natural setting for understanding Moonshine phenomena [7],[8],[9]. The Moonshine Module [10],[4] whose automorphism group is the Monster finite sporadic group $\mathbf{M}$, is an orbifold MCFT constructed by orbifolding the Leech lattice MCFT with respect to the group generated by a reflection involution. The Moonshine Module partition function is the classical elliptic J function which is a hauptmodul for the genus zero modular group $SL(2, \mathbf{Z})$ and is believed to be the unique MCFT with this partition function [4]. Orbifolding the Moonshine Module with respect to the group generated by $g \in \mathbf{M}$ leads naturally to the notion of the orbifold partition function i.e. the Thompson series T_g. Monstrous Moonshine is mainly concerned with the property, conjectured by Conway and Norton [11] and subsequently proved by Borcherds [12], that each Thompson series T_g is a hauptmodul for some genus zero fixing modular group. Assuming the uniqueness of the Moonshine Module, this genus zero property is believed to be equivalent to the following statement [8]: the only orbifold MCFT that can arise by orbifolding with respect to g is either the Moonshine Module itself (for g belonging to a so-called Fricke Monster conjugacy class) or the Leech lattice MCFT (for g belonging to a non-Fricke Monster conjugacy class).

The Generalised Moonshine conjecture of Norton [13] is concerned with modular functions associated with a commuting pair $g, h \in \mathbf{M}$ and asserts that each such modular function is either constant or is a hauptmodul for some genus zero fixing group. No extension of the Borcherds' approach to Monstrous Moonshine has yet been shown to be possible for Generalised Moonshine. We argue that the most natural setting for these conjectures is to consider orbifoldings of the Moonshine Module with respect to the abelian group $\langle g, h\rangle$ generated by g, h [14]. In the cases where $\langle g, h\rangle$ can be generated by a single Monster element, for example when g and h have coprime orders, then Generalised Moonshine follows directly from Monstrous Moonshine. In this paper we consider the case where g is of prime order $p = 2, 3, 5$ and 7 and is of Fricke type and h is of order pk for $k = 1$ or k prime. We confirm Norton's conjecture for modular functions with rational coefficients in these cases by considering orbifold modular properties and some consistency conditions arising from the orbifolding procedure. We also demonstrate a number of other non-trivial aspects of Generalised Moonshine such as properties of the character expansion of Generalised Moonshine functions in terms of irreducible characters for the centraliser of g in $\mathbf{M}$ and constraints on the possible Monster conjugacy classes to which the elements of $\langle g, h\rangle$ may belong.

We begin in Section 2 with a general review of Abelian orbifold constructions in Meromorphic Conformal Field Theory (MCFT). We also briefly review the construction of the Moonshine Module and the relationship between the genus zero property of Thompson series T_g in Monstrous Moonshine for $g \in \mathbf{M}$ and evidence to support the claim that the only possible MCFTs obtainable by orbifolding the Moonshine Module with respect to the group generated by g are the Moonshine Module, for g Fricke, and the Leech lattice MCFT, for g non-Fricke.

In Section 3 we begin with a discussion of general properties for Generalised Moonshine Functions (GMF) following from the orbifold considerations of Section 2. We then prove two theorems concerning constraints that arise from the consistency of orbifolding the Moonshine Module with respect to $\langle g, h\rangle$ under various choices of generators. These constraints are exploited in Section 4 in order to determine the residues of singular cusps of GMFs. In particular if all elements of $\langle g, h\rangle$ are non-Fricke the GMF is constant. We then prove some modular properties of GMFs with rational coefficents in the cases where g is Fricke of prime order $p = 2, 3, 5$ and 7 and h is of order pk for $k = 1$ or k prime. This analysis in part relies on properties of the characters of the centralisers of the Monster related to the Baby Monster, Fischer, Harada-Norton and Held sporadic simple groups. We also highlight the importance of the Monster conjugacy classes to which the elements of $\langle g, h\rangle$ belong in determining the possible singularities of a GMF.

In Section 4 we give a comprehensive analysis of the possible singularity structure of GMFs for the cases under consideration. In each case we demonstrate that either the given singularity structure is inconsistent or else all singularities of the GMF can be identified under some genus zero fixing group for which the GMF is a hauptmodul. Thus we obtain non-trivial constraints on the Monster classes to which the elements of $\langle g, h\rangle$ may belong and verify the Generalised Moonshine conjecture in these cases.

In Appendix A we review the definitions of standard modular groups. In Appendix B we consider the first ten coefficients of a general GMF as character expansions for the Monster centralisers related to the Baby Monster and Fischer centraliser subgroups.

2 Abelian Orbifolds and Monstrous Moonshine

2.1 Self-Dual $C = 24$ Meromorphic CFTs

A Meromorphic CFT (MCFT) or chiral CFT is a CFT whose n-point functions are all meromorphic as described in [2], [3]. A MCFT essentially corresponds to a Vertex Operator Algebra in the pure mathematics literature as reviewed in [4], [5], [6]. We briefly review some of their basic properties. A MCFT $(\mathcal{V}, \mathcal{H})$ consists of a Hilbert space of states $\mathcal{H}$ together with a set of vertex operators $\mathcal{V} \equiv \{V(\psi, z) = \sum_{n\in\mathbf{Z}} \psi_n z^{-n-h_\psi} | \psi \in \mathcal{H}\}$ where h_ψ is the conformal weight (see below) and where each mode ψ_n acts as a linear operator on $\mathcal{H}$. The vertex operators are local meaning that for given $\varphi, \psi \in \mathcal{H}$ and sufficiently large N

$$(V(\psi, z)V(\varphi, w) - V(\varphi, w)V(\psi, z))(z - w)^N = 0. \tag{1}$$

$\mathcal{H}$ contains a distinguished vacuum state $|0\rangle$ such that

$$\begin{aligned} V(|0\rangle, z) &= \mathrm{Id}_{\mathcal{H}}, &(2)\\ \lim_{z\to 0} V(\psi, z)|0\rangle &= \psi. &(3)\end{aligned}$$

$\mathcal{H}$ also contains a Virasoro state ω with vertex operator Laurant expansion $V(\omega, z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2}$ where L_n generates the Virasoro algebra of central charge C

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}C(m^3 - m)\delta_{m,-n}. \tag{4}$$

We will consider here MCFTs of central charge $C = 24$ only. The vector space $\mathcal{H}$ is decomposed into finite dimensional spaces $\mathcal{H}_n$ with non-negative integral L_0 grading n, the conformal weight. Taking these various properties together, the vertex operators then satisfy the Operator Product Expansion (OPE) for $|z| > |w|$

$$\begin{aligned} V(\psi, z)V(\varphi, w) &= V(V(\psi, z - w)\varphi, w) &(5)\\ &= \sum_{n\geq 0}\sum_{\chi\in\mathcal{H}_n} C^{\chi}_{\psi\varphi} V(\chi, w)(z - w)^{n-h_\psi-h_\varphi} &(6)\end{aligned}$$

with $\psi_{h_\varphi-n}(\varphi) = \sum_{\chi\in\mathcal{H}_n} C^{\chi}_{\psi\varphi}\chi$ and where the sum is taken over some basis for $\mathcal{H}_n$ [2], [3].

The automorphism group $\mathrm{Aut}(\mathcal{V})$ of $\mathcal{V}$ is the group of linear transformations $g : \mathcal{H} \to \mathcal{H}$ which preserves the Virasoro state ω and where

$$gV(\psi, z)g^{-1} = V(g\psi, z). \tag{7}$$

The OPE (5) is then invariant under $\mathrm{Aut}(\mathcal{V})$. In the case of the Moonshine Module $\mathcal{V}^\natural$, $\mathrm{Aut}(\mathcal{V}^\natural) = \mathbf{M}$ the Monster group, which has a unitary action on $\mathcal{V}^\natural$. We will assume that $\mathrm{Aut}(\mathcal{V})$ has a unitary action from now on (which is expected physically from unitarity).

A representation $(\mathcal{U}, \mathcal{K})$ of a MCFT $(\mathcal{V}, \mathcal{H})$ consists of a vector space (module) $\mathcal{K}$ together with a set of local vertex operators $\mathcal{U} \equiv \{U(\psi, z)|\psi \in \mathcal{H}\}$ where the modes of $U(\psi, z)$ act as linear operators on $\mathcal{K}$ with $U(|0\rangle, z) = \mathrm{Id}_{\mathcal{K}}$. These operators satisfy the OPE

$$U(\psi, z)U(\varphi, w) = U(V(\psi, z-w)\varphi, w), \quad |z| > |w|. \tag{8}$$

$(\mathcal{U}, \mathcal{K})$ is an irreducible representation if $\mathcal{K}$ contains no non-trivial submodule invariant under the modes of $\{U(\psi, z)\}$. $(\mathcal{V}, \mathcal{H})$ is said to be a Self-Dual MCFT or a Holomorphic VOA if $(\mathcal{V}, \mathcal{H})$ is the unique irreducible representation for itself. We assume that $\mathcal{K}$ is decomposed into Verma modules of the Virasoro algebra of $\mathcal{V}$ with non-negative integral L_0 grading i.e. we consider unitary Virasoro representations. Let $\mathcal{K}_0 \subset \mathcal{K}$ denote the subspace of lowest L_0 grading. Then for an irreducible representation, $\mathcal{K}$ is generated by the action of the modes of $\{U(\psi, z)\}$ on $\mathcal{K}_0$.

The automorphism group $\mathrm{Aut}(\mathcal{U})$ of $\mathcal{U}$ is the group of linear transformations $\hat{g} : \mathcal{K} \to \mathcal{K}$ of the form

$$\hat{g}U(\psi, z)\hat{g}^{-1} = U(g\psi, z) \tag{9}$$

for some $g \in \mathrm{Aut}(\mathcal{V})$. A general element of $\mathrm{Aut}(\mathcal{V})$ may give rise to a mapping between different representations and so we denote by $\mathrm{Aut}_{\mathcal{K}}(\mathcal{V})$ the subgroup of $\mathrm{Aut}(\mathcal{V})$ associated with (9). Clearly $\mathrm{Aut}_{\mathcal{K}}(\mathcal{V})$ acts projectively on $\mathcal{K}$ with a natural homomorphism from $\mathrm{Aut}(\mathcal{U})$ to $\mathrm{Aut}_{\mathcal{K}}(\mathcal{V})$ whose kernel consists of phase multipliers assuming that $\mathrm{Aut}(\mathcal{U})$ acts unitarily on $\mathcal{K}$ i.e. $\mathrm{Aut}(\mathcal{U}) = U(1).\mathrm{Aut}_{\mathcal{K}}(\mathcal{V})$. For an irreducible representation, where $\mathcal{K}_0$ is one dimensional, then the $U(1)$ subgroup of $\mathrm{Aut}(\mathcal{U})$ has the same action on all elements of $\mathcal{K}$ so that

$$\mathrm{Aut}(\mathcal{U}) = U(1) \times \mathrm{Aut}_{\mathcal{K}}(\mathcal{V}), \tag{10}$$

if $\dim(\mathcal{K}_0) = 1$.

Let us now assume that $\mathcal{V}$ is Self-Dual. The characteristic function (or genus one partition function) for $\mathcal{V}$ of central charge $C = 24$ is given by

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - 1}), \; q = e^{2\pi i \tau} \tag{11}$$

where $\tau \in \mathbf{H}$, the upper half complex plane, is the usual elliptic modular parameter. $\mathcal{V}$ has integral grading so that $Z(\tau)$ is invariant under $T : \tau \to \tau + 1$. Self-duality implies that $Z(\tau)$ is invariant under $S : \tau \to -1/\tau$ so that $Z(\tau)$ is invariant under the modular group $SL(2, \mathbf{Z})$, generated by S, T [15]. Hence $Z(\tau)$ is uniquely determined up to an additive constant by $J(\tau)$, the hauptmodul for $SL(2, \mathbf{Z})$ [16]

$$\begin{aligned} Z(\tau) &= J(\tau) + N_0, \\ J(\tau) &= \frac{E_4^3}{\eta^{24}} - 744 = \frac{1}{q} + 0 + 196884q + ..., \end{aligned} \tag{12}$$

where $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$ is the Dedekind eta function, $E_n(\tau)$ is the Eisenstein form of weight n [16] and N_0 is the number of conformal weight 1 operators in $\mathcal{V}$. Examples of such theories are lattice models, which we denote by $\mathcal{V}^\Lambda$, where Λ is one of the Niemeier even self-dual 24 dimensional lattices. Then $Z(\tau) = \Theta_\Lambda/\eta^{24}$ with $\Theta_\Lambda = \sum_{\lambda\in\Lambda} q^{\lambda^2/2}$ the lattice theta function for Λ. In this paper we will be particularly concerned with the Leech lattice for which $N_0 = 24$ and the Moonshine Module $\mathcal{V}^\natural$ for which $N_0 = 0$.

2.2 Twisted Sectors of a Self-Dual MCFT

In this section we briefly review relevant aspects of twisted representation of a MCFT. Let $G = \mathrm{Aut}(\mathcal{V})$, denote the automorphism group (7) for $\mathcal{V}$. Consider $g \in G$ of finite order $o(g) = n$ and define the trace function

$$Z\left[\begin{array}{c} g \\ 1 \end{array}\right](\tau) \equiv \mathrm{Tr}_{\mathcal{H}}(gq^{L_0-1}). \tag{13}$$

Clearly $Z\left[\begin{array}{c} 1 \\ 1 \end{array}\right](\tau) \equiv Z(\tau)$ of (11). In the case where $\mathcal{V} = \mathcal{V}^\natural$, the Moonshine Module, (13) is the Thompson series (see below). Let $\mathcal{H}^{(j)}$ denote the eigenspace of $\mathcal{H}$ for g with eigenvalue ω_n^j for $\omega_n \equiv \exp(2\pi i/n)$. The twisted representation $(\mathcal{V}_g, \mathcal{H}_g)$ of $(\mathcal{V}, \mathcal{H})$ consists of a set of vertex operators $\mathcal{V}_g \equiv \{V_g(\psi, z)|\psi \in \mathcal{H}\}$ with mode expansion

$$V_g(\psi, z) = \sum_{m\in\mathbf{Z}+j/n} \tilde{\psi}_m z^{-m-1}, \quad \psi \in \mathcal{H}^{(j)} \tag{14}$$

whose modes $\tilde{\psi}_m$ are linear operators on $\mathcal{H}_g$. The twisting property corresponds to the monodromy relation $V_g(\psi, e^{2\pi i}z) = V_g(g^{-1}\psi, z)$. Furthermore the twisted vertex operators satisfy the 'twisted' non-meromorphic OPE

$$V_g(\psi, z)V_g(\varphi, w) = \omega_n^{-jb} V_g(V(\psi, z-w)\varphi, w), \tag{15}$$

for $\psi \in \mathcal{H}^{(j)}$ and where $b = 0, 1, 2..., n-1$ labels the sheet for the branched n-fold covering for $(w/z)^{1/n}$. For $(\mathcal{V}, \mathcal{H})$ self-dual, $(\mathcal{V}_g, \mathcal{H}_g)$ always exists and is unique up to isomorphism [9]. Furthermore, under conjugation by any element $x \in G$ then $x(\mathcal{V}_g)x^{-1}$ is isomorphic to $\mathcal{V}_{xgx^{-1}}$.

Considering (15) for $\psi, \varphi \in \mathcal{H}^{(0)}$ we see that $(\mathcal{V}_g, \mathcal{H}_g)$ forms a reducible representation for $(\mathcal{V}^{(0)}, \mathcal{H}^{(0)})$ as in (8) where $\mathcal{V}^{(0)}$ are the vertex operators for $\mathcal{H}^{(0)}$. $\mathcal{H}_g$ can therefore be decomposed into L_0 eigenspaces $\mathcal{H}_g = \bigoplus_{m=0}^{\infty} \mathcal{H}_{g,m}$ where $\mathcal{H}_{g,m}$ has rational L_0 eigenvalue $E_0^g + 1 + m/n$. The subspace $\{\sigma_g^a\}$ for $a = 1, 2, ..., D_g$ of lowest grade $E_0^g + 1$ is called the g-twisted vacuum space. E_0^g is called the vacuum energy and D_g is called the vacuum degeneracy.

The automorphism group, $\mathrm{Aut}(\mathcal{V}_g)$, preserving (15) can be defined in a way similar to (9) as an extension of $C_g = \{h \in G|gh = hg\}$, the centraliser of g in G. Since $\mathcal{V}_g$ is unique for a self-dual MCFT we have $\mathrm{Aut}(\mathcal{V}_g) = U(1).C_g$. If the twisted vacuum is unique ($D_g = 1$) then $\mathrm{Aut}(\mathcal{V}_g) = U(1) \times C_g$ from (10).

Let $\hat{g} \in \mathrm{Aut}(\mathcal{V}_g)$ denote the lifting of g with action on the twisted vacuum as follows

$$\hat{g}\sigma^a_g = \exp(-2\pi i E^g_0)\sigma^a_g, \tag{16}$$

where $\exp(-2\pi i E^g_0)$ is a $U(1)$ phase. Then (14) implies that in general

$$\hat{g}\psi_g = \exp(-2\pi i h_g)\psi_g, \tag{17}$$

where $\psi_g \in \mathcal{H}_g$ has Virasoro grading h_g. Clearly $n|o(\hat{g})$, where $o(\hat{g})$ is the order of $\hat{g}$, since $\hat{g}^n$ is a lifting of the identity element of C_g so that $E^g_0 \in \mathbf{Z}/o(\hat{g})$. We say that g is a Normal element of G if $nE^g_0 \in \mathbf{Z}$ so that $\hat{g}$ is of order n, otherwise we say that g is an Anomalous element of G.

Let g be a normal element of G and consider the twisted spaces $(\mathcal{V}_{g^k}, \mathcal{H}_{g^k})$ for $k = 1, 2, ..., n-1$. Let $\mathcal{H}^{(j)}_{g^k}$ denote the eigenspace of $\mathcal{H}_{g^k}$ with $\hat{g}$ eigenvalue ω^j_n where $\mathcal{H}^{(j)}_{g^k}$ can be further decomposed into L_0 eigenspaces $\mathcal{H}^{(j)}_{g^k,m}$. Then $\{(\mathcal{V}^{(j)}_{g^k}, \mathcal{H}^{(j)}_{g^k})\}$ comprises the n^2 irreducible representations for the MCFT $(\mathcal{V}^{(0)}, \mathcal{H}^{(0)})$ [9]. Clearly $\mathrm{Aut}(\mathcal{V}^{(0)}) \supseteq G_g$ where $G_g = C_g/\langle g\rangle$ so that $\mathrm{Aut}(\mathcal{V}^{(j)}_g) = U(1).G_g$. Note that $\mathrm{Aut}(\mathcal{V}^{(j)}_g)$ depends on j in general. In particular, if the twisted vacuum is unique then $\sigma^1_g \in \mathcal{H}^{(j_0)}_g$ for some j_0 and $\mathrm{Aut}(\mathcal{V}^{(j_0)}_g) = U(1) \times G_g$ from (10) where $\omega^{j_0}_n = \exp(-2\pi i E^g_0)$.

