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FULLY INERT SUBGROUPS OF ABELIAN p-GROUPS

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Abstract. A subgroup H of an Abelian group G is said to be fully inert in G, if for every endomorphism ϕ of G, the factor group $(H + \phi(H))/H$ is finite. This notion arises in the study of the dynamical properties of endomorphisms (entropy). The principal result of this work is that fully inert subgroups of direct sums of cyclic p-groups are commensurable with fully invariant subgroups of the direct sum.

Introduction

All groups discussed in this paper are Abelian, so the word "group" always means an additively written "Abelian group". Motivated by the study of the dynamical properties of an endomorphism ϕ of a group, ϕ-inert subgroups have been introduced in [5], according to the following

Definition 1. Let G be a group, $\phi\colon G \to G$ an endomorphism and H a subgroup of G. H is called *ϕ-inert* if $H \cap \phi(H)$ has finite index in $\phi(H)$, equivalently, if the factor group $(\phi(H) + H)/H$ is finite.

The family of all ϕ-inert subgroups of G obviously contains all the ϕ-invariant subgroups of G, as well as the finite subgroups and the subgroups of finite index. Passing to a "global condition", we have the following notion, also introduced in [5].

Definition 2. A subgroup H of a group G is said to be *fully inert* if it is ϕ-inert for every endomorphism ϕ of G.

Fully inert subgroups represent a common generalization of finite subgroups and of subgroups of finite index, as well as of fully invariant subgroups.

Recall that two subgroups K, H of a group G are said to be commensurable, if both $(K + H)/H$ and $(K + H)/K$ are finite. Commensurability is an equivalence relation; this fact was proved in [7], and also follows readily from Proposition 1.4 of the present paper. This notion is relevant in the investigation of the fully inert subgroups of a given group G, since, as shown in [6], *a subgroup commensurable with some*

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fully inert subgroup of G is also fully inert. Actually, a main question in the investigation of fully inert subgroups, is whether or not every fully inert subgroup of G is commensurable with some fully invariant subgroup.

In two recent papers ([6] and [7]), fully inert subgroups of divisible groups and of free groups have been investigated and characterized. Furthermore, extending *verbatim* the definitions to J_p-modules, where J_p denotes the ring of p-adic integers, fully inert submodules of torsion-free complete J_p-modules have been characterized in [9]. It is worth pointing out a significant difference between fully inert subgroups of free groups and those of divisible groups. In the former case, every fully inert subgroup is commensurable with a fully invariant subgroup while in the latter case there exist fully inert subgroups not satisfying this property. It happens also in the case of torsion-free J_p-modules that fully inert submodules may fail to be commensurable with fully invariant submodules (see [9]).

In this paper we begin the investigation of fully inert subgroups of p-groups. The techniques used are more sophisticated than in the divisible and free cases, reflecting the fact that fully invariant subgroups of p-groups have a more complicated structure. Although our interest in this work is centred on direct sums of cyclic p-groups, we will on occasion present results for fully transitive p-groups, noting that direct sums of cyclic p-groups are always fully transitive. Fully invariant subgroups of fully transitive p-groups may be completely described via the well-known classification due to Kaplansky (see [11, Theorem 25]).

After a preliminary section containing some results on commensurable subgroups of p-groups and of their fully invariant subgroups, the main body of the paper is devoted to proving that fully inert subgroups of direct sums of cyclic p-groups are commensurable with fully invariant subgroups. This result was not unexpected, but its proof is far from straightforward and is significantly more difficult than the corresponding proof in the case of torsion-free J_p-modules H, despite the superficial resemblance of the two cases. A significant complication in the case of p-groups is that the lattice of fully invariant subgroups is vastly more complicated than the corresponding lattice in the torsion-free situation, where the lattice is just the chain $p^\alpha H$ $(\alpha \leq \omega)$. Indeed, to get the result we must consider two cases, each needing quite different discussions and techniques. The first case, handled in Section 2, deals with bounded p-groups, while the second case, handled in Section 3, deals with semi-standard direct sums of cyclic p-groups, that is, direct sums whose Ulm-Kaplansky invariants are all finite. A final step that combines the results obtained in the two cases described above, completes the proof.

In the final section, Section 4, we give an example of a separable p-group containing fully inert subgroups not commensurable with any fully invariant subgroup. For this purpose we use a p-group constructed by Pierce (see Theorem 15.4 in [12]), whose endomorphism ring, modulo the ideal of small endomorphisms, has a particularly simple structure.

1. Notations and preliminary results

For unexplained notations, definitions and standard results on p-groups we refer to the classical volumes of Fuchs [8].

Two subgroups K, H of a group G are said to be commensurable, if both $(K+H)/H$ and $(K+H)/K$ are finite. If H and K are commensurable, we use the notation $H \sim_c K$.

We will show that two subgroups of a p-group are commensurable if and only if they contain a common summand of finite index. We start by considering the case when one group contains the other as a subgroup of finite index; then the following result by Pierce is available (see Lemma 16.5 in [12]).

Proposition 1.1. *Let G be an arbitrary p-group and H a subgroup of G such that G/H is finite. Then G decomposes as $G = F \oplus C$, with F finite, and $H = (F \cap H) \oplus C$.*

An immediate consequence of Proposition 1.1 is the following

Corollary 1.2. *Let $\mathcal{C}$ be a class of p-groups containing the finite groups and closed under taking direct summands and finite direct sums. Let H be a subgroup of finite index of the p-group G. Then $H \in \mathcal{C}$ if and only if $G \in \mathcal{C}$.*

Classes of p-groups satisfying the conditions of the preceding corollary are, for instance, the class of totally projective p-groups (of fixed length) and the class of torsion-complete groups.

We need now a preparatory lemma.

Lemma 1.3. *Let $X = F_1 \oplus C_1 = F_2 \oplus C_2$ be two direct decompositions of the p-group X, with F_1 and F_2 finite. Then $C_1 = F_1' \oplus C$ and $C_2 = F_2' \oplus C$, with F_1' and F_2' finite.*

Proof. Applying Proposition 1.1 with $G = X$ and $H = C_1 \cap C_2$, we obtain $X = F \oplus C$ and $C_1 \cap C_2 = (F \cap C_1 \cap C_2) \oplus C$. Thus C is a summand of both C_1 and C_2, and, clearly, the complements are finite. □

We can now prove the announced result on commensurable subgroups of p-groups.

Proposition 1.4. *Let G be a p-group, and H, K two subgroups of G. Then H is commensurable with K if and only if $H = F \oplus C$ and $K = F' \oplus C$, where F and F' are finite.*

Proof. Assume that $H \sim_c K$. Applying Proposition 1.1 twice, firstly to $H+K$ and H, and then to $H+K$ and K, we obtain two direct decompositions $H+K = F_1 \oplus C_1 = F_2 \oplus C_2$, where F_1, F_2 are finite and $H = (F_1 \cap H) \oplus C_1$, $K = (F_2 \cap K) \oplus C_2$. Applying Lemma 1.3 to $X = H+K$, we get $H = (F_1 \cap H) \oplus F_1' \oplus C$ and $K = (F_2 \cap K) \oplus F_2' \oplus C$. Setting $F = (F_1 \cap H) \oplus F_1'$ and $F' = (F_2 \cap H) \oplus F_2'$, we get the desired decompositions. The converse is obvious. □

Note that the preceding proposition readily implies that commensurability is an equivalence relation.

The following well-known fact will be used repeatedly in our discussion; it has been previously used in [4].

Lemma 1.5. *Suppose that $G = A \oplus B$ and X is a fully invariant subgroup of A. Then there is a subgroup C of B such that $X \oplus C$ is fully invariant in G.*

Proof. Let $C = \langle \delta(x) : x \in X,\ \delta \in \mathrm{Hom}(A,B) \rangle$. We claim that

(i) $\gamma C \subseteq X$ for all $\gamma \in \mathrm{Hom}(B,A)$;

(ii) $\beta C \subseteq C$ for all $\beta \in \mathrm{End}(B)$.

