



Technological University Dublin  
ARROW@TU Dublin

---

Articles

School of Mathematics

---


2006-01-01

## Hamiltonian formulation and integrability of a complex symmetric nonlinear system

Rossen Ivanov

*Technological University Dublin*, [rossen.ivanov@tudublin.ie](mailto:rossen.ivanov@tudublin.ie)

Follow this and additional works at: <https://arrow.tudublin.ie/scschmatart>

 Part of the [Dynamic Systems Commons](#), [Mathematics Commons](#), [Non-linear Dynamics Commons](#), and the [Ordinary Differential Equations and Applied Dynamics Commons](#)

---

### Recommended Citation

R. Ivanov, Hamiltonian formulation and integrability of a complex symmetric nonlinear system, *Phys. Lett. A* 350, 232-235 (2006)

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact [yvonne.desmond@tudublin.ie](mailto:yvonne.desmond@tudublin.ie), [arrow.admin@tudublin.ie](mailto:arrow.admin@tudublin.ie), [brian.widdis@tudublin.ie](mailto:brian.widdis@tudublin.ie).



This work is licensed under a [Creative Commons Attribution-Noncommercial-Share Alike 3.0 License](https://creativecommons.org/licenses/by-nc-sa/3.0/)



# Hamiltonian formulation and integrability of a complex symmetric nonlinear system

Rossen Ivanov<sup>\*1</sup>

*School of Mathematics, Trinity College,  
Dublin 2, Ireland*

*Tel: + 353 - 1 - 608 2898*

*Fax: + 353 - 1- 608 2282*

*\* e-mail: ivanovr@tcd.ie*

## Abstract

The integrability of a complex generalisation of the 'elegant' system, proposed by D. Fairlie [1] and its relation to the Nahm equation and the Manakov top is discussed.

**PACS:** 02.30.Ik, 45.20.Jj, 02.30.Hq

**Key Words:** Integrals of Motion, Lax Pair, Nahm equation, Manakov top.

---

<sup>1</sup>On leave from the Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria.

# 1 A complex nonlinear system with symmetry

The Euler top is probably the best known example of a nonlinear integrable system. Generalizations of this system are under intense investigation for many years. In general, the (generalized) top equations contain quadratic nonlinearity. A problem of the form  $dY/dt = Y^2$ , where  $Y$  is an  $n \times n$  symmetric matrix with vanishing diagonal elements is analyzed in the work of Fairlie [1] and leads to the following  $n$ -dimensional nonlinear system:

$$\frac{du_j}{dt} = \prod_{k \neq j}^n u_k, \quad j = 1, \dots, n. \quad (1)$$

In [1] this system is explicitly integrated. The system (1) for  $n = 4$  appears also in the classification of a class of double-elliptic systems [2]. Other higher dimensional integrable tops have been recently found in [3, 4, 5].

The problem of generalizations of the standard Hamiltonian dynamics to complex dynamical variables often contains many interesting aspects, e.g. [6, 7]. In particular, an integrable complexification of (1) for  $n = 3$  and the related Halphen system are considered in [8]. A natural generalization of these systems is the following system in  $n$  complex dimensions:

$$\frac{d\bar{u}_j}{dt} = \prod_{k \neq j}^n u_k, \quad j = 1, \dots, n, \quad (2)$$

where the bar stands for complex conjugation. The system (2) is also integrable. Indeed, it possesses the following independent algebraic constants of the motion:

$$H \equiv iI_1 = \prod_{k=1}^n \bar{u}_k - \prod_{k=1}^n u_k, \quad (3)$$

$$I_k = |u_1|^2 - |u_k|^2, \quad k = 2, \dots, n. \quad (4)$$

There are  $n$  real integrals,  $I_1, \dots, I_n$ , but they are sufficient for the integration of the system.

The system (2) can be realised as a reduction of the following Hamiltonian system with  $\mathbf{Z}_2$  symmetry:

$$H(p, q) \equiv \prod_{k=1}^n p_k - \prod_{k=1}^n q_k, \quad (5)$$

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} = \prod_{k \neq j}^n q_k$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} = \prod_{k \neq j}^n p_k \quad (6)$$

Considering  $q_k \equiv u_k$  as coordinate variables and  $p_k \equiv \bar{u}_k$  as momentum variables, the system (2) admits a Hamiltonian formulation with a Hamiltonian  $H$  given by (3). The equations of motion (5) and (6) then coincide with (2). Note that in the real case (1)  $H \equiv 0$ .