We next define the trace function for $\mathcal{H}_g = \bigoplus_{j=0}^{n-1} \bigoplus_{m=0}^{\infty} \mathcal{H}^{(j)}_{g,m}$ as follows

$$Z\left[\begin{array}{c} 1 \\ g \end{array}\right](\tau) \equiv \mathrm{Tr}_{\mathcal{H}_g}(q^{L_0-1}) \tag{18}$$

$$= \sum_{j=0}^{n-1} \mathrm{Tr}_{\mathcal{H}^{(j)}_g}(q^{L_0-1}) = D_g q^{E^g_0} + ..., \tag{19}$$

$$\mathrm{Tr}_{\mathcal{H}^{(j)}_g}(q^{L_0-1}) = q^{-j/n} \sum_{m=0}^{\infty} D^{(j)}_{g,m} q^m. \tag{20}$$

where the coefficient $D^{(j)}_{g,m}$ is the dimension of the representation $\rho^{(j)}_{g,m}$ of $\mathrm{Aut}(\mathcal{V}^{(j)}_g)$ defined by $\mathcal{H}^{(j)}_{g,m}$. Then D_g is the dimension of ρ^0_g, the representation of $\mathrm{Aut}(\mathcal{V}^{(j)}_g)$ acting on the twisted vacuum.

We can similarly define the general trace function for $\hat{h} \in \mathrm{Aut}(\mathcal{V}_g)$ lifted from $h \in C_g$ by

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) \equiv \mathrm{Tr}_{\mathcal{H}_g}(\hat{h} q^{L_0-1}) \tag{21}$$

$$= \sum_{j=0}^{n-1} q^{-j/n} \sum_{m=0}^{\infty} \chi^{(j)}_{g,m}(\hat{h}) q^m = \chi^0_g(\hat{h}) q^{E^g_0} + ..., \tag{22}$$

where $\chi^{(j)}_{g,m}(\hat{h}) = \mathrm{Tr}(\rho^{(j)}_{g,m}(\hat{h}))$ denotes a character of $\hat{h} \in \mathrm{Aut}(\mathcal{V}^{(j)}_g)$ and χ^0_g is the character for ρ^0_g. We assume below that a particular choice for $\hat{h}$ can be made which resolves the ambiguity inherent in the notation $Z\left[\begin{array}{c} h \\ g \end{array}\right]$ denoting the trace (21). When $D_g=1$, then $\chi^{(j)}_{g,m}(\hat{h})/\chi^0_g(\hat{h})$ is a character for C_g for $j\neq j_0$ and is a character for G_g for $j=j_0$ where $\omega_n^{j_0}=\exp(-2\pi i E^g_0)$.

For general commuting elements g,h, the trace function (21) transforms under a modular transformation with respect to $\gamma=\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\in SL(2,\mathbf{Z})$ in the following way

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)=\varepsilon(g,h;\gamma)Z\left[\begin{array}{c} h \\ g \end{array}\right]^{\gamma}(\gamma\tau), \tag{23}$$

where $\gamma\tau=\frac{a\tau+b}{c\tau+d}$, $\left[\begin{array}{c} h \\ g \end{array}\right]^{\gamma}\equiv\left[\begin{array}{c} h^a g^b \\ h^c g^d \end{array}\right]$ and where $\varepsilon(g,h;\gamma)$ is a phase multiplier [1], [9]. We will assume that if all elements in $\langle g,h\rangle$, the group generated by g and h, are normal then we may choose a lifting of $h^a g^b$ such that the phase multiplier $\varepsilon(g,h;\gamma)=1$ i.e. there are no global phase anomalies [17]. We will employ the abbreviation h for the chosen lifting $\hat{h}\in\mathrm{Aut}(\mathcal{V}_g)$ of $h\in C_g$ from now on. In general, the absence of this phase multiplier is an essential ingredient in the orbifolding procedure that we discuss below.

Suppose that the g twisted vacuum is one dimensional i.e. $D_g=1$. Let $\phi_g(h)$ denote the lifting of h in its action on this twisted vacuum (and hence giving the extension of h on all of $\mathcal{H}_g$) where in particular $\phi_g(1)=1$ and $\phi_g(g)=\exp(-2\pi i E^g_0)$. We conjecture that $\phi_g(h)$ has the following properties where $g^a h^b$ is a normal class for all a,b:

$$\phi_g(g^a h^b) = \phi_g(g)^a\phi_g(h)^b, \tag{24}$$

$$\phi_g(h) \in \langle\omega_n\rangle. \tag{25}$$

(24) follows from modular invariance with $Z\left[\begin{array}{c} h^b \\ g \end{array}\right](\tau+1)=Z\left[\begin{array}{c} g^{-1}h^b \\ g \end{array}\right](\tau)$ using (17) so that $\phi_g(g^a h^b)=\phi_g(g)^a\phi_g(h^b)$ and the assumption that $\widehat{g^a h^b}=(\hat{g})^a(\hat{h})^b$ for normal classes $g^a h^b$. We will prove (25) assuming (24) for the specific examples of Generalised Moonshine Functions that we consider later on.

2.3 Orbifolding a MCFT

Assume that all elements of $\langle g\rangle\simeq\mathbf{Z}_n$, the abelian group of order n generated by g, are normal elements of $\mathrm{Aut}(\mathcal{V})$. Then $(\mathcal{V}^{(0)},\mathcal{H}^{(0)})$ has n^2 irreducible representations $(\mathcal{V}^{(j)}_{g^k},\mathcal{H}^{(j)}_{g^k})$ for $j,k=0...,n-1$. The $\langle g\rangle$ orbifold MCFT $\mathcal{V}^{\langle g\rangle}_{\mathrm{orb}}$ is the MCFT with Hilbert space $\mathcal{H}^{\langle g\rangle}_{\mathrm{orb}}\equiv\oplus_{k=0}^{n-1}\mathcal{H}^{(0)}_{g^k}$. We assume that we can augment the operators $\mathcal{V}^{(0)}$ with appropriate local operators so that the OPE (5) is satisfied. The characteristic function of $\mathcal{V}^{\langle g\rangle}_{\mathrm{orb}}$ is $Z^{\langle g\rangle}_{\mathrm{orb}}=\frac{1}{n}\sum_{l,k=0}^{n-1}Z\left[\begin{array}{c} g^l \\ g^k \end{array}\right]$ which is a modular invariant from (23) since $\varepsilon(g,h;\gamma)\equiv 1$. Hence $\mathcal{V}_{\rm orb}^{\langle g\rangle}$ is a self-dual MCFT and so $Z_{\rm orb}^{\langle g\rangle}(\tau)=J(\tau)+N_0^{\langle g\rangle}$ as in (12) where $N_0^{\langle g\rangle}$ is the number of conformal weight 1 operators in $\mathcal{V}_{\rm orb}^{\langle g\rangle}$.

We can similarly consider the orbifold MCFT $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ found by orbifolding with respect to the abelian group $\langle g,h\rangle$ of order $|\langle g,h\rangle|$ generated by two commuting elements g,h where all the elements of $\langle g,h\rangle$ are assumed to be normal. $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ has Hilbert space $\mathcal{H}_{\rm orb}^{\langle g,h\rangle}=\mathcal{P}_{\langle g,h\rangle}(\oplus_{v\in\langle g,h\rangle}\mathcal{H}_v)$ where $\mathcal{P}_{\langle g,h\rangle}\equiv\frac{1}{|\langle g,h\rangle|}\sum_{v\in\langle g,h\rangle}v$ denotes the projection with respect to the group $\langle g,h\rangle$. Again we assume that we may augment the MCFT $\mathcal{P}_{\langle g,h\rangle}\mathcal{V}$ with appropriate local vertex operators. $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ then has modular invariant characteristic function $Z_{\rm orb}^{\langle g,h\rangle}=\frac{1}{|\langle g,h\rangle|}\sum_{u,v\in\langle g,h\rangle}Z\left[\begin{array}{c}u\\ v\end{array}\right]$. $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ is therefore a self-dual MCFT and so $Z_{\rm orb}^{\langle g,h\rangle}(\tau)=J(\tau)+N_0^{\langle g,h\rangle}$.

We assume that these orbifold MCFTs further can be considered as various embeddings in a larger non-meromorphic CFT $(\mathcal{V}',\mathcal{H}')$ with Hilbert space $\mathcal{H}'=\oplus_{v\in\langle g,h\rangle}\mathcal{H}_v$. In this CFT, all twisted states are created by vertex operators satisfying some non-meromorphic OPE of the generic form

$$V(\psi,z)V(\varphi,w)=\sum_{\chi}C_{\psi\varphi}^{\chi}V(\chi,w)(z-w)^{h_\chi-h_\psi-h_\varphi},\qquad(26)$$

for $\psi\in\mathcal{H}_g,\quad\varphi\in\mathcal{H}_h$ and $\chi\in\mathcal{H}_{gh}$ and similarly for all commuting pairs in $\langle g,h\rangle$. $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ then consists of all $\mathcal{V}'$ operators invariant under $\langle g,h\rangle$. A rigorous discussion in the case of the reflection automorphism orbifolding of the Leech lattice theory appears in [18].

Consider independent generators g,h of $\langle g,h\rangle$ i.e. $g^A\neq h^B$ for all $A=1,...,o(g)-1$ and $B=1,...,o(h)-1$ where $o(g)$ is the order of g etc. Then $|\langle g,h\rangle|=o(g)o(h)$ and $\mathcal{P}_{\langle g,h\rangle}=\mathcal{P}_g\mathcal{P}_h$, where $\mathcal{P}_g=\frac{1}{o(g)}\sum_{k=0}^{o(g)-1}g^k$ denotes the projection operator with respect to g. The orbifold $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$ can then be considered as a composition of orbifoldings for any independent generators g,h where

$$\begin{aligned}\mathcal{V}_{\rm orb}^{\langle g,h\rangle}&=\mathcal{P}_g(\sum_{k=0}^{o(g)-1}\mathcal{P}_h(\sum_{l=0}^{o(h)-1}\mathcal{V}_{g^kh^l}))\\&=\mathcal{P}_g(\sum_{k=0}^{o(g)-1}(\mathcal{V}_{\rm orb}^{\langle h\rangle})_{g^k}))=(\mathcal{V}_{\rm orb}^{\langle h\rangle})_{\rm orb}^{\langle g\rangle},\end{aligned}\qquad(27)$$

using the uniqueness of the twisted sectors for a self-dual MCFT and the assumed embedding of $\mathcal{V}_{\rm orb}^{\langle g,h\rangle}$, $\mathcal{V}_{\rm orb}^{\langle g\rangle}$ and $\mathcal{V}_{\rm orb}^{\langle h\rangle}$ in $\mathcal{V}'$. Later on we consider the consistency of (27) under the various possible independent choices for the generators of $\langle g,h\rangle$ in proving Theorems 3.1 and 3.2 in Section 3.

2.4 The Moonshine Module and Monstrous Moonshine

The Moonshine module $\mathcal{V}^\natural$ is historically the first example of a self-dual orbifold MCFT [10] and is constructed as a $\mathbf{Z}_2$ orbifolding of $\mathcal{V}^\Lambda$, which denotes the Leech lattice MCFT from now on. The $\mathbf{Z}_2$ automorphism $r \in \mathrm{Aut}(\mathcal{V}^\Lambda)$ is a lifting of the lattice reflection symmetry chosen so that $\mathcal{P}_r\mathcal{H}^\Lambda$ contains no Virasoro level one states. The r-twisted space $\mathcal{H}_r^\Lambda$ has vacuum energy $E_0^r = 1/2 > 0$ (r is a normal element of $\mathrm{Aut}(\mathcal{V}^\Lambda)$) and hence contains no Virasoro level one states. The resulting orbifold MCFT, $\mathcal{V}^\natural \equiv (\mathcal{V}^\Lambda)_{\mathrm{orb}}^{\langle r\rangle}$, therefore has characteristic function $J(\tau)$ (12) with $\mathrm{Aut}(\mathcal{V}^\natural) = \mathbf{M}$, the Monster group [10, 4]. We can identify a 'dual' automorphism $r^* \in \mathbf{M}$ where $\mathcal{P}_r(\mathcal{H}^\Lambda)$ (respectively $\mathcal{P}_r(\mathcal{H}_r)$) is even (respectively odd) under r^* [8]. This is an obvious automorphism of the non-meromorphic OPE (26) for $\mathcal{V}'$ for $g, h \in \langle r\rangle$. Then orbifolding $\mathcal{V}^\natural$ with respect to r^* we recover $\mathcal{V}^\Lambda$. Furthermore, one obtains the Monster centraliser $C_{r^*} = 2_+^{1+24}.\mathrm{Co}_1$ where Co_1 denotes the Conway simple group and 2_+^{1+24} is an extra-special 2-group [19, 10, 4].

It is conjectured that $\mathcal{V}^\natural$ is characterised (up to isomorphism) as the unique self-dual $C = 24$ MCFT with characteristic function $J(\tau)$ [4]. We may consider other $\mathbf{Z}_n$ orbifoldings of $\mathcal{V}^\Lambda$ with characteristic function $J(\tau)$ which should reproduce $\mathcal{V}^\natural$ according to this conjecture. In general, we can classify all automorphisms $a \in \mathrm{Aut}(\mathcal{V}^\Lambda)$ lifted from automorphisms $\overline{a} \in \mathrm{Co}_0$ the Leech lattice automorphism group for which $\mathcal{V}_a^\Lambda$ can be explicitly constructed satisfying the following constraints [8]

(i) $\mathcal{P}_a\mathcal{H}^\Lambda$ contains no Virasoro level one states i.e. $\bar{a}$ is fixed point free.

(ii) $\mathcal{H}_a^\Lambda$ is non-tachyonic (i.e. $E_0^a \geq 0$) and furthermore contains no Virasoro level one states so that

$$E_0^a > 0 \tag{28}$$

(iii) a is a normal element of $\mathrm{Aut}(\mathcal{V}^\Lambda)$.

There are 38 classes of Co_0 obeying these constraints including the 5 prime ordered cases considered by Dong and Mason [20].

For each of these 38 classes, we expect that a self-dual MCFT $\mathcal{V}_{\mathrm{orb}}^{\langle a\rangle}$ with characteristic function $J(\tau)$ exists. Furthermore, we can identify a dual automorphism a^* of order n so that $\mathcal{V}^\Lambda = ((\mathcal{V}^\Lambda)_{\mathrm{orb}}^{\langle a\rangle})_{\mathrm{orb}}^{\langle a^*\rangle}$ and where the a^* centraliser agrees with a corresponding Monster centraliser in all known cases [8]. All of this provides evidence that $(\mathcal{V}^\Lambda)_{\mathrm{orb}}^{\langle a\rangle} \simeq \mathcal{V}^\natural$ in each construction lending weight to the uniqueness conjecture.

Let us now define the Thompson series $T_g(\tau)$ for each $g \in \mathbf{M}$

$$\begin{aligned} T_g(\tau) \quad &\equiv \quad \mathrm{Tr}_{\mathcal{H}^\natural}(g q^{L_0 - 1}) = Z\left[\begin{array}{c} g \\ 1 \end{array}\right](\tau) && (29) \\ &= \quad \frac{1}{q} + 0 + [1 + \chi_A(g)]q + ... && (30) \end{aligned}$$

where $\chi_A(g)$ is the character of the 196883 dimensional adjoint representation for $\mathbf{M}$. The Thompson series for the identity element is $J(\tau)$ of (12), which is the hauptmodul for the genus zero modular group $SL(2, \mathbf{Z})$ as already stated.

Conway and Norton [11] conjectured and Borcherds [12] proved that $T_g(\tau)$ is the hauptmodul for some genus zero fixing modular group Γ_g. This remarkable property is known as Monstrous Moonshine. In general, for g of order n, $T_g(\tau)$ is found to be $\Gamma_0(n)$ invariant up to h^{th} roots of unity where h is an integer with $h|n$ and $h|24$ (see Appendix A for the definition of various standard modular groups). g is a normal element of $\mathbf{M}$ if and only if $h=1$, otherwise g is anomalous. $T_g(\tau)$ is fixed by some $\Gamma_g \supseteq \Gamma_0(N)$ which is contained in the normalizer of $\Gamma_0(N)$ in $SL(2,\mathbf{R})$ where $N = nh$ [11]. This normalizer contains the Fricke involution $W_N : \tau \to -1/N\tau$. All classes of $\mathbf{M}$ can therefore be divided into Fricke and non-Fricke type according to whether or not $T_g(\tau)$ is invariant under the Fricke involution. There are a total of 51 non-Fricke classes of which 38 are normal and there are a total of 120 Fricke classes of which 82 are normal. We now briefly describe how the genus zero properties of Monstrous Moonshine can be understood be using the orbifold ideas reviewed in the last section [7], [8].

For each of the 38 Leech lattice automorphisms a satisfying the conditions (i)-(iii) above we can compute the dual automorphism Thompson series T_{a^*}. This agrees precisely with the genus zero series for the 38 non-Fricke normal classes of the Monster where [8]

$$\begin{aligned} T_{a^*}(\tau) &= \mathrm{Tr}_{\mathcal{H}^\Lambda}(aq^{L_0-1}) - a_1 && (31)\\ &= \frac{1}{\eta_{\bar{a}}(\tau)} - a_1, && (32) \end{aligned}$$

with $\eta_{\bar{a}}(\tau) = \prod_{k|n} \eta(k\tau)^{a_k}$ where a is a lifting of $\overline{a} \in \mathrm{Co}_0$ with characteristic equation $\det(x-\bar{a}) = \prod_{k|n}(x^k-1)^{a_k}$ and where $n = o(a) = o(\bar{a})$. $\{a_k\}$ are called the 'Frame-shape' parameters of $\bar{a}$. We can also identify the other 13 non-Fricke classes which are anomalous and find the corresponding correct genus zero Thompson series [8]. This is further evidence for the assertion that $(\mathcal{V}^\Lambda)^{\langle a\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ implied by the uniqueness conjecture for $\mathcal{V}^\natural$ which we will now assume from now on.

Consider next $f \in \mathbf{M}$, a Fricke element of order n. For normal elements we orbifold $\mathcal{V}^\natural$ with respect to $\langle f\rangle$ to obtain a self-dual MCFT $(\mathcal{V}^\natural)^{\langle f\rangle}_{\text{orb}}$. Assuming $T_f(\tau)$ is a hauptmodul then $(\mathcal{V}^\natural)^{\langle f\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ for every normal Fricke element [8]. The converse is also true, where given that $(\mathcal{V}^\natural)^{\langle f\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ for some $f \in \mathbf{M}$ then T_f is the hauptmodul for a genus zero modular group containing the Fricke involution [8]. In general, assuming the uniqueness conjecture, for all normal elements

$$\mathcal{V}^\Lambda \ \begin{matrix} \overset{\langle a\rangle}{\longrightarrow} \\ \underset{\langle a^*\rangle}{\longleftarrow} \end{matrix} \ \mathcal{V}^\natural \overset{\langle f\rangle}{\longleftrightarrow} \mathcal{V}^\natural \Leftrightarrow T_{a^*}, T_f \text{ are hauptmoduls} \tag{33}$$

where the arrows represent an orbifolding with respect to the indicated group.

For any normal Fricke element $f \in \mathbf{M}$ of order n (33) is equivalent to the following properties for the twisted sector $\mathcal{V}_f$:

(i) The $\mathcal{V}_f$ vacuum is unique so that $D_f = 1$ and has negative vacuum energy $E_0^f = -1/n$. $\mathcal{V}_f$ is then said to be tachyonic.

(ii) If f^r is Fricke then f^s is also Fricke where $(o(f^r), o(f^s)) = 1$ and $n = o(f^r)o(f^s)$.

These conditions are then sufficient to supply all the poles and residues of T_f so that T_f is a hauptmodul for a genus zero fixing group which includes the Fricke involution [7, 8]. The genus zero property for an anomalous class of $\mathbf{M}$, which corresponds to the Harmonic formula of [11], is described in [8].