Assuming for the moment that we have established these claims, consider the subgroup $X \oplus C$ of G. If $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an arbitrary endomorphism of G (with the usual conventions), then $\Delta(X \oplus C) \subseteq (\alpha X + \gamma C) \oplus (\delta X + \beta C)$. Clearly $\alpha X \subseteq X$ by the full invariance of X in A and $\delta X \subseteq C$ by the definition of C. The claims (i) and (ii) above then establish the full invariance of $X \oplus C$ in G.

To establish the first claim, note that if $c \in C$, then $c = \sum \delta_i(x_i)$ for some $x_i \in X$, $\delta_i \colon A \to B$. But then $\gamma(c) = \gamma(\sum \delta_i(x_i)) = \sum \gamma\delta_i(x_i)$ and $\gamma\delta_i \in \mathrm{End}(A)$. Thus $\gamma\delta_i(x_i) \in X$ since the latter is fully invariant in A.

For the second claim, it suffices to note, using the same notation as above, that $\beta\gamma_i \in \mathrm{Hom}(A,B)$ so that $\beta(c) = \sum \beta\gamma_i(x_i) \in C$ by definition. □

Let G be a direct sum of cyclic p-groups. We fix a standing notation, namely, we write

$$G = \bigoplus_{0<n<\kappa} G_n$$

where κ is either a finite ordinal or ω, and G_n is a direct sum of λ_n copies of $\mathbb{Z}(p^{c_n})$, where $0 < c_1 < \cdots < c_n < \dots$, and λ_n $(n < \kappa)$ is a nonzero cardinal. Thus, for $0 < n < \kappa$, we have

$$G_n = \bigoplus_{\alpha<\lambda_n} \mathbb{Z}e_{\alpha,n} \cong \bigoplus_{\lambda_n} \mathbb{Z}(p^{c_n}),$$

where the $e_{\alpha,n}$ are fixed generators of G_n of order p^{c_n}.

The next lemma was proved in [1], with a slightly different formulation; see [1, Theorem 2.8]. It will be repeatedly applied throughout this paper.

Lemma 1.6. *Let* $G = \bigoplus_{0<n<\kappa} G_n$ *be as above, and consider its subgroup* $G' = \bigoplus_{n\in\mathcal{A}} G_n$*, where* $\emptyset \neq \mathcal{A} \subseteq \kappa$*. Then* L *is a fully invariant subgroup of* G' *if and only if* $L = \bigoplus_{n\in\mathcal{A}} p^{h(n)}G_n$*, where the integers* $h(n)$ *satisfy the conditions*

(1) $h(n) \leq c_n$ *for all* $n > 0$ *in* $\mathcal{A}$*;*

(2) $h(i) \leq h(n) \leq h(i) + c_n - c_i$ *for all* $0 < i < n$ *in* $\mathcal{A}$*.*

2. The case of bounded p-groups

In this section we examine the case when the p-group G is bounded. Hence, in the above notation, $\kappa = k + 1$ is finite, and we may write $G = \bigoplus_{n=1}^{k} G_n$. We assume that G is not finite, otherwise our results are devoid of interest.

Denote by π_n the canonical projections of G onto G_n and by $\pi_{\alpha,n}$ the projections onto $\mathbb{Z}e_{\alpha,n}$, for $\alpha < \lambda_n$. The *n-support* of $z \in G$ is defined by $\mathrm{Supp}_n(z) = \{\alpha < \lambda_n : \pi_{\alpha,n}(z) \neq 0\}$.

Let H be a fully inert subgroup of G. We assume that H is not finite. We introduce an *ad hoc* notation: for $n > 0$, let

$$X_n = X_n(H) = \{\alpha < \lambda_n : \pi_{\alpha,n}(z) \neq 0, \text{ for some } z \in H\}.$$

Of course $X_n = \bigcup_{z\in H} \mathrm{Supp}_n(z)$. Note that $H \subseteq \bigoplus_{n\leq k,\, \alpha\in X_n} \mathbb{Z}e_{\alpha,n}$. Since H is not finite, it follows that there is at least one $m \leq k$ such that X_m is an infinite set.

It is also clear that, for each $\alpha \in X_n$, there is $y \in H$ such that $\pi_{\alpha,n}(y) = p^{c_n-1}e_{\alpha,n}$.

To avoid cumbersome repetitions, we also introduce a terminology for endomorphisms: we say that $\phi \in \mathrm{End}(G)$ is the standard endomorphism that extends the assignments $\phi\colon e_{\alpha,n} \mapsto g_\alpha$, for suitable $n \leq k$, $\alpha \in \mathcal{V} \subseteq \lambda_n$, and $g_\alpha \in G$, if ϕ annihilates all the other generators $e_{\beta,i}$ of G ($i \leq k$, $\beta < \lambda_i$).

The following technical lemma will be crucial for our discussion, specifically in proving Steps 1 and 5 of Theorem 2.2.

Lemma 2.1. *In the above notation, let* X_n *be infinite and pick any infinite subset* Y *of* X_n*. Then there exist countably many distinct* $\beta_i \in Y$ *and* $z_i \in H$ *(*$i < \omega$*) such that*

(1) $\beta_i \in \mathrm{Supp}_n(z_i)$ *and* $\beta_i \notin \mathrm{Supp}_n(z_j)$ *for every* $j < i < \omega$*;*

(2) if $g(i)$ *is the minimum exponent such* $0 \neq p^{g(i)}e_{\beta_i,n} \in \pi_{\beta_i,n}H$*, then* $\pi_{\beta_i,n}(z_i) = p^{g(i)}e_{\beta_i,n}$*;*

(3) if $j < i$*, and* $\pi_{\beta_j,n}(z_i) = up^t e_{\beta_j,n}$*,* t *a suitable integer,* u *coprime with* p*, then* $g(j) \leq t$*.*

Proof. We construct the sequence $\{z_i\}_{i<\omega}$ by induction. Pick any $\beta_0 \in Y \subseteq X_n$, and choose $z_0 \in H$ such that $\pi_{\beta_0,n}(z_0) = p^{g(0)}e_{\beta_0,n} \neq 0$, and $g(0)$ is minimum in the sense of (2). Assume that $z_0, \ldots, z_{i-1}$ have been constructed. Choose $\beta_i \in Y$ such that $\beta_i \notin \mathrm{Supp}_n(z_j)$ for every $j < i < \omega$; this is possible, since Y is infinite. Then pick $z_i \in H$ such that $\pi_{\beta_i,n}(z_i) = p^{g(i)} e_{\beta_i,n} \neq 0$, and $g(i)$ is minimum. Take any $j < i$, and let $\pi_{\beta_j,n}(z_i) = up^t e_{\beta_j,n}$, t a suitable integer, u coprime with p. Then $g(j) \leq t$ since, by induction, $g(j)$ was the minimum exponent appearing in $\pi_{\beta_j,n}H$, and u is a unit of J_p. The element z_i satisfies the conditions (1)–(3). □

The following theorem deals with the special case when all the G_n are infinite. It is the crucial stage in deriving the final theorem of this section.

Theorem 2.2. *Let H be a fully inert subgroup of a bounded p-group $G = \bigoplus_{n=1}^{k} G_n$, where each G_n is an infinite direct sum of copies of $\mathbb{Z}(p^{c_n})$ and $0 < c_1 < \cdots < c_k$. Then H is commensurable with a fully invariant subgroup of G.*

Proof. We may assume that H is infinite, since finite subgroups are commensurable with the fully invariant subgroup $\{0\}$. We adopt the notation and conventions that precede the statement. In particular, $G_n = \bigoplus_{\alpha<\lambda_n} \mathbb{Z}e_{\alpha,n} \cong \bigoplus \mathbb{Z}(p^{c_n})$; by hypothesis, each λ_n is infinite.

Now we proceed by steps.

Step 1. Let $m \leq k$ be minimal such that X_m is an infinite set. Then $\lambda_n \setminus X_n$ is a finite set, for every $n \geq m$.

