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On a Nonlocal Nonlinear Schrödinger Equation

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Abstract

We consider a nonlocal nonlinear Schrödinger equation recently proposed by Ablowitz and Musslimani as a theoretical model for wave propagation in PT-symmetric coupled wave-guides and photonic crystals. This new equation is integrable by means of inverse scattering method, i. e. it possesses a Lax pair, infinite number of integrals of motion and exact solutions. We aim to describe here some of the basic properties of the nonlocal Schrödinger equation and its scattering operator. In doing this we shall make use of methods alternative to those applied by Ablowitz and Musslimani which seem to be better suited for treating possible multicomponent generalizations.

1 Introduction

Nonlinear Schrödinger equation (NLS)

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0, \qquad q : \mathbb{R}^2 \to \mathbb{C} \tag{1}$$

is one of classical integrable nonlinear equations. It appears in a variety of physical areas [2, 17] like nonlinear optics, plasma physics, fluid mechanics as well as in a purely mathematical context like differential geometry of curves [10]. Although having been extensively studied and a subject of numerous monographs like [2, 5, 14, 17], NLS is still stimulating further research activity [7, 13, 16]. Finding new integrable reductions of known nonlinear equations like NLS is one important trend in theory of integrable systems. Thus nonlocal nonlinear Schrödinger equation (NNS "±")

$$\mathrm{i}q_t + q_{xx} \pm 2q^2(x,t)q^*(-x,t) = 0 \tag{2}$$

recently introduced by Ablowitz and Musslimani [1], is a significant contribution to that area. Like the local NLS, equation (2) is PT-symmetric, i.e. it is invariant under the parity-time transform

$$x \to -x, \qquad t \to -t, \qquad q \to q^*. \tag{3}$$

This motivated the authors to propose NNS as a mathematical model to describe wave phenomena observed in PT symmetric nonlinear media [6, 11, 12].

The purpose of this report is to study some basic properties of NNS "+" equation and its scattering operator. In doing this we shall make use of covariant approaches [5] being better suited for treating multicomponent generalizations of NNS in a uniform way than the approaches of Ablowitz and Musslimani. The report is organized as follows. Section 2 is dedicated to the direct scattering problem for NNS and spectral properties of the corresponding scattering operator. In next section we demonstrate how one can apply Zakharov-Shabat's dressing method to obtain special solutions to NNS. This way one easily reproduces the breathing solution obtained by Ablowitz and Musslimani [1]. In Section 4 we develop the Hamiltonian formalism for NNS. For that purpose we derive recursion operator which generates the hierarchy of higher nonlinear equations, integrals of motion and symmetries associated with NNS. Then we apply method of diagonalization of Lax pair [3] to describe conserved densities of NNS through a recursion formula generating all of them. We point a Hamiltonian to NNS and a Poisson structure assigned to it. Finally, Section 5 contains conclusions and additional remarks.

2 Direct Scattering Problem

NNS is a S-integrable equation, i.e. it is equivalent to the compatibility condition $[L(\lambda), A(\lambda)] = 0$ for matrix differential operators $L(\lambda)$ and $A(\lambda)$ being chosen in the form:

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x + Q - \lambda\sigma_3, && (4)\\ A(\lambda) &= \mathrm{i}\partial_t + \frac{\mathrm{i}}{2}[\sigma_3, Q_x] + qp\sigma_3 + 2\lambda Q - 2\lambda^2\sigma_3, && (5)\end{aligned}$$

where $\lambda \in \mathbb{C}$ is spectral parameter and matrix coefficients are defined as follows:

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ p(x,t) & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

We shall restrict ourselves with the simplest case of zero boundary conditions

$$\lim_{|x|\to\infty} Q(x,t) = \mathbf{0}$$

for potential Q. More precisely, we shall assume that q and p are Schwartz type functions.