Now we are able to integrate explicitly the system (2). Let  $u_k = e^{i\phi_k} r_k$  where  $r_k, \phi_k$  are real functions and  $\phi = \phi_1 + \phi_2 + \dots + \phi_n$ . Then

$$u_k d\bar{u}_k + \bar{u}_k du_k = r dt$$

where  $r = \prod_{k=1}^n u_k + \prod_{k=1}^n \bar{u}_k$  is a real function. Let

$$r dt = d\tau. \quad (7)$$

Then

$$dr_k^2 = r dt \quad (8)$$

and

$$r_k^2 = \tau + c_k \quad (9)$$

for some real constants  $c_k$ . It is obvious that

$$I_1 = -2 \left( \prod_{k=1}^n r_k \right) \sin \phi \quad (10)$$

$$r = 2 \left( \prod_{k=1}^n r_k \right) \cos \phi \quad (11)$$

and therefore from (7), (9), (10) and (11) we obtain

$$\left( \frac{d\tau}{dt} \right)^2 = 4 \left( \prod_{k=1}^n r_k \right)^2 - I_1^2 = 4 \left( \prod_{k=1}^n (\tau + c_k) \right) - I_1^2. \quad (12)$$

i.e. formally

$$t = \int \frac{d\tau}{\sqrt{4 \left( \prod_{k=1}^n (\tau + c_k) \right) - I_1^2}} \quad (13)$$

For a specific choice of the constants  $c_k$  and  $I_1$ ,  $\tau$  can be expressed via  $t$  through a hyperelliptic function. Furthermore

$$\frac{d}{dt} \left( \frac{u_k}{\bar{u}_k} \right) = i \frac{I_1}{\bar{u}_k^2},$$

i.e.

$$\frac{d\phi_k}{dt} = \frac{I_1}{2r_k^2}, \quad (14)$$

and from (9) and (12) we have formally

$$\phi_k(\tau) = \frac{I_1}{2} \int \frac{d\tau}{(\tau + c_k) \sqrt{4 \left( \prod_{k=1}^n (\tau + c_k) \right) - I_1^2}}. \quad (15)$$

The integrals (13) and (15) can be expressed in terms of hyperelliptic functions.

In order to better understand the underlying symmetry of the system let us rewrite it in terms of real variables as follows. The summation over  $k$  of (14) gives

$$\frac{d\phi}{dt} = \frac{I_1}{2} \sum_{k=1}^n \frac{1}{r_k^2}, \quad (16)$$

which, due to (10) is equivalent to

$$\frac{d\phi}{dt} = -\sin \phi \left( \prod_{k=1}^n r_k \right) \sum_{k=1}^n \frac{1}{r_k^2}. \quad (17)$$

From (8) and (11) we obtain the following set of equations:

$$\frac{dr_k}{dt} = \cos \phi \prod_{i \neq k} r_i. \quad (18)$$

Now it is clear that (2) is equivalent to the  $n + 1$  dimensional real system (17), (18). The reduction to the 'real' version (1) can be achieved as follows. Let us choose for  $\phi$  the initial value  $\phi(0) = 2K\pi$  for some arbitrary integer  $K$ . Then the integral  $I_1 = 0$ , according to (10), and consequently  $\frac{d\phi}{dt} = 0$ , according to (16). Thus the solution in this case is  $\phi(t) = 2K\pi$  and (18) simply coincides with (1).

## 2 Example: $su(3)$ Nahm top ( $n = 3$ )

It is well known that the one-dimensional reductions of the Self-dual Yang Mills (SDYM) equations in four dimensions yield various generalizations of some classical systems as the Euler-Arnold-Manakov and Nahm equations [9, 10]. In [8] the SDYM equations in seven dimensions have been investigated, in the case when their invariance group is the  $\mathbf{G}_2$  subgroup of the rotation group  $SO(7)$ . One of the six-dimensional reductions of these equations leads to (2) with  $n = 3$ :

$$\frac{d\bar{u}_1}{dt} = u_2 u_3 \quad \text{and cyclic permutations,} \quad (19)$$

In [8] the question about the existence of a Lax pair for the system (19) is posed, but the Lax pair is not found. Although this system is in principle known for some time, since it is an interesting one here we pay some more attention to it.

In physical terms, the system (19) describes membrane instantons in three complex dimensions [11, 12]. It can also be seen as a "stationary

point” for the 3–wave equations with a ”blow-up” instability (i.e.  $\frac{\partial u_i}{\partial x} = 0$ ):

$$\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} = i\epsilon \bar{u}_2 \bar{u}_3,$$

where  $v_k$  and  $\epsilon$  are real constants [13, 14, 15]. The equations (19) possess a complete symmetry and in this respect are similar to the Kaup equations, where the derivatives are over the three space variables [16].

