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Extended Camassa-Holm Hierarchy and Conserved Quantities

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Abstract

An extension of the Camassa-Holm hierarchy is constructed in this letter. The conserved quantities of the hierarchy are studied and a recurrent formula for the integrals of motion is derived.

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1   Introduction

The Camassa-Holm equation (CH) [1]

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

where ω is a real constant parameter, describes the unidirectional propagation of shallow water waves over a flat bottom [1, 2]. It firstly appeared in [3] as an equation with a bi-hamiltonian structure. CH is a completely integrable equation [4, 5, 6, 7, 8, 9], describing permanent and breaking waves [10, 11]. Its solitary waves are stable solitons if $\omega > 0$ [12, 13, 14, 15] or peakons if $\omega = 0$ [1, 16]. CH arises also as an equation of the geodesic flow for the H^1 metrics on the Bott-Virasoro group [17, 18, 19]. The bi-Hamiltonian form of (1) is [1, 3, 5]

$$m_t = -(\partial - \partial^3)\frac{\delta H_2[m]}{\delta m} = -(2\omega\partial + m\partial + \partial m)\frac{\delta H_1[m]}{\delta m}, \tag{2}$$

where

$$m = u - u_{xx}, \tag{3}$$

and the Hamiltonians are

$$H_1[m] = \frac{1}{2}\int mu\,\mathrm{d}x, \tag{4}$$

$$H_2[m] = \frac{1}{2}\int (u^3 + uu_x^2 + 2\omega u^2)\mathrm{d}x. \tag{5}$$

The integration is from $-\infty$ to ∞ in the case of Schwartz class functions, and over one period in the periodic case.

In [20] it is shown that CH has an infinite number of local conserved quantities. A scheme for computation of the conservation laws is proposed in [21, 22, 23]. In this contribution we present a scheme, providing an explicit recurrent formula for the infinite sequence of independent integrals of motion for a chain of CH type equations.

The equation (1) admits a Lax pair [1]

$$\Psi_{xx} = \left(\frac{1}{4} + \zeta(m + \omega)\right)\Psi, \tag{6}$$

$$\Psi_t = \left(\frac{1}{2\zeta} - u\right)\Psi_x + \frac{u_x}{2}\Psi. \tag{7}$$

Recently, various multi-component generalizations of the CH equation are under intense investigation, e.g. [9, 24, 25, 26, 27] and probably some others. In this work we also present a construction for a multi-component system, which admits reduction to CH equation.

To this end, instead of the Lax pair (6), (7), we consider a more general one, leading to a hierarchy of Camassa-Holm type:

$$\Psi_{xx} = Q(x, \lambda)\Psi, \tag{8}$$

$$\Psi_t = -U(x, \lambda)\Psi_x + \frac{1}{2}U_x(x, \lambda)\Psi, \tag{9}$$

where

$$Q(x, \lambda) = \lambda^n q_n(x) + \lambda^{n-1}q_{n-1}(x) + \ldots + \lambda q_1(x) + \frac{1}{4}, \tag{10}$$

$$U(x, \lambda) = u_0(x) + \frac{u_1(x)}{\lambda} + \ldots \frac{u_k(x)}{\lambda^k}. \tag{11}$$

The compatibility condition of (8), (9) gives the equation

$$Q_t = \frac{1}{2}U_{xxx} - 2U_x Q - UQ_x, \tag{12}$$

which, due to (10), (11), is equivalent to a chain of n evolution equations with $k+1$ differential constraints for the $n+k+1$ variables q_1, q_2, ..., q_n, u_0, u_1, ..., u_k (n and k are arbitrary natural numbers, i.e. positive integers):

$$q_{n-r,t} = -\sum_{s=\max(0,r-k)}^{r}(2u_{r-s,x}q_{n-s} + u_{r-s}q_{n-s,x}), \qquad r = 0, 1, \ldots, n-1,$$

$$0 = \frac{1}{2}(u_{r,xxx} - u_{r,x}) - \sum_{s=1}^{\min(n,k-r)}(2u_{r+s,x}q_s + u_{r+s}q_{s,x}),$$

$$r = 0, 1, \ldots, k-1,$$

$$0 = \frac{1}{2}(u_{k,xxx} - u_{k,x}). \tag{13}$$

The system (13) is similar to the hydrodynamic chain, studied in a series of papers [28, 29, 30, 31], and to other CH generalizations [9, 24, 26, 27, 32]. Let us now consider the following examples.

Example 1: $k = n = 2$.