To obtain a scalar NNS one has to impose an extra symmetry condition on Q so that p and q are no more independent. For instance, Ablowitz-Musslimani's NNS can be derived if one requires that $p(x,t) = q^*(-x,t)$. This idea can be made precise if one slightly extends the notion of Mikhailov's reduction group [9] by allowing action on x and t. Let us denote by $\{\psi(x,t,\lambda)\}$ the set of all fundamental matrices of the linear problem

$$L(\lambda)\psi(x,t,\lambda) = 0. \tag{7}$$

Let a finite group $\mathrm{G_R}$ act on $\{\psi(x,t,\lambda)\}$, i.e. it maps a solution ψ to the linear problem (7) onto another solution

$$\tilde{\psi}(x,t,\lambda) = \mathcal{K}_{\mathrm{g}}\{\psi[k_{\mathrm{g}}(x,t,\lambda)]\}, \qquad \mathrm{g} \in \mathrm{G_R} \tag{8}$$

where $k_{\mathrm{g}} : \mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2 \times \mathbb{C}$ is a smooth transform and $\mathcal{K}_{\mathrm{g}}$ is a group automorphism[1]. As a result we see that the Lax operator $L(\lambda)$ must fulfill certain algebraic condition, hence the potential acquires certain symmetries. Let us consider an example:

Example 2.1. Ablowitz-Musslimani's reduction

In this case the reduction group is $\mathbb{Z}_2$. It maps an arbitrary fundamental solution ψ onto

$$\tilde{\psi}(x,t,\lambda) = \sigma_1\psi^*(-x,t,-\lambda^*)\sigma_1, \qquad \sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right). \tag{9}$$

Therefore the potential satisfies the symmetry condition

$$Q(x,t) = \sigma_1 Q^*(-x,t)\sigma_1. \tag{10}$$

Hence relation $p(x,t) = q^*(-x,t)$ holds true. $\square$

Most of our considerations in this section are general, i.e. we shall not fix a particular reduction.

Let $\psi(x,t,\lambda)$ be any fundamental solution to Zakharov-Shabat's system. Since $[L(\lambda), A(\lambda)] = 0$ it satisfies

$$A(\lambda)\psi(x,t,\lambda) = \psi(x,t,\lambda)C(\lambda) \tag{11}$$

for some arbitrary matrix $C(\lambda)$. An important class of solutions to (7) is given by Jost solutions ψ_+ and ψ_- defined through:

$$\lim_{x\to\pm\infty} \psi_\pm(x,t,\lambda)\mathrm{e}^{\mathrm{i}\lambda\sigma_3 x} = \mathbb{1}. \tag{12}$$

To make sure that (12) is correct we require that $C(\lambda) = -2\lambda^2\sigma_3$. Scattering matrix is introduced as usual being the transition matrix

$$\psi_-(x,t,\lambda) = \psi_+(x,t,\lambda)T(t,\lambda). \tag{13}$$

between the Jost solutions. One can present the scattering matrix in the following way

$$T(t,\lambda) = \left(\begin{array}{cc} a^+(t,\lambda) & -b^-(t,\lambda) \\ b^+(t,\lambda) & a^-(t,\lambda) \end{array}\right). \tag{14}$$

Its time evolution is driven by the linear equation:

$$\mathrm{i}\partial_t T - 2\lambda^2[\sigma_3, T] = 0. \tag{15}$$

[1] The fundamental solutions take values in $SL(2,\mathbb{C})$

It is easily integrated to give

$$T(t,\lambda) = \mathrm{e}^{-2\mathrm{i}\lambda^2\sigma_3 t} T(0,\lambda) \mathrm{e}^{2\mathrm{i}\lambda^2\sigma_3 t}. \tag{16}$$

Equation (16) represents a linearization of NNS. Due to (16) the functions $a^{\pm}$ do not depend on t hence they could serve as generating functions of integrals of motion for NNS.

The Jost solutions are defined for real λ only. Nevertheless, one can construct from the Jost solutions another pair of fundamental solutions χ^+ and χ^- analytic in the upper half plane $\mathbb{C}_+$ and the lower half plane $\mathbb{C}_-$ respectively [5, 17]. In order to do this one makes use of LDU decomposition of the scattering matrix

$$\begin{aligned}
T(t,\lambda) &= T^{\mp}(t,\lambda) D^{\pm}(\lambda) \left(S^{\pm}(t,\lambda)\right)^{-1}, \qquad (17)\\
T^- &= \begin{pmatrix} 1 & 0 \\ b^+/a^+ & 1 \end{pmatrix}, \quad S^+ = \begin{pmatrix} 1 & 0 \\ b^-/a^+ & 1 \end{pmatrix},\\
T^+ &= \begin{pmatrix} 1 & -b^-/a^- \\ 0 & 1 \end{pmatrix}, \quad S^- = \begin{pmatrix} 1 & 0 \\ -b^+/a^- & 1 \end{pmatrix},\\
D^+ &= \operatorname{diag}\left(a^+, 1/a^+\right), \quad D^-(\lambda) = \operatorname{diag}\left(1/a^-, a^-\right).
\end{aligned}$$

