When the Intrinsic Algebraic Entropy is not Really Intrinsic

Brendan Goldsmith
Technological University Dublin, brendan.goldsmith@tudublin.ie

Luigi Salce
University of Padova

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Brendan Goldsmith and Luigi Salce

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Dedicated to Alberto Facchini on the occasion of his 60th birthday

Abstract

The intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$ of an endomorphism ϕ of an Abelian group G can be computed using fully inert subgroups of ϕ-invariant sections of G, instead of the whole family of ϕ-inert subgroups. For a class of groups containing the groups of finite rank, as well as those groups which are trajectories of finitely generated subgroups, it is proved that only fully inert subgroups of the group itself are needed to compute $\widetilde{\mathrm{ent}}(\phi)$. Examples show how the situation may be quite different outside of this class.

1 Introduction

All the groups considered in this paper are assumed to be Abelian. Following [DGSV], given an endomorphism ϕ of a group G, the intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$ of ϕ is defined by the formula

$$\widetilde{\mathrm{ent}}(\phi) = \sup_{H \in \mathcal{I}_\phi(G)} \widetilde{\mathrm{ent}}(\phi, H),$$

where $\mathcal{I}_\phi(G)$ is the family of the ϕ-inert subgroups of G (so it is a set of subgroups depending on ϕ, which justifies the name "intrinsic");

$$\widetilde{\mathrm{ent}}(\phi, H) = \lim_{n\to\infty} \frac{\log |T_n(\phi, H)/H|}{n}$$

for $H \in \mathcal{I}_\phi(G)$; $T_n(\phi, H) = H + \phi H + ... + \phi^{n-1} H$ is the nth ϕ-trajectory of H. Recall that a subgroup H of the group G is ϕ-inert if $(H + \phi H)/H$ is finite,

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or, equivalently, $T_n(\phi, H)/H$ is finite for all n, while H is called fully inert if the same condition holds for all endomorphisms ϕ of G. The family of the fully inert subgroups of G is denoted by $\mathcal{I}(G)$.

The intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$ is a variation of the classical algebraic entropy $\mathrm{ent}(\phi)$, introduced in [1] and carefully investigated in [8] and in many subsequent papers, whose definition uses the family of the finite subgroups of G instead of the larger family $\mathcal{I}_\phi(G)$. If finite subsets instead of finite subgroups are used, then one obtains the Peters algebraic entropy $h(\phi)$ (see [18] and [5]). When G is a torsion group, that is, a group of (torsion-free) rank zero, then the three entropies $\mathrm{ent}(\phi)$, $\widetilde{\mathrm{ent}}(\phi)$ and $h(\phi)$ coincide (see Proposition 3.6 in [6]).

The relevance of the intrinsic algebraic entropy relies mainly on the fact that it enlightens the Algebraic Yuzvinski Formula recently proved in [14]. Recall that the original Yuzvinski Formula [20] states that the value of the topological entropy of a full solenoidal automorphism coincides with the Mahler measure of the characteristic polynomial over $\mathbb{Q}$ of its dual map, while the Algebraic Yuzvinski Formula states that the value of the Peters entropy $h(\phi)$ of a linear transformation ϕ of a finite-dimensional $\mathbb{Q}$-vector space coincides with the Mahler measure of its own characteristic polynomial; we refer to [18], [5] and [18] for the definition of Mahler measure and for the definition and properties of h and its relationship with ent. In the Mahler measure of a characteristic polynomial a term $\log s$ appears, where the integer s is the least common multiple of the denominators of the coefficients of the polynomial, together with the logarithms of certain eigenvalues. These later have an interpretation in terms of dynamics while in [6] it was proved that the intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$ of such a linear transformation coincides with the term $\log s$.

But the importance of the intrinsic algebraic entropy is also due to the fact that it satisfies most of the usual properties required of a function from the endomorphism ring of an Abelian group to $\mathbb{R}_{\geq 0} \cup \{\infty\}$ in order that it be an algebraic entropy; this includes the Addition Theorem (see Section 2) and a kind of Uniqueness Theorem (see [6]). Furthermore, compared with the Peters entropy h, it maintains an algebraic character, while h is of a more combinatorial nature.

In the computation of the intrinsic algebraic entropy of the endomorphisms of a group G, it would be desirable to be able to restrict the consideration to the trajectories of the fully inert subgroups of G which do not depend on the particular choice of the endomorphism; this also in view of the fact that fully inert subgroups of various classes of Abelian groups have been recently investigated in [7], [9], [15] and [16]. The goal of this paper is to investigate when this restriction is possible.

After a preliminary Section 2, collecting notions and results needed later on, we prove in Section 3 that $\widetilde{\mathrm{ent}}(\phi)$ can be computed using only fully inert subgroups of ϕ-invariant *sections* (that is, subgroups of quotients) of G. Thus these sections depend on ϕ, but their fully inert subgroups do not depend on it.

This does not mean that the following equality holds:

$$(*) \qquad \sup_{H \in \mathcal{I}_\phi(G)} \widetilde{\mathrm{ent}}(\phi, H) = \sup_{H \in \mathcal{I}(G)} \widetilde{\mathrm{ent}}(\phi, H).$$

In fact, this equality is not generally true, as Example 3.1 shows.

In Section 4 we prove that all the endomorphisms ϕ of G satisfy the equality $(*)$ if the group G has finite (torsion-free) rank. This result extends to mixed groups well-known facts valid for torsion and torsion-free groups.

In Section 5 we prove that the equality $(*)$ holds if the group G is the ϕ-trajectory of a finitely generated subgroup, that is, if G is a finitely generated $\mathbb{Z}[X]$-module with the module structure induced by ϕ. Then we extend both results quoted above by showing that the equality $(*)$ still holds provided G contains a fully invariant subgroup H of finite rank such that the factor group G/H is a finitely generated $\mathbb{Z}[X]$-module with the module structure induced by ϕ.

Section 6 starts with some examples concerning groups G of infinite rank. These examples show that completely different values of the entropy function are obtained by taking ϕ-inert subgroups or fully inert subgroups. Then a construction of mixed groups G of uncountable rank is performed, such that the torsion part $t(G)$ is an arbitrary unbounded semi-standard p-group, and the equality $(*)$ above is satisfied for all endomorphisms ϕ of G.

2 Preliminaries

In this section we recall the basic notions and facts used in the paper. By the *rank* of a group G we mean the torsion-free rank, that is, the dimension of the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus torsion groups are exactly the groups of rank zero. The torsion subgroup of a group G will be denoted by $t(G)$.