Proof. Assume, for a contradiction, that $\lambda_n \setminus X_n$ is infinite, for some $n \geq m$. Take infinitely many $\alpha_i < \lambda_n$, $i < \omega$, with $\alpha_i \notin X_n$. Say $A = \langle e_{\alpha_i,n} : i < \omega \rangle$; note that $A \cap H = 0$. Since X_m is infinite, we may construct sequences $\{\beta_i\}_{i<\omega} \subseteq X_m$, and $\{z_i\}_{i<\omega} \subseteq H$ as in Lemma 2.1.

Consider the standard endomorphism ϕ of G that extends the assignments

$$\phi\colon e_{\beta_i,m} \mapsto p^{c_n - c_m} e_{\alpha_i,n}, \ i < \omega.$$

We verify that $\phi H \supseteq p^{c_n - 1}A$. Actually, we will show that $p^{c_n - c_m + g(i)} e_{\alpha_i,n}$ is in ϕH for every $i < \omega$, and so we are then finished, since $g(i) \leq c_m - 1$. The verification is by induction on $i < \omega$. If $i = 0$, then $\phi(z_0) = \phi(p^{g(0)} e_{\beta_0,m})$, since $\beta_i \notin \mathrm{Supp}_n(z_0)$ for every $i > 0$; it follows that $\phi(p^{g(0)} e_{\beta_0,m}) = p^{c_n - c_m + g(0)} e_{\alpha_0,n} \in \phi H$. If $i > 0$, then

$$\phi(z_i) = \phi(p^{g(i)} e_{\beta_i,m}) + \sum_{j<i} \phi(u_j p^{t_j} e_{\beta_j,m}) =$$

$$p^{c_n - c_m}(p^{g(i)} e_{\alpha_i,n} + \sum_{j<i} u_j p^{t_j} e_{\alpha_j,n}),$$

for suitable $t_j \geq g(j)$, u_j units of J_p, again since $\beta_i \notin \mathrm{Supp}_n(z_j)$ for every $j < i$. By induction $p^{c_n - c_m + g(j)} e_{\alpha_j,n} \in \phi H$ for any $j < i$, and therefore we get $p^{c_m - c_n + g(i)} e_{\alpha_i,n} \in \phi H$, as required.

Since $p^{c_n-1}A \cap H = 0$, the group $(p^{c_n-1}A + H)/H$ is infinite, while $(\phi H + H)/H$ is finite, since H is fully inert – a contradiction. $\square$

If m is minimal such that X_m is infinite, then $H = F \oplus H_1$, where $H_1 \subseteq G_m \oplus \cdots \oplus G_k$, and F is a finite group. Thus $H_1 \sim_c H$ and so H_1 is also fully inert. Then, replacing H by H_1 if necessary, we may safely assume that $H \subseteq G_m \oplus \cdots \oplus G_k$. Moreover, in view of Step 1, we may also assume that $\lambda_n = X_n$ for $m \leq n \leq k$. It suffices to add to H the finitely many elements $e_{\alpha,n}$, whenever $\alpha \in \lambda_n \setminus X_n$ $(m \leq n \leq k)$, and note that the resulting enlarged subgroup is again fully inert and commensurable with H.

Step 2. For $m \leq n \leq k$, let $V_n = \langle p^{c_n-1}e_{\alpha,n} : \alpha < \lambda_n \rangle = p^{c_n-1}G_n$. Then the $\mathbb{Z}(p)$-vector space $(V_n + H)/H$ is finite.

Proof. Recall that we may assume $\lambda_n = X_n$ for $m \leq n \leq k$ and so $X_n = \lambda_n$ is infinite. Assume, for a contradiction, that $(V_n + H)/H$ is infinite. Then there exist countably many distinct $p^{c_n-1}e_{\alpha_i,n} \in V_n$ $(i < \omega)$ that are linearly independent modulo H. Let $W = \langle p^{c_n-1}e_{\alpha_i,n} \in V_n : i < \omega \rangle$; note that $W \cap H = 0$. Since X_n is infinite, applying Lemma 2.1 to X_m and arguing as in the proof of Step 1, we may find an endomorphism ϕ of G such that $\phi H \supseteq W$. Since W is infinite and $W \cap H = 0$, as in Step 1 we conclude that $(\phi H + H)/H$ is infinite – a contradiction. $\square$

From the above discussion we conclude that for $m \leq n \leq k$, there exist nonnegative integers $f(n) \leq c_n - 1$, such that

$$|(p^{f(n)}G_n + H)/H| < \infty, \quad m \leq n \leq k.$$

We assume that $f(n)$ is the *minimum* integer that satisfies this finiteness property.

Let $L = \bigoplus_{m \leq n \leq k} p^{f(n)}G_n$; clearly, we also have $|(L + H)/H| < \infty$.

Step 3. Let $K = \langle e_{\beta_i,n} : i < \omega \rangle \subseteq G_n$ where $\beta_i < \lambda_n$ are distinct. If $(p^tK + H)/H$ is finite for some $t < c_n$, then also $(p^tG_n + H)/H$ is finite.

Proof. Let $K_0 = \langle e_{\beta_0,n}, e_{\beta_1,n}, \ldots, e_{\beta_s,n} \rangle \subseteq G_n$ be such that $(p^tK + H)/H \subseteq (p^tK_0 + H)/H$. In particular, $p^te_{\beta_i,n} \in p^tK_0 + H$ for all $i < \omega$. Assume, for a contradiction, that $(p^tG_n + H)/H$ is infinite. Then it contains a countable subgroup and so there exist countably many distinct $\alpha_i < \lambda_n$ $(i < \omega)$ such that $(T + H)/H$ is infinite, where $T = \langle p^te_{\alpha_i,n} : i < \omega \rangle$.

Consider the standard endomorphism ϕ of G that extends the assignments

$$\phi\colon e_{\beta_i,n} \mapsto e_{\alpha_i,n}, \ i < \omega.$$

Then $p^te_{\alpha_i,n} = \phi(p^te_{\beta_i,n}) \in \phi(p^tK_0) + \phi H$ for all $i < \omega$. It follows that $T \subseteq \phi(p^tK_0) + \phi H$. Since $\phi(p^tK_0)$ is finite and H is fully inert, it follows that $(\phi(p^tK_0) + \phi H + H)/H$ is finite, hence also $(T + H)/H$ is finite – a contradiction. $\square$

We want to show that $L = \bigoplus_{m\le n\le k} p^{f(n)}G_n$ is a fully invariant subgroup of $G' = \bigoplus_{m\le n\le k} G_n$. By Lemma 1.6, it suffices to show that the integers $f(n)$ satisfy conditions (1) and (2) of that lemma. Actually, condition (1) is automatic, since $f(n) < c_n$, by our choice of the $f(n)$. The next step verifies that condition (2) also holds.

Step 4. If $m \le n < n+r \le k$, then $f(n) \le f(n+r) \le f(n) + c_{n+r} - c_n$.

Proof. Since the λ_n are all infinite, the cardinal ω is contained in both λ_n and λ_{n+r}. We show firstly that $f(n) \le f(n+r)$. Consider the standard $\phi \in \mathrm{End}(G)$ such that $\phi\colon e_{i,n+r} \mapsto e_{i,n}$, $i < \omega$. Let $K = \langle e_{i,n+r} : i < \omega\rangle \subseteq G_{n+r}$; since $p^{f(n+r)}K \subseteq H$, we get $p^{f(n+r)}\phi K = \langle p^{f(n+r)}e_{i,n} : i < \omega\rangle \subseteq \phi H$. Then $(p^{f(n+r)}\phi K + H)/H$ is finite, since H is fully inert. Then Step 3 shows that $(p^{f(n+r)}G_n + H)/H$ is also finite. From the minimality of $f(n)$ it follows that $f(n) \le f(n+r)$.