Then χ^+ and χ^- are given by:

$$\chi^{\pm} = \psi_- S^{\pm} = \psi_+ T^{\mp} D^{\pm}. \tag{18}$$

Remark 2.2. We shall assume that a^+ and a^- have a finite number of simple zeros in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively. The zeros of a^+ and a^- correspond to pole singularities of χ^+ and χ^- respectively. $\square$

Remark 2.3. Like in the local NLS case [5, 14, 17] fundamental solutions η^+ and η^- can be viewed as solutions to a local Riemann-Hilbert factorization problem

$$\begin{aligned}
\eta^-(x,t,\lambda) &= \eta^+(x,t,\lambda) G(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \qquad (19)\\
G(x,t,\lambda) &= \mathrm{e}^{-\mathrm{i}\lambda\sigma_3 x} \left[S^+(t,\lambda)\right]^{-1} S^-(t,\lambda) \mathrm{e}^{\mathrm{i}\lambda\sigma_3 x}.
\end{aligned}$$

with canonical normalization

$$\lim_{|\lambda|\to\infty} \eta^{\pm}(x,t,\lambda) = \mathbb{1}.\square \tag{20}$$

The fundamental analytic solutions allow one to study spectral properties of the scattering operator. Following [4, 5] we define resolvent operator of $L(\lambda)$ as an operator $R(\lambda)$ satisfying

$$L(\lambda) \circ R(\lambda) = \mathbb{1},$$

where $\circ$ means composition of operators. One can write down $R(\lambda)$ in the form:

$$\left(R(\lambda)\, F\right)(x,t) = \int_{-\infty}^{\infty} \mathcal{R}(x,y,t,\lambda) F(y)\, \mathrm{d}\, y \tag{21}$$

for $F:\mathbb{R}\to\mathbb{C}^2$ being a continuous vector-valued function. The integral kernel of the resolvent can be expressed through the fundamental analytic solutions as follows [5]:

$$\mathcal{R}(x,y,t,\lambda)=\left\{\begin{array}{lr} \mathrm{i}\chi^+(x,t,\lambda)\Theta^+(x-y)\left[\chi^+(y,t,\lambda)\right]^{-1}, & \lambda\in\mathbb{C}_+,\\ -\mathrm{i}\chi^-(x,t,\lambda)\Theta^-(x-y)\left[\chi^-(y,t,\lambda)\right]^{-1}, & \lambda\in\mathbb{C}_- \end{array}\right. \tag{22}$$

where $\Theta^\pm$ are matrix-valued functions defined by

$$\Theta^\pm(x-y)=\theta(\pm(y-x))P-\theta(\pm(x-y))\,(\mathbb{1}-P)\,,\quad P=\mathrm{diag}\,(1,0).$$

The locus in λ-plane where $\mathcal{R}$ is unbounded constitutes the continuous part of spectrum of $L(\lambda)$. It is determined by the requirement $\mathrm{Im}\,\lambda=0$, i.e. it coincides with the real axis in the λ-plane. According to Remark 2.2 the fundamental analytic solutions may have a finite number of pole singularities determined by the zeros of the diagonal elements of $T(t,\lambda)$. These in turn determine pole singularities of $\mathcal{R}$ which form a discrete eigenvalues of $L(\lambda)$, see [4, 5]. In the presence of reduction the discrete eigenvalues go together in certain symmetric configurations. To illustrate this let us consider an example.

Example 2.4. Ablowitz-Musslimani's reduction

In this case the resolvent operator $R(\lambda)$ obeys symmetry condition

$$\sigma_1R^*(-\lambda^*)\sigma_1=R(\lambda) \tag{23}$$

while its kernel satisfies

$$\sigma_1\mathcal{R}^*(-x,y,t,-\lambda^*)\sigma_1=\mathcal{R}(x,y,t,\lambda). \tag{24}$$

Relation (23) means that if μ is a discrete eigenvalue of $L(\lambda)$ so is $-\mu^*$, i.e. eigenvalues are symmetrically located to imaginary axis (in particular, they can lie on the imaginary axis itself).

