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Algebraic Entropies, Hopficity and co-Hopficity of Direct Sums of Abelian Groups

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Abstract: Necessary and sufficient conditions to ensure that the direct sum of two Abelian groups with zero entropy is again of zero entropy are still unknown; interestingly the same problem is also unresolved for direct sums of Hopfian and co-Hopfian groups. We obtain sufficient conditions in some situations by placing restrictions on the homomorphisms between the groups. There are clear similarities between the various cases but there is not a simple duality involved.

Keywords: algebraic entropy; adjoint entropy; direct sums; matrix of endomorphisms; Abelian groups

MSC: Primary 20K30. Secondary 20K25, 37A35

1 Introduction

All groups mentioned in this paper are Abelian groups and are written additively, unless otherwise specified.

The concept of entropy was first introduced to Abelian groups in 1965 in the paper [1]. In that paper, the main idea was sketched in the final section. Later on, Weiss [21] continued the study of entropy in groups and established many basic properties of such an entropy. In 1979 Peters [16] defined a different kind of entropy for automorphisms of discrete groups by taking finite subsets instead of finite subgroups; for torsion groups the resulting entropy is the same as that outlined in [1].

In recent years various "algebraic" entropies in groups have been studied intensively. D. Dikranjan *et al* in [7] developed algebraic entropy based on Weiss's definition. In that paper many basic results, including the important Addition Theorem and the Poincaré-Birkhoff recurrence theorem for algebraic entropy, were established. The note following the recurrence theorem [7, Proposition 2.9] is of interest; it states that a group of zero algebraic entropy is necessarily co-Hopfian. This is an important connection between entropy and (co-)Hopficity. Later on, D. Dikranjan, A. Giordano and L. Salce [6], and B. Goldsmith and K. Gong [9] introduced another kind of entropy in groups; they call such entropy, the adjoint (algebraic) entropy. In these papers finite index subgroups were used instead of finite subgroups, and pre-images instead of images. It is interesting to recall an observation in [9, Corollary 2.22]: a reduced torsion-free group with zero adjoint entropy is necessarily Hopfian; this corollary connects the notions of adjoint entropy and Hopficity in a similar fashion to the connection between algebraic entropy and co-Hopficity.

Various developments of "algebraic" entropies were made meanwhile; for instance, in [19], groups with zero adjoint algebraic entropy were studied in full, while entropies for modules over various classes of rings, have been considered in [22], [18].

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As mentioned above, algebraic entropy and adjoint algebraic entropy have close connections with the concepts of co-Hopficity and Hopficity. Recall the definitions of Hopficity and co-Hopficity: a group is called Hopfian if every surjective endomorphism is an automorphism, and co-Hopfian if every injective endomorphism is an automorphism; or equivalently, a group is called Hopfian it has no proper isomorphic quotient groups and co-Hopfian if it has no proper isomorphic subgroups. These notions were introduced by Baer [2] who used the terminology Q-groups and S-groups, where Q-groups were exactly the Hopfian groups and S-groups were co-Hopfian groups. It should be noted that such properties are defined for arbitrary (non-commutative) groups and it was in that context that Baer initiated their study.

An immediate question which arises is the easily posed, but not so easily answered, "Is the direct sum of two (co-)Hopfian groups again (co-)Hopfian?". This question for Hopfian groups was mentioned in [3] and was immediately answered for Abelian groups by A. L. S. Corner [4], who constructed two torsion-free Abelian Hopfian groups which have non-Hopfian direct sum. Recently Vámos and the first author [11] have shown that for any positive integer n, there exists a torsion-free Hopfian group G such that the direct sum of n copies of G is Hopfian but the direct sum of $n + 1$ copies is not.

In the context of arbitrary groups Hirshon [14] has established many interesting results which guarantee the Hopficity of a direct product of groups: for example, if every proper homomorphic image of A is Abelian and B is a group which satisfies the ascending chain condition for normal subgroups, then $A \times B$ is Hopfian. For arbitrary co-Hopfian groups, Li [15] has established that the direct product of two such groups A, B is again co-Hopfian if the pair is semi-rigid in the sense that either there are no non-trivial homomorphisms from A to B or from B to A. In Section 4 we shall give simple matrix-based proofs for some analogous results in the Abelian context.

Due to the close connections between (co-)Hopficity and the algebraic entropy and adjoint algebraic entropy, it is natural to ask if similar "closure" properties hold for direct sums of Abelian groups with zero (adjoint) algebraic entropy. We shall address such questions in Section 3.

We finish this introduction by noting that our notation is standard and follows that in Fuchs [8].

2 Definitions and Basic results

In this section, we recall the definitions of algebraic entropy and adjoint entropy of groups, and list some useful properties of such entropies. Some basic propositions and auxiliary lemmas are then stated for later use in Section 3. Their proofs can be derived easily from the references given, so proofs will be omitted or only an outline sketch will be provided.

