Remarks on Quadratic Bundles Related to Hermitian Symmetric Spaces

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Abstract. We consider quadratic bundles related to Hermitian symmetric spaces of the type $SU(m+n)/S(U(m)\times U(n))$. We discuss the spectral properties of scattering operator, develop the direct scattering problem associated with it and stress on the effect of reduction on these. By applying a modification of Zakharov-Shabat's dressing procedure we demonstrate how one can obtain reflectionless potentials. That way one is able to generate soliton solutions to the nonlinear evolution equations belonging to the integrable hierarchy associated with quadratic bundles under study.

1. Introduction

Derivative nonlinear Schrödinger equation (DNLS)

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0 \tag{1}$$

is one of classical S-integrable nonlinear evolution equations (NLEE). It occurs in plasma physics to describe the propagation of nonlinear Alfvén waves with circular polarization [18, 19].

The zero curvature representation $[L, A] = 0$ of DNLS was discovered by Kaup and Newell [12] who picked up L and A in the form:

$$L(\lambda) = \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2\sigma_3, \tag{2}$$

$$A(\lambda) = \mathrm{i}\partial_t + \sum_{k=1}^{3} A_k(x,t)\lambda^k - 2\lambda^4\sigma_3, \tag{3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ q^*(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

DNLS is deeply connected to 2-dimensional Thirring model [13, 15] and the Gerdjikov-Ivanov equation [5, 6], both related to certain reductions of quadratic bundle L operator of generic form:

$$L(\lambda) = \mathrm{i}\partial_x + U_0(x,t) + \lambda U_1(x,t) - \lambda^2\sigma_3 \tag{4}$$

for $U_1(x,t)$ being an off-diagonal 2×2 matrix and $U_0(x,t)$ being a traceless 2×2 matrix otherwise arbitrary.

Derivation and study of multicomponent generalizations of classical scalar integrable equations is a trend of current interest [7, 8, 11] in theory of integrable systems. It was pioneered by Manakov [14] who studied 2-component counterpart of nonlinear Schrödinger equation to later become known as Manakov system. A. Fordy et al. developed that idea and established a geometric relation between linear and quadratic bundles and Hermitian symmetric spaces [1, 2, 3]. In [2] multicomponent versions of DNLS like the following one:

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + \frac{2m\mathrm{i}}{n+m}\left(\mathbf{q}\mathbf{q}^\dagger\mathbf{q}\right)_x = 0 \tag{5}$$

for $\mathbf{q}$ being a smooth $n\times m$ matrix-valued function, were derived. Equation (5) is related to symmetric space of the type $SU(m+n)/S(U(m)\times U(n))$ which is to say the operator L has the same form as (2) but now Q is a block $(m+n)\times(m+n)$ matrix of the form

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^T(x,t) \\ \mathbf{q}^*(x,t) & 0 \end{pmatrix}$$

and σ_3 is replaced by matrix $J = \mathrm{diag}\,(\frac{n}{m}\mathbb{1}_m, -\mathbb{1}_n)$ where $\mathbb{1}_m$ is the $m\times m$ unit matrix. Like the scalar DNLS (5) is a Hamiltonian system [2, 20] with Hamiltonian given by

$$H = \int_{-\infty}^{\infty} \mathrm{d}\,x\,\mathrm{tr}\,\left(\mathrm{i}\mathbf{q}^\dagger\mathbf{q}_x - \frac{m}{n+m}(\mathbf{q}^\dagger\mathbf{q})^2\right),$$

provided the Poisson bracket is defined as

$$\{F,G\} = \int_{-\infty}^{\infty} \mathrm{d}\,y\,\mathrm{tr}\,\left(\frac{\delta F}{\delta Q}\partial_y\frac{\delta G}{\delta Q}\right)$$

for F and G being functionals of the potential Q.