3 Special Solutions

In this section we aim at demonstrating how one can apply Zakharov-Shabat's dressing method to construct particular solutions to NNS. We shall start with a brief reminder of the concept underlying the dressing method [17, 18]. Then we shall illustrate all general ideas with an example.

The dressing method is an indirect way to generate solutions to a S-integrable equation, i.e. we construct solutions to an equation starting from a known one

$$Q_0(x,t)=\left(\begin{array}{cc} 0 & q_0(x.t)\\ p_0(x,t) & 0\end{array}\right)$$

called a seed (bare) solution. The latter plays the role of a potential for the scattering operator

$$L_0(\lambda)=\mathrm{i}\partial_x+Q_0(x,t)-\lambda\sigma_3. \tag{25}$$

Let $\psi_0(x,t,\lambda)$ be an arbitrary fundamental solution to the linear problem

$$\mathrm{i}\partial_x\psi_0 + (Q_0 - \lambda\sigma_3)\psi_0 = 0. \tag{26}$$

Now let us construct function $\psi_1(x,t,\lambda) = g(x,t,\lambda)\psi_0(x,t,\lambda)$ and assume it satisfies a similar linear problem

$$\mathrm{i}\partial_x\psi_1 + (Q_1 - \lambda\sigma_3)\psi_1 = 0 \tag{27}$$

for some other potential Q_1 to be found. The multiplier g bears the name dressing factor and satisfies the linear equation:

$$\mathrm{i}\partial_x g + Q_1 g - gQ_0 - \lambda[\sigma_3, g] = 0. \tag{28}$$

Due to Remark 2.3 g must be normalized as follows:

$$\lim_{|\lambda|\to\infty} g(x,t,\lambda) = \mathbb{1}. \tag{29}$$

Then the simplest nontrivial choice possible for the dressing factor is

$$g(x,t,\lambda) = \mathbb{1} + \frac{A(x,t)}{\lambda-\mu}, \qquad \mu\in\mathbb{C} \tag{30}$$

while its inverse is sought in the form

$$[g(x,t,\lambda)]^{-1} = \mathbb{1} + \frac{B(x,t)}{\lambda-\nu}, \qquad \nu\in\mathbb{C}. \tag{31}$$

After substituting the ansatz for g into (28) and set $|\lambda|\to\infty$ we derive an interrelation between the seed potential and the dressed one, namely:

$$Q_1(x,t) = Q_0(x,t) + [\sigma_3, A(x,t)]. \tag{32}$$

The residues A and B are not independent. Indeed, from the identity $gg^{-1} = \mathbb{1}$ one can see that

$$A = -B = (\mu - \nu)P \tag{33}$$

for some projector P ($P^2 = P$). Since P is a projector of rank 1 it can be presented in the form:

$$P = \frac{XF^T}{F^TX}$$

where $X(x,t)$ and $F(x,t)$ are some 2-component column vectors. In order to find them one should analyze equation (28) and its counterpart satisfied by g^{-1}. As a result one can convince himself that X and F are expressed in terms of fundamental solutions ψ_0 and $\tilde{\psi}_0$ to the bare linear problem defined in a vicinity of the poles μ and ν respectively:

$$F^T(x,t) = F_0^T\left[\psi_0(x,t,\mu)\right]^{-1}, \qquad X(x,t) = \tilde{\psi}_0(x,t,\nu)X_0. \tag{34}$$

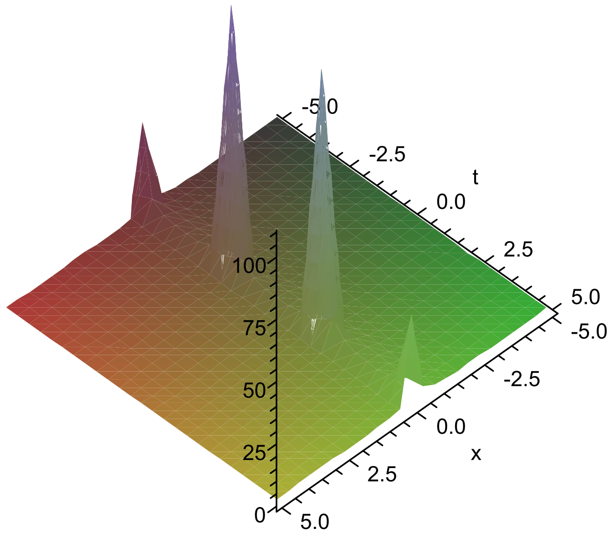

Figure 1: 3D plot of the module of the dressed solution (42) when $\gamma = 1$, $\kappa = 3$ and $\delta = 0$.