Recall the definition of algebraic entropy, see [7]: let G be a group, ϕ an endomorphism of G, then for every positive integer n and every finite subgroup F of G, set

$$T_n(\phi, F) = F + \phi F + \phi^2 F + \cdots + \phi^{n-1} F.$$

The $T_n(\phi, F)$ is called the n^{th}-trajectory of F respect to ϕ, and the subgroup $T(\phi, F) = \sum_{n\geq 0} \phi^n F$ is called the trajectory of F respect to ϕ. Denote the cardinality of $T_n(\phi, F)$ by $|T_n(\phi, F)|$; it was shown in [7] that the following limit does exist

$$\lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n},$$

and we denote it by $H(\phi, F)$. Denote the family of all finite subgroups of G as $\mathcal{F}(G)$, then the algebraic entropy of ϕ is defined as

$$\text{ent}(\phi) = \sup\{H(\phi, F) \mid F \in \mathcal{F}(G)\}$$

and the algebraic entropy of the group G is defined as

$$\text{ent}(G) = \sup\{\text{ent}(\phi) \mid \phi \in \text{End}(G)\}.$$

We mention the following proposition, which appears as [7, Proposition 1.18]; this result shows that, in some case, it suffices to check the algebraic entropy of an endomorphism restricted on the socle:

Proposition 2.1. *(Proposition 1.18, [7]). Let G be a p-group and $G[p]$ its socle. If ϕ is an endomorphism of G, then* $\mathrm{ent}(\phi) > 0$ *implies* $\mathrm{ent}(\phi \restriction G[p]) > 0$, *where $\phi \restriction G[p]$ denotes the restriction of ϕ to $G[p]$.*

This fact together with the so-called Addition Theorem, see [7, Theorem 3.1] and the fact that the socle of a group is fully invariant give the first part of the following useful corollary:

Corollary 2.2. *Let G be a p-group and ϕ an endomorphism of G, then* $\mathrm{ent}(\phi) = 0$ *if and only if* $\mathrm{ent}(\phi \restriction G[p]) = 0$*; moreover,* $\mathrm{ent}(\phi) = 0$ *if and only if the trajectory $T(\phi, F)$ is finite for every finite subsocle F.*

Proof. The last part of the corollary is contained in [7, Proposition 1.3]. □

We now consider adjoint entropy. Recall the definition, see, for example, [6, 9]: let G be a group and N a finite index subgroup of G, ϕ an endomorphism of G, for every fixed natural number n set

$$C_n(\phi, N) = N \cap \phi^{-1}N \cap \phi^{-2}N \cap \cdots \cap \phi^{-(n-1)}N.$$

It is pointed out in papers [6, 9] that $C_n(\phi, N)$ is a finite index subgroup in G. $C_n(\phi, N)$ is called the n^{th}-co-trajectory of N with respect to ϕ, and the subgroup $C(\phi, N) = \cap_{n\geq 0}\phi^{-n}N$ is called the co-trajectory of N with respect to ϕ. Denote $\log|G/C_n(\phi, N)|$ by $I_n(\phi, N)$, then the following limit exists as shown in, for instance, [6, 9]:

$$\lim_{n\to\infty} \frac{I_n(\phi, N)}{n},$$

and we denote the above limit as $I(\phi, N)$. Let $\mathcal{N}(G)$ denote the family of all finite index subgroups of G. Then the adjoint entropy of ϕ is defined as

$$\mathrm{ent}^{\star}(\phi) = \sup\{I(\phi, N) \mid N \in \mathcal{N}(G)\},$$

and the adjoint entropy of G is defined as the supreme

$$\mathrm{ent}^{\star}(G) = \sup\{\mathrm{ent}^{\star}(\phi) \mid \phi \in \mathrm{End}(G)\}.$$

This entropy has many nice properties; here we mention a few which will be useful later. These properties are contained in [6, Corollary 7.7], but we prefer to re-state part of that corollary here for later use:

Proposition 2.3. *(Corollary 7.7, [6]). Let G be a group, ϕ an endomorphism of G and for each prime number p, let $\overline{\phi_p} : G/pG \longrightarrow G/pG$ be the homomorphism induced by ϕ; then* $\mathrm{ent}^{\star}(\phi) = 0$ *if and only if* $\mathrm{ent}^{\star}(\overline{\phi_p}) = 0$ *for every prime number p.*

From this result, the following corollary is immediate:

Corollary 2.4. *For a group A, if A/pA is finite for every prime number p, then* $\mathrm{ent}^{\star}(A) = 0$*. In particular, when A is torsion and each primary component is finite, then* $\mathrm{ent}^{\star}(A) = 0$*; and if A is torsion-free of finite rank, then* $\mathrm{ent}^{\star}(A) = 0$.

The following result is well known and will be used repeatedly in the sequel.

Theorem 2.5. *(Vol II, Sec.106, Theorem 106.1, [8]). Let A, B be two groups. Then* $\mathrm{End}(A \oplus B)$ *is isomorphic to the ring of all matrices*

$$\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$$

where $\alpha \in \mathrm{End}(A)$, $\gamma \in \mathrm{Hom}(B, A)$, $\delta \in \mathrm{Hom}(A, B)$ *and* $\beta \in \mathrm{End}(B)$.

This theorem guarantees that there is no confusion when we say a matrix of this kind is an endomorphism of a direct sum of two groups or vice versa.

The next fact is straightforward:

Lemma 2.6. *For groups A and B, and homomorphisms ϕ and ψ from A to B, if C is a subgroup of B, then* $(\phi+\psi)^{-1}C \geq \phi^{-1}C \cap \psi^{-1}C$.

