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Murrough Golden
*Technological University Dublin*, murrough.golden@dit.ie

G.A.C. Graham
*Applied and Computational Mathematics Program Simon Fraser University, Burnaby, BC, Canada V5A 1S6*



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# The Problem of a Viscoelastic Cylinder Rolling on a Rigid Half-Space

J. M. GOLDEN
School of Mathematical Sciences
Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
jmgolden@maths1.kst.dit.ie

G. A. C. GRAHAM
Applied and Computational Mathematics Program
Simon Fraser University, Burnaby, BC, Canada V5A 1S6
gac@cs.sfu.ca

**Abstract**—The problem of a viscoelastic cylinder rolling on a rigid base, propelled by a line force acting at its centre, is solved in the noninertial approximation. The method used is based on a decomposition of hereditary integrals developed by the authors in previous work, and on the viscoelastic Kolosov-Muskhelishvili equations which are used to generate a Hilbert problem. In this formulation, the problem reduces to a nonsingular integral equation in space and time, which simplifies under steady-state conditions and for exponential decay materials, to algebraic form. There are also two subsidiary conditions.

In the case of a standard linear model, explicit analytic results and numerical examples are given for the pressure function, for surface displacements, and also for hysteretic friction. 

**Keywords**—Hysteretic friction, Viscoelasticity.

## 1. INTRODUCTION

The problem of two viscoelastic cylinders in rolling contact has been treated by Morland, both the case where the cylinders are identical [1] and where they differ in radius and mechanical response [2]. This was one of a class of problems attracting interest at the time which could not be solved by using the classical correspondence principle (see [3] and references cited therein). A related problem was that of a rigid indentor moving over a viscoelastic half-space [3–7]. The results obtained in [5], by a different method, were similar to those of Morland [2].

We consider here the problem of a viscoelastic cylinder on a rigid half-space. This is in fact the same as the problem of two identical cylinders [1]. However, the motivation for this re-examination is that it is a problem which provides a nontrivial application of a general technique developed in [3]. This method allows a systematic approach to solving a wide variety of problems, which contrasts with the early treatments of such problems [1,2,4] where special methodologies had to be employed in each case to complete the solution.

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The relations obtained here agree with those of Morland [2] in a short memory or slow rolling approximation. Explicit results and numerical examples are given for the pressure and surface displacements, and also for the hysteretic friction. The expressions for displacements have not, to our knowledge, been given before. It should be noted that this problem has more recently been treated numerically for simple elastic and viscoelastic and nonlinear behaviour [8–10].

## 2. FUNDAMENTAL EQUATIONS

Let the cylinder, of radius $R$, be moving across a rigid half-space occupying $y < 0$, in the negative $x$ direction (in a coordinate system fixed in space) with speed $V$. It is assumed to be infinitely long in the third direction, with uniform cross-section, so that plane strain conditions prevail. We choose coordinates moving with the cylinder both linearly and rotationally, but eventually switch to a nonrotating frame moving with the cylinder. A line load acting at the centre of the cylinder is assumed to exert a downwards force $W$ counteracted by the upwards pressure of the half-space. Also, the line load has a horizontal component of magnitude $H$ acting in the direction of motion. This is balanced by the hysteretic friction force, which is not known until the problem is solved. Thus, the quantity $H$ is to be determined. The half-space is a distance $d$ below the centre of the cylinder.

We begin by writing down the general viscoelastic Kolosov-Muskhelishvili equations in polar coordinates [11,12], though retaining Cartesian components for displacement. The reason for this is that the component of displacement normal to the half-plane is known in the contact region. The equations are

$$\begin{aligned}
\sigma_{rr} + \sigma_{\theta\theta} &= 2\left[\phi(z,t) + \bar{\phi}(\bar{z},t)\right] = 4\,\mathrm{Re}\,\phi(z,t),\\
\sigma_{rr} + i\sigma_{r\theta} &= \phi(z,t) + \bar{\phi}(\bar{z},t) - \bar{z}\bar{\phi}'(\bar{z},t) - \frac{\bar{z}}{z}\bar{\psi}(\bar{z},t),\\
2[\mu * D'](\underline{r},t) &= iz\left\{[\kappa * \phi](z,t) - \bar{\phi}(\bar{z},t) + \bar{z}\bar{\phi}'(\bar{z},t) + \frac{\bar{z}}{z}\bar{\psi}(\bar{z},t)\right\},\\
D(\underline{r},t) &= u_x(\underline{r},t) + iu_y(\underline{r},t), \qquad D'(\underline{r},t) = \frac{\partial}{\partial\theta}D(\underline{r},t),\\
\underline{r} &= (x,y), \quad \text{or} \quad \underline{r} = (r,\theta), \quad z = x + iy = re^{i\theta}, \qquad |z| \le R,\\
\phi'(z,t) &= \frac{\partial}{\partial z}\phi(z,t), \qquad \bar{\phi}'(\bar{z},t) = \frac{\partial}{\partial \bar{z}}\bar{\phi}(\bar{z},t),\\
[f * g](\underline{r},t) &= \int_{-\infty}^{t} dt' f(t-t')\, g(\underline{r},t').
\end{aligned} \tag{2.1}$$