The choice $u_2 = -1/2$ solves automatically one of the constraints. The other two differential constraints can be easily integrated, giving

$$q_1 = u_1 - u_{1,xx} + \omega_1, \tag{14}$$

$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1,x}^2 - 2u_1 u_{1,xx} + 4\omega_1 u_1 + \omega_2, \tag{15}$$

where $\omega_{1,2}$ are arbitrary constants. The system of equations for u_0, u_1 is

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$$q_{2,t} \; + \; 2u_{0,x}q_2 + u_0q_{2,x} = 0, \tag{16}$$
$$q_{1,t} \; + \; 2u_{0,x}q_1 + u_0q_{1,x} + 2u_{1,x}q_2 + u_1q_{2,x} = 0. \tag{17}$$

Example 2: $k = 1$, $n = 2$.

This system was studied in more details in [25]. In the notations $u_0 \equiv u$, $q_1 \equiv -q$ and $q_2 \equiv \rho^2$, and with the choice $u_1 = -1/2$, the system can be written in the form

$$q_t \; + \; uq_x + 2qu_x - \rho\rho_x = 0, \tag{18}$$
$$\rho_t \; + \; (u\rho)_x = 0, \tag{19}$$

where $q = u - u_{xx} + \omega$ and ω is an arbitrary constant.

Example 3: CH equation

CH can be considered as a reduction from the system $(14) - (17)$. Indeed, one can obtain an integrable reduction of $(14) - (17)$ by taking $u_1 = \omega_1 = 0$. Then $q_1 = 0$, $q_2 = u_0 - u_{0,xx} + \omega_2$ and (16) is exactly the CH equation (1) with $u \equiv u_0$ and $\omega \equiv \omega_2$. Equation (17) is trivially satisfied. Thus, the CH corresponds to a Lax pair with

$$Q(x,\lambda) = \lambda^2 q_2(x) + \frac{1}{4}, \qquad U(x,\lambda) = u_0(x) - \frac{1}{2\lambda^2}. \tag{20}$$

It is not difficult to recover the Lax pair $(6) - (7)$ by identifying $\zeta = \lambda^2$.

CH can also be obtained as a reduction from $(18) - (19)$ by setting $\rho = 0$.

2 Generating Function for the Integrals of Motion

Introducing

$$p = \frac{\Psi_x}{\Psi}, \tag{21}$$

from (8) we obtain (cf. [33, 34, 21])

$$p_x + p^2 = Q(x,\lambda). \tag{22}$$

Then, from (9), (21) and (22) the following conservation law follows:

$$p_t = \left(\frac{1}{2}U_x - pU \right)_x. \tag{23}$$

Therefore $p(x, \lambda)$, regarded as a solution of (22), is the density of the generating function of the conservation laws. We notice that the densities are determined up to a constant. Indeed, if $\alpha(\lambda)$ is an arbitrary function of λ, the quantity $P = p + \alpha$ is also a generating function, since it satisfies

$$P_t = \left(\frac{1}{2} U_x - PU + \alpha U \right)_x. \tag{24}$$

We can use this freedom to fix the convergency properties of the integrals representing the conserved quantities.

Clearly, p is related to the scattering matrix [7] when the Schwartz class of solutions is considered, or to the monodromy matrix in the periodic case [34].

Now it is evident that (23) or (24) represent a parameter-dependent conservation law, which is equivalent to a sequence of infinitely many conservation laws.

Indeed, since λ is an arbitrary (spectral) parameter, one can expand the solution $p(x, \lambda)$ of (22) about $\lambda = \infty$. Let us suppose for simplicity that $n = 2a$ is an even number. (The case when n is odd is handled in a similar manner, e.g. by introducing a new spectral parameter ζ, such that $\lambda = \zeta^2$, see the CH case [21, 23], where $n = 1$). Then

$$p(x, \lambda) = p_a \lambda^a + \ldots + p_1 \lambda + p_0 + + \sum_{s=1}^{\infty} \frac{p_{-s}}{\lambda^s}. \tag{25}$$

There are finitely many positive powers in λ due to their presence in the RHS of (22). From (23) and (25) the following infinite sequence of conservation laws follows (it is assumed $u_r \equiv 0$ for $r > k$):

$$p_{a-r,t} = -\left(\sum_{s=0}^{r} u_{r-s} p_{a-s} \right)_x, \qquad r = 0, 1, \ldots, a - 1,$$

$$p_{-r,t} = \left(\frac{1}{2} u_{r,x} - \sum_{s=0}^{a+r} u_{a+r-s} p_{a-s} \right)_x, \qquad r = 0, 1, \ldots \ . \tag{26}$$