Secondly, we show that $f(n+r) \le f(n) + c_{n+r} - c_n$. Consider the standard $\psi \in \mathrm{End}(G)$ such that $\psi\colon e_{i,n} \mapsto p^{c_{n+r}-c_n}e_{i,n+r}$, $i < \omega$. Then ψH contains $\langle p^{f(n)+c_{n+r}-c_n}e_{i,n+r} : i < \omega\rangle$. Arguing as above, we see that $(p^{f(n)+c_{n+r}-c_n}G_{n+r} + H)/H$ is finite. From the minimality of $f(n+r)$ it follows that $f(n+r) \le f(n) + c_{n+r} - c_n$. □

Now we prove that $H \sim_c L$. Since we have observed above that $(H+L)/H$ is finite, the commensurability follows from the next step.

Step 5. $(H+L)/L$ is a finite group.

Proof. It suffices to show that $(\pi_n H + p^{f(n)}G_n)/p^{f(n)}G_n$ is finite for all $m \le n \le k$. Indeed

$$(H+L)/L \subseteq \Big(\bigoplus_{m\le n\le k} \pi_n H + L\Big)/L = \bigoplus_{m\le n\le k} (\pi_n H + p^{f(n)}G_n)/p^{f(n)}G_n.$$

We assume, for a contradiction, that $(\pi_n H + p^{f(n)}G_n)/p^{f(n)}G_n$ is infinite, for some $m \le n \le k$. Then $(\pi_n H + p^{f(n)}G_n)/p^{f(n)}G_n$ contains a countable group, say $\langle y_i + p^{f(n)}G_n : i < \omega\rangle$, $y_i \in \pi_n H$. In particular, there must be an infinite subset Y of X_n such that, for all $\alpha \in Y$, there exists $g(\alpha) < f(n)$ such that $p^{g(\alpha)}e_{\alpha,n} \in \pi_{\alpha,n}H$. We apply Lemma 2.1 to the set Y, getting countably many distinct $\beta_i \in Y$ and $z_i \in H$ $(i < \omega)$ satisfying conditions (1)–(3) of that lemma. Note that, by the definition of Y, $g(i) < f(n)$ for all $i < \omega$.

Consider the standard endomorphism ϕ of G such that

$$\phi\colon e_{\beta_i,n} \mapsto e_{\beta_i,n},\ i < \omega.$$

An argument, similar to that in the proof of Step 1 (under the present circumstances $c_m = c_n$, $\beta_i = \alpha_i$), shows that all the $p^{g(i)}e_{\beta_i,n}$ lie in ϕH. It follows that $(\langle p^{f(n)-1}e_{\beta_i,n} : i < \omega\rangle + H)/H$ is finite. Using Step 3, we may conclude that $(p^{f(n)-1}G_n + H)/H$ is finite, contrary to the minimality of $f(n)$. We have reached the desired contradiction. □

The final step in our argument is:

Step 6. If $m > 1$ and $s < m$, then $f(m) \geq c_s$.

Proof. Recall that $H \cap G_s = 0$. Pick infinitely many distinct $e_{i,m} \in G_m$ and $e_{i,s} \in G_s$, $i < \omega$. Consider the standard endomorphism ϕ such that $\phi\colon e_{i,m} \mapsto e_{i,s}$. Then $\phi H \supseteq p^{f(m)} \bigoplus_{i<\omega} \mathbb{Z}e_{i,s}$. Since H is fully inert and $H \cap \bigoplus_{i<\omega} \mathbb{Z}e_{i,s} = 0$, this is possible only if $p^{f(m)} \bigoplus_{i<\omega} \mathbb{Z}e_{i,s} = 0$, i.e., $f(m) \geq c_s$. □

An immediate consequence of Step 6 and Lemma 1.6, is that $L = \{0\} \oplus L$ is fully invariant in the whole group $G = \bigoplus_{n=1}^{k} G_n$. Then H is commensurable with a fully invariant subgroup of G, as required. □

Before we give the final theorem of this section, we make an elementary observation: if $H \subseteq X$, X is a direct summand of G, and H is fully inert in G, then H is fully inert in X.

Theorem 2.3. *Let H be a fully inert subgroup of a bounded p-group G. Then H is commensurable with a fully invariant subgroup of G.*

Proof. Assume H is fully inert in $G = \bigoplus_{n=1}^{k} G_n = F \oplus \tilde{G}$, where all the homogeneous components of F are finite (and so F itself is finite) while those of $\tilde{G}$ are all of infinite rank. Now if $H_1 = F + H$, then $H \sim_c H_1$ and so H_1 is fully inert in G. Furthermore $H_1 = H_1 \cap (F \oplus \tilde{G}) = F \oplus (H_1 \cap \tilde{G})$; set $H_2 = H_1 \cap \tilde{G}$. Note that $H_2 \sim_c H_1$, so H_2 is fully inert in G and as observed above, H_2 is fully inert in $\tilde{G}$.

Now, applying Theorem 2.2 to H_2, we get that $H_2 \sim_c \tilde{K}$, where $\tilde{K}$ is fully invariant in $\tilde{G}$. Now, using Lemma 1.5, we can find a fully invariant subgroup K of G such that $K = C \oplus \tilde{K}$ for some subgroup $C \subseteq F$. Note that $\tilde{K} \sim_c K$ and as

$$H \sim_c H_1 \sim_c H_2 \sim_c \tilde{K} \sim_c K,$$

we have that H is commensurable with a fully invariant subgroup of G. □

3. The general case

As usual, if G is a p-group and $\mathbf{u} = (\sigma_n)_{n\geq 0}$ is an increasing sequence of ordinals or symbols ∞, $G(\mathbf{u})$ denotes the fully invariant subgroup of G defined as follows:

$$G(\mathbf{u}) = \{g \in G \mid h(p^n g) \geq \sigma_n,\ n \geq 0\}.$$

Let G be a fully transitive p-group, H an arbitrary subgroup of G. Consider the increasing sequence of ordinals $\mathbf{u}(H) = (\sigma_n)_{n\geq 0}$ defined by

$$\sigma_n = \min\{h(p^n g) \mid g \in H\}.$$

Obviously the inclusion $H \subseteq G(\mathbf{u}(H))$ holds.

Let H be an arbitrary subgroup of a p-group G. We denote by H^* the intersection of the fully invariant subgroups of G containing H; clearly, H^* is the smallest fully invariant subgroup of G containing H. We call it the *fully invariant hull* of H.

It is easy to show that

$$H^* = \langle \phi(x) : \phi \in \mathrm{End}(G), x \in H \rangle = \sum_{\phi \in \mathrm{End}(G)} \phi H.$$

Lemma 3.1. *If G is a fully transitive p-group and $H \subseteq G$, then $H^* = G(\mathbf{u}(H))$.*

Proof. The inclusion $H^* \subseteq G(\mathbf{u}(H))$ is obvious, as $G(\mathbf{u}(H))$ fully invariant in G and contains H. For the reverse inclusion, consider the sequences $U(H)$ and $U(H^*)$. Clearly $U(H) \geq U(H^*)$, where the ordering is taken pointwise; hence $G(U(H)) \subseteq G(U(H^*))$. However, as G is fully transitive and H^* is fully invariant in G, it follows from Kaplansky's classification [11, Theorem 25] that $H^* = G(U(H^*)) \supseteq G(U(H))$, as required. □

We prove two general lemmas on fully inert subgroups that have some independent interest. They will be needed in the discussion that follows.