The final result in this section concerns a fundamental property of the so-called *small* homomorphisms of p-groups; recall the definition: if A, C are p-groups, then a homomorphism $\phi : A \to C$ is said to be small if, given any positive integer e, there exists a positive integer m such that $\phi((p^m A)[p^e]) = 0$. The basic result that we shall need later is the well-known:

Proposition 2.7. *If $B = \bigoplus_{n=1}^{\infty} B_n$, where each B_n is a direct sum of cyclic groups of order p^n, is a basic subgroup of A and ϕ is a small homomorphism $\phi : A \to C$, then the image $\phi(A[p]) = \phi(B_1 \oplus B_2 \cdots \oplus B_m)$ for some integer m.*

Proof. We outline the standard proof. Since ϕ is small, there is an integer m such that $\phi((p^m A)[p]) = 0$. Now by [17, Lemma 16.3], one can write $A = (B_1 \oplus B_2 \cdots \oplus B_m) \oplus H_m$, where $H_m = p^m A + (B_{m+1} \oplus B_{m+2} \cdots \oplus B_k \oplus \dots)$. Hence any element of $A[p]$ is of the form $x+y$ where $x \in (B_1 \oplus B_2 \cdots \oplus B_m)[p]$ and $y \in H_m[p]$. A straightforward calculation shows that a socle element of H_m must have height (in A) at least p^m and hence $\phi(y) = 0$. So $\phi(A[p]) = \phi((B_1 \oplus B_2 \cdots \oplus B_m)[p])$, as required. □

3 Direct sums of groups with zero entropy

The principal objective in this section is to find reasonably general sufficient conditions that will ensure that the direct sum of two groups with zero entropy is again a group of zero entropy. We begin with the situation for p-groups of zero algebraic entropy.

Recall that a homomorphism ϕ from a group A to a group B is called *socle-finite* if the image of the socle of A is finite.

Theorem 3.1. *Let A, B be two p-groups with zero algebraic entropy, and suppose that all homomorphisms from A to B are socle-finite. Then the algebraic entropy of the direct sum of A and B is zero.*

Proof. Let Φ be an endomorphism of $A \oplus B$. Then Φ has a matrix representation $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{End}A$, $\gamma \in \mathrm{Hom}(B, A)$, $\delta \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{End}(B)$. Write $\Phi = \phi + \theta$ where $\phi = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ and $\theta = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right)$. Note that A is invariant under ϕ and so by the Addition Theorem [7, Theorem 3.1], $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction A) + \mathrm{ent}(\bar{\phi})$, where the latter is the induced mapping on the quotient $A \oplus B/A$. Since $\mathrm{ent}(A) = 0$, the first term in the equation above is zero; the second term is also zero since $\bar{\phi}$ is conjugate to an endomorphism of B and $\mathrm{ent}(B) = 0$.

Consider an arbitrary but fixed finite subgroup F of $(A \oplus B)[p]$. If we show that the trajectory of F with respect to Φ is finite, then $\Phi \restriction (A \oplus B)[p]$ has zero algebraic entropy and it will follow from Corollary 2.2 that Φ has zero algebraic entropy.

Since F is a finite subgroup of $(A \oplus B)[p]$, there are finite subgroups F_A, F_B of $A[p]$ and $B[p]$ respectively such that $F \leq F_A \oplus F_B$. Clearly it suffices to show that the trajectory of $F_A \oplus F_B$ with respect to Φ is finite. Note that since δ is socle-finite, $\theta(A[p] \oplus B[p]) = \delta(A[p])$ is finite; let L denote the group $\phi(F_A \oplus F_B)$ and K denote the fixed group $\delta(A[p])$. Clearly, $L + K \leq A[p] \oplus B[p]$.

Now $\Phi(F_A \oplus F_B) = (\phi+\theta)(F_A \oplus F_B) \leq \phi(F_A \oplus F_B) + \theta(F_A \oplus F_B) \leq L + K$; furthermore,

$$\begin{aligned} \Phi^2(F_A \oplus F_B) &\leq \Phi(L+K) \leq (\phi+\theta)(L+K) \leq \phi(L+K) + \theta(L+K) \\ &\leq \phi(L+K) + K \leq T(\phi, L+K). \end{aligned}$$

Similarly, we have $\Phi^3(F_A \oplus F_B) \leq \phi T(\phi, L+K) + \theta T(\phi, L+K) \leq T(\phi, L+K) + K \leq T(\phi, L+K)$. By induction we deduce that $\Phi^n(F_A \oplus F_B) \leq T(\phi, L+K)$. Now since we have already proved that $\mathrm{ent}(\phi) = 0$, thus, by Corollary 2.2, $T(\phi, L+K)$ is finite. Therefore, $T(\Phi, F_A \oplus F_B) \leq T(\phi, L+K)$ is also finite. Hence the proof is complete. □

Recall that a p-group is said to be *semi-standard* if all its finite Ulm invariants are finite. By [7, Proposition 4.1], a necessary condition for a group to have zero algebraic entropy is that it be semi-standard; in particular a group of zero algebraic entropy must have cardinality at most $2^{\aleph_0}$. If A is a group of zero algebraic entropy and thus semi-standard, every small homomorphism from A to any group B is socle-finite, by Proposition 2.7. From this we deduce immediately:

Corollary 3.2. *Let A, B be two p-groups with zero algebraic entropy. If all the homomorphisms from A to B are small, then the algebraic entropy of the direct sum of A and B is again zero. In particular, if B is finite, then the algebraic entropy of the direct sum of A and B is zero.*