Firstly, one can show that (19) is related to the well known Nahm equation. Using the matrices

$$\begin{aligned} T_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -u_1 \\ 0 & \bar{u}_1 & 0 \end{pmatrix}, & T_2 &= \begin{pmatrix} 0 & 0 & \bar{u}_2 \\ 0 & 0 & 0 \\ -u_2 & 0 & 0 \end{pmatrix}, \\ T_3 &= \begin{pmatrix} 0 & -u_3 & 0 \\ \bar{u}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (20)$$

the system (19) can be written in the usual form for the Nahm equation

$$\frac{dT_i}{dt} = \frac{1}{2} \varepsilon_{ijk} [T_j, T_k]. \quad (21)$$

Equation (21) was introduced initially in relation to  $SU(2)$  monopoles on  $\mathbf{R}^3$  [10]. However, the Nahm matrices,  $T_i$  in (21) can take values in various simple Lie algebras (not necessarily  $su(2)$ ). This is explained in details in [17]. Since the matrices (20) are anti-Hermitian, i.e. take values in  $su(3)$ , we will call (19) an  $su(3)$  Nahm top. Now it is straightforward to represent (21) in the Lax form  $\frac{dL}{dt} = [L, M]$  (e.g. see [18, 19]) with

$$L(\zeta) = T_1 + iT_2 - 2\zeta iT_3 + \zeta^2(T_1 - iT_2) \quad (22)$$

$$M(\zeta) = -iT_3 + \zeta(T_1 - iT_2). \quad (23)$$

The computation of  $\det(L(\zeta) - \lambda \mathbf{1})$  immediately gives the integrals of motion (3), (4). For another form of the Lax pair for the Nahm equation see also [12].

The system (19) can also be seen as a Manakov top [20] with a complex inertia tensor. Indeed, consider the following Lax pair:

$$L(\zeta) = [J^2, Q] - \zeta J^2 \quad (24)$$

$$M(\zeta) = [J, Q] - \zeta J, \quad (25)$$

where the cubic root of unity,  $\omega = e^{2\pi i/3}$  is used to define the inertia tensor

$$J = \frac{1}{\omega(\omega - 1)} J_0, \quad J_0 = \text{diag}(\omega, \omega^2, 1); \quad (26)$$

the matrix  $Q$  is defined as  $[J_0, Q] = -(T_1 + T_2 + T_3)$  where  $T_k$  are the matrices (20), i.e.  $Q = -\text{ad}_{J_0}^{-1}(T_1 + T_2 + T_3)$ . Explicitly, the operators  $L$  (24), and  $M$  (25) are (the irrelevant overall factor  $[\omega(\omega - 1)]^{-2}$  in  $L$  is omitted)

$$L(\zeta) = \sum_{k=1}^3 \omega^k T_k - \zeta J_0^2, \quad (27)$$

$$M(\zeta) = -\frac{1}{\omega(\omega - 1)} \left( \sum_{k=1}^3 T_k + \zeta J_0 \right). \quad (28)$$

The characteristic polynomial of  $L(\zeta)$ , (27) is

$$\det L(\zeta) = -\zeta^3 - \zeta \left( \omega |u_1|^2 + \omega^2 |u_2|^2 + |u_3|^2 \right) + (\bar{u}_1 \bar{u}_2 \bar{u}_3 - u_1 u_2 u_3) \quad (29)$$

Its coefficients can be split into real and imaginary parts to give the integrals of motion (3), (4).

The representation of the system (19) as a Manakov top allows immediately the classical  $r$ -matrix to be written [21, 22]. Defining a Poisson bracket

$$\{A \otimes B\} \equiv \sum_{k=1}^3 \left( \frac{\partial A}{\partial u_k} \otimes \frac{\partial B}{\partial \bar{u}_k} - \frac{\partial A}{\partial \bar{u}_k} \otimes \frac{\partial B}{\partial u_k} \right), \quad (30)$$

it is straightforward to check, that

$$\{L(\zeta) \otimes L(\mu)\} = [r(\zeta - \mu), L(\zeta) \otimes \mathbf{1} + \mathbf{1} \otimes L(\mu)], \quad (31)$$

where  $L$  is given by (27), and the classical  $r$  matrix is

$$r(\zeta) = -\frac{P}{\omega(\omega - 1)\zeta}. \quad (32)$$

Here  $P$  is the permutation matrix in  $\mathbb{C}^3 \otimes \mathbb{C}^3$ , i.e.