The 2-vectors X_0 and F_0 are x-independent but evolve with time. It can be shown that their t-evolution is driven by the dispersion law of nonlinear equation through formulas:

$$X_0(t) = \mathrm{e}^{\mathrm{i}f(\nu)t} X_{0,0}, \qquad F_0^T(t) = F_{0,0}^T \mathrm{e}^{-\mathrm{i}f(\mu)t}. \tag{35}$$

We are particularly interested in the case when $Q_0(x,t) = 0$. In this case as a seed fundamental solution we can take

$$\psi_0(x,t,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda\sigma_3 x}. \tag{36}$$

Our further considerations depend on the reduction imposed on Lax pair. Let us assume we have Ablowitz-Musslimani's reduction (9). Then the dressing factor is subject to the symmetry condition:

$$\sigma_1 g^*(-x,t,-\lambda^*)\sigma_1 = g(x,t,\lambda). \tag{37}$$

This means that the poles of the dressing factor and its inverse are imaginary[2], i.e. $\mu = \mathrm{i}\gamma$ and $\nu = -\mathrm{i}\kappa$ while the projector P obeys the equality:

$$\sigma_1 P^*(-x,t)\sigma_1 = P(x,t). \tag{38}$$

[2] In order to ensure proper asymptotic behaviour of the dressed solution we shall require that poles μ ans ν are located at different half planes of λ-plane.

The vectors X and F are given by:

$$X(x,t) = \mathrm{e}^{-\kappa\sigma_3 x} X_0(t), \qquad F(x,t) = \mathrm{e}^{-\gamma\sigma_3 x} F_0(t). \tag{39}$$

Due to (38) X_0 and F_0 have one independent component only

$$X_0 = \sigma_1 X_0^* \quad \Rightarrow \quad X_{0,2} = X_{0,1}^*, \tag{40}$$
$$F_0 = \sigma_1 F_0^* \quad \Rightarrow \quad F_{0,2} = F_{0,1}^*. \tag{41}$$

After substituting all information into (32) we get the following result:

$$q_1(x,t) = \frac{2\mathrm{i}(\gamma+\kappa)\mathrm{e}^{2\gamma x}\mathrm{e}^{4\mathrm{i}\kappa^2 t}}{\mathrm{e}^{2(\gamma+\kappa)x} + \mathrm{e}^{2\mathrm{i}[2(\kappa^2-\gamma^2)t+\delta]}} \tag{42}$$

where $\delta \in \mathbb{R}$ corresponds to the phases of $X_{0,1}$ and $F_{0,1}$. (42) coincides with that found in [1]. This is not a travelling wave — it is a breathing solution, see Fig. 1. The breather develops singularities at $x = 0$ and

$$t_{\mathrm{sing}} = \frac{(2j+1)\pi - 2\delta}{4(\kappa^2-\gamma^2)}, \qquad j \in \mathbb{Z}.$$

One can apply the dressing procedure described above recursively thus generating a series of solutions as shown

$$Q_0 \xrightarrow{g_0} Q_1 \xrightarrow{g_1} Q_2 \to \ldots \xrightarrow{g_{m-1}} Q_m \to \ldots$$

The dressing factor g_k, $k = 0, 1, \ldots$ formally looks as in (30) but instead of the bare solution ψ_0 one uses $k-1$-times dressed fundamental solution

$$\psi_k(x,t,\lambda) = \overleftarrow{\prod_{l=0,\ldots,k-1}} g_l(x,t,\lambda)\psi_0(x,t,\lambda)$$

to construct the residue of the factor g_k. Another way to obtain more complicated solutions is by using a multiple pole dressing factor of the form:

$$g(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{m} \frac{A_k(x,t)}{\lambda - \mu_k}, \qquad \mu_k \in \mathbb{C}. \tag{43}$$