Remark. We recall a well-known realization theorem of Corner [5, Theorem 1.1], which states that there is a family of $2^{2^{\aleph_0}}$ groups G_i such that every homomorphism between different groups of the family is *small*; furthermore, it is proved in [7, Thoerem 5.4] that there is a family of $2^{2^{\aleph_0}}$ groups G_i such that every group of the family has zero algebraic entropy and that each homomorphism between distinct members of the family is small. With these two facts and Corollary 3.2, one can deduce that there is a family of $2^{2^{\aleph_0}}$ groups G_i, each having zero algebraic entropy and such that the direct sum $G_i \oplus G_j$ $(i \neq j)$ again has zero algebraic entropy.

3.1 Adjoint entropy

The arguments in this section are in some sense dual to those for algebraic entropy but involve concepts that are not so well known in the literature. First we make a rather *ad hoc* definition: given a prime p, groups A, B and a homomorphism $\phi : B \to A$, we say that ϕ is *p-quotient finite* if the induced map $\bar{\phi}_p : B/pB \to A/pA$ has finite image. The mapping ϕ is said to be *quotient-finite* if it is p-quotient finite for all primes p. It is straightforward to show that ϕ is quotient-finite if $\phi^{-1}(pA)$ is of finite index in B for all primes p.

The main result in this subsection is:

Theorem 3.3. *Let A, B be two groups with zero adjoint entropy such that every homomorphism ϕ from B to A is quotient-finite, then the adjoint entropy of the direct sum $A \oplus B$ is again zero.*

Proof. Let Φ be an endomorphism of $A \oplus B$. According to Proposition 2.3, in order to prove that the adjoin entropy of Φ is zero, it suffices to show that the induced endomorphism $\overline{\Phi_p} : (A \oplus B)/p(A \oplus B) \longrightarrow (A \oplus B)/p(A \oplus B)$ has zero adjoint entropy for every prime number p. Note that $(A \oplus B)/p(A \oplus B) \cong (A/pA) \oplus (B/pB)$, it can be shown that $\overline{\Phi_p}$ is conjugate to the following endomorphism:

$$\Psi = \begin{pmatrix} \overline{\alpha_p} & \overline{\gamma_p} \\ \overline{\delta_p} & \overline{\beta_p} \end{pmatrix} : (A/pA) \oplus (B/pB) \longrightarrow (A/pA) \oplus (B/pB),$$

where the maps are those induced from $\alpha \in \mathrm{End}A$, $\gamma \in \mathrm{Hom}(B, A)$, $\delta \in \mathrm{Hom}(A, B)$. So $\mathrm{ent}^\star(\overline{\Phi_p}) = \mathrm{ent}^\star(\Psi)$ by [6, Lemma 4.3]; therefore, we only need to show that $\mathrm{ent}^\star(\Psi) = 0$. To reach this, we first note that since we have already assumed $\mathrm{ent}^\star(A) = \mathrm{ent}^\star(B) = 0$, so $\mathrm{ent}^\star(\alpha) = \mathrm{ent}^\star(\beta) = 0$ for every $\alpha \in \mathrm{End}(A)$, $\beta \in \mathrm{End}(B)$; by Proposition 2.3, $\mathrm{ent}^\star(\overline{\alpha_p}) = \mathrm{ent}^\star(\overline{\beta_p}) = 0$ for every prime number p. To make the symbols less tedious, in the rest of the proof, we fix an arbitrary prime number p and we simply write α for the $\overline{\alpha_p} : A/pA \longrightarrow A/pA$, γ for $\overline{\gamma_p}$, etc., also we simply write A, B for A/pA, B/pB respectively.

From Theorem 2.5, we write $\Psi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{End}(A)$, $\gamma \in \mathrm{Hom}(B, A)$, $\delta \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{End}(B)$. Write $\Psi = \phi + \theta$ where $\phi = \left(\begin{smallmatrix} \alpha & 0 \\ \delta & \beta \end{smallmatrix}\right)$ and $\theta = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)$. Now for every fixed finite index subgroup N of $A \oplus B$, set $N_A = N \cap A$, then N_A is a finite index subgroup of A; similarly $N_B = N \cap B$ a finite index subgroup of B. Note that $N_A \oplus N_B \leq N$ and that $N_A \oplus N_B$ is also of finite index in $A \oplus B$. We show that the cotrajectory of $N_A \oplus N_B$ respect to Ψ is stationary eventually, then $C(\Psi, N) \geq C(\Psi, N_A \oplus N_B)$ is also finite index in $A \oplus B$ and stationary eventually, and since N is arbitrarily chosen, then from [9, Lemma 2.7] and its proof, or from [6, Proposition 2.3] one can deduce that the adjoint entropy of Ψ is zero, and therefore the proof is complete. We first show that $\mathrm{ent}^\star(\overline{\phi_p}) = 0$. Note that by the assumption and analysis, $\mathrm{ent}^\star(\alpha) = \mathrm{ent}^\star(\beta) = 0$.