The quantity $\mu(t)$, which vanishes for negative times, is the shear singular viscoelastic function of the material, defined in [3, p. 10]. Also, $\kappa(t)$ is characterized by the fact that

$$\begin{aligned}
\hat{\kappa}(\omega) &= 3 - 4\hat{\nu}(\omega),\\
\hat{\nu}(\omega) &= \frac{\hat{\lambda}(\omega)}{2\left(\hat{\lambda}(\omega) + \hat{\mu}(\omega)\right)},\\
\hat{\mu}(\omega) &= \int_0^\infty dt\, e^{-i\omega t}\mu(t),\\
\hat{\lambda}(\omega) &= \int_0^\infty dt\, e^{-i\omega t}\lambda(t),
\end{aligned} \tag{2.2}$$

where $\lambda(t)$ is the singular viscoelastic function corresponding to the Lamé constant $\lambda$ [3]. Note that $\hat{\nu}(\omega)$ is a generalization of Poisson's ratio, while $\hat{\mu}(\omega)$ is the shear complex modulus of the material. The quantities $\sigma_{rr}$, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$ are the stresses in polar coordinates at position $\underline{r}$ and time $t$, while $u_x$, $u_y$ are the Cartesian displacements. The functions $\phi(z,t)$, $\psi(z,t)$ are the complex potentials, or more precisely their derivatives [3,11,12].

Under the steady-state assumption, which is assumed but initially not consistently used, the stresses and displacements in polar coordinates will depend only on $\theta + \omega t$ rather than on $\theta, t$ separately. This is achieved by allowing the quantities $\phi(z,t)$ and $z^2\psi(z,t)$ to depend only on the combination $ze^{i\omega t}$, where $\omega = V/R$, rather than on $z$ and $t$ separately. This result is derived in Appendix A. It follows from the fact that in fixed (nonrotating) coordinates moving with the cylinder, all quantities must be time-independent under steady-state conditions. The rotating and nonrotating frames are assumed to coincide at $t = 0$.

The boundary conditions are that, on the surface, $\sigma_{r\theta}$ is zero everywhere and $\sigma_{rr}$ is zero outside the contact region. As mentioned earlier, the normal displacement is known in the contact region. We now derive an expression for this latter quantity.

Since coordinates fixed in the cylinder are being used, the contact region will rotate with time. We denote this region by $\mathcal{C}(t) = [a(t), b(t)]$ where $a(t)$ and $b(t)$ are complex numbers of magnitude $R$. Under steady-state conditions

$$a(t) = a_0 e^{-i\omega t}, \qquad b(t) = b_0 e^{-i\omega t}. \tag{2.3}$$

In the interval $\mathcal{C}(t)$, there is a point $s_0(t)$ such that the vector represented by this complex number is perpendicular to the half-space. We write it as $s_0(t) = Re^{i(\theta_0 - \omega t)}$ where $\theta_0$ is the argument of $s_0(0)$. The angle $\theta_0$ is equal to $3\pi/2$ since the rotating and nonrotating frames coincide at $t = 0$. Thus,

$$s_0(t) = -iRe^{-i\omega t}. \tag{2.4}$$

However, the symbol $\theta_0$ will be used often for convenience. The displacement parallel to $s_0(t)$ is known. This is a displacement in the nonrotating frame. A fixed reference point must be chosen where the displacement is taken to be zero in order to eliminate the freedom allowed by rigid translation. The natural point is the centre of the cylinder, but since a singular line force is acting at this point, the displacement there is indeterminate, and it is unsuitable as a point of reference. We choose $s_0(t)$ instead, taking the displacement to be zero at this point. The normal displacement, under this convention, has the form