Following [1], given an endomorphism $\phi : G \to G$ of a group G, its algebraic entropy is defined by the formula

$$\mathrm{ent}(\phi) = \sup_{F \in \mathcal{F}(G)} \mathrm{ent}(\phi, F)$$

where $\mathcal{F}(G)$ is the family of the finite subgroups of G,

$$\mathrm{ent}(\phi, F) = \lim_{n \to \infty} \frac{\log |T_n(\phi, F)|}{n}$$

for $F \in \mathcal{F}(G)$, and $T_n(\phi, F) = F + \phi F + ... + \phi^{n-1} F$ is the nth ϕ-trajectory of F. Algebraic entropy has been the subject of in-depth investigations in a series of papers, starting with [8] (see [4] for more references).

The definition of the intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$, introduced and investigated in [6], replaces the family $\mathcal{F}(G)$ of finite subgroups of G by the larger family $\mathcal{I}_\phi(G)$ of the ϕ-inert subgroups (already defined in the Introduction). In [6] it is proved that, for a ϕ-inert subgroup H of G, the quantity

$$\widetilde{\text{ent}}(\phi, H) = \lim_{n\to\infty} \frac{\log |T_n(\phi, H)/H|}{n}$$

is finite and it coincides with $\log |T_{n+1}(\phi, H)/T_n(\phi, H)|$ for n large enough. Therefore, as recalled in the Introduction, one can define the intrinsic algebraic entropy $\widetilde{\text{ent}}(\phi)$ as the supremum of these quantities, letting H range over $\mathcal{I}_\phi(G)$.

It was also shown in [6] that $\widetilde{\text{ent}}(\phi)$ is upper continuous, that is, if G is the direct limit of a family of ϕ-invariant subgroups G_i, then $\widetilde{\text{ent}}(\phi) = \sup_i \widetilde{\text{ent}}(\phi \restriction G_i)$. An entire section of [6], Section 5, is devoted to the detailed proof of the Addition Theorem for $\widetilde{\text{ent}}$; this states that, if H is a ϕ-invariant subgroup of G, then

$$\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \restriction H) + \widetilde{\text{ent}}(\bar{\phi})$$

where $\bar{\phi} : G/H \to G/H$ is the map induced by ϕ.

One of the main achievements of [6] is a simplified version of the Algebraic Yuzvinski Formula proved in [14], namely, given an endomorphism $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$, $\widetilde{\text{ent}}(\phi) = \log(s)$, where s is the least common multiple of the denominators of the coefficients of the characteristic polynomial of ϕ over $\mathbb{Q}$ (see Theorem 4.2 in [6]).

A basic tool to work with algebraic entropies is the right Bernoulli shift. Given a direct sum $G^{(\mathbb{N})}$ of copies of a group G indexed by the natural numbers, its right Bernoulli shift is the endomorphism $\beta : G^{(\mathbb{N})} \to G^{(\mathbb{N})}$ defined by the rule: $\beta(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots)$. The Bernoulli shift will play a crucial role also in this paper,

Finally, we recall that fully inert subgroups of divisible groups, of free groups and of p-groups, and fully inert submodule of torsion-free J_p-modules have been recently investigated in [7], [9]. [16] and [15] respectively; from a slight different point of view, inertial automorphisms have been recently studied by Dardano-Rinauro in [3].

For the notation and the remaining notions on Abelian groups used later, we refer to [11].

3 Computing the intrinsic algebraic entropy by means of fully inert subgroups of sections

We start with an example which shows that it is not always possible to compute the intrinsic algebraic entropy of an endomorphism ϕ just using the fully inert subgroups instead of the ϕ-inert ones.

Example 3.1. Let $G = \mathbb{Q}^{(\mathbb{N})}$ be a torsion-free divisible group of countable rank. It is easy to show that G has no fully inert subgroups different from $\{0\}$ and itself, so $\sup_{H\in\mathcal{I}(G)} \widetilde{\text{ent}}(\phi, H) = 0$ for any endomorphism ϕ of G. On the other hand, it is proved in [6, Example 3.7] that $\widetilde{\text{ent}}(\beta) = \infty$, where β is the right

Bernoulli shift of G. This depends on the fact that, for each positive integer k, we can find a β-inert subgroup H_k of G such that $\widetilde{\mathrm{ent}}(\beta, H_k) = \log k$. Actually, if $F = \mathbb{Z}^{(\mathbb{N})}$, then $\widetilde{\mathrm{ent}}(\beta) = \widetilde{\mathrm{ent}}(\beta \restriction_F)$, by [6, Corollary 3.14], the groups H_k are subgroups of finite index of F, so they are fully inert in F, and consequently $\widetilde{\mathrm{ent}}(\beta) = \sup_{H \in \mathcal{I}(F)} \widetilde{\mathrm{ent}}(\beta, H)$.

We recall how the intrinsic algebraic entropy of an endomorphism $\phi : G \to G$ can be computed concretely once the Addition Theorem for $\widetilde{\mathrm{ent}}$ has been proved (see [6, Corollary 5.10]). The process entails the following seven steps:

Firstly we focus on the process of calculating $\widetilde{\mathrm{ent}}(\phi)$ where ϕ is an endomorphism of a torsion-free group G.

(I) We construct, by transfinite induction, a chain of pure ϕ-invariant subgroups of G:

$$G_0 < G_1 < G_2 < \ldots < G_\sigma < \ldots < \bigcup_{\sigma<\lambda} G_\sigma = G.$$

Set $G_0 = 0$ and assume G_σ is already defined. Let $\phi_\sigma : G/G_\sigma \to G/G_\sigma$ be the map induced by ϕ. Choose an element $x \in G \setminus G_\sigma$ and form its ϕ_σ-trajectory $T_\sigma = T(\phi_\sigma, x + G_\sigma)$, which is the minimal ϕ_σ-invariant subgroup of G/G_σ containing $x + G_\sigma$. Let $G_{\sigma+1}/G_\sigma$ be the purification of T_σ/G_σ in G/G_σ; it is immediate that $G_{\sigma+1}/G_\sigma$ is still ϕ_σ-invariant in G/G_σ, hence $G_{\sigma+1}$ is ϕ-invariant in G. For λ a limit ordinal, we set $G_\lambda = \bigcup_{\sigma<\lambda} G_\sigma$.

(II) Using the Addition Theorem and the upper continuity of the intrinsic algebraic entropy (see [6]), we get that $\widetilde{\mathrm{ent}}(\phi) = \sum_\sigma \widetilde{\mathrm{ent}}(\phi_\sigma \restriction_{G_{\sigma+1}/G_\sigma})$, where it is understood that, as soon as there are infinitely many non-zero terms in the sum, or one term has value ∞, then the sum has value ∞. Thus to compute $\widetilde{\mathrm{ent}}(\phi)$, it suffices to compute $\mathrm{ent}(\phi_\sigma \restriction_{G_{\sigma+1}/G_\sigma})$ for all σ.