Lemma 3.2. *Let H be a fully inert subgroup of $G = A \oplus B$. Then $H_1 = H \cap A$ is fully inert in A.*

Proof. Take any $\phi \in \mathrm{End}(A)$ and extend it to $\bar{\phi} \in \mathrm{End}(G)$ by setting $\bar{\phi} B = 0$. Then $\bar{\phi} H \supseteq \phi H_1$, and we get

$$(\bar{\phi} H + H)/H \supseteq (\phi H_1 + H)/H \cong \phi H_1/(H \cap \phi H_1) = \phi H_1/(H_1 \cap \phi H_1).$$

Then $(\phi H_1 + H_1)/H_1 \cong \phi H_1/(H_1 \cap \phi H_1)$ is finite, since H is fully inert. We conclude that H_1 is fully inert in A, since ϕ was arbitrary. □

Lemma 3.3. *Let H be a fully inert subgroup of a p-group $G = \bigoplus_{\beta \in I} G_\beta$, where I is a totally ordered set of indices, and let $\pi_\beta \colon G \to G_\beta$ ($\beta \in I$) be the canonical projections. Then there exists $t \in I$ such that $(\sum_{\beta \geq t} \pi_\beta H + H)/H$ is finite.*

Proof. For convenience, we introduce the notation $\pi^t H = \sum_{\beta \geq t} \pi_\beta H$. Assume, for a contradiction, that $(\pi^t H + H)/H$ is infinite for every $t \in I$. For every $n > 0$ we will construct, by induction, the following sequences

(i) indices $t_1 < t_2 < \cdots < t_n < \ldots$ in I;

(ii) elements $a_n \in G$ ($n > 0$), where $a_n \in \pi_{t_n} H$, and $a_n + H \notin \langle a_1 + H, \ldots, a_{n-1} + H \rangle$ for all $n > 0$;

Let us first construct a_1, t_1. Take any $t_1 \in I$; since $(\pi^{t_1} H + H)/H$ is infinite, there exists an element $a_1 \in G$ such that $a_1 \in \pi_{t_1} H \setminus H$.

Assume that for $i \leq n$, the families $t_1, \dots, t_n \in I$, $a_i \in \pi_{t_i} H$ have been constructed. Observe that $(\pi^{t_n+1}H + \langle a_1, \dots, a_n\rangle + H)/H$ is infinite, since it contains the infinite group $(\pi^{t_n+1}H + H)/H$. Hence there exists an index $t_{n+1} > t_n$ and an element $a_{n+1} \in G$ such that $a_{n+1} \in \pi_{t_{n+1}} H$ and $a_{n+1} + H \notin \langle a_1 + H, \dots, a_n + H\rangle$.

The new families obtained by adding t_{n+1}, a_{n+1} to the given ones, satisfy conditions (i) and (ii).

Let $T = \langle a_n : n > 0\rangle$. Clearly the group $(T + H)/H$ is infinite, since $a_n + H \notin \langle a_1 + H, \dots, a_{n-1} + H\rangle$ for all $n > 0$.

Now we set $\psi = \sum_{n>0} \pi_{t_n}$, and observe that this is a well-defined endomorphism of G. Then $(\psi H+H)/H$ contains $(T+H)/H$, hence it is infinite. But this is impossible, since H is fully inert – a contradiction. □

Let now G be an unbounded direct sum of cyclic p-groups. We maintain the notation of the preceding section, so $G = \bigoplus_{0<n<\kappa} G_n$, each $G_n = \bigoplus_{\alpha<\lambda_n} \mathbb{Z}e_{\alpha,n}$ is a direct sum of copies of $\mathbb{Z}(p^{c_n})$, $1 \leq c_1 < c_2 < \dots < c_n < \dots$, and the λ_n are suitable cardinals. Here $\kappa = \omega$, since G is unbounded.

Note that we may order lexicographically the pairs of indices (n, α) of the $e_{\alpha,n}$, where $n > 0$, $\alpha < \lambda_n$; hence we get a totally ordered set of indices I, and we can write $G = \bigoplus_{(n,\alpha)\in I} \mathbb{Z}e_{\alpha,n}$, where I is a totally ordered set. Accordingly, we denote by $\pi_{(n,\alpha)}$ the canonical projections $\pi_{(n,\alpha)} \colon G \to \mathbb{Z}e_{\alpha,n}$.

For each $t < \omega$, let $G^t = \bigoplus_{n\geq t} G_n$, and $G^{<t} = \bigoplus_{1\leq i<t} G_i$. For H any subgroup of G, define $H^t = H \cap G^t$ and denote by H^{*t} the fully invariant hull of H^t in G^t. Recall that, by Lemma 3.2, H^t is fully inert in G^t whenever H is fully inert in G.

Note that $H^{*t} \subseteq H^* \cap G^t$ for every $t > 0$.

Corollary 3.4. *Let the notation be as above. If H is a fully inert subgroup of G, there exists a $t > 0$ such that, for every $r \geq t$, we have $H^r = \bigoplus_{n\geq r} \pi_n H = \bigoplus_{n\geq r,\alpha<\lambda_n} \pi_{(n,\alpha)} H$.*

Proof. In view of Lemma 3.3, there exists an index $(t, \alpha) \in I$, where $t > 0$, $\alpha < \lambda_t$, such that $(\bigoplus_{\beta\geq(t,\alpha)} \pi_\beta H + H)/H$ is finite. Analogously to Lemma 3.3, we use the notation $\pi^{(t,\alpha)}H = \bigoplus_{\beta\geq(t,\alpha)} \pi_\beta H$.

Since $(\pi^{(t,\alpha)}H + H)/H \supseteq (\pi^\gamma H + H)/H$ whenever $\gamma \geq (t, \alpha)$, and $\bigcap_{\gamma\geq(t,\alpha)} \pi^\gamma H = 0$, we conclude that $(\pi^\gamma H + H)/H = 0$, for some $\gamma \geq (t, \alpha)$. Since the order of the indices is lexicographic, we readily see that, without loss of generality, we may assume $\alpha = 0$ and $\gamma = (t, 0)$. Then we get $\pi^{(t,0)}H \subseteq H \cap G^t = H^t$; moreover $H^t \subseteq \bigoplus_{\beta\geq(t,0)} \pi_\beta H = \pi^{(t,0)}H$ is obvious. Since $\bigoplus_{n\geq t} \pi_n H \subseteq \bigoplus_{n\geq t,\alpha<\lambda_n} \pi_{(n,\alpha)}H = \pi^{(t,0)}H$, we get $H^t = \bigoplus_{n\geq t} \pi_n H = \bigoplus_{n\geq t,\alpha<\lambda_n} \pi_{(n,\alpha)}H$.

The general formula follows since $(\pi^{(t,0)}H + H)/H = 0$ implies $(\pi^{\gamma}H + H)/H = 0$, for any index $\gamma \geq (t, 0)$, and we can repeat the above argument. □

We prove a result, of independent interest, on fully invariant hulls. As a consequence, we can derive Lemma 3.6 which is a crucial component of the proof of the key theorem, Theorem 3.7.

Proposition 3.5. *Let $G = \bigoplus_{j\in J} \mathbb{Z}e_j$ be a direct sum of cyclic p-groups, and consider the subgroup $H = \bigoplus_{j\in J} \pi_j H$ of G, where π_j denotes the projection of G onto $\mathbb{Z}e_j$. Then the fully invariant hull H^* of H satisfies the equality $H^* = \bigcup_{\phi\in\mathrm{End}(G)} \phi H$.*

In particular, for every element $a \in H^*$ there exist an element $g \in H$ and an endomorphism ϕ of G such that $\phi(g) = a$.

Proof. To get the equality $\sum_{\phi\in\mathrm{End}(G)} \phi H = H^* = \bigcup_{\phi\in\mathrm{End}(G)} \phi H$, we will prove that, given arbitrary $x, y \in H$ and $\phi, \psi \in \mathrm{End}(G)$, there exist $z \in H$ and $\alpha \in \mathrm{End}(G)$ such that $\phi(x) + \psi(y) = \alpha(z)$. Let us consider the supports $X = \mathrm{Supp}(x)$ and $Y = \mathrm{Supp}(y)$, which are finite subsets of J. We distinguish two cases.

If $X \cap Y = \emptyset$, let $z = x+y$ and define α by extending the assignments: $\alpha(e_j) = \phi(e_j)$, if $j \in X$, $\alpha(e_j) = \psi(e_j)$, if $j \in Y$, and $\alpha(e_j) = 0$, if $j \notin X \cup Y$.

Then $\alpha(x + y) = \alpha(x) + \alpha(y) = \phi(x) + \psi(y)$, as desired.