$$u_n = -R[1 - \cos(\theta - \theta_0 + \omega t)]. \tag{2.5}$$

The quantity $\frac{\partial u_n}{(R\partial\theta)}$ will be of the order of a typical strain, so that if $s = Re^{i\theta}$ is in the contact interval, $(\theta - \theta_0 + \omega t)$ will be small, in fact of the order of strain. The contact region must therefore be small compared to $R$, within the framework of a linear theory. The ratio of tangential displacement to radius $u_t/R$ in the contact region is second order in strain since it is zero at some point in this region and its Taylor expansion about this point is the product of strain and some term of the order of the angle spanned by the contact region [1]. Note that (see (A.1))

$$\frac{D(\underline{r}, t)}{is_0(t)} = \frac{1}{R}[u_t + iu_n]. \tag{2.6}$$

We define

$$w(s,t) = \mathrm{Im}\left[\frac{\partial}{\partial s} D(\underline{r}, t)\right] = \frac{1}{R}\left[u_n' \cos(\theta - \theta_0 + \omega t) - u_t' \sin(\theta - \theta_0 + \omega t)\right], \tag{2.7}$$

where, as before, the prime on the displacement indicates differentiation with respect to $\theta$. Note that the differentiation with respect to $s$ is taken along the cylinder in a positive (counterclockwise) direction. It follows from (2.7) that, to second order in strain,

$$w(s,t) = \frac{1}{R}\frac{\partial}{\partial\theta} u_n = -\sin(\theta - \theta_0 + \omega t) \approx -(\theta - \theta_0 + \omega t), \qquad s \in \mathcal{C}(t). \tag{2.8}$$

We can also write it in the form

$$w(s,t) = -\frac{1}{2i}\left[\frac{s}{s_0(t)} - \frac{s_0(t)}{s}\right]. \tag{2.9}$$

The complex potentials $\phi(z,t)$ and $\psi(z,t)$ are singular at $z = 0$ because of the line load operating there. We impose the condition that the total load per unit length acting at the centre is

$$P(t) = (W - iH)e^{i(\theta_0 - \omega t)} = -(H + iW)e^{-i\omega t}. \tag{2.10}$$

It is clear that this gives the correct form at $t = 0$ where moving coordinates and fixed coordinates are assumed to coincide. The load on a small circle about the origin must balance $P(t)$ so that [12]

$$P(t) = -r\int_0^{2\pi} d\theta(\sigma_{rr} + i\sigma_{r\theta})e^{i\theta}, \qquad r \ll R. \tag{2.11}$$

Applying also the further constraint that the displacements are single-valued, we deduce from equations (2.1) the behaviour of the complex potentials near the origin to be [12]

$$\phi(z,t) \underset{|z|\to 0}{\longrightarrow} \frac{A(t)}{z}, \qquad \psi(z,t) \underset{|z|\to 0}{\longrightarrow} \frac{B(t)}{z}, \tag{2.12}$$

where

$$A(t) = -\frac{P(t)}{2\pi[\hat{\kappa}(-\omega) + 1]}, \qquad B(t) = \frac{\bar{P}(t)\hat{\kappa}(\omega)}{2\pi[\hat{\kappa}(\omega) + 1]}. \tag{2.13}$$

In the interests of simplicity, though it is not necessary for the analytic developments, the proportionality assumption will be adopted under which [3] the viscoelastic material has a unique Poisson's ratio $\nu$. Then, (2.13) becomes

$$\begin{aligned} A(t) &= -\frac{P(t)}{8\pi(1-\nu)} = W_1 e^{-i\omega t}, \\ B(t) &= \frac{\bar{P}(t)(3-4\nu)}{8\pi(1-\nu)} = -\bar{W}_2 e^{i\omega t}, \\ W_1 &= \frac{H + iW}{8\pi(1-\nu)}, \qquad W_2 = W_1(3-4\nu). \end{aligned} \tag{2.14}$$