The orbifold method can also be employed [7] to explain the Conway Norton Power Map Formula, which is a property of the Thompson series, independent of the hauptmodul property:

Power Map Formula: Suppose T_g is invariant under $\Gamma_0(n|h) + e_1,\ e_2, \ldots$. Then for any d, T_{g^d} is invariant under $\Gamma_0(n'|h') + e'_1,\ e'_2, \ldots$, where $n' = n/(n,d)$, $h' = h/(h,d)$ and $e'_1,\ e'_2, \ldots$, are the divisors of n'/h' amongst the numbers e_1, $e_2, \ldots$.

3 Generalised Moonshine

3.1 Properties of Generalised Moonshine Functions

We now consider Generalised Moonshine Functions (GMFs) which are generalised Thompson series depending on two commuting Monster elements of the form

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = \mathrm{Tr}_{\mathcal{H}^\natural_g}(hq^{L_0-1}), \tag{34}$$

for $h \in C_g$ where the action of the lifting of h on $\mathcal{H}^\natural_g$ is also denoted by h as discussed in section 2. Norton has conjectured that [13]:

Generalised Moonshine. *The GMF (34) is either constant or is a hauptmodul for some genus zero fixing group* $\Gamma_{h,g}$.

Note. We denote the fixing group for $Z\left[\begin{array}{c} h \\ g \end{array}\right](o(g)\tau)$ by $\tilde{\Gamma}_{h,g}$ which is obviously conjugate to $\Gamma_{h,g}$.

The properties of (34) that follow from the previous section can be summarised as follows:

(i) When all elements of $\langle g, h\rangle$ are normal elements of $\mathbf{M}$ then for $\gamma \in SL(2,\mathbf{Z})$

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma\tau) = Z\left[\begin{array}{c} h^d g^{-b} \\ h^{-c} g^a \end{array}\right](\tau), \tag{35}$$

Hence $\Gamma_{h,g} \supseteq \Gamma(o(h), o(g))$ defined in Appendix A. In practice $\Gamma(o(h), o(g)) \triangleleft \Gamma_{h,g}$. In particular, since γ and $-\gamma$ act equally we have

$$Z\left[\begin{array}{c} h \\ g \end{array}\right] = Z\left[\begin{array}{c} h^{-1} \\ g^{-1} \end{array}\right], \tag{36}$$

which property is known as charge conjugation invariance.

(ii) Given the uniqueness of the twisted sectors for $\mathcal{V}^\natural$, under conjugation by any element $x \in \mathbf{M}$ then $x(\mathcal{V}^\natural_g)x^{-1}$ is isomorphic to $\mathcal{V}^\natural_{xgx^{-1}}$ so that

$$Z\left[\begin{array}{c} h \\ g \end{array}\right] = \theta(g,h,x) Z\left[\begin{array}{c} xhx^{-1} \\ xgx^{-1} \end{array}\right], \tag{37}$$

$$\theta(g,h,x) = \frac{\phi_g(h)}{\phi_{xgx^{-1}}(xhx^{-1})}. \tag{38}$$

with ϕ_g of (24) and (25).

(iii) For a normal Fricke element f of order n the twisted sector $\mathcal{H}^\natural_f$ is unique with vacuum energy $E_0^f = -1/n$. Hence

$$\phi_f(f) = \omega_n \tag{39}$$

from (16). Recall that the extension determined by the phase $\phi_f(h)$ for each $h \in C_f$ is chosen in order to comply with (35). If h and hf are in the same C_f conjugacy class where $h = x(hf)x^{-1}$ for some $x \in C_f$, then using (35) for $\gamma = T$, (36) and (37) we find that $Z\left[\begin{array}{c} h \\ f \end{array}\right] = \theta(f,h,x) Z\left[\begin{array}{c} hf \\ f \end{array}\right]$ where $\theta(f,h,x) = \omega_n$. In general, conjugating with respect to elements of C_f we find that the GMF is a class function up to such an n^{th} root of unity i.e. $\theta(g,h,x) \in \langle \omega_n \rangle$ for all $x \in C_f$. If, on the other hand, h and hf are not in the same C_f conjugacy class then ϕ_f can be chosen so that

$$\phi_f(xhx^{-1}) = \phi_f(h) \tag{40}$$

for all $x \in C_f$.

From (22) for $h \in C_f$ we have

$$\begin{aligned} Z\left[\begin{array}{c} h \\ f \end{array}\right](\tau) &= q^{-1/n} \sum_{m=0}^{\infty} \chi^{(1)}_{f,m}(h) q^m + \sum_{j=0, j\neq 1}^{n-1} q^{-j/n} \sum_{m=1}^{\infty} \chi^{(j)}_{f,m}(h) q^m \\ &= \phi_f(h)[q^{-1/n} + 0 + \sum s = 1^{\infty} a_{f,s}(h) q^{s/n}]. \end{aligned} \tag{41}$$

The coefficient $a_{f,s}(h) = \chi^{(j)}_{f,m}(h)/\phi_f(h)$ is called a head character for the given GMF where $s = mn - j$. Then $a_{f,s}(h)$ is a character for G_f where $G_f = C_f/\langle f \rangle$ for $s = -1 \bmod n$ and otherwise is a character for C_f. For example, if we choose $f = 2+$ (the $2A$ Monster element) then $C_f = 2.B$ and $G_f = B$, the Baby Monster. Then $a_{f,s}(h)$ is a character for B for odd s and is a character for $2.B$ for even s [21]. This implies that the decomposition of (41) into the irreducible characters of C_g involves only those characters obeying such conditions. This observation is confirmed in Appendix B for $f = 2+$ and $3+$ where explicit character expansions are given.

(iv) For a normal non-Fricke element a^* of order n we find the S transformation of T_{a^*} of (32) results in

$$Z\left[\begin{array}{c} 1 \\ a^* \end{array}\right](\tau) = -a_1 + O(q^{1/n}). \tag{42}$$

i.e. $\mathcal{H}^{\natural}_{a^*}$ has vacuum energy $E_0^{a^*}=0$. Furthermore, from (16)

$$\rho^0_{a^*}(a^*)=\mathbf{1} \tag{43}$$

with vacuum degeneracy $D_{a^*}=\dim(\rho^0_{a^*})=-a_1$. Furthermore if $(a^*)^B$ is Fricke for some B then from (32) we find that $o(a^*)||n$ i.e. $o(a^*)|n$ and $(n,o(a^*))=1$.

(v) The value of (34) at any parabolic cusp a/c with $(a,c)=1$ is determined by the vacuum energy of the $g^a h^{-c}$ twisted sector from (i). If $g^a h^{-c}$ is Fricke then (34) is singular at a/c [13] with residue $\phi_{g^a h^{-c}}(h^d g^{-b})$. (34) is holomorphic at all other points on $\mathbf{H}$. Once these singularities are known, then (34) can be analysed to check whether it is constant or is a hauptmodul for an appropriate genus zero modular group. If all elements of $\langle g,h\rangle$ are Non-Fricke then there are no singular cusps so that $Z\left[\begin{array}{c} h \\ g \end{array}\right]$ is holomorphic on $\mathbf{H}/\Gamma(o(h),o(g))$ and hence is constant. This accounts for the constant GMFs referred to in the Generalised Moonshine Conjecture above. We therefore assume from now on that at least one element of $\langle g,h\rangle$ is Fricke which we chose to be g without loss of generality. The singularities and residues of $Z\left[\begin{array}{c} h \\ g \end{array}\right]$ are then constrained by certain orbifolding constraints which we discuss below in terms of two consistency theorems.

(vi) The operators $\{L_0,L_{-1},L_1\}$ generate an $sl(2,\mathbf{C})$ sub-algebra acting on $\mathcal{H}_g=\bigoplus_{j=0}^{n-1}\bigoplus_{m=0}^{\infty}\mathcal{H}^{(j)}_{g,m}$ where $L_{-1}(\mathcal{H}^{(j)}_{g,m})\subseteq\mathcal{H}^{(j)}_{g,m+1}$ and $L_1(\mathcal{H}^{(j)}_{g,m})\subseteq\mathcal{H}^{(j)}_{g,m-1}$ so that

$$\mathcal{H}^{(j)}_{g,m}=\ker_{\mathcal{H}^{(j)}_{g,m}}L_1\oplus\mathrm{im}_{\mathcal{H}^{(j)}_{g,m-1}}L_{-1}. \tag{44}$$

Hence the corresponding representations for $\mathrm{Aut}(\mathcal{V}_g)$ are related where $\rho^{(j)}_{g,m}=\rho^{(j)}_{g,m-1}+\tilde{\rho}^{(j)}_{g,m}$ where $\tilde{\rho}^{(j)}_{g,m}$ is the representation formed by $\ker_{\mathcal{H}^{(j)}_{g,m}}L_1$. Therefore every character $\chi^{(j)}_{g,m}(h)$ for $h\in\mathrm{Aut}(\mathcal{V}_g)$ obeys the property

$$\chi^{(j)}_{g,m}(h)=\chi^{(j)}_{g,m-1}(h)+\tilde{\chi}^{(j)}_{g,m}(h). \tag{45}$$

In Appendix B we observe this property for the first ten head characters $a_{g,mn-j}(h)=\chi^{(j)}_{g,m}(h)/\phi_g(h)$ for $g=p+$ and $p=2$ and 3. For $p=5$ and 7 this property can be observed in [22],[23].

3.2 Two Consistency Theorems

In this section we prove two theorems which simplify our analysis of GMFs. Their purpose is to identify the phases $\phi_{h^{-c}g^a}(h^d g^{-b})$ for $h^{-c}g^a$ Fricke appearing in (35). This will be achieved by considering the consistency of orbifolding $\mathcal{V}^{\natural}$ with respect to $\langle g,h\rangle$ under various choices of independent generators.

Theorem 3.1. *Let $g,h\in\mathbf{M}$ be independent commuting elements where both g and h are Fricke such that $\phi_g(h)=1$ and where all elements of $\langle g,h\rangle$ are normal. Let u, v be any independent generators for $\langle g,h\rangle$. Then:*

(i) $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$.

(ii) If u is non-Fricke then $\phi_{u^A v^B}(u) \neq 1$ for all Fricke elements $u^A v^B$ except possibly when $o(v^B)||o(v)$ with $(B, o(v)) \neq 1$.

(iii) If u is Fricke then there is a unique A mod $o(u)$ such that $u^A v$ is Fricke with $\phi_{u^A v}(u) = 1$ and $o(u^A v) = o(v)$.

Proof (i) Since h is Fricke and normal we have $\mathcal{V}_{\text{orb}}^{\langle h\rangle} \simeq \mathcal{V}^\natural$ from (33). By assumption $\phi_g(h) = 1$ and hence $(\mathcal{V}_{\text{orb}}^{\langle h\rangle})_g = \mathcal{P}_h(\mathcal{V}_g^\natural \oplus \mathcal{V}_{gh}^\natural \oplus \mathcal{V}_{gh^2}^\natural \oplus ...)$ is tachyonic i.e. has negative vacuum energy. Therefore g acts as a normal Fricke element on $\mathcal{V}_{\text{orb}}^{\langle h\rangle} \simeq \mathcal{V}^\natural$ and orbifolding the latter with respect to g we find that $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$ from (33) again.

(ii) Since u, v are independent $\mathcal{V}_{\text{orb}}^{\langle u,v\rangle} = (\mathcal{V}_{\text{orb}}^{\langle u\rangle})_{\text{orb}}^{\langle v\rangle} \simeq \mathcal{V}^\natural$ from (i). With u Non-Fricke then $\mathcal{V}_{\text{orb}}^{\langle u\rangle} \simeq \mathcal{V}^\Lambda$ and hence $(\mathcal{V}_{\text{orb}}^{\langle u\rangle})_{v^B} = \mathcal{P}_u(\mathcal{V}_{v^B}^\natural \oplus \mathcal{V}_{uv^B}^\natural \oplus \mathcal{V}_{u^2v^B}^\natural \oplus ...)$ is non-tachyonic except possibly when $o(v^B)||o(v)$ and $(B, o(v)) \neq 1$. But if $u^A v^B$ is Fricke, then $\mathcal{V}_{u^A v^B}^\natural$ is tachyonic so that $\mathcal{P}_u = 0$ on the corresponding vacuum sector i.e. $\phi_{u^A v^B}(u) \neq 1$ except possibly when $o(v^B)||o(v)$ and $(B, o(v)) \neq 1$.

(iii) With u Fricke then $\mathcal{V}_{\text{orb}}^{\langle u\rangle} \simeq \mathcal{V}^\natural$. But $(\mathcal{V}_{\text{orb}}^{\langle u\rangle})_{\text{orb}}^{\langle v\rangle} \simeq \mathcal{V}^\natural$ implies that v acts a Fricke element of order $o(v)$ on $\mathcal{V}_{\text{orb}}^{\langle u\rangle}$ since u, v are independent. Hence $(\mathcal{V}_{\text{orb}}^{\langle u\rangle})_v = \mathcal{P}_u(\mathcal{V}_v^\natural \oplus \mathcal{V}_{uv}^\natural \oplus \mathcal{V}_{u^2v}^\natural \oplus ...)$ is tachyonic which is possible iff $\mathcal{P}_u = 1$ on precisely one of the tachyonic vacuum sectors $\mathcal{V}_{u^A v}^{(0)}$ for some unique A mod $o(u)$ where $o(v) = o(u^A v)$. $\square$

Example. Consider the orbifolding of $\mathcal{V}^\natural$ with respect to $\langle f\rangle$ for f Fricke of non-prime order n. Let $r||n$ and consider $u = f^r$ of order $s = n/r$ and $v = f^s$ of order r. Then u, v are independent generators of $\langle f\rangle$ and Theorem 3.1 (iii) with $g = f$ and $h = 1$ implies that if u is Fricke then there is a unique A mod s such that $u^A v = f^{Ar+s}$ is Fricke of order r. But $(r, s) = 1$ implies that $A = 0$ mod s so that v is Fricke. This is the Atkin-Lehner closure property for Thompson series [7] i.e. if f and f^r are Fricke for $r||n$ then f^s is also Fricke for $s = n/r$.

Theorem 3.2. *Let $g, h \in \mathbf{M}$ be independent commuting elements where g is Fricke and h is Non-Fricke such that $\phi_g(h) = 1$ and where all elements of $\langle g,h\rangle$ are normal. Let u, v be any independent generators for $\langle g,h\rangle$. Then:*

(i) $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\Lambda$.

(ii) If u is Fricke then $\phi_{u^A v^B}(u) \neq 1$ for all $u^A v^B$ Fricke except possibly when $o(v^B)||o(v)$ with $(B, o(v)) \neq 1$.

Proof (i) Since g is Fricke $\mathcal{V}_{\text{orb}}^{\langle g\rangle} \simeq \mathcal{V}^\natural$ from (33). Then either $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} = \mathcal{P}_h(\mathcal{V}_{\text{orb}}^{\langle g\rangle} \oplus (\mathcal{V}_{\text{orb}}^{\langle g\rangle})_h \oplus ...) \simeq \mathcal{V}^\natural$ or $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\Lambda$ when h acts as Fricke or Non-Fricke element on $\mathcal{V}_{\text{orb}}^{\langle g\rangle}$ from (33) again. Assume that $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$ and consider the alternative composition of orbifoldings $\mathcal{V}_{\text{orb}}^{\langle g,h\rangle} = \mathcal{P}_g(\mathcal{V}_{\text{orb}}^{\langle h\rangle} \oplus (\mathcal{V}_{\text{orb}}^{\langle h\rangle})_g \oplus ...)$ where since h is non-Fricke, $\mathcal{V}_{\text{orb}}^{\langle h\rangle} \simeq \mathcal{V}^\Lambda$. But the condition $\phi_g(h) = 1$ implies that the vacuum energy of $(\mathcal{V}^{\langle h\rangle}_{\rm orb})_g = \mathcal{P}_h\mathcal{V}_g \oplus ...$ is negative which is impossible according to (28). Therefore $\mathcal{V}^{\langle g,h\rangle}_{\rm orb} \simeq \mathcal{V}^{\Lambda}$.

(ii) For independent u, v, $\mathcal{V}^{\langle u,v\rangle}_{\rm orb} = (\mathcal{V}^{\langle u\rangle}_{\rm orb})^{\langle v\rangle}_{\rm orb} \simeq \mathcal{V}^{\Lambda}$ from (i) where u Fricke implies $\mathcal{V}^{\langle u\rangle}_{\rm orb} \simeq \mathcal{V}^{\natural}$. Hence v is Non-Fricke in its action on $\mathcal{V}^{\langle u\rangle}_{\rm orb}$ and so $(\mathcal{V}^{\langle u\rangle}_{\rm orb})_{v^B} = \mathcal{P}_u(\mathcal{V}^{\natural}_{v^B} \oplus \mathcal{V}^{\natural}_{uv^B} \oplus \mathcal{V}^{\natural}_{u^2v^B} \oplus ...)$ is non-tachyonic except possibly when $o(v^B)||o(v)$ and $(B, o(v)) \neq 1$. But if $u^A v^B$ is Fricke, then $\mathcal{V}^{\natural}_{u^A v^B}$ is tachyonic so that $\mathcal{P}_u = 0$ on the corresponding vacuum sector i.e. $\phi_{u^A v^B}(u) \neq 1$ except possibly when $o(v^B)||o(v)$ and $(B, o(v)) \neq 1$. $\square$

Note. There is no corresponding statement to (iii) in Theorem 3.1. This is because not all self-orbifoldings of the Leech theory are of 'Fricke' type e.g. the involution of Co_0 with frame shape $2^{16}/1^8$ has a lifting $a \in \mathrm{Aut}(\mathcal{V}^{\Lambda})$ with non-tachyonic vacuum energy $E^0_a = 0$ and vacuum degeneracy $D_a = 16$ whereas $(\mathcal{V}^{\Lambda})^{\langle a\rangle}_{\rm orb} \simeq \mathcal{V}^{\Lambda}$.

3.3 Symmetries of Rational GMFs for $g = p+$

We now consider GMFs (34) where g is assumed to be Fricke and of prime order p i.e. $g = p+$ in Conway-Norton notation (see Appendix A) for $p = 2, 3, 5, ..., 31, 41, 47, 59, 71$. If $\langle g, h\rangle = \langle u\rangle$ for some $u \in \mathbf{M}$ then (34) can always be transformed to a regular Thompson series (29) via a modular transformation (35) as follows

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = Z\left[\begin{array}{c} u^A \\ 1 \end{array}\right](\gamma\tau), \tag{46}$$

where $g = u^{-C}$, $h = u^D$ for some C, D and where $A = (C, D)$, $c = C/A$, $d = D/A$ and a, b are chosen so that $ad - bc = 1$. In particular, this is always possible when p and $o(h)$ are coprime. An example of this phenomenon is discussed later on in Section 4.4. In these cases, the genus zero property for the GMF therefore follows from that for a regular Thompson series (29). The GMFs with non-trivial genus zero behaviour then occur for $h \in C_g$ where $o(h) = pk$ for $k \geq 1$. These only exist for $p \leq 13$. Furthermore, from now on we restrict our analysis to $k = 1$ or k prime alone.