(III) Note that, for each ordinal σ, the group $G_{\sigma+1}/T_\sigma$ is torsion, and T_σ/G_σ is ϕ_σ-invariant in G/G_σ. So, again by the Addition Theorem, $\widetilde{\mathrm{ent}}(\phi_\sigma \restriction_{G_{\sigma+1}/G_\sigma})$ is the sum of $\widetilde{\mathrm{ent}}(\phi_\sigma \restriction_{T_\sigma/G_\sigma})$ plus the intrinsic algebraic entropy of the endomorphism induced on the quotient $G_{\sigma+1}/T_\sigma$ by ϕ; the latter group being torsion, the computation of the intrinsic algebraic entropy of this endomorphism can be achieved using only finite, hence fully inert, subgroups of the section $G_{\sigma+1}/T_\sigma$ of G.

Setting $A = T_\sigma/G_\sigma$, $\bar{x} = x + G_\sigma$, and $\psi = \phi_\sigma$, it remains to compute $\widetilde{\mathrm{ent}}(\psi)$, where $A = T(\psi, \bar{x})$ is a section of the original group G and a cyclic torsion-free ψ-trajectory. At this stage we distinguish two cases, according as $\mathrm{rk}(A)$ is finite or countable.

(IV) Let $\psi : A \to A$ be an endomorphism of the torsion-free group A of finite rank. Then $\widetilde{\mathrm{ent}}(\psi) = \widetilde{\mathrm{ent}}(\tilde{\psi})$, where $\tilde{\psi} : D(A) \to D(A)$ is the unique extension of ψ to the divisible hull $D(A)$ of A (see Corollary 3.4 of [6]), and $\widetilde{\mathrm{ent}}(\tilde{\psi}) = \widetilde{\mathrm{ent}}(\psi, F)$, where F is a free subgroup of A of full rank, hence fully inert in A, by Lemma 2.3 of [7]. So again, $\widetilde{\mathrm{ent}}(\psi)$ is calculated using fully inert subgroups of a section of G. (Recall that in Theorem 4.2 of [6] it was proved that $\widetilde{\mathrm{ent}}(\psi, F) = \log s$, where s is the least common multiple of the denominators of the coefficients of the characteristic polynomial of $\tilde{\psi}$ over $\mathbb{Q}$ (a variation of the Algebraic Yuzvinski Formula proved in [14]).

(V) Let $\psi : A \to A$ be an endomorphism of the countable rank torsion-free group A, and assume that $A = T(\psi, x)$ is a cyclic ψ-trajectory. Then A is free and ψ acts as the right Bernoulli shift on it; furthermore, for each positive integer k, there exists a fully inert subgroup H_k of A such that $\widetilde{\mathrm{ent}}(\psi, H_k) = \log k$. Consequently $\widetilde{\mathrm{ent}}(\psi) = \infty$. Again, $\widetilde{\mathrm{ent}}(\psi)$ may be computed just using fully inert subgroups of A. Note that $\widetilde{\mathrm{ent}}(\psi) = \widetilde{\mathrm{ent}}(\tilde{\psi})$, where $\tilde{\psi} : D(A) \to D(A)$, and $D(A)$ has no non-trivial fully inert subgroups, as noted in Example 3.1.

As observed in stages (III) - (V) above, the calculation of $\widetilde{\mathrm{ent}}(\phi)$ can be accomplished using fully inert subgroups of ϕ-invariant sections of G.

Now we return to the general, that is, not necessarily torsion-free, case. So suppose that G' is an arbitrary group with torsion subgroup tG' and torsion-free quotient $G = G'/tG'$ and ϕ' is an endomorphism of G'.

(VI) As $t(G')$ is fully invariant in G', we can compute $\widetilde{\mathrm{ent}}(\phi' \restriction_{t(G')})$ which equals $\mathrm{ent}(\phi' \restriction_{t(G')})$; to do this, we have to calculate only the values of $\mathrm{ent}(\phi', F)$, letting F range through the family of finite, and hence fully inert, subgroups of G'. Thus $\widetilde{\mathrm{ent}}(\phi' \restriction_{t(G')})$ may be calculated using only fully inert subgroups of sections of G'.

(VII) Since, by the Addition Theorem, $\widetilde{\mathrm{ent}}(\phi') = \widetilde{\mathrm{ent}}(\phi' \restriction_{t(G')}) + \widetilde{\mathrm{ent}}(\bar{\phi}')$, the computation $\widetilde{\mathrm{ent}}(\phi')$ will be complete if we compute $\widetilde{\mathrm{ent}}(\bar{\phi}')$, where $\bar{\phi}' : G \to G$ is the induced endomorphism of the torsion-free group $G = G'/t(G')$.

We have already seen in steps (I) - (V) that the intrinsic algebraic entropy of an endomorphism of the torsion-free group G can be accomplished using only fully inert subgroups of sections of G. But since any ϕ-invariant section of G is certainly a ϕ'-invariant section of G', we have established:

Theorem 3.2. *Let G be an arbitrary group and $\phi : G \to G$ an endomorphism of G. Then the intrinsic algebraic entropy $\widetilde{\mathrm{ent}}(\phi)$ can be computed using only fully inert subgroups of ϕ-invariant sections of G.*

It is convenient at this point to set:

$$\bar{\mathrm{ent}}(\phi) = \sup_{H \in \mathcal{I}(G)} \widetilde{\mathrm{ent}}(\phi, H).$$

Obviously the following inequalities always hold:

$$\mathrm{ent}(\phi) \leq \bar{\mathrm{ent}}(\phi) \leq \widetilde{\mathrm{ent}}(\phi).$$

We are tempted to call $\bar{\mathrm{ent}}(\phi)$ the *fully inert algebraic entropy* of ϕ. Even though we shall adopt this terminology from now on, one could be very critical of it, since $\bar{\mathrm{ent}}(\phi)$ does not satisfy the usual properties satisfied by typical entropy functions like ent or $\widetilde{\mathrm{ent}}$, as the next remark emphasizes.

Remark 3.3. ¿From Example 3.1 it follows that $\bar{\text{ent}}$ is neither monotone under taking restrictions to invariant subgroups, nor upper continuous, and that $\bar{\text{ent}}$ does not satisfy the Addition Theorem. For, if β is the right Bernoulli shift of $G = \mathbb{Q}^{(\mathbb{N})}$, we have seen that $\bar{\text{ent}}(\beta) = 0$, because G has no non-trivial fully inert subgroups. On the other hand, $F = \mathbb{Z}^{(\mathbb{N})}$ is a β-invariant subgroup of G, and we have also seen that $\bar{\text{ent}}(\beta \restriction F) = \infty$, so $\bar{\text{ent}}$ is not monotone and the Addition Theorem does not hold. Furthermore, G is the union of its fully invariant subgroups $\frac{1}{n!}F$, and $\bar{\text{ent}}(\beta \restriction \frac{1}{n!}F) = \infty$ for all n, consequently $\bar{\text{ent}}$ is not upper continuous.