From [9, Proposition 2.18] and its proof or from [6, Proposition 2.3] one can deduce that $\mathrm{ent}^*(\phi) = 0$ and so the induced endomorphism $\overline{\phi_p}$ also has zero adjoint entropy by Proposition 2.3; hence the adjoint entropy of its conjugate $\begin{pmatrix} \overline{\alpha_p} & 0 \\ \delta_p & \beta_p \end{pmatrix}$ is also zero by [6, Lemma 4.3].

For simplicity, denote $N_A \oplus N_B$ by L. Then by Lemma 2.6, $\Psi^{-1}L = (\phi + \theta)^{-1}L \geq \phi^{-1}L \cap \theta^{-1}L$. Notice that $\theta^{-1}L \geq \mathrm{Ker}\theta$, and the latter is a finite index subgroup of $A \oplus B$: in fact, a standard verification shows that $\mathrm{Ker}\theta \geq A \oplus \mathrm{Ker}\gamma$. By the condition that γ is quotient-finite we deduce that the kernel of γ is a finite index subgroup of B, thus $A \oplus \mathrm{Ker}\gamma$ is of finite index in $A \oplus B$ and so is $\mathrm{Ker}\theta$. Notice that $\phi^{-1}L \geq C(\phi, L)$ and therefore, $\Psi^{-1}L \geq C(\phi, L) \cap \mathrm{Ker}\theta$. Furthermore, $\Psi^{-2}L \geq (\phi + \theta)^{-1}(C(\phi, L) \cap \mathrm{Ker}\theta)$, again by Lemma 2.6, and so we have

$$\begin{aligned}(\phi + \theta)^{-1}\Big(C(\phi, L) \cap \mathrm{Ker}\theta\Big) &\geq \phi^{-1}\Big(C(\phi, L) \cap \mathrm{Ker}\theta\Big) \cap \theta^{-1}\Big(C(\phi, L) \cap \mathrm{Ker}\theta\Big) \\ &\geq \phi^{-1}\Big(C(\phi, L) \cap \mathrm{Ker}\theta\Big) \cap \mathrm{Ker}\theta;\end{aligned}$$

the latter inclusion relationship holds since $\mathrm{Ker}\theta$ is always contained in a pre-image of a subgroup under θ; in other words, for every subgroup C of $A \oplus B$, $\mathrm{Ker}\theta \leq \theta^{-1}C$. Notice that $C(\phi, L) \cap \mathrm{Ker}\theta$ is a finite index subgroup of $A \oplus B$, we denote it by M. Moreover, since $\mathrm{ent}^*(\phi) = 0$, from [9, Lemma 2.7] and its proof, or from [6, Proposition 2.3], this gives that $C(\phi, M)$ is also a finite index subgroup of $A \oplus B$, hence $\Psi^{-2}L \geq \phi^{-1}M \cap \mathrm{Ker}\theta \geq C(\phi, M) \cap \mathrm{Ker}\theta$. Furthermore, notice that $\mathrm{Ker}\theta \geq M \geq C(\phi, M)$, thus, $\Psi^{-2}N \geq C(\phi, M)$. We continue in the same way and deduce:

$$\begin{aligned}\Psi^{-3}N &\geq (\phi + \theta)^{-1}(C(\phi, M)) \geq \phi^{-1}C(\phi, M) \cap \theta^{-1}C(\phi, M) \\ &\geq \phi^{-1}C(\phi, M) \cap \mathrm{Ker}\theta \geq C(\phi, M) \cap \mathrm{Ker}\theta \geq C(\phi, M).\end{aligned}$$

So in general, for each $n \geq 1$, $\Psi^{-n}L \geq C(\phi, M)$; thus, $C_n(\Psi, L) \geq N \cap C(\phi, M)$. In other words, the cotrajectory of Ψ with respect to $L = N_A + N_B$ is stationary eventually. This completes the proof. □

Corollary 3.4. *If A is a group of zero adjoint entropy and B is group such that B/pB is finite for all primes p, then $A \oplus B$ again has zero adjoint entropy. In particular, if B is finite or B is torsion-free of finite rank, then $A \oplus B$ has zero adjoint entropy.*

Proof. If B/pB is finite for all primes p, then it follows from Corollary 2.4 that $\mathrm{ent}^*(B) = 0$. The finiteness of B/pB for all primes p ensures that every map from $B \to A$ is quotient-finite. The particular cases follow easily since in each case it is clear that B/pB is finite for all primes p. □

We remark that the condition that B/pB be finite does not restrict one to groups of finite rank; for example, it is a well-known result of Griffith [13] that a torsion-free group B has the property that $r_p(B) = \dim_{\mathbb{Z}/p\mathbb{Z}}(B/pB) \leq 1$ if, and only if, B is isomorphic to a pure subgroup of the group $J = \prod_p J_p$.

4 Direct sums of (co-)Hopfian groups

In this section we investigate the (co-)Hopficity of a direct sum sum of certain (co-)Hopfian groups. The results are similar to those obtained in the previous section for groups of zero entropy, but we note that the results we obtain here are not immediate corollaries of those in the previous section. Some results in this section were previously obtained by Hirshon [14] and Li [15]. Here in this note proofs to these results are given in a systematic way by using representative of a homomorphism of matrix. This approach is transparent and natural.