Assume now that $X \cap Y = I$ is non empty; let $X' = X \setminus I$ and $Y' = Y \setminus I$. Note that, since $\mathbb{Z}e_i$ is a cyclic p-group, for each $i \in I$ either $\pi(x) = r\pi_i(y)$, for some $r \in \mathbb{Z}$, or $s\pi_i(x) = \pi_i(y)$, for some $s \in \mathbb{Z}$. Therefore we can split I as $I = I_1 \cup I_2$, where $I_1 \cap I_2 = \emptyset$, $\pi_i(x) = r_i\pi_i(y)$ for $i \in I_1$, and $\pi_i(y) = s_i\pi_i(x)$ for $i \in I_2$, for suitable $r_i, s_i \in \mathbb{Z}$. Let $x' = \sum_{i\in X'} \pi_i(x)$ and $y' = \sum_{i\in Y'} \pi_i(y)$, and set

$$z = x' + y' + \sum_{i\in I_1} \pi_i(y) + \sum_{i\in I_2} \pi_i(x).$$

Note that all the above summands of z are elements of H, since $H = \bigoplus_j \pi_j H$, so also $z \in H$.

We define the endomorphism α of G by the assignments:

(1) $\alpha(e_j) = \phi(e_j)$ if $j \in X'$, $\alpha(e_j) = \psi(e_j)$, if $j \in Y'$, $\alpha(e_j) = 0$ if $j \notin X \cup Y$;

(2) $\alpha(e_i) = (r_i\phi + \psi)(e_i)$ for $i \in I_1$, $\alpha(e_i) = (\phi + s_i\psi)(e_i)$, for $i \in I_2$.

Then we get

$$\alpha(z) = \alpha(x') + \alpha(y') + \sum_{i\in I_1} \alpha\pi_i(y) + \sum_{i\in I_2} \alpha\pi_i(x) =$$

$$\phi(x') + \psi(y') + \sum_{i\in I_1} (r_i\phi + \psi)\pi_i(y) + \sum_{i\in I_2} (\phi + s_i\psi)\pi_i(x) =$$

$$\phi(x') + \psi(y') + \phi \sum_{i\in I_1} r_i\pi_i(y) + \psi \sum_{i\in I_1} \pi_i(y) + \phi \sum_{i\in I_2} \pi_i(x) + \psi \sum_{i\in I_2} s_i\pi_i(x) =$$

$$\phi(x') + \phi\sum_{i\in I_1}\pi_i(x) + \phi\sum_{i\in I_2}\pi_i(x) + \psi(y') + \psi\sum_{i\in I_1}\pi_i(y) + \psi\sum_{i\in I_2}\pi_i(y) =$$
$$\phi\sum_{i\in X}\pi_i(x) + \psi\sum_{i\in Y}\pi_i(y) = \phi(x) + \psi(y)$$

as desired.

Once we have $H^* = \bigcup_{\phi\in\mathrm{End}(G)}\phi H$, the final assertion of our statement is obvious. ☐

Lemma 3.6. *Let $G = \bigoplus_n G_n$ be a direct sum of cyclic p-groups, where $G_n = \bigoplus_{\alpha<\lambda_n}\mathbb{Z}e_{\alpha,n}$, and let H be a fully inert subgroup of G. Then there exists an integer $t > 0$ such that, for any $s \geq t$ and every element $a \in H^{*s}$, there exist an element $g \in H^s$ and an endomorphism ϕ of G^s such that $\phi(g) = a$.*

Proof. By Corollary 3.4, there exists an integer $t > 0$ such that

$$H^s = \bigoplus_{n\geq s,\alpha<\lambda_n}\pi_{(n,\alpha)}H,$$

for any $s \geq t$. Since H^s is fully inert in G^s, we are in the position to apply Proposition 3.5 to G^s, H^s and H^{*s}, reaching the desired conclusion. ☐

We remark that, in the statement of the preceding lemma, we may safely replace $\phi \in \mathrm{End}(G^t)$ by $\phi \in \mathrm{End}(G)$, as every endomorphism of G^t trivially extends to an endomorphism of G.

The basic idea in the proof of the next theorem is the same as in Lemma 3.3, but the argument is considerably more delicate.

Theorem 3.7. *Let H be a fully inert subgroup of the direct sum of cyclic p-groups G. Then there exists $t > 0$ such that $(H^{*t} + H)/H$ is finite.*

Proof. Assume, for a contradiction, that $(H^{*t} + H)/H$ is infinite for every $t > 0$. For every $n > 0$ we will construct, by induction, the following families, increasing by inclusion:

(i) integers $t_1 < t_2 < \cdots < t_n < t_{n+1}$;

(ii) elements $a_1, a_2, \ldots, a_n \in H^*$, where $a_i \in H^{*t_i}$, and $a_i + H \notin \langle a_1 + H, \ldots, a_{i-1} + H\rangle$ for all $i \leq n$;

(iii) $g_1, \ldots, g_n \in H$, where $g_i \in H^{t_i}$;

(iv) $\psi_1, \ldots, \psi_n \in \mathrm{End}(G)$, such that $\mathrm{Supp}(\psi_i) \subseteq G_{t_i} \oplus \cdots \oplus G_{t_{i+1}-1}$ and $\psi_i(g_i) = a_i$, for all $i \leq n$.

Let $t > 0$ be the integer furnished by Lemma 3.6.

We start with $t_1 = t$, so that $H^{*t_1} = H^{*t}$. Since $(H^{*t} + H)/H$ is infinite, we may pick $a_1 \in H^{*t} \setminus H$. Now we apply Lemma 3.6 for the case $s = t$, to find an element $g_1 \in H^t$ and an endomorphism ϕ_1 of G^t such that $\phi_1(g_1) = a_1$. Say $g_1 \in G_t \oplus \cdots \oplus G_{t_2-1}$ for some $t_2 > t_1 = t$, and let ψ_1 be the endomorphism of G^t which coincides with ϕ_1 on $G_t \oplus \cdots \oplus G_{t_2-1}$, and vanishes elsewhere.

Assume that for $i \leq n$, the families $t_1, \dots, t_n, t_{n+1} \in \mathbb{Z}$, $a_i \in H^{*t_i}$, $g_i \in H^{t_i}$ and $\psi_i \in \mathrm{End}(G)$ satisfying the conditions (i)–(iv), have been constructed.

Observe that $(H^{*t_{n+1}} + \langle a_1, \dots, a_n \rangle + H)/H$ is infinite, since it contains the infinite group $(H^{*t_{n+1}} + H)/H$. Hence there exists $a_{n+1} \in H^{*t_{n+1}}$ such that $a_{n+1} + H \notin \langle a_1 + H, \dots a_n + H \rangle$. We apply Lemma 3.6 for the case $s = t_{n+1}$. We may find $g_{n+1} \in H^{t_{n+1}}$ and $\phi_{n+1} \in \mathrm{End}(G^{t_n+1})$ such that $\phi_{n+1}(g_{n+1}) = a_{n+1}$. Say $g_{n+1} \in G_{t_{n+1}} \oplus \cdots \oplus G_{t_{n+2}-1}$, for some integer $t_{n+2} > t_{n+1}$, and let ψ_{n+1} be the endomorphism of G which coincides with ϕ_{n+1} on $G_{t_{n+1}} \oplus \cdots \oplus G_{t_{n+2}-1}$, and vanishes elsewhere. Note that, by construction, the endomorphisms ψ_{n+1} and ψ_i, $i \leq n$ have disjoint supports.

The new families, obtained adding t_{n+2}, a_{n+1}, g_{n+1}, ψ_{n+1} to the given ones, satisfy conditions (i)–(iv).

Let $T = \langle a_n : n > 0 \rangle$. Clearly the group $(T + H)/H$ is infinite, since $a_n + H \notin \langle a_1 + H, \dots, a_{n-1} + H \rangle$ for all $n > 0$.