All afore-mentioned facts make a deeper study of quadratic bundles associated with symmetric spaces very important. Our purpose here is to shed some light on certain basic properties of equation (5) and the corresponding Lax pair. In doing so we shall partially extend some results already published in [20]. The report is structured as follows. Second section is preliminary in its nature. We discuss some basic properties of the scattering operator L and the linear problem $L\psi = 0$ related to the nonlinear evolution equation (5) to be used further in text. In next section we show how one can modify Zakharov-Shabat's dressing method for the case of quadratic bundles. This allows one to generate special types of solutions in an algebraic manner. Last section contains summary of our results and some additional remarks.

2. Quadratic bundles and Hermitian symmetric spaces

Let us consider the Lax pair:

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x + \lambda Q(x,t) - \lambda^2 J, & (6)\\ A(\lambda) &= \mathrm{i}\partial_t + \sum_{k=1}^{2N}\lambda^k A_k(x,t), & (7)\end{aligned}$$

where $Q(x,t)$, J and $A_k(x,t)$, $k=1,\dots 2N$ are $(m+n)\times(m+n)$ traceless Hermitian matrices. Moreover, L and A are subject to additional $\mathbb{Z}_2$ reduction [16, 17]

$$\mathbf{C}L(-\lambda)\mathbf{C} = L(\lambda), \qquad \mathbf{C}A(-\lambda)\mathbf{C} = A(\lambda), \qquad \mathbf{C} = \mathrm{diag}\,(\mathbb{1}_m, -\mathbb{1}_n). \tag{8}$$

The constant matrix $\mathbf{C}$ defines an action of Cartan's involution connected with symmetric space $SU(m+n)/S(U(m)\times U(n))$ in the Lie algebra $\mathfrak{sl}(m+n)$, see [10] for more details. That way the Lax pair (6), (7) is related toa space of the type **A.III**. Cartan's involution induces a $\mathbb{Z}_2$ grading in $\mathfrak{sl}(m+n)$ as follows:

$$\mathfrak{sl}(m+n) = \mathfrak{sl}^0(m+n) + \mathfrak{sl}^1(m+n), \tag{9}$$

where

$$\mathfrak{sl}^\sigma(m+n) = \{X \in \mathfrak{sl}(m+n) |\ \mathbf{C}X\mathbf{C} = (-1)^\sigma X\}, \qquad \sigma = 0,1$$

are eigensubspaces of the adjoint action of $\mathbf{C}$. Due to reduction condition(8) the potential Q has the block structure:

$$Q(x,t) = \left(\begin{array}{cc} 0 & \mathbf{q}^T(x,t) \\ \mathbf{q}^*(x,t) & 0 \end{array}\right),$$

where $\mathbf{q}(x,t)$ is a $n\times m$ rectangular matrix and J is picked up in the form

$$J = \mathrm{diag}\,(n_1, n_2, \ldots, n_m, -1\!\!1_n), \qquad \sum_{k=1}^{m} n_k = n. \tag{10}$$

From now on we shall assume that $\mathbf{q}$ is an infinitely smooth function to obey boundary conditions

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{0}.$$

Let us now consider auxiliary linear problem

$$L(\lambda)\psi(x,t,\lambda) = \mathrm{i}\partial_x\psi + \lambda(Q - \lambda J)\psi = 0. \tag{11}$$

The function ψ is viewed as a fundamental set of solutions hence $\psi(x,t,\lambda)$ is a unimodular matrix. Since L and A commute we have as well

$$A\psi = \mathrm{i}\partial_t\psi + \sum_{k=1}^{2N} \lambda^k A_k\psi = \psi f. \tag{12}$$

Quantity

$$f(\lambda) = \lim_{x\to\pm\infty} \sum_{k=1}^{2N} \lambda^k A_k(x,t)$$

is called dispersion law of NLEE and it carries all essential characteristics of NLEE.