The complex potential $\psi(z,t)$ in (2.1) can be eliminated by the use of an analytic continuation argument for problems with circular symmetry, in a manner broadly analogous to that for half-space problems [3,11–13]. One obtains

$$\begin{aligned} \sigma_{rr} + \sigma_{\theta\theta} &= 2\left[\phi(z,t) + \bar{\phi}(\bar{z},t)\right], \\ \sigma_{rr} + i\sigma_{r\theta} &= \phi(z,t) - \frac{R^2}{|z|^2}\phi\left(\frac{R^2}{\bar{z}},t\right) + \left[1 - \frac{R^2}{|z|^2}\right]\left[\bar{\phi}(\bar{z},t) - \bar{z}\bar{\phi}'(\bar{z},t)\right], \\ 2\left[\mu * D'\right](\underline{r},t) &= iz\left[[\kappa * \phi](z,t) + \phi\left(\frac{R^2}{\bar{z}},t\right) + \frac{\bar{z}}{z}\left[1 - \frac{|z|^2}{R^2}\right]\bar{\psi}(\bar{z},t)\right], \\ \bar{\psi}(\bar{z},t) &= \frac{R^2}{\bar{z}^2}\left[\phi\left(\frac{R^2}{\bar{z}},t\right) + \bar{\phi}(\bar{z},t) - \bar{z}\bar{\phi}'(\bar{z},t)\right]. \end{aligned} \tag{2.15}$$

Let the cylinder cross-section be denoted by $S^+$ and the area external to this by $S^-$. The quantity $\phi(z,t)$ is analytic in $S^+$ except for a simple pole at the origin. Its behaviour at zero and at large $|z|$ must be such as to ensure that $\psi(z,t)$ also has a simple pole at the origin. We put

$$\begin{aligned} \phi(z,t) &\underset{|z|\to 0}{\longrightarrow} \frac{A(t)}{z} + A_0 + A_1(t)z + \cdots, \\ \phi(z,t) &\underset{|z|\to \infty}{\longrightarrow} B_{-1}(t)z + B_0 + \frac{B_1(t)}{z} + \cdots, \end{aligned} \tag{2.16}$$

where $A(t)$ is given by (2.14). The quantities $A_0, B_0$ are independent of time for the steady-state problem by virtue of the fact that $\phi(z,t)$ depends only on $ze^{i\omega t}$. Equation (2.16) and the requirement that $\psi(z,t)$, given by the last relation of (2.15), has a simple pole at the origin gives the relations

$$R^2 B_{-1}(t) = -2\bar{A}(t), \qquad B_0 = -\bar{A}_0, \qquad B_1(t) = \bar{B}(t). \tag{2.17}$$

The quantity $A_0$ is not yet determined. We can choose it, without loss of generality, to be real. Adding an imaginary term simply rotates the entire system rigidly through a certain angle.

The second equation of (2.15) gives that on the cylinder surface

$$\sigma_{rr} = \phi^+(s,t) - \phi^-(s,t), \qquad s = Re^{i\theta}, \tag{2.18}$$

where $\phi^\pm(s,t)$ are the limits of $\phi(z,t)$ from $S^\pm$, respectively. Note that we have incorporated the no shear condition on the cylinder surface. Taking the limit of the complex conjugate of the second relation of (2.15) and comparing with (2.18) gives that

$$\phi^+(s,t) + \bar{\phi}^-(\bar{s},t) = \phi^-(s,t) + \bar{\phi}^+(\bar{s},t). \tag{2.19}$$

It follows that the function

$$\phi_c(z,t) = \phi(z,t) + \bar{\phi}\left(\frac{R^2}{z},t\right) \tag{2.20}$$

is continuous across the cylinder surface. Using (2.16) and (2.17), we deduce that

$$\phi_c(z,t) + \frac{A(t)}{z} + \frac{\bar{A}(t)}{R^2}z \tag{2.21}$$

is analytic everywhere and vanishes at infinity, so that by Liouville's theorem

$$\phi(z,t) = -\bar{\phi}\left(\frac{R^2}{z},t\right) - g(z,t), \qquad g(z,t) = \frac{A(t)}{z} + \frac{\bar{A}(t)}{R^2}z. \tag{2.22}$$

The third equation of (2.15) gives that on the cylinder surface

$$2\left[\mu * \frac{\partial}{\partial s}D\right](s,t) = \left[\kappa * \phi^+\right](s,t) + \phi^-(s,t). \tag{2.23}$$

Subtracting the complex conjugate of this relation from itself gives, with the aid of (2.22),

$$iv_1(s,t) = \phi^+(s,t) + \phi^-(s,t), \qquad v_1(s,t) = v(s,t) + ig(s,t), \tag{2.24}$$

the quantity $v(s,t)$ being given by

$$v(s,t) = [l * w](s,t), \tag{2.25}$$

and $g(s,t)$, a real quantity, being the function $g(z,t)$, defined by (2.22), on the cylinder surface. Also, $w(s,t)$ is given by (2.7) and $l(t)$, zero for $t < 0$, is defined by the relation

$$\hat{l}(\omega) = \frac{4\hat{\mu}(\omega)}{1+\hat{\kappa}(\omega)} = \frac{\hat{\mu}(\omega)}{1-\nu}. \tag{2.26}$$