We now make the further restriction of considering rational GMFs i.e. those with q expansion (41) with rational (and therefore integral) coefficients. No rational GMFs for $o(h) = pk$ occur for $p = 11$ and 13 from explicit calculations of the head characters. We therefore only consider $p = 2, 3, 5$ and 7 from now on. This firstly implies that the GMF for $p > 2$ has leading q expansion in normalised function form [26]

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = \frac{1}{q^{1/p}} + 0 + O(q^{1/p}), \tag{47}$$

i.e. $\phi_g(h) = 1$. We may also assume this normalised function form for $p = 2$ by relabelling as discussed below. This choice is also sufficient to ensure that g, h

are independent generators of $\langle g, h\rangle$ since otherwise $h^k = g^A$ for some $A \neq 0$ mod p which implies the contradictory relation $(\phi_g(h))^k = (\phi_g(g))^A = \omega_p^A$ from (24). Then the centraliser C_g and sporadic finite simple group $G_g = C_g/\langle g\rangle$ are as follows:

$g = p+$	C_g	G_g	Name
2+	2.B	B	Baby Monster
3+	3.Fi	Fi	Fischer
5+	$5 \times$ HN	HN	Harada-Norton
7+	$7 \times$ He	He	Held

Table 1. $g = p+$ centralisers for $p = 2, 3, 5, 7$.

Each element $l \in G_g$ is the image of p distinct elements, which we denote by $(g^a, l) \in C_g$ for $a = 1, \ldots p$ where $(g^{a_1}, l_1).\,(g^{a_2}, l_2) = (g^{a_2+a_2+c(l_1,l_2)}, l_1 l_2)$ with non-trivial cocycle $c(l_1, l_2)$ for $p = 2, 3$. If $(g^{a_1}, l_1) \overset{C_g}{\sim} (g^{a_2}, l_2)$ then clearly $l_1 \overset{G_g}{\sim} l_2$ and hence the conjugacy classes of C_g can be labelled (non-uniquely) by the conjugacy classes of G_g as in the ATLAS [24]. We define the $(1, G_g)$ classes of C_g to be those classes displayed in the ATLAS. For $p = 2, 3$ where C_g is a non-trivial extension of G_g then $h \overset{C_g}{\sim} gh$ for some elements $h \in C_g$ and the corresponding $(1, G_g)$ class contains elements of the form (g^a, l) for all a. Any G_g irreducible character χ^{G_g} becomes a C_g irreducible character with $\chi(g^a, l) = \chi^{G_g}(l)$ for $a = 1, \ldots p$ whereas for the remaining C_g irreducible characters we have $\chi^{C_g}(g^a, l) = \omega_p^a \chi^{C_g}(1, l)$. The irreducible characters for the $(1, G_g)$ classes are given unambiguously in the ATLAS for $p > 2$ [24]. For $p = 2$, we define the $(1, G_g)$ classes to be those classes with characters as given in the ATLAS for 2.B.

We will now restrict our attention to those elements h where $o(h) = pk$ for $k = 1$ and k prime and h an element of a $(1, G_g)$ class of C_g. We will find all the corresponding GMFs below using orbifold considerations and demonstrate that they must all have genus zero fixing groups. Towards this aim we firstly prove a theorem concerning the phase multipliers $\phi_g(h)$.

Theorem 3.3. *For $g = p+$, $p = 2, 3, 5$ and 7 and $o(h) = pk$, $k = 1$ or k prime and assuming (24) then for h normal $\phi_g(h) \in \langle\omega_p\rangle$.*

Proof. If $k = 1$ then the result is obvious from (24) since $\phi_g(h)^p = \phi_g(h^p) = 1$. It is necessary to separately consider (a) $k = p$ and (b) $k \neq p$, $k > 1$.

(a) Assume $k = p$. For $p = 2$ using charge conjugation (36) we have $\phi_g(h) = \phi_g(h^{-1})$ since $g^{-1} = g$ so that $\phi_g(h) \in \langle\pm 1\rangle$ for all normal h. However, note that with $h = 4|2+$, an anomalous Monster class, we have $g = h^2 = 2+$ and h projects down to the $2C$ class of B. However in this case $\phi_g(h)^2 = \phi_g(h^2) = \phi_g(g) = -1$ and hence $\phi_g(h) = \pm i$ so that (36) doesn't hold. This anomalous case is discussed later on in Section 4.5.

For $p > 2$, $H = h^p$ is of order p and hence $\phi_g(H) \in \langle\omega_p\rangle$. From (24) we have $\phi_g(H) = \phi_g(h)^p$. On the other hand $\phi_g(xHx^{-1}) = \phi_g(xhx^{-1})^p$ for all $x \in C_g$. But then $\theta(g, H, x) = \theta(g, h, x)^p = 1$ and so from (38) $\phi_g(H) = \phi_g(xHx^{-1})$. For $p > 2$ we also find that H must be an element of a class of type $(1, G_g)$. For $p = 5$ and 7 this is obvious since the extension of G_g to C_g is trivial. For $p = 3$, all classes of order 3 in Fi lift to classes of order 3 in 3.Fi from the ATLAS [24] and hence the same result follows. From the ATLAS it is also clear that every class of order p and of type $(1, G_g)$ is conjugate to at least one its powers. Hence $\phi_g(H) = \phi_g(H^a)$ for some $a \neq 1 \bmod p$. But since $\phi_g(H) \in \langle \omega_p \rangle$ it follows that $\phi_g(H) = \phi_g(H)^a = 1$ for $p > 2$. Hence $\phi_g(h) \in \langle \omega_p \rangle$ as claimed.

(b) Lastly consider $k \neq p$, $k > 1$. Then $H = h^p$ is of order k coprime to p with $\phi_g(H) \in \langle \omega_k \rangle$ since $\phi_g(H)^k = \phi_g(H^k) = 1$. But in this case H and g are coprime and hence the GMF $Z \begin{bmatrix} H \\ g \end{bmatrix}$ can be directly found from the Thompson series for a normal class of order pk i.e. either (i) $H = k+$ with $gH = pk+$ or (ii) $H = k-$ with $gH = pk + p$ from the power map formula.

(i) $H = k+$ and $gH = pk+$. Since gH is Fricke we have $\mathcal{V}_{\mathrm{orb}}^{\langle g,H \rangle} = \mathcal{V}_{\mathrm{orb}}^{\langle gH \rangle} \simeq \mathcal{V}^\natural$. Consider $(\mathcal{V}_{\mathrm{orb}}^{\langle H \rangle})_g = \mathcal{P}_H(\mathcal{V}_g^\natural \oplus \mathcal{V}_{gH}^\natural \oplus \ldots \oplus \mathcal{V}_{gH^{k-1}}^\natural)$ which must be a tachyonic Fricke twisted sector of $\mathcal{V}_{\mathrm{orb}}^{\langle H \rangle}$ of order p since $\mathcal{V}_{\mathrm{orb}}^{\langle H \rangle} \simeq \mathcal{V}^\natural$. However, gH^a is of order p for $a = 0 \bmod p$ only so that $\mathcal{P}_H(\mathcal{V}_g^\natural)$ is tachyonic and hence $\phi_g(H) = 1$.

(ii) $H = k-$ and $gH = pk + p$. Since gH is Non-Fricke we have $\mathcal{V}_{\mathrm{orb}}^{\langle gH \rangle} \simeq \mathcal{V}^\Lambda$. Hence $Z \begin{bmatrix} gH \\ 1 \end{bmatrix}(\tau)$ is the known Thompson series

$$T_{pk+p}(\tau) = \left[\frac{\eta(\tau)\eta(p\tau)}{\eta(k\tau)\eta(pk\tau)} \right]^{\frac{24}{(k-1)(p+1)}} + \frac{24}{(k-1)(p+1)}, \tag{48}$$

from (32). This is invariant under the Atkin-Lehner transformation W_p (A.5) of Appendix A which we can choose as $W_p = \begin{pmatrix} p & 1 \\ ckp & dp \end{pmatrix}$ where $dp - ck = 1$ for coprime p, k. Then

$$\begin{aligned} Z \begin{bmatrix} gH \\ 1 \end{bmatrix}(\tau) &= Z \begin{bmatrix} gH \\ 1 \end{bmatrix}(W_p(\tau)) \\ &= Z \begin{bmatrix} H \\ g \end{bmatrix}(p\tau) = \frac{1}{q} + 0 + \ldots \end{aligned} \tag{49}$$

so that $\phi_g(H) = 1$ and hence $\phi_g(h) \in \langle \omega_p \rangle$. Note that this argument can also be directly used in (i) if we consider the W_p invariance for T_{pk+}. □

Corollary 3.4. *With g, h as above in Theorem 3.3 then*

(a) If h and gh are conjugate in C_g we can choose $\phi_g(h) = 1$ for $p = 2, 3$.

(b) If h and gh are not conjugate in C_g then for $p = 3, 5, 7$ we have $\phi_g(h) = 1$ iff h is an element of a $(1, G_g)$ class of C_g.

Proof. (a) As noted in Section 3.1 (iii) above if h and gh are conjugate in C_g (and hence both belong to a $(1, G_g)$ class) then $Z \begin{bmatrix} gh \\ g \end{bmatrix} = \omega_p Z \begin{bmatrix} h \\ g \end{bmatrix}$ and similarly for any $g^a h$ for all a. Hence from Theorem 3.3 $\phi_g(g^a h) = 1$ for some a so that after the relabeling $g^a h \to h$ we choose $\phi_g(h) = 1$.

(b) If h and gh are not conjugate in C_g then h is conjugate to h^a for some $a \neq 1 \bmod pk$ iff h is a class of type $(1, G_g)$ from the Atlas [24]. Furthermore from (40) for $p > 2$ it follows that $\phi_g(h) = \phi_g(h^a)$ for some $a \neq 1 \bmod pk$ iff $\phi_g(h) = 1$ and hence the result follows. $\square$

Remark. For $p = 2$ if h and gh are not conjugate in C_g then we may choose $\phi_g(h) = 1$ in conjunction with the definition of the $(1, G_g)$ classes for 2.B.

Corollary 3.5. *For $p = 3, 5, 7$ then $Z \begin{bmatrix} h \\ g \end{bmatrix}$ rational implies $\phi_g(h) = 1$ and h is a member of a class of type $(1, G_g)$.*

Proof. From Theorem 3.3 $\phi_g(h) \in \langle \omega_p \rangle$ which is rational for $\phi_g(h) = 1$ only for $p > 2$. Hence from Corollary 3.4 we have h is a member of a class of type $(1, G_g)$. $\square$

We make the following useful observations concerning the irreducible characters for the groups of Table 1 which can be checked by inspecting the appropriate ATLAS character tables [24].

Lemma 3.6. *The irreducible characters χ for the groups C_{p+} of Table 1 enjoy the following properties for $(1, G_g)$ classes:*

(a) *Any irrationality for χ is quadratic i.e. for $h \in C_{p+}$, $\chi(h) = a \pm \sqrt{b}$ for some $a, b \in \mathbf{Q}$ where $b = 0$ for rational $\chi(h)$.*

(b) *The number, N_χ, of independent irreducible characters of a given dimension is $N_\chi = 1$ or 2 or in the case of the Held group possibly $N_\chi = 3$. If $N_\chi = 2$ then the two characters, χ and $\bar{\chi}$ (say), are irrational and algebraically conjugate i.e. $\bar{\chi}(h) \equiv a \mp \sqrt{b}$ for all h in the notation of (a). In the case of the Held group there are three characters of dimension 1275 where two of the characters are irrational and algebraically conjugate and the other is rational.*

Note. *A character is said to be irrational if it is irrational for at least one conjugacy class. Otherwise, it is said to be rational. Clearly if $N_\chi = 1$ then the irreducible character χ is rational since $\bar{\chi}$ is also an irreducible character.*

Examples. The Fischer group has two irreducible algebraically conjugate characters χ_6, χ_7 of dimension 1603525 and two inequivalent conjugacy classes $23A$, $23B$ [24] such that $\chi_6(23A) = \chi_7(23B) = (-1 + i\sqrt{23})/2$ and $\chi_6(23B) = \chi_7(23A) = (-1 - i\sqrt{23})/2$.

Theorem 3.7. *Consider $g = p+$ for $p = 3, 5, 7$ and $o(h) = pk$, $k = 1$ or k prime. Suppose that $xC_gx^{-1} = C_g$ for some $x \in \mathbf{M}$ and that the GMF $Z \begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ is rational. Then*

$$Z \begin{bmatrix} h \\ g \end{bmatrix} = Z \begin{bmatrix} xhx^{-1} \\ g \end{bmatrix}. \tag{50}$$

Proof. Since the GMF is rational from Corollary 3.5 then $\phi_g(h) = 1$ and h is a member of a $(1, G_g)$ class. Furthermore h is conjugate to h^a for some $a \neq 1 \bmod pk$ from an inspection of the ATLAS character tables [24]. Hence xhx^{-1} is conjugate to $(xhx^{-1})^a$ and is also member of a $(1, G_g)$ class of C_g. Therefore $\phi_g(xhx^{-1}) = 1$ also. For any irreducible character χ of C_g and for all $h \in C_g$, $\chi^x(h) \equiv \chi(xhx^{-1})$ is another irreducible character of the same dimension. Furthermore $xC_gx^{-1} = C_g$ implies that the number of elements in the conjugacy classes for h and xhx^{-1} in C_g are equal. We now show that either $\chi^x = \chi$ or $\bar{\chi}$, the algebraic conjugate. Suppose that $\chi^x \neq \chi$. From Lemma 3.6 then either (i) they are both irrational and algebraically conjugate with $\chi^x = \bar{\chi}$ or else (ii) $G_g = \mathrm{He}$ and χ is of dimension 1275 where one is irrational χ, say, and the other χ^x is the unique rational character of dimension 1275. However the character table [24] for He reveals that (ii) cannot be true since there are no two conjugacy classes of He with the same number of elements for which a 1275 dimensional character χ is irrational and the other χ^x is rational. Hence $\chi^x = \bar{\chi}$ if $\chi^x \neq \chi$.

The head character $a_{g,mp-j}(h)$ coefficient of $q^{m-j/p}$ in (41) is rational by assumption and is some integer linear combination of the irreducible characters for C_g. If a given irreducible character $\chi(h)$ is irrational then χ and $\bar{\chi}$ must appear with the same (possibly zero) multiplicity and $\chi + \bar{\chi} = \chi^x + (\bar{\chi})^x$ whereas if $\chi(h)$ is rational then $\chi(h) = \chi^x(h)$. It is therefore clear that $a_{g,mp-j}(h) = a_{g,mp-j}(xhx^{-1})$ for each head character and hence the result follows. $\square$

Theorem 3.8. *Consider $g = p+$ for $p = 2, 3, 5, 7$ and $o(h) = pk$, $k = 1$ or k prime. If the GMF $Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ is rational then*

$$Z\begin{bmatrix} h \\ g \end{bmatrix} = Z\begin{bmatrix} h^d \\ g^a \end{bmatrix} \tag{51}$$

where $(d, pk) = 1$ and $(a, p) = 1$.

Proof. Each head character $a_{g,mp-j}(h)$ of (41) is rational by assumption. This implies that $a_{g,mp-j}(h) = a_{g,mp-j}(h^d)$ for each d such that $(d, pk) = 1$ e.g. [25]. Hence $Z\begin{bmatrix} h \\ g \end{bmatrix} = Z\begin{bmatrix} h^d \\ g \end{bmatrix}$ since $\phi_g(h^d) = 1$. Furthermore, for $p = 3, 5, 7$, $g = p+$ is conjugate to g^a in $\mathbf{M}$ for all $(a, p) = 1$ so that $g = xg^ax^{-1}$ for some $x \in \mathbf{M}$. Clearly $C_g = C_{g^a} = xC_gx^{-1}$ so that $xhx^{-1} \in C_g$ and h is a member of a $(1, G_{g^a})$ class of C_{g^a}. Hence Theorem 3.7 implies (50) and $\phi_g(xhx^{-1}) = 1$. Using (37) we then find $Z\begin{bmatrix} h \\ g^a \end{bmatrix} = Z\begin{bmatrix} h \\ g \end{bmatrix}$ since $\phi_{g^a}(h) = 1$. $\square$

Corollary 3.9. *Consider $g = p+$ for $p = 2, 3, 5, 7$ with $o(h) = pk$ for $k = 1$ or k prime. If $Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ is rational then $\Gamma_0^0(pk, p) \subseteq \Gamma_{h,g}$ i.e. $\Gamma_0(p^2k) \subseteq \tilde{\Gamma}_{h,g}$.*

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^0(pk, p)$ we have $b = 0 \bmod p$ and $c = 0 \bmod pk$ as described in Appendix A with $ad - bc = 1$. Hence the RHS of (35) gives

$Z\left[\begin{array}{c} h^d \\ g^a \end{array}\right](\tau)$ with $(d, pk) = 1$ and $(a, p) = 1$. The result then follows from Theorem 3.8. $\square$

Note. We conjecture that a more general version of Theorem 3.8 holds, namely, if a GMF is rational then (51) holds with $(d, o(h)) = 1$ and $(a, o(g)) = 1$ so that the GMF is $\Gamma_0^0(o(h), o(g))$ invariant.

Theorem 3.8 further implies that the singularity structure is restricted to the Fricke classes of $\langle g, h\rangle$ inequivalent under the equivalence relation $g^A h^B \sim g^{aA} h^{dB}$ where $(d, pk) = (a, p) = 1$ for $(A, B) = 1$. We call the equivalence classes under the relation $g^A h^B \sim g^{aA} h^{dB}$ for $(d, pk) = 1$ and $(a, p) = 1$ the class structure of the commuting pair g, h. We will see below that in all cases considered the class structure together with Theorems 3.1, 3.2 and 3.8 uniquely determine the GMF and its genus zero property.

4 The Genus Zero Property for Some Rational GMFs for $g = p+$

We now come to the main purpose of this paper which is to demonstrate the genus zero property for Generalised Moonshine Functions (GMFs) (34) for rational GMFs with $g = p+$ and $o(h) = pk$ for $k = 1$ or k prime. We will show how the orbifold considerations discussed in the previous sections allow us to demonstrate that either the given Fricke singularity structure is inconsistent or else all singularities of the GMF can be identified under some genus zero fixing group for which the GMF is a hauptmodul. We conjecture that the method described can be used to demonstrate the genus zero property for general GMFs including irrational cases. We begin by discussing four examples of possible Fricke class structures for $\langle g, h\rangle$ where in the first two examples we demonstrate the genus zero property and in the second two examples, demonstrate that the given Fricke class structure is impossible. These examples of class structures are by no means exhaustive but are nevertheless of general applicability.

Theorem 4.1. *For $g = p+$ for $p = 2, 3, 5, 7$ with $o(h) = pk$ for $k = 1$ or k prime with $\phi_g(h) = 1$ then the following class structures give rise to a rational GMF with genus zero fixing group $\tilde{\Gamma}_{h,g}$ for $Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau)$ as follows:*

I *g Fricke and all other classes Non-Fricke then $\tilde{\Gamma}_{h,g} = p^2k-$.*

II *g, h Fricke and all other classes Non-Fricke then $\tilde{\Gamma}_{h,g} = p^2k + p^2k$.*

Note. *See Appendix A for the modular group notation.*

Proof. I. Here $Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau)$ has a unique pole at the cusp $\tau = i\infty$ ($q = 0$) which is invariant under $\Gamma_0(p^2k)$, from Corollary 3.9, and is therefore the hauptmodul for $\Gamma_0(p^2k)$ which must be of genus zero. This restricts p^2k to the possible values $4, 8, 9, 12, 18, 25$.