A special case of solutions to (11) are Jost solutions defined as follows:

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda^2 Jx} = 1\!\!1, \qquad \lambda^2 \in \mathbb{R}. \tag{13}$$

Due to the special choice of the right handside of (12) the definition of the Jost solutions is correct. The transition matrix

$$T(t,\lambda) = \hat{\psi}_+(x,t,\lambda)\psi_-(x,t,\lambda), \qquad \hat{\psi} \equiv \psi^{-1}$$

between the Jost solutions is called scattering matrix. It can be proven that the scattering matrix evolves with time according to:

$$\mathrm{i}\partial_t T + [f(\lambda), T] = 0 \quad \Rightarrow \quad T(t,\lambda) = \mathrm{e}^{\mathrm{i}f(\lambda)t}T(0,\lambda)\mathrm{e}^{-\mathrm{i}f(\lambda)t}.$$

The Jost solutions are defined for λ lying on real and imaginary axis. To see this one introduces functions $\xi_\pm = \psi_\pm \exp(\mathrm{i}\lambda^2 Jx)$ to satisfy modified linear equation

$$\mathrm{i}\partial_x \xi_\pm + \lambda Q \xi_\pm - \lambda^2 [J, \xi_\pm] = 0. \tag{14}$$

Equivalently $\xi_\pm$ can be viewed as solutions to the following integral equations of Voltera type

$$\xi_\pm(x,\lambda) = \mathbb{1} + \mathrm{i}\lambda \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}\lambda^2 J(x-y)} Q(y) \xi_\pm(y,\lambda) \mathrm{e}^{\mathrm{i}\lambda^2 J(x-y)} \mathrm{d}\, y. \tag{15}$$

By analysing it one can see that analytic continuation outside the real and imaginary axis is possible for the first and the last columns of $\xi_\pm$ only. Quite similarly to $\mathfrak{sl}(2)$ case [5] starting from the Jost solutions one is able to construct another set of solutions to have such analytic properties. Namely, the following theorem holds true:

Theorem 1 *There exists a pair of solutions χ^+ and χ^- analytic for $\operatorname{Im}\lambda^2 > 0$ (i.e. in the first and third quadrant in λ-plane and $\operatorname{Im}\lambda^2 < 0$ (in the second and the forth quadrants). They can be constructed from the Jost solutions as follows:*

$$\chi^\pm(x,\lambda) = \psi_-(x,\lambda) S^\pm(\lambda) = \psi_+(x,\lambda) T^\mp(\lambda) D^\pm(\lambda).$$

where the matrices $S^\pm(\lambda)$, $T^\pm(\lambda)$ and $D^\pm(\lambda)$ are

$$\begin{aligned}
S^+(\lambda) &= \begin{pmatrix} \mathbb{1}_m & \mathbf{s}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix}, \qquad T^+(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{t}_+^T(\lambda) \\ \mathbf{0} & \mathbb{1}_n \end{pmatrix}, \\
S^-(\lambda) &= \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{s}_-(\lambda) & \mathbb{1}_n \end{pmatrix}, \qquad T^-(\lambda) = \begin{pmatrix} \mathbb{1}_m & \mathbf{0}^T \\ \mathbf{t}_-(\lambda) & \mathbb{1}_n \end{pmatrix}, \\
D^+(\lambda) &= \begin{pmatrix} d_m^+(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^+(\lambda) \end{pmatrix}, \quad D^-(\lambda) = \begin{pmatrix} d_m^-(\lambda) & \mathbf{0}^T \\ \mathbf{0} & d_n^-(\lambda) \end{pmatrix}.
\end{aligned}$$

The latter are involved in generalised LDU decomposition

$$T(\lambda) = T^\mp(\lambda) D^\pm(\lambda) \hat{S}^\pm(\lambda)) \tag{16}$$

of the scattering matrix[1].

The reductions imposed on our Lax operators give rise to certain symmetry conditions on the Jost solutions, the scattering matrix and fundamental analytic solutions [16, 17], as follows:

$$\begin{aligned}
\hat{\psi}_\pm^\dagger(x,\lambda^*) &= \psi_\pm(x,\lambda), \qquad \hat{T}^\dagger(\lambda^*) = T(\lambda), & (17)\\
\mathbf{C}\psi_\pm(x,-\lambda)\mathbf{C} &= \psi_\pm(x,\lambda), \qquad \mathbf{C}T(-\lambda)\mathbf{C} = T(\lambda), & (18)\\
\left[\chi^+(x,\lambda^*)\right]^\dagger &= \hat{\chi}^-(x,\lambda), \quad \mathbf{C}\chi^\pm(x,-\lambda)\mathbf{C} = \chi^\pm(x,\lambda). & (19)
\end{aligned}$$