As we saw at steps (IV), (V) and (VI), the equality $\bar{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi)$ holds when $\phi : G \to G$ is an endomorphism of either a torsion group, or of a torsion-free group of finite rank (in which case $\bar{\text{ent}}(\phi)$ is finite), or of a torsion-free cyclic ϕ-trajectory of infinite rank (in which case $\bar{\text{ent}}(\phi) = \infty$). The following natural question arises: are there more groups G for which the equality $\bar{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi)$ holds, either for all endomorphisms ϕ, or for some suitable ϕ? We will give some answers to this question in the rest of the paper.

4 Groups all of whose endomorphisms have intrinsic and fully inert algebraic entropies coinciding

In this section we give an answer to the question posed at the end of the preceding section. We will use the following notation: if $\phi : G \to G$ is an endomorphism of the group G, we denote the factor group $G/t(G)$ by $\bar{G}$ and the map induced by ϕ on it by $\bar{\phi}$. We need a lemma, proved in [6], which relates ϕ-inert (respectively, fully inert) subgroups of G with those of $\bar{G}$.

Lemma 4.1. (1) *Let H be a subgroup of the group G, and let $\phi : G \to G$ an endomorphism. Then H is ϕ-inert in G if and only if $\bar{H} = (H + t(G))/t(G)$ is $\bar{\phi}$-inert in $\bar{G} = G/t(G)$ and $t(H + \phi H)/t(H)$ is finite (equivalently, $t(T_n(\phi, H))/t(H)$ is finite for each n). The second condition is automatically satisfied if H contains $t(G)$.*

(2) *A subgroup K of a group G is fully inert if and only if $\bar{K} = (K + t(G))/t(G)$ is fully inert in $\bar{G} = G/t(G)$ and $t(K + \psi K)/t(K)$ is finite for every endomorphism ψ of G.*

Proof. Part (1) follows from Lemma 2.3 of [6], while (2) is a direct consequence of (1). □

The main result of this section is the following theorem, which extends to mixed groups what we have already seen for torsion-free groups.

Theorem 4.2. *Let G be a group of finite rank and $\phi : G \to G$ an endomorphism. Then $\bar{ent}(\phi) = \widetilde{ent}(\phi)$.*

Proof. First assume that $\widetilde{ent}(\phi) = \infty$. Then the Addition Theorem ensures that

$$\widetilde{ent}(\phi) = \widetilde{ent}(\phi \restriction_{t(G)}) + \widetilde{ent}(\bar{\phi}).$$

As $\widetilde{ent}(\bar{\phi})$ is finite, as we saw in step (IV) of Section 3, then we must have $\widetilde{ent}(\phi \restriction_{t(G)}) = \infty = ent(\phi \restriction_{t(G)}) = ent(\phi)$; therefore, also $\bar{ent}(\phi) = \infty$.

Assume now $\widetilde{ent}(\phi)$ finite. Then both $\widetilde{ent}(\phi \restriction_{t(G)})$ and $\widetilde{ent}(\bar{\phi})$ are finite, so

$$\widetilde{ent}(\phi \restriction_{t(G)}) = \widetilde{ent}(\phi, F) = ent(\phi, F)$$

where F is a suitable finite subgroup of G, and

$$\widetilde{ent}(\bar{\phi}) = \widetilde{ent}(\bar{\phi}, \bar{H}),$$

where $\bar{H} = H/t(G)$ is a free subgroup of $\bar{G}$ of full rank, and hence is fully inert in $\bar{G}$ (see the proof of Theorem 4.2 in [6]). Then $H = Z \oplus t(G)$ for a free subgroup Z of G of full rank. We will show that the subgroup $K = Z \oplus F$ is fully inert in G and that $\widetilde{ent}(\phi) = \widetilde{ent}(\phi, K)$, from which the claim clearly follows. Since $K + t(G) = H$, it follows from Lemma 4.1 that K is fully inert in G if and only if $t(K + \psi K)/t(K)$ is finite for every endomorphism ψ of G. A simple computation shows that

$$t(K + \psi K)/t(K) = (t(Z + \psi Z) + F + \psi F)/F$$

and so $t(K + \psi K)/t(K)$ is finite, since $Z + \psi Z$ is finitely generated, hence its torsion part is finite, and ψF is trivially finite. Thus we have proved that K is fully inert in G.

In order to show that $\widetilde{ent}(\phi) = \widetilde{ent}(\phi, K)$, we recall that it was proved in [6] that:

(i) if H is a finitely generated ϕ-inert subgroup of G, then $\widetilde{ent}(\phi \restriction T(\phi, H)) = \widetilde{ent}(\phi, H)$ (Proposition 3.15 (c) of [6]);

(ii) if H is a finitely generated subgroup of G, then

$$\widetilde{ent}(\phi \restriction T(\phi, H)) = \widetilde{ent}(\phi \restriction t(T(\phi, H))) + \widetilde{ent}(\bar{\bar{\phi}})$$

where $\bar{\bar{\phi}}$ is the endomorphism of $T(\phi, H)/t(T(\phi, H))$ induced by ϕ (Proposition 5.8 of [6]).

Using these two facts, applied to the finitely generated fully inert subgroup $K = Z \oplus F$, we have:

$$\widetilde{ent}(\phi, K) = \widetilde{ent}(\phi \restriction T(\phi, K)) = \widetilde{ent}(\phi \restriction t(T(\phi, K))) + \widetilde{ent}(\bar{\bar{\phi}}).$$

But $\widetilde{ent}(\phi \restriction t(T(\phi, K))) = ent(\phi, F) = ent(\phi)$ and $\widetilde{ent}(\bar{\bar{\phi}}) = \widetilde{ent}(\bar{\phi}, \bar{H}) = \widetilde{ent}(\bar{\phi})$, so the conclusion follows from the equality $\widetilde{ent}(\phi) = ent(\phi) + \widetilde{ent}(\bar{\phi})$. $\square$

The following example provides a family of $2^{\aleph_0}$ non-isomorphic reduced torsion-free groups G_σ of countable rank, with the same endomorphism ring A, such that $\widetilde{\mathrm{ent}}(\phi) = 0$ for every endomorphism $\phi \in A$. In particular, $\widetilde{\mathrm{ent}}(\phi) = 0 = \bar{\mathrm{ent}}(\phi)$, thus these groups G_σ furnish more examples related to Theorem 4.2.