In this case the inverse of g is given by:

$$[g(x,t,\lambda)]^{-1} = \mathbb{1} + \sum_{k=1}^{m} \frac{B_k(x,t)}{\lambda - \nu_k}, \qquad \nu_k \in \mathbb{C} \tag{44}$$

and the dressed solution is obtained through the following formula:

$$Q_1 = Q_0 + \sum_{k=1}^{m} [\sigma_3, A_k]. \tag{45}$$

Each residue of the factor (43) and its inverse has the decomposition

$$A_k = X_k F_k^T, \qquad B_k = Y_k G_k^T, \quad k = 1, \dots, m$$

for X_k, F_k, Y_k and G_k being complex vector-valued functions. Like in the single pole case, F_k and Y_k are expressed through the seed solution:

$$F_k^T(x,t) = F_0^T \left[\psi_0(x,t,\mu_k)\right]^{-1}, \qquad Y_k(x,t) = \tilde{\psi}_0(x,t,\nu_k) Y_{0,k}$$

while the vectors X_k and G_k can be found by solving the linear system:

$$Y_k = \sum_l \frac{F_l^T Y_k}{\mu_l - \nu_k} X_l,$$
$$F_k = \sum_l \frac{Y_l^T F_k}{\nu_l - \mu_k} G_l.$$

Therefore the procedure to derive a solution in that case can be described using the diagram below:

$$\psi_0 \to F_k, Y_k \to X_k, G_l \to A_k \to Q_1.$$

4 Hamiltonian Formulation

Like the local NLS, equation (2) is an infinite dimensional Hamiltonian system. In that section we shall consider the Hamiltonian properties of (2). We shall start with an analytic description of the hierarchy of higher integrable equations in terms of recursion operator. The recursion operator generates all integrals of motion and symmetries of an equation and thus serves as an integrability criterion [5].

Let us consider the generic Lax pair

$$\begin{aligned} L(\lambda) &= \mathrm{i}\partial_x + Q(x,t) - \lambda\sigma_3, && (46)\\ A(\lambda) &= \mathrm{i}\partial_t + \sum_{k=0}^{N} A_k(x,t)\lambda^k && (47)\end{aligned}$$

for $Q(x,t)$ being into the form (6). Since the compatibility condition of (46) and (47) holds identically with respect to λ it yields to the following series of recurrence relations for coefficients A_k, $k = 0, \dots, N$

$$\begin{aligned} \lambda^{N+1} &: \quad [\sigma_3, A_N] = 0, && (48)\\ \lambda^{N} &: \quad \mathrm{i}\partial_x A_N + [Q, A_N] = [\sigma_3, A_{N-1}], && (49)\\ &\quad \dots \\ \lambda^{k} &: \quad \mathrm{i}\partial_x A_k + [Q, A_k] = [\sigma_3, A_{k-1}], && (50)\\ &\quad \dots \\ \lambda^{0} &: \quad \mathrm{i}\partial_x A_0 - \mathrm{i}\partial_t Q + [Q, A_0] = 0. && (51)\end{aligned}$$

By starting to resolve them from the highest term towards the lowest one, one can determine all coefficients [5]. In doing this it is convenient to make use of the splitting

$$A_k(x,t) = A_k^{\perp}(x,t) + a_k(x,t)\sigma_3$$

of each coefficient into a nondiagonal part $A_k^{\perp}$ and a diagonal one proportional to σ_3. Proceeding that way we see from (48) that $A_N = c_N\sigma_3$ for some constant c_N while the nondiagonal part of (49) leads to $A_{N-1}^{\perp} = -c_N Q$. Similarly, the nondiagonal part of the generic recurrence relation (50) allows one to express $A_{k-1}^{\perp}$ through $A_k^{\perp}$ as follows:

$$A_{k-1}^{\perp} = \frac{\mathrm{i}}{4}\left[\sigma_3, \partial_x A_k^{\perp}\right] - a_k Q. \tag{52}$$

On the other hand, from its diagonal part one finds the coefficient a_k to be equal to

$$a_k = c_k + \frac{\mathrm{i}}{2}\int_{\pm\infty}^{x} \mathrm{d}\, y \mathrm{tr}\, \left([Q(y), A_k^{\perp}(y)]\sigma_3\right) \tag{53}$$