We recall for later purposes that there is a well-known strong hypothesis which ensures (co-)Hopficity of a direct sum as the following proposition. It can be considered as a special case of [10, Proposition 2.3.]:

Proposition 4.1. *If A, B are (co-)Hopfian and either* $\mathrm{Hom}(A, B) = 0$ *or* $\mathrm{Hom}(B, A) = 0$, *then $G = A \oplus B$ is (co-)Hopfian.*

We begin by investigating sums of Hopfian groups; as heretofore the groups are additively written Abelian groups.

We first investigate the Hopficity of a direct sum of a Hopfian group and a cyclic p-group. The first observation is

Proposition 4.2. *If A, B are Hopfian groups and $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ represents a surjective endomorphism of $G = A \oplus B$, then ϕ is invertible if one of α, β is surjective.*

Proof. Without loss in generality assume α is surjective. Since A is a Hopfian group and α is epic on A, then α is invertible. Note that for any δ,

$$\begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \delta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that $\begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\delta \\ 0 & 1 \end{pmatrix}$ is invertible. Hence as a result, ϕ is invertible if and only if $\Delta = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\delta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & \beta-\gamma\alpha^{-1}\delta \end{pmatrix}$ is invertible; note that Δ is, of course, surjective.

To prove Δ is invertible, it suffices to show that $\beta - \gamma\alpha^{-1}\delta$ is surjective since the Hopficity of B would imply it is invertible. To see this, pick any element $b \in B$. By the surjectivity of Δ we have elements $x \in A, y \in B$ such that $\Delta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$. Thus we see the following identities

$$\begin{cases} 1(x) = 0, \\ \gamma\alpha^{-1}(x) + (\beta - \gamma\alpha^{-1}\delta)(y) = b. \end{cases}$$

Hence $x = 0$ and $(\beta - \gamma\alpha^{-1}\delta)(y) = b$. Thus $\beta - \gamma\alpha^{-1}\delta$ is surjective and invertible. Set $(\beta - \gamma\alpha^{-1}\delta)^{-1} = \epsilon$. By the following equations

$$\begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & \beta - \gamma\alpha^{-1}\delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\epsilon\gamma\alpha^{-1} & \epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ -\epsilon\gamma\alpha^{-1} & \epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma\alpha^{-1} & \beta - \gamma\alpha^{-1}\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that Δ is invertible and so is ϕ. □

For the later reference we list:

Corollary 4.3. *If A, B are Hopfian groups and $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ represents a surjective endomorphism of $G = A \oplus B$, then ϕ in invertible if one of δ, γ is zero.*

Proof. Without loss in generality it suffices to handle the case when $\delta = 0$. We claim that α is surjective and thus by Proposition 4.2 ϕ is invertible.

For any element $a \in A$, since ϕ is surjective, then there exists elements $x \in A, y \in B$ such that $\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$. That is

$$\begin{cases} \alpha(x) = a, \\ \gamma(x) + \beta(y) = b. \end{cases}$$

From the first equation and the arbitrariness of a, we see α is surjective. □

We can deduce the following:

Proposition 4.4. *(see [14]). Suppose that A is a Hopfian group and B is a cyclic p-group generated by the element b, then the direct sum $G = A \oplus B$ is also Hopfian.*

Proof. Let $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix}$ be an epimorphism of $G = A \oplus B$, where β is an integer. We divide the proof into two cases:

i) At least one of α, $\beta 1_B$ is surjective. In this case, by Proposition 4.2, ϕ is invertible.

ii) Both α, $\beta 1_B$ are not surjective. It is easy to see that p divides β. Since ϕ is surjective, then we have the necessary condition:

$$\alpha(A) + \delta(B) = A.$$

Since α is not surjective, we claim that $\delta(b) \notin \alpha(A)$. For if $\delta(b) \in \alpha(A)$, then, since $\delta(B)$ is generated by $\delta(b)$, $\delta(B) \le \alpha(A)$ forcing $\alpha(A) = A$, contrary to the fact that α is not surjective. Thus we can write $\alpha(A) \cap \delta(B) = \langle p^s \delta(b) \rangle$, $s \ge 1$.

Since $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ is invertible, then clearly $\Delta = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} \alpha+\delta\gamma & \delta \\ \gamma+\beta 1_B \gamma & \beta 1_B \end{pmatrix} = \begin{pmatrix} \alpha+\delta\gamma & \delta \\ \gamma+\beta\gamma & \beta 1_B \end{pmatrix}$ is surjective. Thus, we have the following necessary condition:

$$(\alpha + \delta\gamma)(A) + \delta(B) = A. \tag{1}$$

Since ϕ is surjective, there exist elements $x \in A$, $\lambda b \in B$ such that $\phi \begin{pmatrix} x \\ \lambda b \end{pmatrix} = \begin{pmatrix} \delta(b) \\ b \end{pmatrix}$. Hence we have the following

$$\begin{cases} \alpha(x) = (1 - \lambda)\delta(b), \\ \gamma(x) = (1 - \beta\lambda)b. \end{cases}$$

By the condition $\alpha(A) \cap \delta(B) = \langle p^s \delta(b) \rangle$, p divides $1 - \lambda$. From a straightforward computation we see $(\alpha + \delta\gamma)(x) = \{(1 - \lambda) - \beta\lambda + 1\}\delta(b)$. On the other hand, p divides $1 - \lambda$ and p divides β, thus $(1 - \lambda) - \beta\lambda + 1$ is relatively prime to p, and hence is an automorphism of the cyclic p-group $\alpha(A) \cap \delta(B)$. Hence $\delta(b) \in (\alpha + \delta\gamma)(A)$ and by the necessary condition (1), $\alpha + \delta\gamma$ is a surjection. Thus it is invertible. Hence Δ is invertible by Proposition 4.2, and so is $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix}$. □

By induction, we deduce the following

Theorem 4.5. *(see [14]). Suppose that A is a Hopfian group and B is a finite group, then the direct sum $G = A \oplus B$ is also Hopfian.*

We can extend Theorem 4.5, assuming the group B finitely generated; it is worthwhile to note that finitely generated groups are Hopfian (this is true for finitely generated modules over any commutative ring, by a celebrated result by Vasconcelos, see [20].