As a simple consequence of their construction the fundamental analytic solutions χ^+ and χ^- are interrelated through:

$$\chi^+(x,\lambda) = \chi^-(x,\lambda) G(\lambda) \tag{20}$$

for some function $G(\lambda) = \hat{S}^-(\lambda) S^+(\lambda)$. This means that they can be viewed as solutions to a local Riemann-Hilbert problem [9, 21] with boundary given by real and imaginary axis

[1] The decomposition (16) is called generalised since all factors have a block structure which respects the splitting (9), that is $d_n^\pm(\lambda)$ are $n \times n$ matrices for example while $\mathbf{s}_\pm(\lambda)$ are $n \times m$ matrices.

in λ-plane. To be more precise solutions to a local Riemann-Hilbert problem are functions $\eta^{\pm} = \chi^{\pm} \exp(\mathrm{i}\lambda^2 Jx)$ which satisfy (14). The latter pair of solutions can be expanded [9]

$$\eta^{\pm}(x,\lambda) = \eta_0^{\pm}(x) + \frac{\eta_1^{\pm}(x)}{\lambda} + \frac{\eta_2^{\pm}(x)}{\lambda^2} + \dots \tag{21}$$

Substituting the asymptotic expansion (21) into (14) leads to the following sequence of recurrence relations

$$\begin{array}{rl} \lambda^2 & [J, \eta_0^{\pm}(x)] = 0, \\ \lambda & Q(x)\eta_0^{\pm}(x) - [J, \eta_1^{\pm}(x)] = 0, \\ \lambda^0 & \mathrm{i}\partial_x \eta_0^{\pm}(x) + Q(x)\eta_1^{\pm}(x) - [J, \eta_2^{\pm}(x)] = 0, \\ \dots & \dots \end{array} \begin{array}{r} (22) \\ (23) \\ (24) \\ \end{array}$$

It is seen from (22) that $\eta_0^{\pm} \in \mathfrak{sl}^0(m+n)$ and it depends on x due to the second term in (24). All this tells us that the Riemann-Hilbert problem is not canonically normalized.

One application of the fundamental analytic solutions is in spectral theory of the scattering operator $L(\lambda)$ [20]. The resolvent of $L(\lambda)$ is defined by

$$L(\lambda) \circ R(\lambda) = \mathbb{1},$$

where $\circ$ means composition of linear operators. $R(\lambda)$ is an integral operator of the form

$$(R(\lambda)\,F)\,(x,t) = \int_{-\infty}^{\infty} \mathcal{R}(x,y,t,\lambda) F(y)\,\mathrm{d}\,y$$

for $F : \mathbb{R} \to \mathbb{C}^n$ being a continuous function. The following theorem holds true:

Theorem 2 *The kernel $\mathcal{R}(x,y,t,\lambda)$ is expressed through the fundamental analytic solutions as follows:*

$$\mathcal{R}(x,y,\lambda) = \left\{ \begin{array}{ll} \mathrm{i}\chi^{+}(x,\lambda)\Theta^{+}(x-y)\hat{\chi}^{+}(y,\lambda), & \mathrm{Im}\,\lambda^2 > 0 \\ -\mathrm{i}\chi^{-}(x,\lambda)\Theta^{-}(x-y)\hat{\chi}^{-}(y,\lambda), & \mathrm{Im}\,\lambda^2 < 0 \end{array} \right. \tag{25}$$

where $\Theta^{\pm}$ are matrix-valued functions given by

$$\Theta^{\pm}(x-y) = \theta(\pm(y-x))P - \theta(\pm(x-y))\,(\mathbb{1} - P)$$

and

$$P = \left(\begin{array}{cc} \mathbb{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array} \right)$$

being a constant projector. The kernel $\mathcal{R}$ is a meromorphic function in $\mathbb{C}$ with a finite number of poles to form the discrete spectrum of the scattering operator L.