II. Here we have singular behaviour in both the h and g twisted sectors, corresponding to cusps at $\tau = 0$ and $\tau = i\infty$. From Theorem 3.1 (i) using $\phi_g(h) = 1$ it follows that $\mathcal{V}_{\mathrm{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$. Using Theorem 3.1 (iii) with $u = g$, $v = h$ it follows that there exists a unique A such that $g^A h$ is Fricke of order pk where $\phi_{g^A h}(g) = 1$. But $g^A h$ must be therefore conjugate to h i.e. $A = 0 \bmod p$ and $\phi_h(g) = 1$. Therefore

$$Z\left[\begin{array}{c} g^{-1} \\ h \end{array}\right](\tau) = q^{-1/pk} + 0 + O(q^{1/pk}). \tag{52}$$

Define $\hat{W}_k$: $\tau \to -1/k\tau$ which is conjugate to the Fricke involution W_{p^2k} for $\Gamma_0(p^2k)$ (see Appendix A). $\hat{W}_k$ normalizes $\Gamma_0^0(pk,p)$ and interchanges the cusps at 0 and $i\infty$.

Let $f(\tau) = Z\left[\begin{array}{c} h \\ g \end{array}\right](\hat{W}_k\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = Z\left[\begin{array}{c} g^{-1} \\ h \end{array}\right](k\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = O(q^{1/p})$. $f(\tau)$ is $\Gamma_0^0(pk,p)$ invariant without poles on $\mathbf{H}/\Gamma_0^0(pk,p)$ and hence is constant and equal to zero. Hence $\Gamma_{h,g} = \langle \Gamma_0^0(pk,p), \hat{W}_k\rangle$ i.e. $\tilde{\Gamma}_{h,g} = p^2k + p^2k$. This restricts p^2k to the possible values $4, 8, 9, 12, 18, 20, 25, 27$. $\square$

Theorem 4.2. *For $g = p+$ for $p = 2, 3, 5, 7$ and $o(h) = pk$ for $k = 1$ or k prime with $\phi_g(h) = 1$ the following class structures are impossible for $k > 1$.*

I *g, gh and h Fricke where g is the only Fricke class of order p.*

II *g, gh^k and h Fricke where h is the only Fricke class of order pk.*

Proof. **I** From Theorem 3.1 (i) $\mathcal{V}_{\mathrm{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$ and from (iii) with $u = gh$ Fricke and $v = h^k$ of order p it follows that $u^A v = (gh)^A h^k$ is Fricke for some A with $o(u^A v) = p$. Hence $u^A v = g^A h^{A+k}$ must be conjugate to g being the only Fricke class of order p i.e. $A = -k \bmod pk$ so that $u^A v = g^{-k}$ with $\phi_{g^{-k}}(gh) = 1$. However from (39) we also have $\phi_{g^{-k}}(g^{-k}) = \omega_p$ which implies that $(\phi_{g^{-k}}(h))^k = \phi_{g^{-k}}((gh)^k)\phi_{g^{-k}}(g^{-k}) = \omega_p$. But Theorem 3.8 implies that since $\phi_g(h) = 1$ then $\phi_{g^{-k}}(h) = 1$ leading to a contradiction. Hence this conjugacy class structure is impossible.

II Again $\mathcal{V}_{\mathrm{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$. As in Theorem 4.1 II we can firstly show that $\phi_h(g) = 1$. Then applying Theorem 3.1 (iii) with $u = gh^k$ Fricke and $v = h$ it follows that $u^A v = (gh^k)^A h$ is Fricke for some A where $o(u^A v) = pk$. Hence $u^A v = g^A h^{kA+1}$ must be conjugate to h being the only Fricke class of order pk i.e. $A = 0 \bmod p$ and $u^A v = h$ with $\phi_h(gh^k) = 1$. But from (39) we have $\phi_h(h) = \omega_{pk}$ so that $\phi_h(g) = (\phi_h(h))^{-k} = \omega_p^{-1}$. This contradicts our earlier statement that $\phi_h(g) = 1$. Hence this class structure is also impossible. $\square$

4.1 Case A: Rational GMFs for $o(h) = p$, $k = 1$

The analysis when $o(g) = o(h) = p$ with prime p has been previously reported in [14]. The class structure of $\langle g, h\rangle$ is now determined by g, h and gh. The cusps of $\mathbf{H}/\Gamma_0^0(p)$ are at $\{i\infty, 0, 1, 2, ..., p-1\}$ corresponding to the sectors twisted by

g, h and $g^s h \sim gh$ $(s = 1, 2, ..., p-1)$ respectively with $\Gamma_0^0(p) \equiv \Gamma_0^0(p,p)$ [7], [14]. There are then four possible cases that may occur.

Theorem 4.3. *For $g = p+$ and h of order p with $\phi_g(h) = 1$ each possible class structure gives rise to a genus zero fixing group $\tilde{\Gamma}_{h,g}$ for rational $Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau)$ as follows:*

I.A *g Fricke. $\tilde{\Gamma}_{h,g} = p^2-$.*

II.A *g, h Fricke. $\tilde{\Gamma}_{h,g} = p^2+$.*

III.A *g, gh Fricke. $\tilde{\Gamma}_{h,g} = p-$.*

IV.A *g, h and gh Fricke. $\tilde{\Gamma}_{h,g} = p|p-$ for $p = 2, 3$ or $5||5+$ for $p = 5$.*

Note. *Here every element of $\langle g, h\rangle$ is Non-Fricke unless otherwise stated. Table 3 below shows all possible examples and the GMFs actually observed. See Appendix A for the modular group notation.*

Proof. I.A This follows from Theorem 4.1 I. This restricts the possible values for p to $p = 2$, 3 and 5. $p = 5$ does not arise in practice - see Appendix B [22],[23].

II.A This follows from Theorem 4.1 II. This restricts the possible values for p to $p = 2$, 3, 5, 7. $p = 7$ does not arise in practice - see Appendix B [22],[23].

III.A Here we have singular behaviour in the g and gh twisted sectors. Consider the $SL(2, \mathbf{Z})$ transformation $\gamma_p \equiv STS = \left(\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array}\right)$ which normalises $\Gamma(p) \equiv \Gamma(p,p)$ and is of order p in $\Gamma(p)$. We show that the cusps $\{i\infty, 1, 2, ..., p-1\}$ are identified under γ_p. From (35) we have

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_p\tau) = \phi_{gh}(h)q^{-1/p} + 0 + O(q^{1/p}). \tag{53}$$

First we prove that $\phi_{gh}(h) = 1$. According to Theorem 3.2 (i) $\mathcal{V}_{\mathrm{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^{\Lambda}$. From Theorem 3.2 (ii) with $u = g$ Fricke and $v = h$ it follows that $\phi_{gh}(g) \neq 1$ so that $\phi_{gh}(g) = \omega_p^s$ for some $s \neq 0 \bmod p$. However with $u = g^a h$ Fricke for $a \neq 0 \bmod p$ and $v = h$ it also follows that $\phi_{gh}(g^a h) \neq 1$. (If $a = 1 \bmod p$ then from (39) $\phi_{gh}(gh) = \omega_p$ whereas if $a \neq 1 \bmod p$ then we can choose A such that $aA = 1 \bmod p$ and $B = (a-1)A \bmod p$, $B \neq 0 \bmod p$, so that $(g^a h)^A (h)^B = gh$ and then apply Theorem 3.2 (ii)). Let $\phi_{gh}(h) = \omega_p^r$ then $\phi_{gh}(g^a h) = \omega_p^{as+r} \neq 1$. If $r \neq 0 \bmod p$ then since $s \neq 0 \bmod p$ we can always find a such that $as = -r \bmod p$ which is a contradiction. Hence $\phi_{gh}(h) = 1$.

From (53) it follows that $f(\tau) \equiv Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_p\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = O(q^{1/p})$ is $\Gamma(p)$ invariant and is non-singular at $q = 0$. One can easily check that the other cusps of f are similarly non-singular so that f is holomorphic on $\mathbf{H}/\Gamma(p)$ and is therefore zero. Hence $\Gamma_{h,g} = \langle\Gamma_0^0(p), \gamma_p\rangle = \Gamma_0^0(1,p)$ invariant i.e. $\tilde{\Gamma}_{h,g} = \Gamma_0(p)$.

This is possible for $p = 2, 3, 5, 7$. Note that for $p = 7$ this is the only rational class.

IV.A In this case all twisted sectors are Fricke and all cusps are singular. For $p = 2$, 3 and 5 there is exactly one rational class of order p in G_{p+} not already identified in cases I.A to III.A above. We consider each p in turn.

Consider $p = 2$. From Theorem 3.1 (i) $\mathcal{V}_{\mathrm{orb}}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$ and from (iii) with $u = h$ and $v = g$ there exists a unique A mod 2 such that $h^A g$ is Fricke and where $\phi_{h^A g}(h) = 1$ i.e. $A = 0$ and $\phi_{gh}(h) = -1$. From (39) $\phi_{gh}(gh) = -1$ and so $\phi_{gh}(g) = 1$. Similarly, from Theorem 3.1 (iii) with $u = g$ and $v = h$ there exists a unique A mod 2 such that $g^A h$ is Fricke and $\phi_{g^A h}(g) = 1$ i.e. $A = 1$ and $\phi_h(g) = -1$. From (39) $\phi_h(h) = -1$ and so $\phi_h(gh) = -1$. We summarise these phases in Table 2.

Phase	g	h	gh
ϕ_g	-1	1	-1
ϕ_{gh}	1	-1	-1
ϕ_h	-1	-1	1

Table 2.

Using (35) we then find that $Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ is also invariant under ST of order 3 which normalises $\Gamma(2)$ and so is the hauptmodul for $\langle \Gamma_0^0(2), ST\rangle$ of level two and index two in $SL(2, \mathbf{Z})$ i.e. $\tilde{\Gamma}_{h,g} = 2|2$.

For $p = 3, 5$ we can show that $\phi_h(g) = 1$. From Theorem 3.8, $Z\begin{bmatrix} h \\ g \end{bmatrix} = Z\begin{bmatrix} h \\ g^{-1} \end{bmatrix}$ which implies after an S transformation that $Z\begin{bmatrix} g^{-1} \\ h \end{bmatrix} = Z\begin{bmatrix} g \\ h \end{bmatrix}$ so that $\phi_h(g) = (\phi_h(g))^{-1}$. But $\phi_h(g) \in \langle \omega_p\rangle$ implies $\phi_h(g) = 1$ for $p > 2$. Hence $Z\begin{bmatrix} g \\ h \end{bmatrix}(\tau) = q^{-1/p} + 0 + O(q^{1/p})$. For $p = 3$ we can consider Theorem 3.1 (iii) with $u = g^2 h$ and $v = g^2$. Then $\phi_{g^{2A+2}h^A}(g^2 h) = 1$ for a unique A. However $\phi_{g^2}(g^2 h) = \omega_3^2$ and $\phi_{h^2}(g^2 h) = \omega_3^2$ and hence $A = 1$ so that $\phi_{gh}(g^2 h) = 1$ and hence $\phi_{gh}(g) = \phi_{gh}(h) = \omega_3^2$ since $\phi_{gh}(gh) = \omega_3$ from (39). Hence the phase residues of all the singular cusps are now known. For $p = 5$ we observe that h is an element of the $5A$ class of $G_g = \mathrm{HN}$ being the only remaining available rational class not identified in cases I.A to III.A above. But $\phi_h(g) = 1$ implies that g is an element of the $5A$ class of the isomorphic group $G_h = \mathrm{HN}$. Hence $Z\begin{bmatrix} g \\ h \end{bmatrix} = Z\begin{bmatrix} h \\ g \end{bmatrix}$ which leads to $\phi_{gh}(g) = \phi_{gh}(h) = \omega_5^3$. A similar argument can also be used for $p = 3$. Thus we find $\phi_{gh}(g) = \phi_{gh}(h) = \omega_p^{(p+1)/2}$ for $p = 3, 5$.

For $p = 3$, $\gamma_2 = T^{-1}ST$ is of order 2 and interchanges the cusps $\{\infty, 0\} \leftrightarrow \{2, 1\}$ whereas S interchanges $\{\infty, 1\} \leftrightarrow \{0, 2\}$. We then find using the given phase residues that $\Gamma_{h,g} = \langle \Gamma_0^0(3), \gamma_2, S\rangle$ which is of level 3 and index 3 in $SL(2, \mathbf{Z})$ i.e. $\tilde{\Gamma}_{h,g} = 3|3$.

For $p = 5$, let $\gamma_3 = TST^3$ which is of order 3 in $\Gamma_0^0(5)$ and cyclically permutes the cusps $\{\infty, 0\} \to \{1, 4\} \to \{2, 3\}$ whereas S interchanges the cusps $\{\infty, 1, 2\} \leftrightarrow \{0, 4, 3\}$. Hence $\Gamma_{h,g} = \langle \Gamma_0^0(5), S, \gamma_3\rangle$ which is of level 5 and index 5 in $SL(2, \mathbf{Z})$

i.e. $\tilde{\Gamma}_{h,g} = 5||5+$. $\square$

We summarise cases I–IV.A in Table 3 where all possible examples are indicated. Examples that do not arise in explicit calculations are marked with an asterix and all Fricke classes are in boldface.

Case	h	gh	$\tilde{\Gamma}_{h,g}$	Examples
I.A	$p-$	$p-$	$\Gamma_0(p^2)$	$p = 2, 3, 5^*$
II.A	$\mathbf{p}+$	$p-$	$\Gamma_0(p^2)+$	$p = 2, 3, 5, 7^*$
III.A	$p-$	$\mathbf{p}+$	$\Gamma_0(p)-$	$p = 2, 3, 5, 7$
IV.A	$\mathbf{p}+$	$\mathbf{p}+$	$p\|p-$ or $p\|\|p+$	$2\|2-$, $3\|3-$, $5\|\|5+$

Table 3. * indicates that no such GMF occurs. Fricke classes are in boldface

4.2 Case B: Rational GMFs for $o(h) = p^2$, $k = p$

The class structure of $\langle g, h\rangle$ is now described by g, h, gh and gh^p where from Theorem 7.1 of Appendix A the cusps of $\mathbf{H}/\Gamma_0^0(p^2, p)$ are at $\{i\infty,\ 0,\ s,\ \frac{s}{p}\}$ for $s = 1, 2, ..., p-1$ corresponding to the sectors twisted by g, h, $g^s h \sim gh$ and $g^s h^p \sim gh^p$ for $s =$ 1,2,...,$p-1$ respectively. Furthermore no such $h \in C_g$ exists for $p = 7$ whereas for $p = 5$ only two algebraically conjugate classes of order 25 exist with irrational characters and irrational GMFs. Hence we consider $p = 2, 3$ only. There are then eight possible cases that may occur as follows.

Theorem 4.4. *For $p = 2, 3$ with $g = p+$ and h of order p^2 with $\phi_g(h) = 1$ some possible class structures gives rise to a genus zero fixing group $\tilde{\Gamma}_{h,g}$ for rational $Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau)$ while others are impossible as follows:*

I.B *g Fricke. $\tilde{\Gamma}_{h,g} = p^3-$.*

II.B *g, h Fricke. $\tilde{\Gamma}_{h,g} = p^3+$.*

III.B *g, gh^p Fricke. $\tilde{\Gamma}_{h,g} = p^2-$ or $p^2|p$.*

IV.B *g, gh and gh^p Fricke. Impossible.*

V.B *g, h and gh Fricke. Impossible.*

VI.B *g, h and gh^p Fricke. Impossible.*

VII.B *g and gh Fricke. $\tilde{\Gamma}_{h,g} = 8\frac{1}{2}+$ for $p = 2$. Impossible for $p = 3$.*

VIII.B *g, h, gh and gh^p Fricke. $\tilde{\Gamma}_{h,g} = 4|2+$ or $4|2+2'$ for $p = 2$, $\tilde{\Gamma}_{h,g} = 9|3+$ or $9||3+$ for $p = 3$.*

Note. *Every element of $\langle g, h\rangle$ is Non-Fricke unless otherwise stated. Table 5 below shows all possible examples and the GMFs actually observed. See Appendix A for the modular group notation.*

Proof. I.B This follows from Theorem 4.1 I.

II.B This follows from Theorem 4.1 II.

III.B We have singular behaviour in the g and gh^p twisted sectors. Let $\phi_{gh^p}(h) = \omega_p^b$ for some b using (39). Consider $\gamma_p^{(b)} \equiv ST^pST^b = \begin{pmatrix} 1 & b \\ -p & 1-pb \end{pmatrix}$ which normalises and is of order p in $\Gamma(p^2, p)$. We have from (35) that

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_p^{(b)}\tau) = \phi_{gh^p}(h^{1-pb}g^{-b})q^{-1/p} + 0 + O(q^{1/p}) \qquad (54)$$

Then $\phi_{gh^p}(h^{1-pb}g^{-b}) = (\phi_{gh^p}(gh^p))^{-b}\phi_{gh^p}(h) = 1$ due to (39). From (54) it follows that $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_p\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = O(q^{1/p})$ is holomorphic on $\mathbf{H}/\Gamma(p^2, p)$ and is therefore zero. Thus $\Gamma_{h,g} = \langle \Gamma_0^0(p^2, p), \gamma_p^{(b)} \rangle$. If $b = 0 \bmod p$ then $\Gamma_{h,g} = \Gamma_0^0(p, p)$ and $\tilde{\Gamma}_{h,g} = \Gamma_0(p^2)$ invariant of genus zero. If $b \neq 0 \bmod p$ then $\Gamma_{h,g}$ has index p in $\Gamma_0^0(p, 1) = \Gamma_0(p)$ and we find $\tilde{\Gamma}_{h,g} = p^2|p-$ of genus zero.

IV.B From (39) $\phi_{gh}(gh) = \omega_{p^2}$ so that $\phi_{gh}(h^p) = \omega_p$. From Theorem 3.2 (ii) with $u = g$ Fricke and $v = h$ it follows that $\phi_{gh}(g) \neq 1$ so that $\phi_{gh}(g) = \omega_p^s$ for some $s \neq 0 \bmod p$ since g is order p. Taking $u = g^a h^p$ Fricke for any $a \neq 0 \bmod p$ and $v = h$ it also follows that $\phi_{gh}(g^a h^p) \neq 1$ (similarly to Theorem 4.3 III.A). Choose a such that $as = -1 \bmod p$ so that $\phi_{gh}(g^a h^p) = \omega_p^{as+1} = 1$ which is a contradiction. Hence this class structure is impossible.

V.B From Theorem 4.2 I this class structure is impossible.

VI.B From Theorem 4.2 II this class structure is impossible.

VII.B From (39) $\phi_{gh}(gh) = \omega_{p^2}$. From Theorem 3.2 (ii) with $u = g$ Fricke and $v = h$ it follows that $\phi_{gh}(g) \neq 1$. For $p = 2$ these conditions imply $\phi_{gh}(g) = -1$ and $\phi_{gh}(h) = \omega_4^{-1}$. Consider $\gamma_2^{(2)} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ which normalizes $\Gamma_0^0(4, 2)$ and is of order 2. We then find that $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_2^{(2)}(\tau)) = Z\left[\begin{array}{c} gh^2 \\ gh \end{array}\right](2\tau) = q^{-1/2} + O(q^{1/2})$. Hence we find that $\Gamma_{h,g} = \langle \Gamma_0^0(4, 2), \gamma_2^{(2)} \rangle$ i.e. $\tilde{\Gamma}_{h,g} = 8\frac{1}{2}+$ of genus zero.