Let us now expand the solution $p(x, \lambda)$ of (22) about $\lambda = 0$:

$$p(x, \lambda) = p_0 + \sum_{s=1}^{\infty} p_s \lambda^s. \tag{27}$$

Note that the expansion coefficients p_s in (27) are not the same as those in (25). From (23) and (27) another infinite sequence of conservation laws follows:

$$p_{r,t} = -\left(\sum_{s=0}^{k} u_s p_{s+r}\right)_x \qquad r = 1, 2, \ldots,$$

$$p_{0,t} = \left(\frac{1}{2}u_{0,x} - \sum_{s=0}^{k} u_s p_s\right)_x, \tag{28}$$

and in addition, some of the constraints can be rewritten in the form:

$$\left(\frac{1}{2}u_{r,x} - \sum_{s=0}^{k-r} u_{r+s}p_s\right)_x = 0, \qquad r = 1, 2, \ldots, k. \tag{29}$$

In general, there are two different solutions of (22). However, these solutions do not represent independent integral densities of conserved quantities, due to the following result, showing that each of the solutions can be expressed linearly through the other one, up to an exact derivative:

Lemma 1 *If $p^+(x, \lambda)$ and $p^-(x, \lambda)$ are two different solutions of (22), then there exists a function $f(x, \lambda)$, such that*

$$p^+(x, \lambda) = -p^-(x, \lambda) + f_x(x, \lambda). \tag{30}$$

Proof. Writing (22) for $p^+(x, \lambda)$ and $p^-(x, \lambda)$ and subtracting the two equations we obtain immediately $f(x, \lambda) = -\ln[p^+(x, \lambda) - p^-(x, \lambda)]$.

3   Computation of the Conserved Quantities

In order to find the integral densities, one needs to compute explicitly the expansion coefficients in (25), (27). In this Section we will illustrate the method at the system from Example 1. (The conservation laws for the CH, Example 3, follow immediately via the described reduction.) Clearly, one can apply an analogous procedure to any particular case of the system (13).

Before going into the actual computations, the following observation is in order. Exactly as in [7, 35], using only (16) one can prove that $q_2(x,t)$ does not change sign if $q_2(x,0)$ does not. The idea of proof is as follows. Consider the diffeomorphism of the line $\varphi(x,t)$, such that

$$\varphi_t = u_0(\varphi(x,t),t), \qquad \varphi(x,0) = x. \tag{31}$$

The solution of (31) is unique and represents an increasing diffeomorphism ($\varphi_x > 0$ for all t) of $\mathsf{R} \ni x$ [35]. Then, using (16) and (31), one can check that

$$q_2(\varphi(x,t),t)\varphi_x^2(x,t) = q_2(x,0) \tag{32}$$

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and the claim easily follows. Then, for simplicity, in order to make sense of the expressions like $\sqrt{q_2(x,t)}$, $1/\sqrt{q_2(x,t)}$, etc; we assume that the initial data is such that $q_2(x,0)$ does not change sign, i.e. $q_2(x,0) > 0$.

Also, in the case when u_0, u_1, ... are Schwartz class functions, it may happen that the integral density at $x \to \pm\infty$ is not zero but a constant. Then, in order to make sense of the integral, the density obviously should be reduced by the same constant, cf. (24).

The equation (22) is

$$p_x + p^2 = \frac{1}{4} + \lambda q_1 + \lambda^2 q_2, \tag{33}$$

and admits a solution of the form (25):

$$p = p_1\lambda + p_0 + \sum_{s=1}^{\infty} \frac{p_{-s}}{\lambda^s}, \tag{34}$$

where $p_1 = \pm\sqrt{q_2}$, i.e. there are two solutions of (33). Since these two solutions do not produce independent integral densities due to the Lemma 1, we proceed with, say, $p_1 = \sqrt{q_2}$. The first nontrivial integral from here is

$$h_1 \equiv \int p_1 dx = \int \sqrt{q_2} dx. \tag{35}$$

From (33) and (34) we have

$$2p_1 p_0 + p_{1,x} = q_1, \qquad p_0 = \frac{q_1}{2\sqrt{q_2}} - \frac{q_{2,x}}{4q_2}. \tag{36}$$

Neglecting the exact derivative, we obtain the integral

$$h_0 = \frac{1}{2} \int \frac{q_1}{\sqrt{q_2}} dx. \tag{37}$$

The next equation,

$$p_0^2 + 2p_1 p_{-1} + p_{0,x} = \frac{1}{4}, \tag{38}$$

gives

$$p_{-1} = \frac{1}{32}\left(\frac{4}{\sqrt{q_2}} + \frac{q_{2,x}^2}{q_2^{5/2}} - \frac{4q_1^2}{q_2^{3/2}} \right) + \left(\frac{q_{2,x}}{8q_2^{3/2}} - \frac{q_1}{4q_2} \right)_x, \tag{39}$$