The proof of theorem 2 is quite similar to that one in the case of linear bundles this is why we shall skip it. The reader can find a detailed exposition of the proof in [4, 9]. As a direct consequence of theorem 2 one derives:

Corollary 1 *The spectrum of the scattering operator L comprises a continuous part and a discrete part. The continuous part of the spectrum is determined by requirement*

$$\mathrm{Im}\,\lambda^2 J = 0, \tag{26}$$

i.e. it coincides with the real and the imaginary axis in the spectral λ-plane. The discrete spectrum belongs to orbits of the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. all discrete eigenvalues go together in quadruplets $\{\pm\mu_k, \pm\mu_k^*\}_{k=1}^r$.

As in linear bundle case the fundamental analytic solutions allow one to construct nonlinear Fourier transform to map the potential Q onto spectral data. For that purpose one introduces the so-called adjoint solutions (or squared solutions)

$$\mathcal{E}^{\pm}_{ij}(x,\lambda) = \left(\chi^{\pm}(x,\lambda)E_{ij}\hat{\chi}^{\pm}(x,\lambda)\right)^{\perp},$$

which satisfy the adjoint representation of the linear problem (11):

$$\mathrm{i}\partial_x \mathcal{E}^{\pm}_{ij} + \lambda\left[Q - \lambda J, \mathcal{E}^{\pm}_{ij}\right] = 0, \tag{27}$$

where $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$ are Weyl generators of the Lie algebra $\mathfrak{sl}(m+n)$ and $\perp$ stands for taking the block off-diagonal part of matrix.

3. Dressing Method and Special Solutions

This section is dedicated to an effective way of finding particular solutions of NLEE whose scattering problem is (11). The method we shall discuss here is dressing Zakharov-Shabat's method [21, 22]. Conceptually the dressing method is an indirect way of integration, i.e. it generates a solution to a given NLEE starting from a known one by substantially using auxiliary problem associated with it. Let ψ_0 be a fundamental solution to problem

$$L_0\psi_0 = \mathrm{i}\partial_x\psi_0 + \lambda(Q_0 - \lambda J)\psi_0 = 0, \tag{28}$$

where

$$Q_0(x) = \begin{pmatrix} 0 & \mathbf{q}_0^T(x) \\ \mathbf{q}_0^*(x) & 0 \end{pmatrix} \tag{29}$$

for some known $n\times m$ matrix $\mathbf{q}_0$ satisfying (5). Next we apply a gauge transform $\psi_0 \to \psi_1 = g\psi_0$ such that the auxiliary linear system remains covariant, i.e. we have

$$L_1\psi_1 = \mathrm{i}\partial_x\psi_1 + \lambda(Q_1 - \lambda J)\psi_1 = 0, \tag{30}$$

where Q_1 has the form (29) but for some other $n \times m$ matrix $\mathbf{q}_1$ to be found. Then the dressing factor g satisfies:

$$\mathrm{i}\partial_x g + \lambda Q_1\, g - \lambda g Q_0 - \lambda^2[J, g] = 0. \tag{31}$$

Similarly, by comparing linear problems:

$$A_0(\lambda)\psi_0 = \mathrm{i}\partial_t\psi_0 + \sum_{k=1}^{2N}\lambda^k A^{(0)}_k\psi_0 = \psi_0 f(\lambda), \tag{32}$$

$$A_1(\lambda)\psi_1 = \mathrm{i}\partial_t\psi_1 + \sum_{k=1}^{2N}\lambda^k A^{(1)}_k\psi_1 = \psi_1 f(\lambda) \tag{33}$$

one derives another p.d.e. for the dressing factor

$$\mathrm{i}\partial_t g + \sum_{k=1}^{2N}\lambda^k A^{(1)}_k g - g\sum_{k=1}^{2N}\lambda^k A^{(0)}_k = 0. \tag{34}$$

Due to constraints (17)–(19) the dressing factor must obey the following symmetry conditions:

$$\hat{g}^{\dagger}(x,t,\lambda^*) = g(x,t,\lambda), \tag{35}$$

$$\mathbf{C}g(x,t,-\lambda)\mathbf{C}^{-1} = g(x,t,\lambda). \tag{36}$$

The gauge transform acts on all fundamental solution including the Jost solutions. To ensure that the dressed Jost solutions have proper asymptotics (30) should be modified into:

$$\psi_{0,\pm} \to \psi_{1,\pm} = g\psi_{0,\pm}\,\hat{g}_{\pm}\,, \qquad g_{\pm} := \lim_{x\to\pm\infty} g\,. \tag{37}$$