Now we set $\psi = \sum_{n>0} \psi_n$, and observe that this is a well defined endomorphism of G, since, by construction, the ψ_n have pairwise disjoint supports. Then $(\psi H + H)/H$ contains $(T + H)/H$, hence it is infinite. But this is impossible, since H is fully inert – a contradiction. □

Corollary 3.8. *Let the notation be as above. If H is a fully inert subgroup of G, there exists a $t > 0$ such that*

$$H^t = \bigoplus_{n \geq t} \pi_n H = H^{*t}$$

In particular, we get the direct decomposition $H = H_1 \oplus H^{*t}$, where $H_1 = H \cap G^{<t}$.

Proof. In view of Theorem 3.7 and Corollary 3.4, there exists $t > 0$ such that $(H^{*t} + H)/H$ is finite and $H^r = \bigoplus_{n \geq r} \pi_n H$ for every $r \geq t$. From the descending chain of finite groups

$$(H^{*t} + H)/H \supseteq (H^{*t+1} + H)/H \supseteq \cdots ,$$

and $\bigcap_{n \geq t} H^{*n} = 0$, we conclude that $(H^{*s} + H)/H = 0$, for some $s \geq t$. Without loss of generality, we assume that $s = t$. Then we get $H^{*t} \subseteq H \cap G^t = H^t = \bigoplus_{n \geq t} \pi_n H \subseteq H^{*t}$.

Finally, since $G^{<t} \oplus \bigoplus_{n \geq t} \pi_n H \supseteq H$, the modular law yields the direct decomposition $H = H_1 \oplus \bigoplus_{n \geq t} \pi_n H = H_1 \oplus H^t = H_1 \oplus H^{*t}$. □

We note that $H_1 = H \cap G^{<t}$ is a fully inert subgroup of the bounded group $G^{<t}$, by Lemma 3.2.

Theorem 3.9. *A fully inert subgroup H of a semi-standard p-group G is commensurable with a fully invariant subgroup of G.*

Proof. By Corollary 3.8, $H = H_1 \oplus H^{*t}$ for a suitable $t > 0$, where, due to the hypothesis that G is semi-standard, H_1 is finite. Since H^{*t}

is fully invariant in G^t, by Lemma 1.5 there exists a finite subgroup C of $G^{<t}$ such that $A = C \oplus H^{*t}$ is fully invariant in G, so H^{*t} is commensurable with A. Since H^{*t} has finite index in H, we conclude that also H is commensurable with A. □

In order to prove the general result, we need to apply Lemma 1.6 several times. For the convenience of the reader, we recall the content of that lemma: the subgroup $L = \bigoplus_{n\in\mathcal{A}} p^{h(n)}G_n$ of G is fully invariant in $G' = \bigoplus_{n\in\mathcal{A}} G_n$ ($\emptyset \neq \mathcal{A} \subseteq \kappa$) if and only if the integers $h(n)$ satisfy the conditions

(1) $h(n) \leq c_n$ for all $n > 0$ in $\mathcal{A}$;
(2) $h(i) \leq h(n) \leq h(i) + c_n - c_i$ for all $0 < i < n$ in $\mathcal{A}$.

Theorem 3.10. *A fully inert subgroup H of a direct sum of cyclic p-groups G is commensurable with a fully invariant subgroup of G.*

Proof. By Theorem 2.3 we may assume that G is unbounded. We adopt the previous notation. By Corollary 3.8, we have $H = H_1 \oplus H^{*t}$, for some $t > 0$, where $H_1 = H \cap G^{<t}$ is fully inert in the bounded group $G^{<t}$, by Lemma 3.2. By Theorem 2.3 it follows that H_1 is commensurable with a fully invariant subgroup of $G^{<t}$, say $L = \bigoplus_{0<i<t} p^{f'(i)}G_i$, where the $f'(i)$ satisfy the conditions (1), (2) of Lemma 1.6, for $0 < i < t$. On the other hand, H^{*t} is fully invariant in G^t, hence $H^{*t} = \bigoplus_{n\geq t} p^{f(n)}G_n$, where the $f(n)$ satisfy conditions (1), (2) of Lemma 1.6, for $t \leq n < \omega$.

Clearly $H \sim_c L \oplus H^{*t}$, hence, to reach the desired conclusion, it suffices to show that the subgroup

$$H' = L \oplus H^{*t} = \bigoplus_{0<i<t} p^{f'(i)}G_i \oplus \bigoplus_{n\geq t} p^{f(n)}G_n,$$

which is fully inert in G since it is commensurable with H, is commensurable with a fully invariant subgroup of G.

Step 1. If $p^{f'(s)}G_s$ is infinite, for some $s < t$, then $f'(s) \leq f(n) \leq f'(s) + c_n - c_s$ for almost all $n \geq t$.

Proof. For all $n \geq t$ we pick $e_{0,n} \in G_n$. Since λ_s contains ω, we may pick infinitely many distinct $e_{n,s} \in G_s$, $t \leq n < \omega$.

Consider the standard endomorphism ϕ of G such that $\phi\colon e_{0,n} \mapsto e_{n,s}$ $(n \geq t)$. Let $X = \langle p^{f(n)}e_{n,s} : n \geq t\rangle$. Then $\phi H' \supseteq X$, and so $(X + H')/H' \subseteq (\phi H' + H')/H'$ is finite, since H' is fully inert. Assume that $f(n) < f'(s)$ if and only if $n \in \mathcal{B}$ (where $\mathcal{B}$ might be empty). Since

$$(X + H')/H' \cong \bigoplus_{n\in\mathcal{B}} \mathbb{Z}(p^{f'(s)-f(n)}),$$

we readily conclude that $\mathcal{B}$ is a finite set, i.e., $f(n) \geq f'(s)$ for almost all $n \geq t$.

Consider now the standard endomorphism ψ such that $\psi\colon e_{n,s} \mapsto p^{c_n-c_s}e_{0,n}$ $(n \geq t)$. Let $Z = \langle p^{f'(s)+c_n-c_s}e_{0,n} : n \geq t\rangle$; then $(Z +$

$H')/H' \subseteq (\psi H' + H')/H'$ is finite. Assume that $f(n) > f'(s) + c_n - c_s$ if and only if $n \in \mathcal{C}$. Since

$$(Z + H')/H' \cong \bigoplus_{n \in \mathcal{C}} \mathbb{Z}(p^{f(n)-f'(s)-c_n+c_s}),$$

we conclude that $\mathcal{C}$ is finite, hence $f'(s) + c_n - c_s \geq f(n)$ for almost all $n \geq t$. Thus we have seen that $f'(s) \leq f(n) \leq f'(s) + c_n - c_s$ for every $n \in \omega \setminus \mathcal{B} \cup \mathcal{C}$. $\square$

Step 2. If $p^{f'(s)}G_s$ and $p^{f(n)}G_n$ are both infinite, for some $1 \leq s < t$, $n \geq t$, then $f'(s) \leq f(n) \leq f'(s) + c_n - c_s$.

Proof. Since both λ_s and λ_n contain ω, we may pick infinitely many distinct $e_{i,s} \in G_s$ and $e_{i,n} \in G_n$, $i < \omega$.

We firstly assume, for a contradiction, that $f'(s) > f(n)$; let $W = \langle p^{f(n)}e_{i,s} : i < \omega \rangle$. Then the group $(W + H')/H' \cong \bigoplus_\omega \mathbb{Z}(p^{f'(s)-f(n)})$ is infinite. Consider the standard endomorphism ϕ such that $\phi\colon e_{i,n} \mapsto e_{i,s}$. Then $\phi H' \supseteq W$, and so $(\phi H' + H')/H' \supseteq (W + H')/H'$ is infinite – a contradiction, since H' is fully inert.

Secondly, we assume, for a contradiction, that $f(n) > f'(s) + c_n - c_s$; let $Y = \langle p^{f'(s)+c_n-c_s}e_{i,n} : i < \omega \rangle$. Like above, the group $(Y + H')/H'$ is infinite. Consider the standard endomorphism ψ such that $\psi\colon e_{i,s} \mapsto p^{c_n-c_s}e_{i,n}$. Then $\psi H' \supseteq Y$, and so $(\psi H' + H')/H' \supseteq (Y + H')/H'$ is infinite – a contradiction. $\square$

Let us set $h(i) = f'(i)$, for $1 \leq i < t$ and $h(n) = f(n)$, for $n \geq t$, whence $H' = \bigoplus_{n>0} p^{h(n)}G_n$.