Equation (2.18) gives that $\phi(z,t)$ is continuous across the cylinder outside the contact region. Though $v_1(s,t)$ is not known at this stage, we proceed as if it were known in the contact interval. Thus, equation (2.24) gives a relationship between $\phi^\pm(s,t)$ on the contact interval. Therefore, equations (2.18) and (2.24) constitute a Hilbert problem for $\phi(z,t)$. Its solution with no singularities at $a(t)$ and $b(t)$ is [11]

$$\begin{aligned}\phi(z,t) &= X(z,t)\left[\frac{1}{2\pi}\int_{a(t)}^{b(t)} ds\frac{v_1(s,t)}{(s-z)X^+(s,t)} + \frac{A(t)}{\rho z} - \frac{2\bar{A}(t)}{R^2}\right],\\ X(z,t) &= \{[z-a(t)][z-b(t)]\}^{1/2},\\ \rho(t) &= X(0,t) = [a(t)b(t)]^{1/2},\end{aligned} \tag{2.27}$$

where $X^+(s,t)$ is the limiting value of $X(z,t)$ as $z \to s$ from $S^+$. The function $X(z,t)$ is conventionally chosen to be the branch behaving as $z$ for large $|z|$. It is discontinuous across the cylinder surface over the contact interval $[a(t), b(t)]$. The added terms in (2.27) may be deduced from (2.16) and (2.17). Using the integrals (B.3) in Appendix B, we can rewrite (2.27) in the form

$$\phi(z,t) = \frac{X(z,t)}{2\pi}\int_{a(t)}^{b(t)} ds\frac{v(s,t)}{X^+(s,t)(s-z)} + \frac{3}{2}X(z,t)\left[\frac{A(t)}{\rho z} - \frac{\bar{A}(t)}{R^2}\right] - \frac{1}{2}g(z,t), \tag{2.28}$$

where $g(s,t)$ is defined by (2.22). It may be checked that (2.28) obeys (2.22), with the aid of the relation

$$\bar{X}\left(\frac{R^2}{z},t\right) = \frac{R^2}{\rho z}X(z,t), \tag{2.29}$$

from which it follows for $s \in C(t)$ that

$$\bar{X}^-(\bar{s},t) = -\bar{X}^+(\bar{s},t) = \frac{R^2}{\rho s}X^+(s,t). \tag{2.30}$$

Equation (2.22) together with the third relation of (2.17) gives that

$$R^2 A_1(t) = -\left[\bar{A}(t) + B(t)\right]. \tag{2.31}$$

The third relation of (2.17) gives the condition

$$\begin{aligned}\frac{1}{2\pi}\int_{a(t)}^{b(t)} ds\frac{v(s,t)}{X^+(s,t)}[c(t)-s] &= \bar{B}(t) + \frac{A(t)}{2} + \frac{3c(t)}{2\rho}A(t) - \frac{3(a-b)^2}{16R^2}\bar{A}(t),\\ c(t) &= \frac{a(t)+b(t)}{2}.\end{aligned} \tag{2.32}$$

This relation can be simplified somewhat. The last term on the right is of higher order than the others and will be neglected. Also, consider the first relation of (B.7). The left-hand side must vanish at least as rapidly as $|b-a|$ in the limit as $a \to b$. This tells us which branch of the square root function $\rho$ must be on. Thus,

$$c + \rho = \frac{((b-a)/2)^2}{c-\rho} \approx \frac{(b-a)^2}{4(a+b)}, \tag{2.33}$$

so that the left-hand side is of order $(b-a)^2$. Therefore, the right-hand side of (2.32) becomes, by virtue of (2.13), $\bar{B}(t) - A(t) = P(t)/(2\pi)$ and we can write the relation as

$$\int_{a(t)}^{b(t)} ds\frac{v(s,t)[c(t)-s]}{X^+(s,t)} = P(t). \tag{2.34}$$

This equation corresponds to the condition in the elastic version of the theory relating applied load to the size of the contact interval, as will be perceived more clearly below. A second condition results from the fact that $A_0$ and, therefore, $B_0$ is real,