Hence the scattering matrix and the fundamental analytic solutions transform into:

$$T_0 \quad \to \quad T_1 = g_+\, T_0\, \hat{g}_-, \tag{38}$$

$$\chi_0^{\pm} \quad \to \quad \chi_1^{\pm} = g\chi_0^{\pm}\hat{g}_-. \tag{39}$$

Relations (39) and (25) lead to the conclusion that the resolvent kernel transform as:

$$\mathcal{R}_1(x,y,t,\lambda) = g(x,t,\lambda)\mathcal{R}_0(x,y,t,\lambda)\hat{g}(y,t,\lambda). \tag{40}$$

If g does not depend on λ then it follows straight from (31) and (34) that it is constant. To have non-trivial dressing we assume that g depends on the spectral parameter. At this point we recall that $\chi_0^{\pm}$ and $\chi_1^{\pm}$ satisfy Riemann-Hilbert problem with no canonical normalization. This means that the asymptotic value of the dressing factor as $|\lambda| \to \infty$ should be a nonvanishing function of x and t. The simplest possible type is a dressing factor with simple poles only. Due to constraint (36) we pick up g in the form:

$$g(x,t,\lambda) = \mathbb{1} + \sum_{j=1}^{r} \frac{\lambda}{\mu_j}\left(\frac{B_j(x,t)}{\lambda-\mu_j} + \frac{\mathbf{C}B_j(x,t)\mathbf{C}}{\lambda+\mu_j}\right), \qquad \operatorname{Re}\mu_j \neq 0,\ \operatorname{Im}\mu_j \neq 0. \tag{41}$$

while according to (35) its inverse reads:

$$\hat{g}(x,t,\lambda) = \mathbb{1} + \sum_{j=1}^{r} \frac{\lambda}{\mu_j^*}\left(\frac{B_j^{\dagger}(x,t)}{\lambda-\mu_j^*} + \frac{\mathbf{C}B_j^{\dagger}(x,t)\mathbf{C}}{\lambda+\mu_j^*}\right). \tag{42}$$

Setting $|\lambda| \to \infty$ in (31) and taking into account (41) and (42) we get the following interrelation

$$Q_1 = Q_0 + \sum_{j=1}^{r}[J, B_j - \mathbf{C}B_j\mathbf{C}]A^{\dagger} \tag{43}$$

between the seed solution Q_0 and dressed one. The matrix $A(x,t)$ is the asymptotic value of g given by:

$$A = \mathbb{1} + \sum_{j=1}^{r} \frac{1}{\mu_j}\left(B_j + \mathbf{C}B_j\mathbf{C}\right).$$

Thus to obtain a new solution one needs to know the residues of g. The latter are found from identity $g\hat{g} = \mathbb{1}$. Indeed, after evaluating the residue of $g\hat{g}$ at $\lambda = \mu_k$ we obtain:

$$B_k\left\{\mathbb{1} + \sum_{j} \frac{\mu_k}{\mu_j^*}\left(\frac{B_j^{\dagger}}{\mu_k-\mu_j^*} + \frac{\mathbf{C}B_j^{\dagger}\mathbf{C}}{\mu_k+\mu_j^*}\right)\right\} = 0, \qquad k = 1,\ldots,r. \tag{44}$$

To ensure that we shall obtain non-trivial result we assume the residues are degenerate matrices, i.e. we have $B_k = X_k F_k^T$ for some rectangular matrices $X_k(x,t)$ and $F_k(x,t)$. The factors X_k can be found by solving linear system:

$$F_k^* = \sum_{j=1}^{r} \frac{\mu_k^*}{\mu_j}\left(X_j\frac{F_j^T F_k^*}{\mu_j-\mu_k^*} - \mathbf{C}X_j\frac{F_j^T\mathbf{C}F_k^*}{\mu_j+\mu_k^*}\right). \tag{45}$$