We define $\mathcal{S}$ to be the set of $s < t$ such that $p^{h(s)}G_s = p^{f'(s)}G_s$ is infinite.

Using Step 1, we see that the integers $h(n)$, $h(s)$ ($n \geq t$, $s \in \mathcal{S}$) satisfy conditions (1), (2) of Lemma 1.6, for all $s \in \mathcal{S}$ and exactly for all $n \geq t$ that don't lie in a suitable finite subset $\mathcal{D}$ of ω. Moreover, Step 2 shows that $p^{h(j)}G_j$ is finite whenever $j \in \mathcal{D}$.

We define $\mathcal{E} = \{i < t : i \notin \mathcal{S}\} \cup \mathcal{D}$. The above discussion shows that $\bigoplus_{j \in \mathcal{E}} p^{h(j)}G_j$ is finite, and that the integers $h(n)$ satisfy conditions (1), (2) for all $n \in \mathcal{A} = \omega \setminus \mathcal{E}$. Since, by Lemma 1.6, $N = \bigoplus_{n \in \mathcal{A}} p^{h(n)}G_n$ is fully invariant in $\bigoplus_{n \in \mathcal{A}} G_n$, by Lemma 1.5 there exists $C \subseteq \bigoplus_{j \in \mathcal{E}} G_j$ such that $N \oplus C$ is fully invariant in G. Since C is finite by construction, by Proposition 1.4 we conclude that $H' = N \oplus \bigoplus_{j \in \mathcal{E}} p^{h(j)}G_j$ is commensurable with $N \oplus C$. $\square$

We remark that, in general, a fully inert subgroup H of a direct sum G of cyclic p-groups is not commensurable with H^* (i.e., H^*/H is infinite), even when in the situation where G is semi-standard.

Example 3.11. Let B be a standard p-group so that $B = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$ where each e_i is of order p^i. Then the socle of B is $B[p] = \bigoplus_{i=1}^{\infty} \langle p^{i-1} e_i \rangle$. Let $H = B[p] + \langle e_2 \rangle = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \bigoplus_{i=3}^{\infty} \langle p^{i-1} e_i \rangle$; then H is a finite extension of $B[p]$ and hence is fully inert in B.

For each $j > 2$, there is an endomorphism ϕ_j of B where $\phi_j(e_2) = p^{j-2} e_j$ and ϕ_j maps the remaining basis elements to 0. Since $e_2 \in H$, the images $\phi_j(e_2)$ are all in H^* and so $H^* \supseteq \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \bigoplus_{i=3}^{\infty} \langle p^{i-2} e_i \rangle$ and hence $H^*/H \supseteq (\langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \bigoplus_{i=3}^{\infty} \langle p^{i-2} e_i \rangle)/(\langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \bigoplus_{i=3}^{\infty} \langle p^{i-1} e_i \rangle)$ is infinite.

4. Fully Inert Subgroups of p-groups not commensurable with fully invariants

The purpose of this final short section is to give an example of a separable p-group containing fully inert subgroups not commensurable with fully invariant subgroups.

Consider any Abelian p-group G whose basic subgroups are semi-standard. Say $B = \bigoplus_n B_n$ is basic in G, where B_n is a finite direct sum of copies of $\mathbb{Z}(p^n)$.

Recall that an endomorphism θ of a p-group G is called *small* if for every $k \geq 1$ there is an integer m (depending on k), such that $\theta((p^m G)[p^k]) = 0$. We will denote by $E_s(G)$ the ideal of $\mathrm{End}(G)$ consisting of the small endomorphisms of G.

We firstly observe the following fact.

Lemma 4.1. *If θ is a small endomorphism of G then the image $\theta(B[p])$ is finite. In particular, $(B[p]+\theta(B[p]))/B[p]$ is finite for all $\theta \in E_s(G)$.*

Proof. As θ is small, there is an integer N such that $\theta((p^N G)[p]) = 0$. However, if $x \in B_{N+i}[p]$ $(i \geq 1)$, then $x \in (p^N G)[p]$, and so the image of $B[p]$ under θ is just the finite image $\theta((B_1 \oplus \cdots \oplus B_N)[p])$. □

Theorem 4.2. *Let G be a separable p-group of cardinality $2^{\aleph_0}$, with semi-standard basic group B, such that $\mathrm{End}(G) = J_p \cdot 1_G \oplus E_s(G)$. Then the subgroup $B[p]$ of G is fully inert in G but it is not commensurable with any fully invariant subgroup of G.*

Proof. If ϕ is any endomorphism of G then ϕ has the form $\phi = r \cdot 1_G + \theta$ for some $r \in J_p$, $\theta \in E_s(G)$. Consequently $B[p] + \phi(B[p]) = B[p] + \theta(B[p])$ and so the quotient $(B[p] + \phi B[p])/B[p]$ is finite by Lemma 4.1. Thus $B[p]$ is fully inert in G.

Suppose, for a contradiction, that $B[p]$ is commensurable with some fully invariant subgroup K of G. Then, since $(K + B[p])/B[p]$ is finite, we get $|K + B[p]| = |B[p]| = \aleph_0$. So K must be countable. Since G is separable, it is fully transitive and so by Kaplansky's classification (see [11, Theorem 25]), K must have the form $K = G(\mathbf{u})$ for some U-sequence $\mathbf{u} = (u_0, u_1, \dots)$. In particular, $K \supseteq (p^{u_0}G)[p]$. However, $|(p^{u_0}G)[p]| = |p^{u_0}G|$ and since $|G/p^{u_0}G| = |B/p^{u_0}B| = \aleph_0$, we have that $|G| = |p^{u_0}G| = |(p^{u_0}G)[p]| = 2^{\aleph_0}$. But this immediately contradicts the fact that K is countable. Thus $B[p]$ is not commensurable with any fully invariant subgroup of G, as required. □

Note that in the above theorem, there is nothing special about $B[p]$; one could use $B[p^k]$ for any $k \geq 1$.

The Abelian p-groups satisfying the requirements of Theorem 4.2 are the so-called Pierce-like groups. They were first constructed in [12, Theorem 15.4]; see also Corner's construction of such groups [3, Theorem 4.1].

References

[1] K. Benabdallah, B. Eisenstadt, J. Irwin, E. Poluianov, *The structure of large subgroups of primary abelian groups*, Acta Math. Acad. Sci. Hungar. 21 (1970) 421–435.

[2] J. Cohen and H. Gluck, *Stacked bases for modules over principal ideal domains*, J. Algebra 14 (1970) 493–505.

[3] A. L. S. Corner, *On endomorphism rings of primary Abelian groups*, Quart. J. Math. Oxford 20 (1969) 277–296.

[4] P. Danchev and B. Goldsmith, *On socle-regularity and some notions of transitivity for Abelian p-groups*, J. Comm. Algebra 3 (2011) 301–319.

[5] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, *Intrinsic algebraic entropy*, submitted.

[6] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, *Fully inert subgroups of divisible Abelian groups*, J. Group Theory 16(6) (2013), 915–939.

[7] D. Dikranjan, L. Salce, P. Zanardo, *Fully inert subgroups of free Abelian groups*, to appear in Periodica Math. Hungarica.

[8] L. Fuchs, Infinite Abelian Groups, Vol. I and II, Academic Press, New York, 1970 and 1973.

[9] B. Goldsmith, L. Salce, P. Zanardo, *Fully inert submodules of torsion-free J_p-modules*, submitted.

[10] P. Hill, C. Megibben, *Generalization of the stacked bases theorem*, Trans. Amer. Math. Soc. 312 (1989), 377–402.

[11] I. Kaplansky, Infinite Abelian Groups, Univ. Michigan Press, Ann Arbor, 1954.

[12] R.S. Pierce, *Homomorphisms of primary abelian groups*, Topics in Abelian Groups, 215-310, Chicago, Illinois, 1963.

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