For $p = 3$ we will show that the orbifolding procedure is not consistent. Since $h^3 = 3-$ then from (32) we have $Z\left[\begin{array}{c} h^3 \\ 1 \end{array}\right](\tau) = \left[\frac{\eta(\tau)}{\eta(3\tau)}\right]^{12} + 12$ i.e. $\rho_{h^3}^0(h^3) = \mathbf{1}$ and $\chi_{h^3}^0(h^3) = 12$ from (43). Since $gh = 9-$ then $\mathcal{V}_{\mathrm{orb}}^{\langle gh \rangle} \simeq \mathcal{V}^\natural$ and the constant term of $\mathrm{Tr}_{\mathcal{H}_{(gh)^A}}(\mathcal{P}_{gh} q^{L_0-1})$ vanishes for all A. Taking $A = 3$ we have $\frac{1}{9}\sum_{A=0}^{8} \chi_{h^3}^0((gh)^A) = \frac{1}{3}[12 + \chi_{h^3}^0(gh) + \chi_{h^3}^0((gh)^{-1}))] = 0$. Hence $\chi_{h^3}^0(gh) + \chi_{h^3}^0((gh)^{-1}) = -12$. Since $h = 9-$ then $\mathcal{V}_{\mathrm{orb}}^{\langle h \rangle} \simeq \mathcal{V}^\Lambda$ and the constant term of $\mathrm{Tr}_{\mathcal{H}_{\mathrm{orb}}^{\langle h \rangle}}(q^{L_0-1})$ is 24. Hence $\frac{1}{9}\sum_{A=0}^{8}\sum_{B=1}^{8} \chi_{h^B}^0(h^A) = 24$. But $\rho_h^0(h) = \mathbf{1}$ with $\chi_h^0(h) = 3$ from (43). From this it follows that $\chi_{h^3}^0(h) + \chi_{h^3}^0(h^2) = -3$.

From Theorem 4.3 I.A we have $Z\left[\begin{array}{c} h^3 \\ g \end{array}\right](3\tau) = \left[\frac{\eta(\tau)}{\eta(9\tau)}\right]^3 + 3$ and after an S transformation we have $Z\left[\begin{array}{c} g^2 \\ h^3 \end{array}\right](\tau/3) = 3 + 27\left[\frac{\eta(\tau)}{\eta(\tau/9)}\right]^3 = 3 + O(q^{1/9})$ i.e.

$\chi^0_{h^3}(g) = \chi^0_{h^3}(g^2) = 3$. Using this information we find an inconsistency in the orbifolding with respect to $\langle g, h\rangle$ where in particular, $\chi^0_{h^3}(\mathcal{P}_{\langle g,h\rangle}) = \sum_{u\in\langle g,h\rangle} \chi^0_{h^3}(u) = -1$ which must be a non-negative quantity being the dimension of the h^3 twisted vacuum space invariant under $\langle g, h\rangle$. Hence this class structure is impossible for $p = 3$.

VIII.B We firstly consider $p = 2$. From Theorem 3.1 (i) it follows that $\mathcal{V}^{\langle g,h\rangle}_{\mathrm{orb}} \simeq \mathcal{V}^\natural$ and from (ii) with $u = gh$ Fricke and $v = g$ it follows that there exists a unique A such that $u^A v = g^{A+1}h^A = 2+$ with $\phi_{g^{A+1}h^A}(gh) = 1$. This implies that $g^{A+1}h^A$ is conjugate to either g or gh^2 i.e. $A = 0$ or $2 \bmod 4$. If $A = 0 \bmod 4$ then $\phi_g(gh) = 1$ which contradicts the known values $\phi_g(h) = 1$ and $\phi_g(g) = -1$. Hence $A = 2 \bmod 4$ and $\phi_{gh^2}(gh) = 1$, $\phi_{gh^2}(gh^2) = -1$ so that $\phi_{gh^2}(g) = \phi_{gh^2}(h) = -1$.

Similarly from Theorem 3.1 (iii) with $u = gh^2$ Fricke and $v = h^{-1}$ it follows that there exists a unique A such that $u^A v = g^A h^{2A-1} = 4+$ with $\phi_{g^A h^{2A-1}}(gh^2) = 1$. This implies that $g^A h^{2A-1}$ must be conjugate to either h or gh i.e. $A = 0$ or $1 \bmod 2$. Suppose $A = 0 \bmod 2$. Then $\phi_h(gh^2) = 1$ and so $\phi_h(h) = i$ implies that $\phi_h(g) = -1$. Furthermore, from Theorem 3.1 (iii) we have $\phi_{gh}(gh^2) \neq 1$ which means $\phi_{gh}(h) \neq -i$ since $\phi_{gh}(gh) = i$. Hence $\phi_{gh}(h) = i$ (since $gh = 4+$) and hence $\phi_{gh}(g) = 1$. If on the other hand $A = 1 \bmod 2$ then $\phi_{gh}(gh^2) = 1$ and so $\phi_{gh}(gh) = i$ implies that $\phi_{gh}(h) = -i$ and $\phi_{gh}(g) = -1$. Furthermore, from Theorem 3.1 (iii) we have $\phi_h(gh^2) = -1$ (since gh^2 is of order 2) which means $\phi_h(g) = 1$ since $\phi_h(h) = i$. Hence all residues of singular cusps are determined for $p = 2$ for both values of A. In particular note that $\phi_h(g) = -1$ for $A = 0 \bmod 2$ and $\phi_h(g) = 1$ for $A = 1 \bmod 2$.

Now we can apply an analysis similar to III.B and using the above information to obtain $\langle\Gamma^0_0(4, 2), \gamma^{(-1)}_2\rangle$ invariance with $b = -1$, $p = 2$ in $\gamma^{(b)}_p$ of III.B. We then find that $\hat{w}_2 : \tau \to -1/2\tau$ either fixes or negates $Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ depending on the sign of $\phi_h(g)$. Hence $\Gamma_{h,g} = \langle\Gamma^0_0(4, 2), \gamma^{(-1)}_2, \hat{w}_2\rangle$ or $\langle\Gamma^0_0(4, 2), \gamma^{(-1)}_2, \hat{w}'_2\rangle$ of genus zero i.e. $\tilde{\Gamma}_{h,g} = 4|2+$ or $4|2 + 2'$ where $2'$ indicates that the GMF is negated by w_2.

Consider $p = 3$. As in IV.A we find that the rationality of the GMF implies that $\phi_h(g) = 1$. Next $\mathcal{V}^{\langle g,h\rangle}_{\mathrm{orb}} \simeq \mathcal{V}^\natural$ and from Theorem 3.1 (ii) with $u = gh$ Fricke and $v = g$ it follows that there exists a unique A such that $u^A v = g^{A+1}h^A$ is Fricke with $\phi_{g^{A+1}h^A}(gh) = 1$. This implies that $g^{A+1}h^A$ is conjugate to either g or gh^3 i.e. $A = 0 \bmod 3$. We let $b = -A/3$. If $b = 0 \bmod 3$ then $\phi_g(gh) = 1$ which contradicts the known values $\phi_g(h) = 1$ and $\phi_g(g) = \omega_3$. If $b = 1 \bmod 3$ then $\phi_{gh^6}(gh) = 1$ and $\phi_{gh^6}(gh^6) = \omega_3$ implies that $\phi_{gh^6}(g) = \phi_{gh^6}(h^{-1}) = \omega_3$ so that $\phi_{gh^3}(g) = \phi_{gh^3}(h) = \omega_3$. If $b = -1 \bmod 3$ then $\phi_{gh^3}(gh) = 1$ and $\phi_{gh^3}(gh^3) = \omega_3$ implies that $\phi_{gh^3}(g) = \omega_3$ and $\phi_{gh^3}(h) = \omega_3^{-1}$. Thus we find $\phi_{gh^3}(g) = \omega_3$ and $\phi_{gh^3}(h) = \omega_3^b$ for $b = \pm 1 \bmod 3$.

We now show that $\phi_{gh}(g) = \omega_3^b$ and $\phi_{gh}(h) = \omega_9^{1-3b}$. Firstly we have $\phi_{gh^3}(g^{-b}h) = 1$ and hence as in IV.A we find that the rationality of the GMF implies that $\phi_{g^{-b}h}(gh^3) = 1$. Conjugating as in (40) we then find that $\phi_{gh}(gh^{-3b}) = 1$.

However $\phi_{gh}(gh)=\omega_9$ implies that $\phi_{gh}(g)=\omega_3^b$ and $\phi_{gh}(h)=\omega_9^{1-3b}$ given that $b=\pm 1 \bmod 3$. Hence all residues of singular cusps are now determined as follows:

Phase	g	h
ϕ_g	ω_3	1
ϕ_h	1	ω_9
ϕ_{gh}	ω_3^b	ω_9^{1-3b}
ϕ_{gh^3}	ω_3	ω_3^b

Table 4.

Similarly to III.B and using the Table 4 we then obtain $\langle \Gamma_0^0(9,3), \gamma_3^{(b)}, \hat{w}_3 \rangle$ invariance with $b=\pm 1$ and $p=3$ for $\gamma_p^{(b)}$ of III.B. This corresponds to $\tilde{\Gamma}_{h,g}=9|3+$ for $b=-1 \bmod 3$ and $\tilde{\Gamma}_{h,g}=9||3+$ for $b=1 \bmod 3$ both of genus zero. $\square$

We summarise cases I–VIII.B in Table 5 where all possible examples are indicated.

Case	h	gh	gh^p	$\tilde{\Gamma}_{h,g}$	Examples
I.B	p^2-	p^2-	$p-$	p^3-	$p=2$
II.B	$\mathbf{p}^2+$	p^2-	$p-$	p^3+	$p=2,3$
III.B	p^2-	p^2-	$\mathbf{p}+$	p^2-, $p^2\|p-$	$p=2,3$
IV.B	p^2-	$\mathbf{p}^2+$	$\mathbf{p}+$	Impossible	-
V.B	$\mathbf{p}^2+$	$\mathbf{p}^2+$	$p-$	Impossible	-
VI.B	$\mathbf{p}^2+$	p^2-	$\mathbf{p}+$	Impossible	-
VII.B	p^2-	$\mathbf{p}^2+$	$p-$	$8\frac{1}{2}+$	$p=2$
VIII.B	$\mathbf{p}^2+$	$\mathbf{p}^2+$	$\mathbf{p}+$	$4\|2+$, $4\|2+2'$, $9\|3+$, $9\|\|3+$	$p=2,3$

Table 5. Fricke classes are in boldface.

4.3 Case C: Rational GMFs for $o(h)=pk$, k prime, $k\neq p$

For $o(h)=pk$, k prime, $k\neq p$, the class structure of $\langle g,h\rangle$ is described by g, h, gh, gh^p, h^k and gh^k. From Theorem 7.1 of Appendix A the cusps of $\mathbf{H}/\Gamma_0^0(pk,p)$ are at $\{i\infty, 0, s, \frac{1}{p}, \frac{p}{k}, \frac{s}{k}\}$ for $s=1,2,...,p-1$ corresponding to the sectors twisted by g, h, $g^s h\sim gh$, gh^p, h^k and $g^s h^k \sim gh^k$ for $s=$ 1,2,...,$p-1$ respectively. h^k and gh^k are determined by the power map formula for h and gh. There are then thirteen possible cases that may occur as follows.

Case	h	gh	gh^p	h^k	gh^k
I.C	$pk-$	$pk-$	$pk+p$	$p-$	$p-$
II.C	$\mathbf{pk+pk}$	$pk-$	$pk+p$	$p-$	$p-$
III.C	$pk+p$	$pk-$	$pk+p$	$\mathbf{p}+$	$p-$
IV.C	$pk+p$	$pk+k$	$\mathbf{pk}+$	$p-$	$p-$
V.C	$\mathbf{pk}+$	$pk+k$	$\mathbf{pk}+$	$\mathbf{p}+$	$p-$
VI.C	$pk-$	$pk+p$	$pk+p$	$p-$	$\mathbf{p}+$
VII.C	$\mathbf{pk+pk}$	$pk+p$	$pk+p$	$p-$	$\mathbf{p}+$
VIII.C	$pk+p$	$pk+p$	$pk+p$	$\mathbf{p}+$	$\mathbf{p}+$
IX.C	$pk-$	$\mathbf{pk+pk}$	$pk+p$	$p-$	$p-$
X.C	$\mathbf{pk+pk}$	$\mathbf{pk+pk}$	$pk+p$	$p-$	$p-$
XI.C	$pk+p$	$\mathbf{pk+pk}$	$pk+p$	$\mathbf{p}+$	$p-$
XII.C	$pk+k$	$\mathbf{pk}+$	$\mathbf{pk}+$	$p-$	$\mathbf{p}+$
XIII.C	$\mathbf{pk}+$	$\mathbf{pk}+$	$\mathbf{pk}+$	$\mathbf{p}+$	$\mathbf{p}+$

Table 6. Possible classes with Fricke classes shown in boldface

Theorem 4.5. *With $g = p+$ and h of order pk for k prime $k \neq p$ with $\phi_g(h) = 1$ some class structures give rise to a genus zero fixing group $\tilde{\Gamma}_{h,g}$ for rational $Z\begin{bmatrix} h \\ g \end{bmatrix}(p\tau)$ while others are impossible as follows:*

I.C *g Fricke. $\tilde{\Gamma}_{h,g} = p^2k-$.*

II.C *g, h Fricke. $\tilde{\Gamma}_{h,g} = p^2k + p^2k$.*

III.C *h^k Fricke. $\tilde{\Gamma}_{h,g} = p^2k + p^2$.*

IV.C *gh^p Fricke. $\tilde{\Gamma}_{h,g} = p^2k + k$ for $p = 2, 3, 5, 7$ or $4k + k'$ for $p = 2$.*

V.C *g, h, gh^p and h^k Fricke. $\tilde{\Gamma}_{h,g} = p^2k+$.*

VI.C *g and gh^k Fricke. $\tilde{\Gamma}_{h,g} = pk-$.*

VII.C *g, h and gh^k Fricke. Impossible.*

VIII.C *g, h^k and gh^k Fricke. $\tilde{\Gamma}_{h,g} = pk|p-$ for $p = 2, 3$. Impossible for $p = 5, 7$.*

IX.C *g and gh Fricke. $\tilde{\Gamma}_{h,g} = 4k\frac{1}{2} + 4k$ when $p = 2$. Impossible for $p = 3, 5, 7$.*

X.C *g, h and gh Fricke. Impossible.*

XI.C *g, gh and h^k Fricke. $\tilde{\Gamma}_{h,g} = \Gamma[9k\tilde{\ }a]$ when $p = 3$. Impossible for $p = 2, 5, 7$.*

XII.C *g, gh, gh^p and gh^k Fricke. $\tilde{\Gamma}_{h,g} = pk + k$.*

XIII.C *All Fricke. $\tilde{\Gamma}_{h,g} = 2k|2+k, 2k|2+k'$ for $p = 2$, $\tilde{\Gamma}_{h,g} = pk|p+$ for $p = 2, 3$ and $\tilde{\Gamma}_{h,g} = pk||p+$ for $p = 3, 5$.*

Note. *Table 12 below shows all possible examples and the GMFs actually observed. Every element of* $\langle g, h\rangle$ *is Non-Fricke unless otherwise stated. See Appendix A for the modular group notation.*

Proof. I.C This follows from Theorem 4.1 I.

II.C This follows from Theorem 4.1 II.

III.C Since h^k is Fricke and $\phi_g(h^k) = (\phi_g(h))^k = 1$ then from Theorem 3.1 (i) it follows that $\mathcal{V}_{\rm orb}^{\langle g,h^k\rangle} \simeq \mathcal{V}^\natural$. Then from (iii) with $u = g$ Fricke and $v = h^k$ it follows that there exists a unique A such that $g^A h^k$ is Fricke and $\phi_{g^A h^k}(g) = 1$. But $g^A h^k$ must be conjugate to g or h^k i.e. $A = 0 \bmod p$ and $\phi_{h^k}(g) = 1$. Furthermore since $(k, p) = 1$ and $\phi_{h^k}(h^p) \in \langle\omega_p\rangle$ it follows that $\phi_{h^k}(h^p) = 1$.

Consider the $SL(2, \mathbf{Z})$ transformation $\hat{W}_{p^2} = \begin{pmatrix} p & b \\ k & pd \end{pmatrix}$, $p^2 d - bk = 1$, conjugate to W_{p^2}, the Atkin-Lehner involution for $\Gamma_0(p^2 k)$ from Appendix A. Then from (35) $Z\begin{bmatrix} h \\ g \end{bmatrix}(\hat{W}_{p^2}\tau) - Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau) = (\phi_{h^k}(h^{-pd}g^b) - 1)q^{-1/p} + 0 + O(q^{1/p}) = O(q^{1/p})$ is $\Gamma_0^0(pk, p)$ invariant without poles on $\mathbf{H}\,/\Gamma_0^0(pk, p)$ and hence is constant and equal to zero. Therefore $\Gamma_{h,g} = \langle\Gamma_0^0(pk, p),\ \hat{W}_{p^2}\rangle$ is a genus zero group i.e. $\tilde{\Gamma}_{h,g} = p^2 k + p^2$.

IV.C We have singular behaviour in the gh^p and g twisted sectors corresponding to the cusps $\tau = 1/p$ and $\tau = i\infty$. Consider the transformation $\hat{W}_k = \begin{pmatrix} -k & pb \\ pk & -kd \end{pmatrix} = \begin{pmatrix} -1 & pb \\ p & -kd \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $\det(\hat{W}_k) = k$; $kd - bp^2 = 1$. $\hat{W}_k$ is conjugate to $W_k = \begin{pmatrix} -k & b \\ p^2 k & -kd \end{pmatrix}$ the Atkin-Lehner involution for $\Gamma_0(p^2 k)$ of Appendix A. Then from (35) $Z\begin{bmatrix} h \\ g \end{bmatrix}(\hat{W}_k\tau) = \phi_{gh^p}(h^{kd}g^{bp})q^{-1/p} + O(q^{1/p}) = \phi_{gh^p}(h^{kd})q^{-1/p} + O(q^{1/p})$.

If $p = 2$ then $o(h^k) = 2$ and hence $\phi_{gh^2}(h^k) = \pm 1$. Then analogously to Theorem 4.4 VIII.B we conclude that $Z\begin{bmatrix} h \\ g \end{bmatrix}(\hat{W}_k\tau) = \pm Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau)$ and $\tilde{\Gamma}_{h,g} = 4k + k$ or $4k + k'$ both of genus zero where k' indicates that the GMF is negated by W_k. For $p > 2$ we will now show that $\phi_{gh^p}(h^k) = 1$. From Theorem 3.8 we have $Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau) = Z\begin{bmatrix} h^{1+ak} \\ g \end{bmatrix}(\tau)$ for $(1 + ak, pk) = 1$. But for all a we have $(1 + ak, k) = 1$ and either $(1 + k, p) = 1$ or $(1 + 2k, p) = 1$ and $1 + 2k < pk$ i.e. we can choose $a = 1$ or 2. Applying $\hat{W}_k$ we then find that $\phi_{gh^p}(h^{kd}) = \phi_{gh^p}(h^{kd(1+ak)})$ i.e. $(\phi_{gh^p}(h^{k^2}))^{ad} = 1$. However from Theorem 3.3 $\phi_{gh^p}(h^k) \in \langle\omega_p\rangle$ and $(ad, p) = 1$ and so $\phi_{gh^p}(h^k) = 1$. Hence $Z\begin{bmatrix} h \\ g \end{bmatrix}(\hat{W}_k\tau) - Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau) = O(q^{1/p})$ is without poles on $\mathbf{H}\,/\Gamma_0^0(pk, p)$ and hence is constant and equal to zero. So $\Gamma_{h,g} = \langle\Gamma_0^0(pk, p),\ \hat{W}_k\rangle$ or $\tilde{\Gamma}_{h,g} = p^2 k + k$ of genus zero for $p = 3, 5, 7$.