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and thus we obtain the integral

$$h_{-1} = \frac{1}{32} \int \left(\frac{4}{\sqrt{q_2}} + \frac{q_{2,x}^2}{q_2^{5/2}} - \frac{4q_1^2}{q_2^{3/2}} \right) \mathrm{d}x. \tag{40}$$

It is not difficult to derive the general recurrent formula from (33) and (34):

$$h_j = \int p_j \mathrm{d}x, \qquad j = 1, 0, -1, ..., \tag{41}$$

$$p_1 = \sqrt{q_2}, \quad p_0 = \frac{q_1}{2\sqrt{q_2}} - \frac{q_{2,x}}{4q_2}, \quad p_{-1} = \frac{1}{2p_1}\left(\frac{1}{4} - p_0^2 - p_{0,x} \right), \tag{42}$$

$$p_{-j} = -\frac{1}{2p_1}\left(\sum_{i=0}^{j-1} p_{-i}p_{-j+i+1} + p_{-j+1,x} \right), \qquad j \geq 2. \tag{43}$$

Now from (42), (43) and (23) one can express the conservation laws in differential form (26) with $a = 1$:

$$p_{1,t} = -(u_0 p_1)_x,$$

$$p_{0,t} = \left(\frac{1}{2}u_{0,x} - u_0 p_0 - u_1 p_1 \right)_x,$$

$$p_{-1,t} = \left(\frac{1}{2}u_{1,x} + \frac{1}{2}p_1 - u_1 p_0 - u_0 p_{-1} \right)_x,$$

$$p_{-j,t} = \left(\frac{1}{2}p_{-j+2} - u_0 p_{-j} - u_1 p_{-j+1} \right)_x, \qquad j \geq 2. \tag{44}$$

The reduction of (41) – (44) to the CH case (Example 3) is straightforward via $q_1 = u_1 = 0$. All even densities, p_0, p_{-2}, ... are exact derivatives and do not produce independent integrals of motion. Each of the remaining odd densities p_1, p_{-1}, p_{-3}, ... produces only one independent integral of motion. Thus the CH integrals are h_1, h_{-1}, h_{-3}, ...

Let us now take the expansion (27). Since in this case $u_2 = -1/2$, from (29) for $r = k$ it follows that $p_{0,x} = 0$ and then from (22), $p_0 = \pm 1/2$. Due to the Lemma 1, we consider here only the first possibility, i.e. $p_0 = 1/2$:

$$p = \frac{1}{2} + \sum_{s=1}^{\infty} p_s \lambda^s. \tag{45}$$

Note that in (45) p_1 is not the same as in (34). From (33) and (45) we obtain

$$p_1 + p_{1,x} = q_1, \tag{46}$$

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which due to (14) has a solution

$$p_1 = u_1 - u_{1,x} + \omega_1, \tag{47}$$

leading to the integral (h_1 is not the same as in (35))

$$h_1 = \int u_1 \mathrm{d}x. \tag{48}$$

The next equation from (33) and (45) is

$$p_2 + p_{2,x} + p_1^2 = q_2. \tag{49}$$

Using (15), (47) one can verify that

$$p_2 = u_0 - u_{0,x} + 2u_1^2 - 2u_1 u_{1,x} + 2\omega_1 u_1 - \omega_1^2 + \omega_2. \tag{50}$$

Then the next independent integral is

$$h_2 = \int (u_0 + 2u_1^2)\mathrm{d}x + 2\omega_1 h_1. \tag{51}$$

The equation for p_3 is

$$p_3 + p_{3,x} + 2p_1 p_2 = 0, \tag{52}$$

giving

$$h_3 = \int p_3 \mathrm{d}x = -2 \int p_1 p_2 \mathrm{d}x, \tag{53}$$

where p_1 and p_2 are given by (47), (50). We notice that formally (see (52))

$$p_3 = -(1 + \partial_x)^{-1} 2p_1 p_2. \tag{54}$$

The equation for p_4 is

$$p_4 + p_{4,x} + 2p_1 p_3 + p_2^2 = 0, \tag{55}$$

and thus

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$$h_4 = \int p_4 \mathrm{d}x = -\int (2p_1 p_3 + p_2^2)\mathrm{d}x. \qquad (56)$$