On the other hand the matrix factors $F_k(x,t)$ are expressed in terms of fundamental solutions of the bare linear problem, namely:

$$F_k^T(x,t) = F_{0,k}^T(t)\hat{\psi}_0(x,t,\mu_k), \tag{46}$$

where $F_{0,k}^T$ are constants of integration, i.e. they depend on t only, see [20] for more detailed explanations. $F_{0,k}$ turn out to be exponential functions on time and one can propose the following simple rule

$$F_{k,0}^T \to F_{k,0}^T\, \mathrm{e}^{-\mathrm{i}f(\mu_k)t}, \tag{47}$$

where $f(\lambda)$ is the dispersion law of NLEE.

In order to illustrate those general constructions let us consider in more detail the case when the dressing factor has a single pair of simple poles only ($r=1$), i.e. it looks like:

$$g(x,\lambda) = \mathbb{1} + \frac{\lambda B(x)}{\mu(\lambda-\mu)} + \frac{\lambda \mathbf{C} B(x)\mathbf{C}}{\mu(\lambda+\mu)}, \qquad \operatorname{Re}\mu \neq 0,\ \operatorname{Im}\mu \neq 0. \tag{48}$$

Assume now that $Q_0 = 0$. Therefore the bare solution ψ_0 is a plane wave:

$$\psi_0(x,t,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda^2 Jx} \tag{49}$$

and the T matrix is simply equal to the unit $(m+n)\times(m+n)$ matrix. Taking into account (33) and the fact that the asymptotic values of g as $x\to\pm\infty$ are block-diagonal matrices we see that dressed scattering matrix T_1 is a block diagonal matrix. This situation generalizes in a natural way the concept of a reflectionless potential for the $\mathfrak{sl}(2)$ case.

Now linear system (45) is easily solved for X to give

$$X = \frac{\mu}{\mu^*}\left(\frac{F^T F^*}{\mu-\mu^*} - \frac{F^T \mathbf{C} F^*}{\mu+\mu^*}\mathbf{C}\right)^{-1} F^*, \tag{50}$$

where F is expressed in terms the seed solution. We shall restrict ourselves here with the case when $\mathrm{rank} B = 1$, that is X and F are $m+n$-vectors. Then after collecting all information from (50) and (47) and plug it into (43) the reflectionless potential acquires the form:

$$\begin{aligned} q_{1,\,ji}(x) \quad &= \quad Q_{1,\,ij+m}(x) = \sum_{k=m+1}^{m+n} \frac{2\mathrm{i}(n_i+1)\rho\sin(2\varphi)\mathrm{e}^{-\mathrm{i}\sigma_{ik}(x)}\mathrm{e}^{-\theta_{ik}(x)}}{\mathrm{e}^{-2\mathrm{i}\varphi}\sum_{p=1}^{m}\mathrm{e}^{-2\theta_{pk}(x)} + \sum_{p=m+1}^{m+n}\mathrm{e}^{2\xi_{0,pk}}} \times \\ &\left(\delta_{kj+m} - \frac{2\mathrm{i}\sin(2\varphi)\mathrm{e}^{\mathrm{i}(\delta_{j+m}-\delta_k-2\varphi)}\mathrm{e}^{2\xi_{0,j+mk}}}{\mathrm{e}^{-2\mathrm{i}\varphi}\sum_{p=1}^{m}\mathrm{e}^{-2\theta_{pk}(x)} + \sum_{p=m+1}^{m+n}\mathrm{e}^{2\xi_{0,pk}}}\right), \end{aligned} \tag{51}$$

where

$$\begin{aligned} \theta_{pk}(x) \quad &= \quad (n_p+1)\rho^2\sin(2\varphi)x - \xi_{0,\,pk}, \qquad \xi_{0,\,pk} = \ln|F_{0,p}/F_{0,k}|, \qquad \delta_p = \arg F_{0,p} \\ \sigma_{pk}(x) \quad &= \quad (n_p+1)\cos(2\varphi)x + \delta_p - \delta_k - \varphi, \qquad \mu = \rho\exp(\mathrm{i}\varphi). \end{aligned}$$