V.C From Theorem 3.3 (i) $\mathcal{V}_{\rm orb}^{\langle g,h\rangle} \simeq \mathcal{V}^\natural$ since h is Fricke. From (iii) with $u = h^k$ Fricke $v = gh^p$ $\langle u, v\rangle = \langle g, h\rangle$ and from (iii) it follows that there exists a unique A such that gh^{Ak+p} is Fricke with $\phi_{gh^{Ak+p}}(h^k) = 1$. But then gh^{Ak+p}

must be conjugate to gh^p. Therefore $A = p$ with $\phi_{gh^p}(h^k) = 1$. As in III.C above and Theorem 4.3 II.A we can also show that $\phi_{h^k}(h^p) = 1$ and $\phi_h(g) = 1$. Using the standard argument we find that $\tilde{\Gamma}_{h,g} = p^2k+$ of genus zero.

VI.C Consider the $SL(2,\mathbf{Z})$ transformation $\gamma_p = ST^kS$ which is of order p in and normalises $\Gamma(pk,p)$. From (35) we have

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_p\tau) = \phi_{gh^k}(h)q^{-1/p} + 0 + O(q^{1/p}) \tag{55}$$

First we will prove that $\phi_{gh^k}(h) = 1$. From Theorem 3.2 (i) $\mathcal{V}_{\rm orb}^{\langle g,h^k\rangle} \simeq \mathcal{V}^\Lambda$. With $u = g$ Fricke and $v = h^k$ from (ii) it follows that $\phi_{gh^k}(g) = \omega_p^s \neq 1$. With $u = g^ah^k$ Fricke for $a \neq 0$ mod p and $v = h^k$ it also follows that $\phi_{gh^k}(g^ah^k) \neq 1$ (Here as in Theorem 4.3 III.A we can choose A and B such that $u^Av^B = gh^k$ for any $a \neq 0$ mod p and apply Theorem 3.2 (ii)). Let $\phi_{gh^k}(h) = \omega_p^r$ then $\phi_{gh^k}(g^ah) = \omega_p^{as+kr} \neq 1$. If $r \neq 0$ mod p then since $s \neq 0$ mod p we can always find a such that $as = -kr$ mod p which is a contradiction. Hence $\phi_{gh^k}(h) = 1$. Now from (55) it follows that $\Gamma_{h,g} = \langle\Gamma_0^0(pk,p),\gamma_p\rangle \equiv \Gamma_0^0(k,p)$ of genus zero and $\tilde{\Gamma}_{h,g} = \Gamma_0(pk)$ where all cusps $\tau = i\infty$, $\frac{s}{k}$, ($s = 1,2,...,p-1$) are identified under γ_p.

VII.C This class structure is impossible as shown in Theorem 4.2 II.

VIII.C From Theorem 4.3 IV.A we must have $p = 2, 3$ or 5. Let us first consider $p = 2$. All Fricke classes are of order 2 and since $k \neq p = 2$ and using Table 2 of Theorem 4.3 IV.A we obtain:

Phase	g	h
ϕ_g	-1	1
ϕ_{gh^k}	1	-1
ϕ_{h^k}	-1	-1

Table 7.

Using Table 7 and (35) it is easy to check that $\Gamma_{h,g} = \langle\Gamma_0^0(2k,2),\gamma_2^{(k)}\rangle$ of genus zero with $\gamma_2^{(k)} = \left(\begin{array}{cc} a & b \\ ck & d \end{array}\right)$, $a = 0$ mod 2, $b, c, d \neq 0$ mod 2; $\det\gamma_2^{(k)} = 1$. $\gamma_2^{(k)}$ is of order 3 in and normalises $\Gamma_0^0(2k,2)$. Then $\tilde{\Gamma}_{h,g} = 2k|2-$.

Let us now examine the $p = 3$ case. Since g, h^k and gh^k are all Fricke and of order 3 from Theorem 4.3 IV.A we have $\phi_{gh^k}(g) = \phi_{gh^k}(h^k) = \omega_3^2$ and $\phi_{h^k}(g) = 1$. Then using $\phi_u(h^k) = (\phi_u(h))^k$ for $u = gh^k, h^k$ we obtain the following residues for the singular cusps

Phase	g	h^k	h
ϕ_g	ω_3	1	1
ϕ_{gh^k}	ω_3^2	ω_3^2	ω_3^2 when $k = 1$ mod 3 ω_3 when $k = -1$ mod 3
ϕ_{h^k}	1	ω_3	ω_3 when $k = 1$ mod 3 ω_3^2 when $k = -1$ mod 3

Table 8.

Using Table 8 and (35) it is easy to check that $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_{3i}^{(k)}\tau) = q^{-1/3} + 0 + O(q^{1/3})$ for $i = 1$, 2 with $\gamma_{31}^{(k)} \equiv T\gamma_{3,k}^{\pm 1}$, $\gamma_{32}^{(k)} \equiv T^2\gamma_{3,k}^{\pm 2}$, when $k = \pm 1$ mod 3 and with $\gamma_{3,k} \equiv ST^{-k}S$. $\gamma_{3i}^{(k)}$ is of order 2 in and normalises $\Gamma_0^0(3k,3)$. Following the standard argument we then find that $\Gamma_{h,g} = \langle \Gamma_0^0(3k,3), \gamma_{31}^{(k)}, \gamma_{32}^{(k)} \rangle$ of genus zero where the cusps $\{i\infty, \frac{1}{k}, \frac{2}{k}, \frac{3}{k}\}$ are identified. Then $\tilde{\Gamma}_{h,g} = 3k|3-$ for $k = 2, 5$ are the only possible such modular groups for k prime.

With $p = 5$ since g, h^k and gh^k are all Fricke and of order 5 then from Theorem 4.3 IV.A and in a similar way to that shown above we obtain the following residues for the singular cusps

Phase	g	h^k	h
ϕ_g	ω_5	1	1
ϕ_{gh^k}	ω_5^3	ω_5^3	$\omega_5^{\mp 2}$ when $k = \pm 1 \bmod 5$ $\omega_5^{\mp 1}$ when $k = \pm 2 \bmod 5$
ϕ_{h^k}	1	ω_5	$\omega_5^{\pm 1}$ when $k = \pm 1 \bmod 5$ $\omega_5^{\mp 2}$ when $k = \pm 2 \bmod 5$

Table 9.

Using Table 9 and (35) we can show that $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_{5i}^{(k)}\tau) = q^{-1/5} + 0 + O(q^{1/5})$ for $i = 1,\ 2$ with $\gamma_{51}^{(k)} = T^{\pm 2}\gamma_{5,k}$ when $k = \pm 1 \bmod 5$, $\gamma_{51}^{(k)} = T^{\pm 1}\gamma_{5,k}$ when $k = \pm 2 \bmod 5$ and $\gamma_{52}^{(k)} = T^{\mp 2}\gamma_{5,k}^{-1}$ when $k = \pm 1 \bmod 5$, $\gamma_{52}^{(k)} = T^{\mp 1}\gamma_{5,k}^{-1}$ when $k = \pm 2 \bmod 5$ where $\gamma_{5,k} = ST^{-k}S$. $\gamma_{5i}^{(k)}$ is of order 3 in and normalises $\Gamma_0^0(5k,5)$. Following the standard argument we then find that $\Gamma_{h,g} = \langle \Gamma_0^0(5k,5), \gamma_{51}^{(k)}, \gamma_{52}^{(k)} \rangle$ of genus zero where the cusps $\{i\infty, \frac{1}{k}, \frac{2}{k}, \frac{3}{k}, \frac{4}{k}, \frac{5}{k}\}$ are identified. However no such genus zero modular group exists for k prime.

IX.C In this case h is a $pk-$ element which is possible only if $h = 6-$ or $10-$. So there are four possibilities: (1) $p = 3$, $k = 2$ (2) $p = 5$, $k = 2$ (3) $p = 2$, $k = 3$ and (4) $p = 2$, $k = 5$. Following detailed arguments similar to Theorem 4.4 VII.B we find that (1) and (2) lead to the contradictory property $\chi_{h^p}(\mathcal{P}_{\langle g,h \rangle}) < 0$.

Consider $p = 2$ with $k = 3, 5$. Since h is non-Fricke then from Theorem 3.2 (ii) with $u = g$ Fricke and $v = h$ we have $\phi_{gh}^0(g) \neq 1$ and hence $\phi_{gh}^0(g) = -1$. Consider $\gamma_2^{(k)} = \left(\begin{array}{cc} 1 & 1-k \\ -1 & k \end{array}\right)\left(\begin{array}{cc} k & 0 \\ 0 & 1 \end{array}\right)$ which normalises and is of order 2 in $\Gamma_0^0(2k,2)$. Then

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_2^{(k)}\tau) = \phi_{gh}^0((gh)^k g^{-1})q^{-1/2} + 0 + O(q^{1/2}).$$

However since $\phi_{gh}^0(gh) = \omega_{2k}$ from (39) and $\phi_{gh}^0(g) = -1$ so we have $\phi_{gh}^0((gh)^k g^{-1}) = 1$. Hence $\Gamma_{h,g} = \langle \Gamma_0^0(2k,2), \gamma_2^{(k)} \rangle$ of genus zero or $\tilde{\Gamma}_{h,g} = 4k\frac{1}{2} + 4k$ [22]. Thus $\tilde{\Gamma}_{h,g} = 12\frac{1}{2} + 12$ in case (3) and $\tilde{\Gamma}_{h,g} = 20\frac{1}{2} + 20$ in case (4) both of genus zero.

X.C From Theorem 4.2 I this class structure is impossible.

XI.C Since $g, h^k = p+$ and $gh^k = p-$ we find from Theorem 4.3 II.A that (1) $p = 2$ (2) $p = 3$ or (3) $p = 5$. Let $\phi_{gh}(g) = \omega_3^{\alpha}$ and $\phi_{gh}(h) = \omega_{pk}^{\beta}$. From (39) $\phi_{gh}(gh) = \omega_{pk}$ and so $\alpha k + \beta = 1 \bmod pk$. Since $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle} \simeq \mathcal{V}^{\Lambda}$ using Theorem 3.2 with $u = g$ Fricke and $v = h$ we have $\phi_{gh}(g) \neq 1$ so that $\alpha \neq 0 \bmod p$.

Furthermore, with $u = h^k$ Fricke and $v = gh^{1-k}$ we find $\phi_{gh}(h^k) \neq 1$ and therefore $\beta \neq 0 \bmod p$.

(1) $p = 2$. Since $\alpha \neq 0 \bmod 2$ we have $\alpha = 1$ and $\beta = 1 - k = 0 \bmod 2$ which is a contradiction to $\beta \neq 0 \bmod 2$ since $k \neq p = 2$. Therefore this class structure is impossible.

(2) $p = 3$. Here h is of type $3k+3$ so that $k = 2$ or 7 only. Since $h^k = 3+$ we find from 4.3 II.A that $\phi_{h^k}(g) = 1$. From (39) $\phi_{h^k}(h^k) = \omega_3$ so that $\phi_{h^k}(h) = \omega_3^2$ for $k = 2$ and $\phi_{h^k}(h) = \omega_3$ for $k = 7$.

For $k = 2$ the constraints $\alpha, \beta \neq 0 \bmod 3$ imply that $\phi_{gh}(g) = \omega_3$, $\phi_{gh}(h) = \omega_6^{-1}$ so that $\phi_{gh}(g^{-1}h^{-2}) = 1$. Consider $\gamma_4(\tau) = TST^2(2\tau)$. γ_4 which normalises and is of order 4 in $\Gamma_0^0(6,3)$ and acts on the cusps $\{i\infty, 1, 2, 3/2\}$ corresponding to the $\{g, gh, g^2h, h^2\}$ Fricke-twisted sectors in the following way: $i\infty \to 2 \to 3/2 \to 1 \to i\infty$. Then $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_4\tau) = Z\left[\begin{array}{c} g^{-1}h^{-2} \\ gh \end{array}\right](2\tau) = 1.q^{-1/3} + 0 + O(q^{1/3})$ and $Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma_4^2\tau) = Z\left[\begin{array}{c} gh^3 \\ h^2 \end{array}\right](\tau) = 1.q^{-1/3} + 0 + O(q^{1/3})$ so that in the usual way we find that $\Gamma_{h,g} = \langle \Gamma_0^0(6,3), \gamma_4 \rangle$ of genus zero so that $\tilde{\Gamma}_{h,g} = \Gamma[18\tilde{\ }a] \equiv \langle \Gamma_0(18), T^{1/3}W_{18}T^{1/3} \rangle$ whose hauptmodul has q expansion denoted by $18z$ [26] or $18\tilde{\ }a$ [N2].

For $k = 7$ the constraints $\alpha, \beta \neq 0 \bmod 3$ imply that $\phi_{gh}(g) = \omega_3^2$ and $\phi_{gh}(h) = \omega_{21}^8$ so that $\phi_{gh}(gh^{-7}) = 1$. Consider $\gamma_1 = \frac{1}{\sqrt{7}}\left(\begin{array}{cc} 7 & -13 \\ -7 & 14 \end{array}\right)$ and $\gamma_2 = \frac{1}{\sqrt{7}}\left(\begin{array}{cc} 14 & -13 \\ -7 & 7 \end{array}\right)$ both of which normalise and are of order 2 in $\Gamma_0^0(21,3)$. γ_1, γ_2 act on the cusps $\{i\infty, 1, 2, 3/7\}$ corresponding to the $\{h, gh, g^2h, h^7\}$ Fricke-twisted sectors in the following way: γ_1: $\{i\infty, 2\} \longleftrightarrow \{1, 3/7\}$ and γ_2: $\{i\infty, 1\} \longleftrightarrow \{2, 3/7\}$. Then in the usual way we find that $\Gamma_{h,g} = \langle \Gamma_0^0(21,3), \gamma_1, \gamma_2 \rangle$ of genus zero so that $\tilde{\Gamma}_{h,g} = \Gamma[63\tilde{\ }a]$ the modular group whose hauptmodul has q expansion denoted by $63\tilde{\ }a$ [N2].

(3) $p = 5$. Similarly to Theorem 4.4 VII.B we find the contradictory property $\chi_{h^5}^0(\mathcal{P}_{\langle g,h \rangle}) < 0$ so that this class structure is impossible.

XII.C From Theorem 3.2 (i) $\mathcal{V}_{\rm orb}^{\langle g,h \rangle} \simeq \mathcal{V}^\Lambda$. Then from (ii) with $u = g$ Fricke and $v = h$ it follows that $\phi_{gh}(g) = \omega_p^s$ for $s \neq 0 \bmod p$. Taking $u = g^a h^k$ Fricke for $a \neq 0 \bmod p$ and $v = h$ it also follows that $\phi_{gh}(g^a h^k) \neq 1$. Let $\phi_{gh}(h^k) = \omega_p^r$ then if $r \neq 0 \bmod p$ we can find a such that $as = -r \bmod p$ so that $\phi_{gh}(g^a h^k) = 1$ which is impossible. Hence $\phi_{gh}(h^k) = 1$. Similarly, from Theorem 3.2 (ii) with $u = g$ Fricke and $v = h$ it follows that $\phi_{gh^p}(g) \neq 1$ and taking $u = g^a h^k$ Fricke for $a \neq 0 \bmod p$ and $v = h$ it follows that $\phi_{gh^p}(g^a h^k) \neq 1$. As above we then find that $\phi_{gh^p}(h^k) = 1$. Finally, as in VI.C, we can also prove that $\phi_{gh^k}(h) = 1$.

Together these results can be used to show that $\Gamma_{h,g} = \langle \Gamma_0^0(pk, p), \gamma_p, \hat{W}_k \rangle$ of genus zero where $\gamma_p = ST^kS$ as defined in VI.C and $\hat{W}_k = \left(\begin{array}{cc} ak & pb \\ ck & kd \end{array}\right) = \left(\begin{array}{cc} a & pb \\ c & kd \end{array}\right)\left(\begin{array}{cc} k & 0 \\ 0 & 1 \end{array}\right)$, $\det(\hat{W}_k) = k$; $akd - bcp = 1$. Therefore we find that

$\tilde{\Gamma}_{h,g} = pk + k$.

XIII.C In this case all twisted sectors are Fricke and all cusps are singular. This case is closely related to case VIII.C. Since $g, h^k, gh^k = p+$ from Theorem 4.3 IV.A we must have $p = 2, 3$ or 5. Furthermore, for each such p there is exactly one rational class of order pk in G_{p+} not already identified in cases I.C to XII.C above. We consider each p in turn.

When $p = 2$ by repeated use of Theorem 3.1 (iii) and (39) we obtain the following residues which augment Table 7 above:

Phase.	g	h	h^k
ϕ_g	-1	1	1
ϕ_h	± 1	ω_{2k}	-1
ϕ_{gh}	∓ 1	$\mp\omega_{2k}$	± 1
ϕ_{gh^2}	-1	$\mp(-\omega_{2k})^{(1-k)/2}$	∓ 1
ϕ_{h^k}	-1	-1	-1
ϕ_{gh^k}	1	-1	-1

Table 10. Phases for $p = 2$.

Using Table 10 and following the usual argument we can then show that $\tilde{\Gamma}_{h,g} = 2k|2 + k$ or $2k|2 + k'$ where the GMF is either fixed (upper signs in Table 10) or negated (lower signs in Table 10) by the involution $\hat{w}_k : \tau \to -1/k\tau$.

When $p = 3, 5$ the phase residues are presented in Table 11 which augment Tables 8 and 9 above.

Phase	g	h^k	h	Parameters
ϕ_g	ω_p	1	1	
ϕ_h	1	ω_p	ω_{pk}	
ϕ_{gh}	ω_p^{λ}	$\omega_p^{1-k\lambda}$	$\omega_{pk}^{1-k\lambda}$	$\lambda = -k \bmod p$, $p = 3$ $\lambda = -2k \bmod 5$, $k = \pm 1 \bmod 5$, $p = 5$ $\lambda = 2k \bmod 5$, $k = \pm 2 \bmod 5$, $p = 5$
ϕ_{gh^p}	ω_p^{α}	1	$\omega_k^{\beta^2}$	$\alpha k + \beta p = 1$
ϕ_{h^k}	1	ω_p	ω_p^{γ}	$\gamma k = 1 \bmod p$
ϕ_{gh^k}	$\omega_p^{(p+1)/2}$	$\omega_p^{(p+1)/2}$	ω_p^{δ}	$\delta k = \frac{p+1}{2} \bmod p$

Table 11. Phases for $p = 3, 5$.