Example 4.3. Consider the ring $\prod_{\mathbb{N}} \mathbb{Z}$, endowed with the pointwise operations. Let A be the subring of $\prod_{\mathbb{N}} \mathbb{Z}$ generated by the identity $\mathbf{1} = (1, 1, 1, ...)$ and by the direct sum $\oplus_{\mathbb{N}}\mathbb{Z}$. The additive group of the ring A is reduced, torsion-free and countable. By a celebrated theorem of Corner, there exists a family of $2^{\aleph_0}$ non-isomorphic reduced torsion-free countable groups G_σ such that $\mathrm{End}_{\mathbb{Z}}(G_\sigma) \cong A$. Every element $\phi \in A$ is of the form $\phi = n\mathbf{1} + \sum_{j\in J} m_j e_j$, where n, m_j are integers, J is a finite subset of $\mathbb{N}$, and $e_i = (0, ..., 0, 1, 0, ...)$ is the element of $\oplus_{\mathbb{N}}\mathbb{Z}$ with i-th coordinate 1 and the other coordinates 0. Each e_i is idempotent, hence integral over $\mathbb{Z}$, therefore also ϕ is integral over $\mathbb{Z}$, hence $\phi^k = a_{k-1}\phi^{k-1} + ... + a_1\phi + a_0$ for suitabel integers a_i. Let now F be a finite subset of G_σ; then, for every $n > k$ we have that

$$\phi^n(F) \subseteq a_{k-1}\phi^{k-1}(F) + ... + a_1\phi(F) + a_0F$$

and consequently $h(\phi, F) = 0$ (h is the Peters entropy as extended in [4]). Thus $h(\phi) = 0$, and since $\widetilde{ent}(\phi) \le h(\phi)$, by Proposition 3.6 of [6], we obtain that $\widetilde{ent}(\phi) = 0$.

The preceding example shows that the class of groups G such that $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$ for all $\phi \in \mathrm{End}_{\mathbb{Z}}(G)$ is strictly larger than the class of groups of finite rank. More examples of groups with this property will be constructed in Section 6. The next result shows that free groups of countable rank do not share this property.

Proposition 4.4. *Let G be a free group of countable rank. Then there exists an endomorphism $\phi : G \to G$ such that:*

(1) ϕ acts as the right Bernoulli shift on a subgroup T of countable rank of G (hence $\widetilde{\mathrm{ent}}(\phi \restriction T) = \infty$, and consequently $\widetilde{\mathrm{ent}}(\phi) = \infty$);

(2) $\bar{\mathrm{ent}}(\phi) = 0$.

Proof. Let $G = \oplus_{n\ge0}\, e_n\mathbb{Z}$ and let T be the subgroup of G defined as follows:

$$T = \oplus_{n\ge0}\, n!\, e_n\mathbb{Z}.$$

Consider the endomorphism $\phi : G \to G$ defined by setting

$$\phi(e_n) = (n + 1)e_{n+1} \qquad \text{for all} \quad n.$$

Since

$$\phi(n!\, e_n) = n!(n + 1)e_{n+1} = (n + 1)!\, e_{n+1},$$

ϕ acts on T as the right Bernoulli shift. Therefore $\widetilde{\mathrm{ent}}(\phi \restriction T) = \infty$ (see step (V) in Section 3 of the quoted paper). In order to show that $\bar{\mathrm{ent}}(\phi) = 0$, we must prove that, given any fully inert subgroup H of G, $\widetilde{\mathrm{ent}}(\phi, H) = 0$. From [DSZ] we know that H is commensurable with kG for some integer k; this implies that $H \leq K$, where $K = (\oplus_{n \leq r} e_n\mathbb{Z}) \oplus (\oplus_{n>r} ke_n\mathbb{Z})$ for a suitable index r. It is enough now to prove that $\widetilde{\mathrm{ent}}(\phi, K) = 0$. But for m large enough we have that

$$\phi^m(H) \leq H$$

hence the conclusion $\widetilde{\mathrm{ent}}(\phi, H) = 0$ immediately follows.

□

5 Endomorphisms whose intrinsic and fully inert algebraic entropies coincide

We adopt now the classical point of view (see for instance Chapter 12 in [17]) of looking at a group G with an endomorphism $\phi : G \to G$ as a $\mathbb{Z}[X]$-module, with the action of X induced by ϕ, that is, given a polynomial $f(X) \in \mathbb{Z}[X]$, $f(X) \cdot g = f(\phi)(g)$, for every $g \in G$. The connection of this point of view with algebraic entropy was developed in [19] in the more general setting of arbitrary modules. In accordance with the notation in that paper, we shall denote by G_ϕ the group G endowed with the $\mathbb{Z}[X]$-module structure induced by ϕ; abusing notation, if H is a ϕ-invariant subgroup of G, we shall denote it by H_ϕ when viewed as submodule of G_ϕ. Recall that G_ϕ is cyclic (respectively, finitely generated), if $G = T(\phi, F)$ is the trajectory of a cyclic (respectively, finitely generated) subgroup F.

If G_ϕ is finitely generated, its submodule $t(G)_\phi$ is also finitely generated, hence $t(G)$ is a bounded group; consequently $G = t(G) \oplus K$, where K is a torsion-free summand of G, not ϕ-invariant in general (see Lemma 5.7 in [6]).

The next theorem shows that the endomorphisms $\phi : G \to G$ such that G_ϕ is a finitely generated $\mathbb{Z}[X]$-module satisfy $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$. It is worthwhile remarking that this result is independent of Theorem 4.2; in fact, there exist finitely generated $\mathbb{Z}[X]$-modules of infinite rank (e.g., $\mathbb{Z}_\beta^{(\mathbb{N})}$, where β is the right Bernoulli shift), and, conversely, there exist endomorphisms $\phi : G \to G$ of groups G of finite rank such that G_ϕ is not finitely generated (e.g., any non-zero endomorphism of $\mathbb{Q}$).

Theorem 5.1. *Let G be a group and $\phi : G \to G$ an endomorphism. If G_ϕ is finitely generated, then $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$.*

Proof. We may assume G is of infinite rank, otherwise the conclusion follows from Theorem 4.2. First assume that G is a torsion-free group. The hypothesis that G_ϕ is finitely generated ensures that G is the sum of finitely many cyclic ϕ-trajectories $T_i = T(\phi, x_i)$ $(i = 1, 2, \cdots, n)$, where at least one of them, say T_1, has infinite rank. We can write $G = T_1 + T$ where T is a finitely generated submodule of G_ϕ. Proposition 5.6 in [6] ensures that $\widetilde{\mathrm{ent}}(\phi) = \infty = \widetilde{\mathrm{ent}}(\phi \restriction_{T_1})$; thus our goal is to prove that $\bar{\mathrm{ent}}(\phi) = \infty$.

Note that $(T_1)_\phi \cong \mathbb{Z}[X]$ and T_1 is a free group of countable rank, on which ϕ acts as the right Bernoulli shift. Thus Example 3.7 in [6] shows that, for all integers $k > 1$, the subgroups of T_1

$$H_k =< x_1 > +kT_1 =< x_1 > \oplus k\phi(T_1)$$

satisfy $\widetilde{\mathrm{ent}}(\phi, H_k) = \log k$ and are fully inert in T_1 as they are commensurable with kT_1.