Theorem 4.6. *(see [14]). If A is Hopfian, and B is finitely generated, then $A \oplus B$ is Hopfian.*

Proof. Clearly it suffices, using Theorem 4.5 and induction, to assume the $B = \langle b \rangle \cong \mathbb{Z}$.

As above we write the endomorphism of $A \oplus B$ as a matrix of endomorphism as follows: $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ where α is an endomorphism on A, δ a homomorphism from B to A, γ a homomorphism from A to B and β an endomorphism on B. To prove the theorem, we suppose ϕ is a surjection and $\gamma(A) \neq 0$; if $\gamma(A) = 0$, the result is immediate.

By the isomorphism theorem $A/\mathrm{Ker}\gamma \cong \gamma(A) \le B$ is an infinite cyclic group. Hence

$$A = \mathrm{Ker}\gamma \oplus \langle a \rangle$$

for some torsion-free element $a \in A$.

On the other hand, $\frac{A}{A\cap\phi(A)} \cong \frac{A+\phi(A)}{\phi(A)} \le \frac{G}{\phi(A)} \cong \frac{G}{\phi^{-1}(\phi A)}$. Now we consider $\phi^{-1}(\phi A)$; clearly, $A \subset \phi^{-1}(\phi A)$. So we may write $\phi^{-1}(\phi A)$ as $\phi^{-1}(\phi A) = A \oplus \langle nb \rangle$ for some integer n. Hence there are two possibilities:

i) $n = 0$, this means that $A = \phi^{-1}(\phi A)$. Hence $\mathrm{Ker}\phi \subset A$. In this case $\mathrm{Ker}\phi = \mathrm{Ker}\gamma \cap \mathrm{Ker}\alpha$.

ii) $n \neq 0$, then $\frac{G}{\phi(A)} \cong \frac{G}{\phi^{-1}(\phi A)} = \frac{G}{A\oplus\langle nb \rangle}$ are finite cyclic groups. Thus $\frac{A}{A\cap\phi(A)}$ is a finite cyclic group of order m and so $A = (A \cap \phi(A)) + \langle a' \rangle$ for some $a' \in A$ and $ma' \in A \cap \phi(A)$.

It is easy to see that $A \cap \phi(A) = \alpha(\text{Ker}\gamma)$ – for if $a \in \phi(A)$ then $\binom{a}{0} = \phi\binom{a_1}{0}$ for some $a_1 \in A$. Thus $a = \alpha(a_1)$ and $\gamma(a_1) = 0$; i.e. $a \in \alpha(\text{Ker}\gamma)$, $A \cap \phi(A) \subset \alpha(\text{Ker}\gamma)$. The reverse inclusion is clear.

Hence we have

$$A = \alpha(\text{Ker}\gamma) + \left\langle a' \right\rangle.$$

Now it is clear that $\mu : A \longrightarrow A$ with $\mu(x + ka) = \alpha(x) + ka'$ (k any integer) is an endomorphism of A and in fact it is a surjection. By hopficity of A we see μ is an automorphism; note that $\mu \restriction \text{Ker}\gamma = \alpha \restriction \text{Ker}\gamma$.

Thus $\text{Ker}\gamma \cap \text{Ker}\alpha = \{0\}$, so in case i), $\text{Ker}\phi = 0$ and hence ϕ is an automorphism.

In case ii), $\mu(x + ma) = \alpha(x) + ma' = \alpha(x) + \alpha(x') = \alpha(x + x')$ for some $x' \in \text{Ker}\gamma$ because $ma' \in \alpha(\text{Ker}\gamma)$ by the condition. But $\mu(x + x') = \alpha(x + x') = \mu(x + ma)$. By the injectivity of μ we see $x + x' = x + ma$, that is, $x' = ma$. Thus $x' = ma = 0$ for $A = \text{Ker}\gamma \oplus \langle a \rangle$ is a direct sum. Hence we deduce that $m = 0$ – which is impossible. □

4.1 Co-Hopfian groups

Co-Hopfian groups are in some sense weakly dual to Hopfian groups but it does not seem easy to deduce results about co-Hopfian groups directly from those known for Hopfian groups - and *vice versa*. However given the similarity in the statement of results, we omit some proofs in this subsection, referring the reader to the second author's doctoral thesis [12]. Some results in this subsection are previously appeared in Li [15].