These phase residues are determined by the use of Theorem 3.1 for $p = 3, 5$. For example for $p = 5$ and $k = 2 \bmod 5$ we have $\phi_{gh^{k+1}}(gh^k) = \omega_5^a$ for some a since $o(gh^k) = 5$. We will show that $a = 0 \bmod 5$. Let $G = h^k$ and $H = g^{\alpha}h^{5\beta}$ where $\alpha k + 5\beta = 1$ so that $g = H^k$ and $h = G^{\alpha}H^{5\beta}$. Then we have $\phi_{gh^{k+1}}(gh^k) = \phi_{H^k(G^{\alpha}H^{5\beta})^{k+1}}(H^k G) = \phi_{G^{\alpha+1}H^{k+5\beta}}(GH^k)$. Furthermore $\phi_G(H) = 1$ using $\phi_{h^k}(g) = 1$ from Theorem 4.3 IV.A so that $H \in G_G$. But H belongs to the same unique rational class in $G_G = G_{5+}$ as h does in G_g and hence $Z\begin{bmatrix} h \\ g \end{bmatrix} = Z\begin{bmatrix} H \\ G \end{bmatrix}$. This implies that $\phi_{gh^{k+1}}(gh^k) = \phi_{g^{\alpha+1}h^{k+5\beta}}(gh^k)$. But $k = 2 \bmod 5$ implies that $\alpha = 3 \bmod 5$ and $k + 5\beta = (1 + 3k) \bmod 5k$. Hence $\omega_5^a = \phi_{g^{-1}h^{1+3k}}(gh^k) = \phi_{gh^{k+1}}(g^{-1}h^{bk})$ where $b(1 + 3k) = (k + 1) \bmod 5k$ using (51). But $b = -1 \bmod 5$ so that $\omega_5^a = \phi_{gh^{k+1}}(g^{-1}h^{-k}) = \omega_5^{-a}$. Hence $a = 0 \bmod 5$ so that $\phi_{gh^{k+1}}(gh^k) = 1$. By use of (51) and (39) we obtain $\phi_{gh}(g) = \omega_5^{-1}$ and

$\phi_{gh}(h)=\omega_{5k}^{1+k}$. We can similarly determine the other entries in Table 11.

For $p=3$ we may follow case VIII.C and use Table 11 to find in the usual way that $\Gamma_{h,g}=\langle\Gamma_0^0(3k,3),\gamma_{31}^{(k)},\gamma_{32}^{(k)},\hat{w}_k\rangle$ of genus zero i.e. $\tilde{\Gamma}_{h,g}=3k|3+$ for $k=1 \bmod 3$ which exists for $k=7,13$ and $\tilde{\Gamma}_{h,g}=3k||3+$ for $k=-1 \bmod 3$ which exists for $k=2,5$.

For $p=5$ case we may follow case VIII.C and use Table 11 to find in the usual way that $\Gamma_{h,g}=\langle\Gamma_0^0(5k,5),\gamma_{51}^{(k)},\gamma_{52}^{(k)},\hat{w}_k\rangle$ of genus zero i.e. $\tilde{\Gamma}_{h,g}=5k||5+$ for $k=\pm 2 \bmod 5$ which exists for $k=2,3,7$. $\tilde{\Gamma}_{h,g}$ does not exist for $k=\pm 1 \bmod 5$.

Table 12 summarises the results for cases I-XIII.C:

Cases	$\tilde{\Gamma}_{h,g}$	$p=2$	$p=3$	$p=5$	$p=7$
I.C	p^2k-	$12-$	$18-$	-	-
II.C	p^2k+p^2k	$12+12$, $20+20$	$18+18$	$50+50^*$	-
III.C	p^2k+p^2	$12+4$, $20+4$	$18+9$	-	-
IV.C	p^2k+k $4k+k'$ $(p=2)$	$12+3$ $28+7$ $12+3'$ $20+5'$	$18+2$	-	-
V.C	p^2k+	$12+$, $20+$, $28+$, $44+$, $92+$	$18+$ $45+$	$50+$	-
VI.C	$pk-$	$6-$, $10-$	$6-$	$10-$	-
VII.C	Impossible	-	-	-	-
VIII.C	$pk\|p-$	$6\|2-$, $14\|2-^*$	$6\|3-$ $15\|3-^*$	-	-
IX.C	$4k\frac{1}{2}+4k$, $p=2$	$12\frac{1}{2}+12$, $20\frac{1}{2}+20$	-	-	-
X.C	Impossible	-	-	-	-
XI.C	$\Gamma[p^2k\tilde{\ }a]$	-	$k=2,7$	-	-
XII.C	$pk+k$	$6+3$, $10+5$, $14+7$, $22+11$ $46+23$,	$6+2$ $15+5$ $33+11$	$10+2$	$21+3$
XIII.C	$2k\|2+k$ $2k\|2+k'$ $pk\|p+$ $pk\|\|p+$	$6\|2+3$ $6\|2+3'$ $10\|2+5'$ $14\|2+7$ $14\|2+7'$ $22\|2+11'$ $26\|2+13'$ $34\|2+17'$ $38\|2+19'$	$6\|\|3+$ $15\|\|3+$ $21\|3+$ $39\|3+$	$10\|\|5+$ $15\|\|5+$ $35\|\|5+$	-

Table 12. * indicates that no such GMF occurs.

4.4 Case D: Anomalous Classes $h=pk|p+...$

There are a number of GMFs for $g=p+$ for $p=2,3$ for which $o(h)=pk$ but h is a member of an anomalous **M** class i.e. $h=pk|p+...$ Then the hauptmodul property for the GMF can be demonstrated as follows. The Thompson series $T_h(\tau)$ (29) has the following property [11] :

$$[T_h(\tau/p)]^p = T_{h^p}(\tau) + \text{const.} \tag{56}$$

and similarly for GMF functions we have [8], [14]

$$\left[Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau)\right]^p = Z\left[\begin{array}{c} h^p \\ g \end{array}\right](p\tau) + \text{const.} \tag{57}$$

We then obtain the following examples of such GMFs:

I.D $p = 2$ with $h = 4|2-$ and $h^2 = 2-$. Then $Z\left[\begin{array}{c} h^2 \\ g \end{array}\right](2\tau)$ is a hauptmodul either for $\Gamma_0(4)$ when $gh^2 = 2-$ or for $\Gamma_0(2)$ when $gh^2 = 2+$ from Table 3. Therefore from (57) $\Gamma_{h,g} = 8|2-$ or $4|2-$ where only the first example arises in practice.

II.D $p = 2$ with $h = 4|2+$ and $h^2 = 2+$. Then $Z\left[\begin{array}{c} h^2 \\ g \end{array}\right](2\tau)$ is a hauptmodul for $\Gamma_0(4)+$ when $gh^2 = 2-$ from Table 3. Therefore $\Gamma_{h,g} = 8|2+4$ or $8|2+4'$. If $gh^2 = 2+$ then $Z\left[\begin{array}{c} h^2 \\ g \end{array}\right](2\tau)$ is the hauptmodul for $2|2$ but no such genus zero modular group commensurable with $SL(2,\mathbf{Z})$ exists fixing $\left(Z\left[\begin{array}{c} h^2 \\ g \end{array}\right](2\tau) + \text{const}\right)^{1/2}$.

III.D $p = 3$ with $h = 3|3$ and $h^3 = 1$. Then $Z\left[\begin{array}{c} h^3 \\ g \end{array}\right](p\tau)$ is the hauptmodul for $\Gamma_0(3)+$ and therefore $\Gamma_{h,g} = 9|3+$.

IV.D $p = 3$ with $h = 6|3-$ and $h^3 = 2-$. Then $gh^3 = 6+3$ (since $(gh^3)^2 = 3+$ and $(gh^3)^3 = 2-$) so that $Z\left[\begin{array}{c} h^3 \\ g \end{array}\right](p\tau)$ is the hauptmodul for $\Gamma_0(6)+3$ and therefore $\Gamma_{h,g} = 18|3+3$.

V.D $p = 3$ with $h = 21|3+$ and $h^3 = 7+$. Then $gh^3 = 21+$ and $Z\left[\begin{array}{c} h^3 \\ g \end{array}\right](p\tau)$ is the hauptmodul for $\Gamma_0(21)+$ and therefore $\Gamma_{h,g} = 63|3+$.

VI.D $p = 3$ with $h = 39|3+$ and $h^3 = 13+$. Then $gh^3 = 39+$ and $Z\left[\begin{array}{c} h^3 \\ g \end{array}\right](p\tau)$ is the hauptmodul for $\Gamma_0(39)+$ and therefore $\Gamma_{h,g} = 117|3+$.

4.5 Case E: Anomalous classes of type $4k|n+2,...$ associated with the Baby monster

Consider $h = 4k|n+2,... \in \mathbf{M}$ for $k = 1$ or k prime. There are eight such classes in the Monster. Then $g = h^{2k} = 2+$ and hence h is of order $4k$ in $C_g = 2.\mathrm{B}$ but is of order $2k$ in $G_g = \mathrm{B}$. The corresponding GMF is then

$$\begin{aligned} Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) &= Z\left[\begin{array}{c} h \\ h^{2k} \end{array}\right](\tau) \\ &= \varepsilon(g,h;ST^{2k}S) Z\left[\begin{array}{c} h \\ 1 \end{array}\right](ST^{2k}S\tau) \end{aligned} \tag{58}$$

where $\varepsilon(g,h;ST^{2k}S)$ is a phase that must be present since h is anomalous. This GMF is directly related to a standard Thompson series and is therefore a hauptmodul. We have the following examples: $h = 4|2+$, $8|4+$, $12|2+$, $12|2+2$, $20|2+$, $28|2+$, $52|2+$ and $68|2+$.

5 Conclusions

We have shown how Generalised Moonshine can be understood within an abelian orbifolding setting and have explicitly demonstrated the genus zero property for rational Generalised Moonshine Functions (GMFs) arising in a number of non-trivial cases. We have also discussed other aspects of Generalised Moonshine such as properties of the head character expansion of GMFs and constraints on the Monster classes for products of commuting Monster elements. The orbifold methods developed in this paper can in principle be extended to analyse all GMFs towards proving the genus zero property in general. Examples of GMFs with irrational coefficients using these methods appear in [IT].

6 Acknowledgments

We thank Simon Norton for providing us with invaluable information on replicable series and for a number of useful discussions. We also thank Geoffrey Mason and Rex Dark for useful discussions. We also acknowledge funding from Enterprise Ireland under the Basic Research Grant Scheme.

7 Appendix A. Modular Groups in Monstrous Moonshine

In this appendix we the describe various modular groups associated with Thompson series and Generalised Moonshine Functions (GMFs). Let $\Gamma = SL(2, \mathbf{Z})$ denote the full modular group. We define the following standard subgroups of Γ:

$$\Gamma_0(N) \equiv \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma,\ c = 0 \bmod N \right\}, \tag{A.1}$$

$$\Gamma_0^0(m,n) \equiv \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma,\, b = 0 \bmod n,\, c = 0 \bmod m \right\}, \tag{A.2}$$

$$\Gamma(m,n) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \bmod n & 0 \bmod n, \\ 0 \bmod m & 1 \bmod m \end{pmatrix} \right\}. \quad \text{(A.3)}$$

We also use the notation $\Gamma_0^0(m) \equiv \Gamma_0^0(m,m)$ and $\Gamma(m) \equiv \Gamma(m,m)$. Note that $\Gamma_0(mn)$ and $\Gamma_0^0(m,n)$ are conjugate where $\Gamma_0(mn) = \theta_n \Gamma_0^0(m,n)\theta_n^{-1}$ with $\theta_n \equiv \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$.

The normalizer $\mathcal{N}(\Gamma_0(N)) = \left\{\rho \in SL(2,\mathbf{R}) \mid \rho\Gamma_0(N)\rho^{-1} = \Gamma_0(N)\right\}$ is also required to describe Monstrous Moonshine [11]. Let h be an integer where $h^2|N$, (h^2 divides N) and let $N = nh$. Then we define the following sets of matrices.

$\Gamma_0(n|h)$: The group of matrices of the form

$$\begin{pmatrix} a & \frac{b}{h} \\ cn & d \end{pmatrix},\ \det = 1, \quad \text{(A.4)}$$

where a, b, c, $d \in \mathbf{Z}$. For h the largest divisor of 24 for which $h^2|N$, $\Gamma_0(n|h)$ forms a subgroup of $\mathcal{N}(\Gamma_0(N))$. For $h = 1$, $\Gamma_0(n|h) = \Gamma_0(n)$.

W_e: The set of matrices for a given positive integer e

$$\begin{pmatrix} ae & b \\ cN & de \end{pmatrix},\ \det = e,\ e||N, \quad \text{(A.5)}$$

where a, b, c, $d \in \mathbf{Z}$. $e||N$ denotes the property that $e|N$, and the $(e, N/e) = 1$. The set W_e forms a single coset of $\Gamma_0(N)$ in $\mathcal{N}(\Gamma_0(N))$ with $W_1 = \Gamma_0(N)$. It is straightforward to show (up to scale factors) that

$$W_e^2 = 1 \bmod (\Gamma_0(N)),\ W_{e_1}W_{e_2} = W_{e_2}W_{e_1} = W_{e_3} \bmod (\Gamma_0(N)), \quad \text{(A.6)}$$

where $e_3 = e_1e_2/(e_1,e_2)$. The coset W_e is referred to as an Atkin-Lehner (AL) involution for $\Gamma_0(N)$. The simplest example is the Fricke involution W_N with coset representative $\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ which generates $\tau \to -1/N\tau$ and interchanges the cusp points at $\tau = i\infty$ and $\tau = 0$. For $e \neq n$ we can choose the coset representative $\begin{pmatrix} e & b \\ N & de \end{pmatrix}$ where $ed - bN/e = 1$ which interchanges the cusp points at $\tau = i\infty$ and $\tau = e/N$.

w_e: The set of matrices for a given positive integer e of the form

$$\begin{pmatrix} ae & \frac{b}{h} \\ cn & de \end{pmatrix},\ \det = e,\ e||\frac{n}{h}, \quad \text{(A.7)}$$

where a, b, c, $d \in \mathbf{Z}$. The set w_e is called an Atkin-Lehner (AL) involution for $\Gamma_0(n|h)$. The properties (A.6) are similarly obeyed by w_e with $\Gamma_0(N)$ replaced by $\Gamma_0(n|h)$.

$\mathcal{N}(\Gamma_0(N))$: The Normalizer of $\Gamma_0(N)$ in $SL(2,\mathbf{R})$ is constructed by adjoining to $\Gamma_0(n|h)$ all its AL involutions w_{e_1}, w_{e_2}, ... where h is the largest divisor of 24 with $h^2|N$ and $N = nh$ [11].

$\Gamma_0(n|h)+e_1, e_2,...$: This denotes the group obtained by adjoining to $\Gamma_0(n|h)$ a particular subset of AL involutions w_{e_1}, w_{e_2}, ... and forms a subgroup of $\mathcal{N}(\Gamma_0(N))$.

$\Gamma_0(n|h)+$: This denotes the group obtained by adjoining to $\Gamma_0(n|h)$ all its AL involutions and forms a subgroup of $\mathcal{N}(\Gamma_0(N))$.

Sometimes we abbreviate $\Gamma_0(n|h)+e_1, e_2,...$ by $n|h+e_1, e_2,...$ and $\Gamma_0(n|h)+$ by $n|h+$ (respectively $n+e_1, e_2,...$ and $n+$ for $h=1$).

$n||h+$: This denotes the modular group containing $\Gamma_0(nh)$ having index h in $\theta_h(\Gamma_0(\frac{n}{h})+)\theta_h^{-1}$. This conjugate contains the transformation $\tau \to \tau + m/h$ for all m which takes the corresponding hauptmodul to h distinct ones labelled by the value of m, $0 \le m < h$ [26].

The following theorem [22] gives us information about the cusp points of $\Gamma_0(N)$:

Theorem 7.1. *For N square-free, all $\Gamma_0(N)$ inequivalent cusps can be represented by the rational numbers a/b such that $b > 0$, $b|N$, $(a,b) = 1$ and $0 < a < N/b$; two cusps a/b and a_1/b_1 being $\Gamma_0(N)$ equivalent if and only if $b = b_1$ and $a = a_1$ mod $(b, N/b)$.*

8 Appendix B. Identification of Generalised Moonshine Functions

In this appendix we consider the explicit form of the head character expansion of the GMF with $\phi_g(h) = 1$

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](o(g)\tau) = \frac{1}{q} + 0 + \sum_{s=1}^{\infty} a_s(h) q^s \tag{B.1}$$

for $g = p+$ for $p = 2,3,5,7$ (see Table 1) and with $h \in C_{p+}$ of order $o(h) = kp$ for $k = 1$ and k prime. Here $a_s(h) \equiv a_{g,s}(h)$ of (41). From (45) we therefore expect that

$$a_s(h) = a_{s-p}(h) + \tilde{\chi}_s(h) \tag{59}$$

for some character $\tilde{\chi}_s$ of C_{p+}.

For $p = 2,3$ we give the first 10 coefficients $a_s(h)$ of (B.1) in terms of the irreducible characters of C_{p+} from the ATLAS [24]. The irreducible expansion for $p = 5,7$ appear in [22], [23]. Then using [22], [23], [26] and [N2] we may identify the genus zero fixing group $\tilde{\Gamma}_{h,g}$ in each case considered.

When $g = 2+$ then $C_{p+} = 2.\mathrm{B}$ the double cover of the Baby Monster B then the first 10 head characters are:

$$
\begin{aligned}
a_1 &= \chi_1+\chi_2, \quad a_2=\chi_{185}, \quad a_3 = 2\chi_1+\chi_2+\chi_3+\chi_4, \quad a_4 = 2\chi_{185}+\chi_{186}, \\
a_5 &= 3\chi_1+3\chi_2+2\chi_3+\chi_4+\chi_6+\chi_7, \quad a_6 = 4\chi_{185}+2\chi_{186}+\chi_{187}, \\
a_7 &= 6\chi_1+5\chi_2+4\chi_3+3\chi_4+\chi_5+2\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}, \\
a_8 &= 8\chi_{185}+4\chi_{186}+3\chi_{187}+\chi_{188}, \\
a_9 &= 8\chi_1+10\chi_2+7\chi_3+4\chi_4+2\chi_5+5\chi_6+4\chi_7+2\chi_8+2\chi_9 \\
&\quad +2\chi_{10}+\chi_{11}+\chi_{12}+\chi_{14}+\chi_{16}+\chi_{17}, \\
a_{10} &= 14\chi_{185}+9\chi_{186}+7\chi_{187}+3\chi_{188}+\chi_{189}+\chi_{192}. \qquad (60)
\end{aligned}
$$

Clearly the property (59) is observed. Furthermore for s odd, a_s is a character for B whereas for s even, a_s is a character for 2.B for which g is represented by -1 as discussed in section 3.1 (iii).

When $g = 3+$ then $C_{p+} = 3.\mathrm{Fi}$, the triple cover of the Fischer group Fi, then the first 10 head characters are:

$$
\begin{aligned}
a_1 &= \chi_{109}, \quad a_2 = \chi_1+\chi_2, \quad a_3 = \chi_{109}+\chi_{110}, \quad a_4 = \chi_{109}+\chi_{110}+\chi_{112}, \\
a_5 &= 3\chi_1+2\chi_2+\chi_3+\chi_8, \qquad a_6 = 2\chi_{109}+2\chi_{110}+\chi_{112}+\chi_{113}, \\
a_7 &= 3\chi_{109}+3\chi_{110}+\chi_{111}+\chi_{112}+\chi_{113}+\chi_{114}, \\
a_8 &= 4\chi_1+5\chi_2+2\chi_3+\chi_5+2\chi_8+\chi_9+\chi_{10}+\chi_{13}, \\
a_9 &= 5\chi_{109}+5\chi_{110}+\chi_{111}+2\chi_{112}+3\chi_{113}+\chi_{114}+\chi_{116}+\chi_{119}, \\
a_{10} &= 6\chi_{109}+6\chi_{110}+\chi_{111}+4\chi_{112}+4\chi_{113}+2\chi_{114}+\chi_{115}+\chi_{116} \\
&\quad +\chi_{118}+\chi_{119}+\chi_{122}. \qquad (61)
\end{aligned}
$$

Clearly the property (59) is again observed. Furthermore for $s = 2 \quad 3$, a_s is a character for Fi otherwise a_s is a character for 3.Fi, the triple cover as discussed in section in Section 3.1 (iii).

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