In order to obtain the soliton solution for matrix DNLS one needs to recover the t-dependence. This can be done by using (47) and taking into account that

$$f_{\mathrm{MDNLS}}(\lambda) = -\frac{n+m}{m}\lambda^4 J\,,$$

which leads to the following rule:

$$\begin{aligned}
\delta_p \quad &\to \quad \delta_p + \left\{ \begin{array}{ll} \frac{(n+m)n}{m^2}\rho^4\cos(4\varphi)t, & p = 1,\dots,m, \\ -\frac{n+m}{m}\rho^4\cos(4\varphi)t, & p = m+1,\dots,m+n, \end{array} \right. \\
\xi_{0,\,pk} \quad &\to \quad \left\{ \begin{array}{ll} \xi_{0,\,pk} - \left(\frac{n+m}{m}\right)^2\rho^4\sin(4\varphi)t, & p = 1,\dots,m, \\ \qquad \xi_{0,\,pk}, & p = m+1,\dots,m+n. \end{array} \right.
\end{aligned}$$

The result we have just obtained represents a generalization of Kaup-Newell's soliton derived in the case of the scalar DNLS equation [12].

It is clear that one can recursively apply the dressing procedure we demonstrate here, to build a whole sequence of exact solutions to a NLEE.

4. Conclusions

We have formulated the direct scattering problem for quadratic bundles related to Hermitian symmetric spaces of the type **A.III** in terms Jost solutions, scattering matrix, fundamental analytic solutions etc. We have discussed the spectral properties of the scattering operator L. In this sense our results generalize those obtained by Gerdjikov et al. in [5, 6] treating the scalar case as well as author's in [20] treating spaces of the type $SU(n)/S(U(1)\times U(n-1))$.

Zakharov-Shabat's dressing method has been adapted to quadratic bundles of aforementioned type. This allowed us to establish an algebraic procedure for construction of reflectionless potentials. For this to be done it suffices to use a dressing factor with simple poles. As a special case we have considered in more detail the simplest case when the dressing factor has a couple of poles. All reflectionless potentials give rise to multisoliton solutions to certain NLEEs. For instance, applying simple time recovery procedure we have easily produced one soliton solutions to the multicomponent DNLS (5). This result naturally generalizes the classical result by Kaup and Newell [12] for the soliton solution to scalar DNLS. The latter can be derived by using a dressing factor chosen in the form:

$$g(x,t,\lambda) = \mathbb{1} + \frac{\lambda B(x,t)}{\mu(\lambda-\mu)} + \frac{\lambda\sigma_3 B(x,t)\sigma_3}{\mu(\lambda+\mu)}\,.$$

The results presented here could be extended in several directions. Firstly, one can search for solutions of different type, say rational type solutions. One possible way to find rational solutions consists in using a factor g whose poles lie in the continuous spectrum of L, i.e. $\mu\in\mathrm{i}\mathbb{R}$. In this "degenerate" case one can obtain the following result for the scalar DNLS:

$$q(x,t) = \frac{4\mathrm{i}\mu^3(x+4\mu^2 t)}{\left[2\mathrm{i}\mu^2(x+4\mu^2 t)-1\right]^2}\,\mathrm{e}^{-2\mathrm{i}\mu^2(x+2\mu^2 t)}. \tag{52}$$

It is seen that it obeys zero boundary conditions as $x\to\pm\infty$. As in the case of linear bundles it is also possible to derive rational solutions but obeying more complicated non-trivial background conditions.

Another meaningful direction of further developments is to study quadratic bundles associated with other types Hermitian symmetric spaces or to put it even in a more general setting, bundles related to homogeneous spaces like the Lax operator given below:

$$L(\lambda) = \mathrm{i}\partial_x + U_0 + \lambda U_1 - \lambda^2 J\,, \tag{53}$$

where U_0 splits into a diagonal and off-diagonal part, U_1 is strictly off-diagonal and J is a diagonal matrix. It is evident that the theory of complete quadratic bundles like this one is more complicated than in the case we have considered in that report.

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