Proposition 4.7. *(see [12]). If A, B are co-Hopfian groups and* $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ *represents an injective endomorphism of* $G = A \oplus B$*, then* ϕ *is invertible if one of* α, β *is injective.*

For later reference we note:

Corollary 4.8. *(see [12]). If A, B are co-Hopfian groups and* $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ *represents an injective endomorphism of* $G = A \oplus B$*, then* ϕ *is invertible if one of* δ, γ *is zero.*

We can deduce the following:

Proposition 4.9. *(see [15]). Suppose that A is a co-Hopfian group and B is a cyclic p-group generated by the element b, then the direct sum* $G = A \oplus B$ *is again co-Hopfian.*

Proof. Let $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix}$ be a monomorphism on $G = A \oplus B$, β is an integer. We divide the proof into two cases:

i) At least one of α, $\beta 1_B$ is injective. In this case, by Proposition 4.7, ϕ is invertible.

ii) Both α, $\beta 1_B$ are not injective. Suppose the order of b is $o(b) = p^n$. Clearly p divides β. Since ϕ is injective, we claim $\text{Ker}\alpha \cap \text{Ker}\gamma = \{0\}$ –for if $x \in \text{Ker}\alpha \cap \text{Ker}\gamma$, then $\phi\binom{x}{0} = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix}\binom{x}{0} = \binom{0}{0}$. Because of the injectivity of ϕ, we have $x = 0$ as required.

As $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ is invertible, then so is $\Delta = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta 1_B \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \begin{pmatrix} \alpha+\delta\gamma & \delta \\ \gamma+\beta\gamma & \beta 1_B \end{pmatrix}$. Similarly, we have the necessary condition $\text{Ker}(\alpha + \delta\gamma) \cap \text{Ker}(\gamma + \beta\gamma) = \{0\}$.

Since we already supposed $\text{Ker}\alpha \neq \{0\}$, pick $a \in \text{Ker}\alpha$ and $a \neq 0$. We show that $\text{Ker}(\alpha+\delta\gamma) = \{0\}$. Suppose on the contrary that $0 \neq x$ and $x \in \text{Ker}(\alpha + \delta\gamma)$. Then we note that

$$\begin{cases} (\gamma + \beta\gamma)(x) = sb \neq 0, \\ \gamma(a) = tb \neq 0, \end{cases}$$

thus p^n does not divide s, t. Let l be the least common multiply of s, t and assume that $us = vt = l$. From a result in elementary number theory, p^n does not divides l, so that $lb \neq 0$. Now we compute

$$\begin{aligned}(\alpha + \delta\gamma)(ux - va) + \delta(lb) &= (\alpha + \delta\gamma)(ux) - (\alpha + \delta\gamma)(va) + l\delta(b)\\ &= 0 - v(\delta\gamma)(a) + l\delta(b)\\ &= -v\delta(tb) + l\delta(b) = -vt\delta(b) + l\delta(b)\\ &= -l\delta(b) + l\delta(b) = 0.\end{aligned}$$

For the other component we have

$$\begin{aligned}(\gamma + \beta\gamma)(ux - va) + \beta 1_B(lb) &= (\gamma + \beta\gamma)(ux) - \gamma(va) - (\beta\gamma)(va) + l\beta b\\ &= usb - vtb - \beta vtb + l\beta b\\ &= lb - lb - l\beta b + l\beta b = 0.\end{aligned}$$

Thus we have the following identity:

$$\Delta \begin{pmatrix} ux - va \\ lb \end{pmatrix} = \begin{pmatrix} \alpha + \delta\gamma & \delta \\ \gamma + \beta\gamma & \beta 1_B \end{pmatrix} \begin{pmatrix} ux - va \\ lb \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, by the injectivity of Δ, we see that $lb = 0$-a contradiction to $lb \neq 0$. So $\mathrm{Ker}(\alpha + \delta\gamma) = \{0\}$, i.e. $\alpha + \delta\gamma$ is injective and thus is invertible. By Proposition 4.7, Δ is invertible and hence so is ϕ. □

By induction, we have:

Theorem 4.10. *(see [15]). Suppose that A is a co-Hopfian group and B is a finite group, then the direct sum $G = A \oplus B$ is again co-Hopfian.*

Note that the direct sum of a co-Hopfian group and a finitely generated group need not be co-Hopfian - consider the group $\mathbb{Q} \oplus \mathbb{Z}$! However, the direct sum of a co-Hopfian group and a finitely co-generated group, does have the desired property. It is worthwhile to note that finitely cogenerated groups are co-Hopfian.

Theorem 4.11. *(see [15]). Suppose that A is a co-Hopfian group and B is a finitely co-generated Abelian group, then the direct sum $G = A \oplus B$ is again co-Hopfian.*

Proof. Write A as a direct sum of a reduced subgroup R and a divisible subgroup D_1, and each summand is therefore a co-Hopfian group; thus the divisible group D_1 is a direct sum of finitely many copies of the group $\mathbb{Z}(q^\infty)$ for various (possibly infinitely many) primes q. The group B is the direct sum of a finite group F and finitely many copies of $\mathbb{Z}(q^\infty)$ for finitely many primes q; call this latter group D_2. Since, for any fixed prime q, the direct sum of finitely many copies of $\mathbb{Z}(q^\infty)$ is co-Hopfian, it follows from Proposition 4.1 that $D_1 \oplus D_2$ is co-Hopfian. Furthermore, the group $R \oplus F$ is co-Hopfian by Theorem 4.10. Another application of Proposition 4.1 yields the desired result that $A \oplus B$ is co-Hopfian. □

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