$$P(x \otimes y) = y \otimes x, \quad x, y \in \mathbb{C}^3. \quad (33)$$

### 3 Example: $so(4)$ Nahm top ( $n = 4$ )

As a second example we consider the system (1) with  $n = 4$ . It is known that with a change of variables [2] it can be represented as two  $so(3)$  Nahm tops. Since  $so(4) = so(3) \oplus so(3)$  it is natural to expect that (1) with  $n = 4$  is an  $so(4)$  Nahm top. Indeed, using the  $so(4)$  parametrization given in [23] we can define the following  $so(4)$  Nahm matrices:

$$\begin{aligned}
T_1 &= \begin{pmatrix} 0 & 0 & 0 & u_1 u_4 \\ 0 & 0 & -u_2 u_3 & 0 \\ 0 & u_2 u_3 & 0 & 0 \\ -u_1 u_4 & 0 & 0 & 0 \end{pmatrix}, \\
T_2 &= \begin{pmatrix} 0 & 0 & u_1 u_3 & 0 \\ 0 & 0 & 0 & u_2 u_4 \\ -u_1 u_3 & 0 & 0 & 0 \\ 0 & -u_2 u_4 & 0 & 0 \end{pmatrix}, \\
T_3 &= \begin{pmatrix} 0 & -u_3 u_4 & 0 & 0 \\ u_3 u_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 u_2 \\ 0 & 0 & -u_1 u_2 & 0 \end{pmatrix}. \tag{34}
\end{aligned}$$

With the help of (34), one can write (1) explicitly in the canonical form (21). Now the Lax pair (22), (23) follows automatically. We also note that the construction

$$L = \sum_{k=1}^3 \omega^k T_k, \tag{35}$$

$$M = -\frac{1}{\omega(\omega-1)} \sum_{k=1}^3 T_k \tag{36}$$

also provides a Lax pair for this system, indeed, without a spectral parameter.

The Lie group interpretation, the Lax pairs etc. for the system (2) with  $n \geq 4$  is a very interesting problem, which deserves a further investigation.

## Acknowledgements

The author is grateful to V. Gerdjikov, G. Grahovski and N. Kostov for many valuable discussions. Funding from Science Foundation Ireland, Grant 04/BR6/M0042 is gratefully acknowledged.

## References

- [1] D. Fairlie, Phys. Lett. A 119 (1987) 438.
- [2] H. Braden, A. Gorsky, A. Odesskii, V. Rubtsov, Nucl. Phys. B 633 (2002) 414; hep-th/0111066.
- [3] D. Fairlie, T. Ueno, Phys. Lett. A 248 (1998) 132.



- [4] T. Ueno, *Phys. Lett. A* 245 (1998) 373.
- [5] D. Fairlie, T. Ueno, *J. Phys. A* 31 (1998) 7785.
- [6] V.S. Gerdjikov, A. Kyuldjiev, G. Marmo, G. Vilasi, *Eur. Phys. J. B* 29 (2002) 177.
- [7] V.S. Gerdjikov, A. Kyuldjiev, G. Marmo, G. Vilasi, *Eur. Phys. J. B* 38 (2004) 635.
- [8] K. Sfetsos, *Nucl. Phys. B* 629 (2002) 417; hep-th/0112117.
- [9] S. Chakravarty, M.J. Ablowitz, P.A. Clarkson, *Phys. Rev. Lett.* 65 (1990) 1085.
- [10] W. Nahm, The construction of all self-dual multimonopoles by the ADHM method, in: *Monopoles in Quantum Field Theory*, N.S Craigie, P. Goddard and W. Nahm (eds.), World Scientific, 1982, pp.87-95.
- [11] E.G. Floratos, G.K. Leontaris, *Phys. Lett. B* 545 (2002) 190.
- [12] T. Curtright, D. Fairlie, C. Zachos, *Phys. Lett. B* 405 (1997) 37.
- [13] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskii, *Theory of solitons: the inverse scattering method*, Plenum, N.Y., 1984.
- [14] V.E. Zakharov, S.V. Manakov, Exact theory of resonant interaction of wave packets in nonlinear media, INF preprint 74-41, Novosibirsk (1975) (In Russian).
- [15] E.B. Gledzer, F.B. Dražanskii, A.M. Obukhov, *Systems of hydrodynamic type*, Nauka, Moscow, 1981 (In Russian).
- [16] D.J. Kaup, *J. Math. Phys.* 22 (1981) 1176.
- [17] T. Brzezinski, H. Merabet, *Czechoslovak J. Phys.* 47 (1997) 1101; hep-th/9709070.
- [18] N.J. Hitchin, *Commun. Math. Phys.* 89 (1983) 145.
- [19] N.J. Hitchin, N.S. Manton, M.K. Murray, *Nonlinearity* 8 (1995) 661.
- [20] S.V. Manakov, *Funct. Anal. Pril.* 10 (1976) 93 (In Russian).
- [21] L.D. Faddeev, L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin, 1987.
- [22] P.P. Kulish and E.K. Sklyanin, *Zap. Nauchn. Sem. LOMI* 77 (1978) 134 (In Russian); *J. Sov. Math.* 22 (1983) 1627.
- [23] L.A. Bordag, A.B. Yanovski, *J. Phys. A: Math. Gen.* 29 (1996) 5575.