Let T_1^* be the purification of T_1 in G. By Proposition 2.5 of [6], T_1^* is ϕ-invariant in G, so $(T_1^*/T_1)_\phi$ is a finitely generated module and a torsion group. From this it follows that T_1^*/T_1 is a bounded group, so there exists a minimal positive integer k_0 such that $k_0 T_1^* \leq T_1$. Furthermore, for each integer k the equality $kG \cap T_1^* = kT_1^*$ holds, by the purity of T_1^*. Consequently, for each integer k multiple of k_0, we have:

$$(**) \qquad kT_1 \leq kG \cap T_1 \leq kG \cap T_1^* = kT_1^* \leq \frac{k}{k_0} T_1.$$

The subgroup $A_k =< x_1 > +kG$ is a fully inert subgroup of G, being commensurable with kG. We claim that, for all integers k, a multiple of k_0,

$$\widetilde{\mathrm{ent}}(\phi, A_k) \geq \log \frac{k}{k_0}$$

from which the desired equality $\bar{\mathrm{ent}}(\phi) = \infty$ obviously follows. The following equalities clearly hold:

$$A_k \cap T_1 = (< x_1 > +kG) \cap T_1 =< x_1 > +(kG \cap T_1) =< x_1 > \oplus C_1$$

where $C_1 \leq T_1$, and the inclusions $(**)$ imply that $H_k \leq A_k \cap T_1 \leq H_{k/k_0}$

For every positive integer n and for every k a multiple of k_0 we have:

$$T_n(\phi, H_k) = (\oplus_{i \leq n} < x_i >) \oplus k\phi^n T_1$$

$$T_n(\phi, A_k \cap T_1) = (\oplus_{i \leq n} < x_i >) \oplus C_n$$

$$T_n(\phi, H_k) = (\oplus_{i \leq n} < x_i >) \oplus \frac{k}{k_0} \phi^n T_1$$

where C_n is a suitable summand of T_1 satisfying the inequalities

$$k\phi^n T_1 \leq C_n \leq \frac{k}{k_0} \phi^n T_1.$$

Now the factor group

$$\frac{T_n(\phi, A_k \cap T_1)}{A_k \cap T_1} = \frac{(\oplus_{i \leq n} < x_i >) \oplus C_n}{< x_1 > \oplus C_1}$$

has as epimorphic image the factor group

$$\frac{(\oplus_{2 \leq i \leq n} < x_i >) \oplus C_n}{(\oplus_{2 \leq i \leq n} < \frac{k}{k_0} x_i >) \oplus C_n}$$

whose cardinality is $(\frac{k}{k_0})^{n-1}$. Taking logarithms, dividing by n and computing the limit for $n \to \infty$, we get $\widetilde{\mathrm{ent}}(\phi, A_k \cap T_1) \geq \log \frac{k}{k_0}$.

But Lemma 3.10 (b) of [6] applied to the ϕ-invariant subgroup T_1 implies that

$$\widetilde{\mathrm{ent}}(\phi, A_k) \geq \widetilde{\mathrm{ent}}(\phi \restriction_{T_1}, A_k \cap T_1),$$

so we are done.

It remains to consider the case of G a mixed group; as G_ϕ is finitely generated, $G = B \oplus K$ for a bounded group B and a torsion-free group K (see Lemma 5.7 of [6]). The group K has infinite rank, as we are assuming G of infinite rank. The map ϕ induces a map $\bar{\phi} : G/B \to G/B$; since $G/B \cong K$, G/B contains, as we have proved above, a fully inert subgroup $\bar{A}_k$ for every multiple k of a fixed integer k_0, such that

$$\widetilde{\mathrm{ent}}(\bar{\phi}, \bar{A}_k) \geq \log \frac{k}{k_0}.$$

Now $\bar{A}_k = (B \oplus A_k)/B$ for a suitable subgroup A_k of K, and $B \oplus A_k$ is fully inert in G, by Lemma 4.1. By Lemma 3.10 of [6],

$$\widetilde{\mathrm{ent}}(\phi, B \oplus A_k) \geq \widetilde{\mathrm{ent}}(\bar{\phi}, \bar{A}_k),$$

therefore, being k arbitrary large, we deduce that $\bar{\mathrm{ent}}(\phi) = \infty$. □

It is worth remarking on the difference between Theorem 4.2 and Theorem 5.1. The first one says that for all the endomorphisms ϕ of a group of finite rank G the equality $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$ holds; the latter result says that all groups G which are finitely generated $\mathbb{Z}[X]$-modules with the module structure induced by an endomorphism ϕ satisfy the above equality.

In both cases, we have two Serre classes of modules: in the first case, the Serre class of groups of finite rank (which includes all torsion groups); in the latter case, the Serre class of finitely generated $\mathbb{Z}[X]$-modules. In order to enlarge the classes of groups for which the above equality of entropies holds, one could be tempted to put together the two Serre classes above using extensions. The next example shows that one kind of extension is not possible.

Example 5.2. Extension of a cyclic $\mathbb{Z}[X]$-module by a finite rank group, with an endomorphism ϕ such that $\widetilde{\mathrm{ent}}(\phi) = \infty$ and $\bar{\mathrm{ent}}(\phi) = 0$.

Consider the group $G = \mathbb{Q}^{(\mathbb{N})}$ endowed with the right Bernoulli shift β. We have seen in Example 3.1 that $\widetilde{\mathrm{ent}}(\beta) = \infty$ and $\bar{\mathrm{ent}}(\beta) = 0$. The subgroup $H = \mathbb{Z}^{(\mathbb{N})}$ is a cyclic $\mathbb{Z}[X]$-submodule of G_β; the cokernel G/H is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{(\mathbb{N})}$, which has rank zero, being a torsion group. Note that the following equalities hold:

$$\widetilde{\mathrm{ent}}(\beta \restriction_H) = \bar{\mathrm{ent}}(\beta \restriction_H) = \infty,$$

$$\widetilde{\mathrm{ent}}(\bar{\beta}) = \bar{\mathrm{ent}}(\bar{\beta}) = \mathrm{ent}(\bar{\beta}) = \infty,$$

where, as usual, $\bar{\beta}$ is the endomorphism of G/H induced by β.

On the other hand, the latter kind of extension works, under the additional hypothesis that the subgroup of finite rank is not only a $\mathbb{Z}[X]$-submodule, but is also fully invariant.