After substituting (53) into (52) we obtain the recursion formula:

$$A_{k-1}^{\perp} = \Lambda_{\pm} A_k^{\perp} - c_k Q. \tag{54}$$

The integro-differential operators

$$\Lambda_{\pm} = \frac{\mathrm{i}}{4}[\sigma_3, \partial_x(.)] + \frac{\mathrm{i}Q}{2}\int_{\pm\infty}^{x} \mathrm{d}\, y \mathrm{tr}\, \left(Q[\sigma_3, (.)]\right) \tag{55}$$

are named recursion operators. By using them one can describe the integrable hierarchy of equations associated with the operator (46) in the following way

$$\frac{\mathrm{i}}{4}[\sigma_3, Q_t] + f(\Lambda_{\pm})Q = 0 \tag{56}$$

where

$$f(\Lambda_{\pm}) = \sum_{k=0}^{N} c_k \Lambda_{\pm}^k, \qquad c_k \in \mathbb{C}. \tag{57}$$

The polynomial $f(\lambda) = \sum_k c_k \lambda^k$ is the dispersion law of nonlinear equation. The NNS equation is obtained from (56) when $f(\lambda) = -2\lambda^2$.

Each member of the integrable hierarchy (56) is a Hamiltonian equation. So there is an infinite family of integrals of motion which can be regarded as Hamiltonians. We shall demonstrate how one can apply the method of diagonalization of Lax pair [3] to derive the integrals of motion of NNS. For this to be done one applies a gauge transform

$$\mathcal{P}(x,t,\lambda) = \mathbb{1} + \sum_{k=1}^{\infty} \mathcal{P}_k(x,t)\lambda^{-k}, \tag{58}$$

where $\mathcal{P}_k(x,t)$ are chosen as off-diagonal 2×2 matrices to fix the natural ambiguity that would appear otherwise. The $L-A$ pair transforms into

$$\begin{aligned}\mathcal{L} &= \mathcal{P}^{-1}L\mathcal{P} = \mathrm{i}\partial_x - \lambda\sigma_3 + \sum_{k=0}^{\infty}\mathcal{L}_k(x,t)\lambda^{-k},\\ \mathcal{A} &= \mathcal{P}^{-1}A\mathcal{P} = \mathrm{i}\partial_t + \sum_{k=-N}^{\infty}\mathcal{A}_k(x,t)\lambda^{-k}.\end{aligned}$$

We require that the coefficients $\mathcal{L}_k(x,t)$ and $\mathcal{A}_k(x,t)$ are diagonal matrices. Then the zero curvature condition for the transformed Lax pair simplifies to give:

$$\partial_t\mathcal{L}_k - \partial_x\mathcal{A}_k = 0. \tag{59}$$

From those equations we deduce that the diagonal elements of $\mathcal{L}_k$ play the role of conserved quantities while those of $-\mathcal{A}_k$ are the corresponding currents. To find $\mathcal{L}_k$ one considers the relation

$$\mathcal{P}\mathcal{L} = L\mathcal{P}. \tag{60}$$

After comparing the coefficients before equal powers of λ in (60) we obtain the following infinite series of recurrence relations

$$\lambda^0 \;:\; \mathcal{L}_0 - \mathcal{P}_1\sigma_3 = Q - \sigma_3\mathcal{P}_1, \tag{61}$$

$$\lambda^{-1} \;:\; \mathcal{L}_1 + \mathcal{P}_1\mathcal{L}_0 - \mathcal{P}_2\sigma_3 = \mathrm{i}\mathcal{P}_{1,x} + Q\mathcal{P}_1 - \sigma_3\mathcal{P}_2, \tag{62}$$

$$\dots$$

$$\lambda^{-k} \;:\; \mathcal{L}_k + \sum_{l=1}^{k}\mathcal{P}_k\mathcal{L}_{k-l} - \mathcal{P}_{k+1}\sigma_3 = \mathrm{i}\mathcal{P}_{k,x} + Q\mathcal{P}_k - \sigma_3\mathcal{P}_{k+1}, \tag{63}$$

$$\dots$$

To resolve them one splits each of them into a diagonal and a nondiagonal part. For example, the first relation leads to

$$\mathcal{L}_0 = 0, \qquad Q = [\sigma_3, \mathcal{P}_1]. \tag{64}$$

From the latter equality we find for $\mathcal{P}_1$ the following result:

$$\mathcal{P}_1 = \frac{1}{4}[\sigma_3, Q]. \tag{65}$$

Proceeding in the same way from the generic relation (63) we derive the recursion formula:

$$\mathcal{L}_k = Q\mathcal{P}_k \tag{66}$$

where

$$\mathcal{P}_k = \frac{1}{4}\mathrm{ad}_{\sigma_3}\left(\mathrm{i}\mathcal{P}_{k-1,x} - \sum_{l=1}^{k-1}\mathcal{P}_l\mathcal{L}_{k-1-l}\right). \tag{67}$$