In order to express $\int p_1 p_3 \mathrm{d}x$ only via the known p_1 and p_2 we proceed as follows, using (52), (47) and neglecting total derivatives:

$$
\begin{aligned}
\int p_1 p_3 \mathrm{d}x &= -2\int p_1^2 p_2 \mathrm{d}x - \int p_1 p_{3,x}\mathrm{d}x = \\
&-2\int p_1^2 p_2 \mathrm{d}x - \int (u_1 - u_{1,x} + \omega_1)p_{3,x}\mathrm{d}x = \\
&-2\int p_1^2 p_2 \mathrm{d}x - \int (u_1 p_{3,x} + u_{1,x}(2p_1 p_2 + p_3))\mathrm{d}x = \\
&-2\int p_1^2 p_2 \mathrm{d}x - \int ((u_1 p_3)_x + 2u_{1,x}p_1 p_2)\mathrm{d}x = \\
&-2\int p_1 p_2 (u_{1,x} + p_1)\mathrm{d}x = -2\int p_1 p_2 (u_1 + \omega_1)\mathrm{d}x = \\
&-2\int p_1 p_2 u_1 \mathrm{d}x + \omega_1 h_3.
\end{aligned}
\qquad (57)
$$

Finally,

$$h_4 = \int (4u_1 p_1 p_2 - p_2^2)\mathrm{d}x - 2\omega_1 h_3. \qquad (58)$$

The densities for the higher integrals are, in general, nonlocal.

To summarize: the conserved quantities are

$$h_j = \int p_j \mathrm{d}x, \qquad j = 1, 2, \ldots, \qquad (59)$$

where according to (33) and (45) the integral densities p_j can be computed recurrently:

$$p_j = -(1 + \partial_x)^{-1} \sum_{i=1}^{j-1} p_i p_{j-i}, \qquad j \geq 3, \qquad (60)$$

where p_1 and p_2 are given in (47), (50), e.g. see (54). Again from (60) and (23), the conservation laws can be expressed in differential form (28):

$$p_{j,t} = \left(\frac{1}{2}p_{j+2} - u_1 p_{j+1} - u_0 p_j\right)_x, \qquad j = 1, 2, \ldots \; . \qquad (61)$$

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The reduction to CH (Example 3) goes as follows. With $q_1 = u_1 = \omega_1 = 0$ we have (47) $p_1 = 0$ and from (50), bearing in mind that in this case $u_0 \equiv u$, $\omega_2 \equiv \omega$,

$$p_2 = u - u_x + \omega. \tag{62}$$

Then, clearly all odd integral densities are zero. Furthermore, h_2 gives the CH integral $H_0 = \int m \mathrm{d}x$ and h_4 leads to H_1 [recall (4)]. Now we can verify that the reduction of h_6 leads to the second CH Hamiltonian H_2 [recall (5)].

The equation for p_6 with the reduction is

$$p_6 + p_{6,x} + 2p_2 p_4 = 0, \tag{63}$$

and correspondingly

$$h_6 = -2 \int p_2 p_4 \mathrm{d}x. \tag{64}$$

The equation for p_4, (55) now is

$$p_4 + p_{4,x} + p_2^2 = 0, \tag{65}$$

and therefore, in order to compute $\int p_2 p_4 \mathrm{d}x$ we proceed as follows. Multiplying (65) by p_2 and then using (65) and (62) we obtain:

$$\int p_2 p_4 \mathrm{d}x = -\int p_2^3 \mathrm{d}x - \int p_2 p_{4,x} \mathrm{d}x =$$
$$-\int p_2^3 \mathrm{d}x - \int (u - u_x + \omega) p_{4,x} \mathrm{d}x =$$
$$-\int p_2^3 \mathrm{d}x - \int (u p_{4,x} + u_x (p_2^2 + p_4)) \mathrm{d}x =$$
$$-\int p_2^3 \mathrm{d}x - \int ((u p_4)_x + u_x p_2^2) \mathrm{d}x =$$
$$-\int p_2^2 (u_x + p_2) \mathrm{d}x = -\int p_2^2 (u + \omega) \mathrm{d}x =$$
$$-\int (u^3 + u u_x^2 + 2\omega u^2) \mathrm{d}x - 3\omega^2 H_0 - 2\omega H_1,$$

leading to the independent integral

$$H_2 = \frac{1}{2} \int (u^3 + u u_x^2 + 2\omega u^2) \mathrm{d}x.$$

In other words, the CH conserved quantities are reproduced by the even integrals h_2, h_4, h_6, ...

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