Theorem 5.3. *Let H be a fully invariant subgroup of finite rank of a group G. Let $\phi : G \to G$ be an endomorphism such that G_ϕ/H_ϕ is finitely generated; then $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$.*

Proof. We can assume that G/H has infinite rank, by Theorem 4.2. Under our assumptions, as we have seen in the proof of Theorem 5.1, G/H contains a free subgroup of countable rank on which the map $\bar{\phi}$ induced by ϕ acts as the right Bernoulli shift; therefore $\widetilde{\mathrm{ent}}(\bar{\phi}) = \bar{\mathrm{ent}}(\bar{\phi}) = \infty$. Consequently $\widetilde{\mathrm{ent}}(\phi) = \infty$ and we must check that $\bar{\mathrm{ent}}(\phi) = \infty$. Again by the proof of Theorem 5.1, for every positive integer k multiple of a fixed integer k_0, the factor group G/H contains a fully inert subgroup $A_k = G_k/H$ such that $\widetilde{\mathrm{ent}}(\bar{\phi}, A_k) \geq \log \frac{k}{k_0}$. The hypothesis that H is fully invariant implies, as is easily checked, that G_k is fully inert in G. Now Lemma 3.10 of [6] shows that $\widetilde{\mathrm{ent}}(\phi, G_k) \geq \widetilde{\mathrm{ent}}(\bar{\phi}, A_k) \geq \log \frac{k}{k_0}$, therefore $\bar{\mathrm{ent}}(\phi) = \infty$, as desired. □

In view of Theorem 5.3, we are led to consider the class of $\mathbb{Z}[X]$-modules G_ϕ containing a fully invariant subgroup H of finite rank such that the factor module G_ϕ/H_ϕ is finitely generated. This class obviously contains the Serre class of the groups of finite rank, as well as the Serre class of the finitely generated $\mathbb{Z}[X]$-modules. Both these inclusions are proper, as the following example shows.

Example 5.4. Consider the group $G = \mathbb{Q} \oplus \mathbb{Z}^{(\mathbb{N})}$ and its endomorphism ϕ, which acts as the right Bernoulli shift on $\mathbb{Z}^{(\mathbb{N})}$, and as the identity map $1_\mathbb{Q}$ on $\mathbb{Q}$. Obviously G has infinite rank, and G_ϕ is not finitely generated, because $\mathbb{Q}_\phi$ is not such. On the other hand, $\mathbb{Q}$ is a fully invariant subgroup of G of rank 1, and $G_\phi/\mathbb{Q}_\phi$ is finitely generated.

6 Groups of uncountable rank all of whose endomorphisms have intrinsic and fully inert algebraic entropies coinciding

In this section we start providing some examples of endomorphisms of groups of infinite rank, which show that different behaviours can arise when the intrinsic and fully inert algebraic entropies of an endomorphism are compared.

The first example shows that there exist arbitrarily large torsion-free groups all of whose subgroups are fully invariant, hence fully inert, and consequently all their endomorphisms have intrinsic algebraic entropy zero.

Example 6.1. It is well known (see Theorem 3.5 in [EM]) that there exist torsion-free groups G of arbitrarily large rank such that $\mathrm{End}(G) = \mathbb{Z}$, so every endomorphism of G is the multiplication by an integer. Consequently, all subgroups of G are fully invariant, hence also fully inert. It follows that, given any $\phi \in \mathrm{End}(G)$, $\widetilde{\mathrm{ent}}(\phi) = 0 = \bar{\mathrm{ent}}(\phi)$.

We have seen in Example 3.1 that the right Bernoulli shift β on $\mathbb{Q}^{(\mathbb{N})}$ satisfies $\bar{\mathrm{ent}}(\beta) = 0$ and $\widetilde{\mathrm{ent}}(\beta) = \infty$. The next example improves that result, showing that an endomorphism can have finite intrinsic algebraic entropy, different from its fully inert algebraic entropy.

Example 6.2. Let $G = \oplus_n D_n$, where $D_n \cong \mathbb{Q}$ for every n, be a torsion-free divisible group of countable rank. We have seen that $\bar{\mathrm{ent}}(\phi) = 0$ for all endomorphisms ϕ of G. For every n choose a rational number a_n/b_n, with $(a_n, b_n) = 1$, and let $\psi_n : D_n \to D_n$ be the endomorphism given by the multiplication by a_n/b_n. Let $\psi : G \to G$ be the endomorphism $\psi = \oplus_n \psi_n$. Obviously, each summand D_n is ψ-invariant in G, so, by step (II) in Section 3, $\widetilde{\mathrm{ent}}(\psi) = \sum_n \widetilde{\mathrm{ent}}(\psi_n)$. But $\widetilde{\mathrm{ent}}(\psi_n) = \log b_n$, by [6], therefore, for suitable choices of the rational numbers a_n/b_n, we can get $\widetilde{\mathrm{ent}}(\psi)$ to be an arbitrary value $\log k$ (for an integer $k \geq 1$), or ∞.

The main goal in this section is to show that the class of the $\mathbb{Z}[X]$-modules G_ϕ containing a fully invariant subgroup H of finite rank such that the factor module G_ϕ/H_ϕ is finitely generated, does not contain all the $\mathbb{Z}[X]$-modules G_ϕ such that $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$. In fact, the groups in this class are necessarily countable modulo their torsion subgroups, while the groups constructed below, endowed with the $\mathbb{Z}[X]$-module structure induced by any endomorphism ϕ, have uncountable rank.

We start with the construction, based on an idea of Corner, of a family of mixed groups. A similar construction has been utilized previously in the construction of certain mixed groups which are Hopfian – see [13] and [14].

Suppose that T is an arbitrary unbounded semi-standard reduced p-group and let $A = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$. Then A is the cotorsion-completion of T and $A/T \cong \mathrm{Ext}(\mathbb{Q}, T)$ is torsion-free divisible. Then, as shown in [13], A/T is torsion-free divisible of rank $2^{\aleph_0}$. Note that if $\phi : A \to T$ is any homomorphism, then the image $\phi(A)$ is both cotorsion and torsion, and hence is bounded [11, Corollary 54.4].

Let H be a maximal proper pure subgroup of J_p containing $\mathbb{Z}_p$, then H has cardinality $2^{\aleph_0}$ and $J_p/H \cong \mathbb{Q}$. Note that there are $2^{2^{\aleph_0}}$ non-isomorphic such maximal pure subgroups. Since every endomorphism of H extends to an endomorphism of J_p, it must be multiplication by a p-adic integer. Moreover, this multiplication must induce an endomorphism on the quotient $J_p/H \cong \mathbb{Q}$, and so it must be both a p-adic integer and a rational integer i.e. it is in $\mathbb{Z}_p$. Conversely since J_p is q-divisible for all primes $q \neq p$, any multiplication by an element of $\mathbb{Z}_p$ is an endomorphism of H. Thus we have a pure subgroup H of the group of p-adic integers J_p such that H contains the subgroup $\mathbb{Z}_p$ of integers localized at p and $\mathrm{End}(H) = \mathbb{Z}_p$. Moreover, as H has rank $2^{\aleph_0}$, we have that $H/\mathbb{Z}_p$ is torsion-free divisible of rank $2^{\aleph_0}$.