The first three conserved densities obtained from (66) read:

$$\mathcal{C}_1 = pq, \qquad \mathcal{C}_2 = \frac{\mathrm{i}}{2}(pq_x - qp_x), \qquad \mathcal{C}_3 = pq_{xx} + (pq)^2. \tag{68}$$

For one to obtain a true conserved quantity for NNS we need to impose an extra reduction. In the case of NNS of the Ablowitz-Musslimani type one has to require that $p(x,t) = q^*(-x,t)$ while for local NLS we have the interrelation $p(x,t) = q^*(x,t)$ holding true. The corresponding integrals of motion are given from the formula:

$$I_a(t) = \int_{-\infty}^{\infty} \mathcal{C}_a(x,t)\,\mathrm{d}\,x, \qquad a = 1, 2, \ldots. \tag{69}$$

If we define a Poisson bracket through:

$$\{F, G\} = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}\,y \left(\frac{\delta F}{\delta q(y)}\frac{\delta G}{\delta p(y)} - \frac{\delta F}{\delta p(y)}\frac{\delta G}{\delta q(y)}\right) \tag{70}$$

for F and G being functionals of q and p then $\mathcal{C}_3$ is a Hamiltonian density for NNS. In other words, NNS can be cast into the following form:

$$q_t = \{q, H\} \tag{71}$$

where

$$H(t) = \int_{-\infty}^{\infty} \mathcal{C}_3(x,t)\,\mathrm{d}\,x$$

is the Hamiltonian functional corresponding to $\mathcal{C}_3$.

5 Conclusion

We have formulated and discussed the direct scattering problem for the scalar NNS. We have shown that in a quite similar manner to the local NLS case one can introduce Jost solutions, scattering matrix, fundamental analytic solutions etc. All that machinery allows one to study spectral properties of the scattering operator by constructing its resolvent operator. Like for the local NLS the operator L has a continuous spectrum which coincides with real axis and a discrete spectrum of points symmetrically located with respect to imaginary axis.

We have applied Zakharov-Shabat's dressing method to linear bundles with nonlocal reduction imposed. It has proved to be sufficient to use dressing factors with simple poles. As a special case we have considered in more detail the simplest case when the dressing factor has a single simple pole. This allowed us to construct solutions in a way alternative to the approach used by Ablowitz and Musslimani [1]. In order to find more complicated solutions one can either dress several times using a single pole factor or can use a dressing factor with multiple poles.

We have derived recursion operator for linear bundles with nonlocal reductions. It allows one to generate all higher equations belonging to the integrable hierarchy under consideration. We have shown that there exist an infinite number of integrals of motion for NNS. A recursion formula to generate all conserved densities has been obtained by using the Lax pair diagonalization method. We have explicitly calculated the first three of them. A Poisson bracket to establish Hamiltonian formulation of NNS has been given.

The results presented here can be extended in several directions. First, one may consider potentials obeying more complicated boundary conditions, say constant nonzero boundary conditions or time dependent boundary conditions (nontrivial background). Such solutions could play an important role similar to that of Peregrine or Ma solutions for the local NLS.

Another promising direction of further developments is study multicomponent NNS and the corresponding linear bundles associated with Hermitian symmetric spaces. An example of a multicomponent NNS related to symmetric spaces of the type **A**.**III** is given by:

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{q}(x,t)(\mathbf{q}^{\dagger}(-x,t)\mathbf{q}(x,t)) = 0,$$

where $\mathbf{q}$ is a matrix-valued smooth function. It has recently become known that certain nonlinear Schrödinger equations related to **A**.**III** and **BD**.**I** symmetric spaces find applications in Bose-Einstein condensation [8, 15] so their nonlocal counterparts could find similar applications too.

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