The groups A/T and $H/\mathbb{Z}_p$ are isomorphic, fixing such an isomorphism we form the pullback of A and H with kernels T and $\mathbb{Z}_p$. The resulting group G is a subgroup of the direct sum $A \oplus H$ and satisfies

$$G/T \cong H,\ G/\mathbb{Z}_p \cong A.$$

Since G/T is torsion-free, T is the torsion subgroup of G. Note that G/T is *reduced* in this case. Let ϵ be an arbitrary endomorphism of G, then ϵ induces a mapping $\bar{\epsilon}$ of $G/T \cong H$ and so $\bar{\epsilon}$ is multiplication by a rational n/m where n, m are coprime and $p \nmid m$. Let $\psi = m\epsilon$. Thus, $\psi - n1_G$ induces the zero map on G/T and so $(\psi - n1_G)(G) \leq T$, in particular $(\psi - n1_G)(\mathbb{Z}_p) \leq T$. However, every homomorphic image of $\mathbb{Z}_p$ in T is cyclic, and so bounded, so there exists an integer $r \geq 0$ such that $p^r(\psi - n1_G)(\mathbb{Z}_p) = 0$. Thus, the endomorphism $p^r(\psi - n1_G)$ of G annihilates $\mathbb{Z}_p$ and so passes to the quotient inducing a map: $G/\mathbb{Z}_p \cong A \to T$; as noted above the choice of A means this image is also bounded. So replacing r by a larger integer if necessary, we may suppose that this image is zero i.e. $p^r(\psi - n1_G) = 0$.

Thus $\psi - n1_G$ is bounded. Now $\psi = m\epsilon$, where m is an integer relatively prime to p. Since T is a p-group, m acts as an automorphism of T; furthermore, multiplication by m is an automorphism of H by construction of H. Hence by the 5-Lemma, multiplication by m is an automorphism of G and so we can consider multiplication by n/m as an endomorphism of G. Thus ϵ can be regarded as an element of $\mathbb{Z}_p 1_G \oplus Bd(G)$. Since the latter are always endomorphisms of G in this situation, we have shown:

Theorem 6.3. *If T is an arbitrary semi-standard unbounded reduced p-group, then there is a mixed group G, which satisfies*

(1) $T = t(G)$ is the torsion subgroup of G;

(2) the quotient G/T is isomorphic to a pure subgroup H, of cardinality $2^{\aleph_0}$, of the group of the p-adic integers J_p (and, in particular, is reduced of uncountable rank);

(3) $\mathrm{End}_{\mathbb{Z}}(G) = \mathbb{Z}_p 1_G \oplus Bd(G)$, where $Bd(G)$ denotes the two-sided ideal of the endomorphisms of bounded image.

Now let G be a group constructed as in Theorem 6.3; such groups are called *Corner mixed groups* in [14]. We claim that $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$ for all $\phi \in \mathrm{End}_{\mathbb{Z}}(G)$. Assume first that $\phi = r + \sigma$ where r is an integer and σ is a bounded endomorphism. By the Addition Theorem we have:

$$\widetilde{\mathrm{ent}}(\phi) = \widetilde{\mathrm{ent}}(\phi\restriction_{t(G)}) + \widetilde{\mathrm{ent}}(\bar{\phi})$$

where $\widetilde{\mathrm{ent}}(\phi\restriction_{t(G)}) = \mathrm{ent}(\phi) = 0$, because $t(G)$ is semi-standard and σ is bounded (see [8]). Since $\bar{\phi}$ coincides with the multiplication by r, also $\widetilde{\mathrm{ent}}(\bar{\phi}) = 0$; therefore $\widetilde{\mathrm{ent}}(\phi) = 0$, consequently also $\bar{\mathrm{ent}}(\phi) = 0$ and the equality $\widetilde{\mathrm{ent}}(\phi) = \bar{\mathrm{ent}}(\phi)$ holds.

Assume now that $\phi = \frac{r}{s} + \sigma$, where r, s are coprime integers, $s > 1$, $(s, p) = 1$, and σ is a bounded endomorphism. Let us denote, as usual, by $\bar{\phi}$ the map induced by ϕ on the factor group $G/t(G) = H$; note that $\bar{\phi} = \frac{r}{s} \cdot 1_H$. As before we have $\widetilde{\text{ent}}(\phi) = \widetilde{\text{ent}}(\phi \restriction_{t(G)}) + \widetilde{\text{ent}}(\bar{\phi})$, where $\widetilde{\text{ent}}(\phi \restriction_{t(G)}) = \text{ent}(\phi) = 0$.

Now H contains free subgroups F of arbitrarily large finite rank k; recalling that $\bar{\phi} = \frac{r}{s} \cdot 1_H$, an easy computation shows that, for all $n \geq 1$, $|T_n(\bar{\phi}, F)/F| = s^{(n-1)k}$. In particular, F is $\bar{\phi}$-inert in H, and, by Lemma 4.1, $t(G) \oplus F$ is ϕ-inert in G. Finally we have:

$$\widetilde{\text{ent}}(\phi, t(G) \oplus F) \geq \widetilde{\text{ent}}(\bar{\phi}, F) = k \cdot \log s;$$

as k is an arbitrary positive integer and $s > 1$, we deduce that $\widetilde{\text{ent}}(\phi) = \infty$.

In order to prove that also $\bar{\text{ent}}(\phi) = \infty$, we can use the preceding argument, but proving that $t(G) \oplus F$ is fully inert in G or, equivalently, that F is fully inert in H. But we can apply the same argument used for $\phi = \frac{r}{s} + \sigma$ to an arbitrary endomorphism $\psi = \frac{a}{b} + \tau$, where a, b are coprime integers, $(b, p) = 1$, and τ is a bounded endomorphism of G, showing that $t(G) \oplus F$ is fully inert in G for an arbitrary free subgroup F of finite rank of H (independently of the fact that $b > 1$).

Thus we have established:

Theorem 6.4. *If ϕ is an arbitrary endomorphism of a Corner mixed group, then $\widetilde{\text{ent}}(\phi) = \bar{\text{ent}}(\phi)$.*

Remark 6.5. The Corner mixed groups constructed in Theorem 6.3 provide a positive answer to the following question posed by Breaz in [2]: does a mixed group of infinite rank G exist, such that $\text{End}_{\mathbb{Z}}(G)$ has finite rank and some primary component of G is unbounded? Notice that the groups G in Theorem 6.3 have as torsion subgroup $t(G) = G_p$ an arbitrary semi-standard unbounded p-group, and that $\text{End}_{\mathbb{Z}}(G)$ has rank one, since $Bd(G)$ is its torsion part.

The following question remains open: are there satisfactory characterizations of the groups G such that $\widetilde{\text{ent}}(\phi) = \bar{\text{ent}}(\phi)$ for all $\phi \in \text{End}_{\mathbb{Z}}(G)$, and of the $\mathbb{Z}[X]$-modules G_ϕ such that $\widetilde{\text{ent}}(\phi) = \bar{\text{ent}}(\phi)$.

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Address of the authors:

B. Goldsmith, Dublin Institute of Technology, Aungier Street, Dublin 2, Ireland - brendan.goldsmith@dit.ie

L. Salce, Dipartimento di Matematica, Via Trieste 63, 35121 Padova, Italy - salce@math.unipd.it