$$\begin{aligned} 0 &= \mathrm{Im}\left\{\frac{1}{2\pi}\int_{a(t)}^{b(t)} ds \frac{v(s,t)}{X^+(s,t)} - \frac{3}{2}\left[\frac{A(t)}{\rho} + \frac{c(t)}{R^2}\bar{A}(t)\right]\right\} \\ &\approx \mathrm{Im}\left\{\frac{1}{2\pi}\int_{a(t)}^{b(t)} ds \frac{v(s,t)}{X^+(s,t)} - \frac{3A(t)}{\rho}\right\}, \end{aligned} \tag{2.35}$$

where (2.33) has been used and the term proportional to $(a-b)^2\bar{A}(t)$ neglected. This constraint reduces to the correct requirement of symmetry of the contact patch in the elastic limit. If $A_0$ were allowed to have an imaginary part of second order (in the contact patch angle) or greater, the elastic limit would not be correct (see (4.39)).

If the quantity $v(s,t)$ is known in the contact interval, then all other quantities can be determined. In particular, we write down expressions for the contact pressure $p(s,t) = -\sigma_{rr}(\underline{r},t)$, $s \in C(t)$, and also $v(s,t)$, $s \notin C(t)$. This latter quantity is important in the theoretical developments of the next section. From (2.18) and (2.28) we obtain, with the aid of the Plemelj formulae,

$$p(s,t) = -\frac{X^+(s,t)}{\pi}\int_{a(t)}^{b(t)} ds' \frac{v(s',t)}{(s'-s)X^+(s',t)} - 3X^+(s,t)\left[\frac{A(t)}{\rho s} - \frac{\bar{A}(t)}{R^2}\right], \tag{2.36}$$

where the integral is understood to be a principal value. We have ([12], see also (2.11))

$$\frac{1}{2\pi i}\int_{a(t)}^{b(t)} ds\, p(s,t) = \frac{P(t)}{2\pi} = \bar{B}(t) - A(t), \tag{2.37}$$

by virtue of (2.13). With the aid of (B.7) and (B.8), one may show that

$$\frac{1}{2\pi i}\int_{a(t)}^{b(t)} ds\, p(s,t) = \frac{1}{2\pi}\int_{a(t)}^{b(t)} ds \frac{v(s,t)}{X^+(s,t)}[c-s] - \frac{3}{2}A(t) - \frac{3c}{2\rho}A(t) + \frac{3(a-b)^2}{16R^2}\bar{A}(t), \tag{2.38}$$

which demonstrates that (2.37) is equivalent to (2.32). The quantity $P(t)$, given by (2.37), is of order $(b(t)-a(t)) = R\epsilon$ multiplied by the pressure, where $\epsilon$ is a typical strain.

Outside of $C(t)$, $X(z,t)$ is continuous across the cylinder surface and we have from (2.24) and (2.28) that

$$\begin{aligned} v(s,t) &= \frac{X(s,t)}{\pi i}\int_{a(t)}^{b(t)} ds' \frac{v(s',t)}{(s'-s)X^+(s',t)} + G(s,t), \\ G(s,t) &= -3iX(s,t)\left[\frac{A(t)}{\rho s} - \frac{\bar{A}(t)}{R^2}\right]. \end{aligned} \tag{2.39}$$

The quantity $v(s,t)$ is continuous across the ends of the contact interval. This follows from the behaviour of Cauchy integrals at the end points [14].

We need also a general expression for the hysteretic friction force, which arises as a result of energy dissipation in the viscoelastic material. The rate at which work is done by the line force at the centre of the cylinder is $-HV$ where $H$ is the horizontal component of that line force, introduced earlier, and $V$ is the speed of motion. The force $H$ must be canceled by the horizontal component of the force due to contact between the cylinder and the half space. The rate at which the contact stresses do work is given by

$$\dot{E} = -R\int_{\theta_a(t)}^{\theta_b(t)} d\theta\, p(s,t)\dot{u}_r(s,t) = HV, \tag{2.40}$$

where $u_r(s,t)$ is the radial component of displacement and $\theta_a(t)$, $\theta_b(t)$ are the arguments of the complex quantities $a(t)$, $b(t)$, respectively. Since $u_r(s,t)$ is a function only of $\theta + \omega t$ (see Appendix A) we can replace the time derivative by $\omega(\frac{\partial}{\partial\theta})$, giving

$$H = -\int_{\theta_a(t)}^{\theta_b(t)} d\theta\, p(s,t)\frac{\partial}{\partial\theta}u_r(s,t). \tag{2.41}$$

The third relation of (A.2) expressing polar displacements in terms of Cartesian displacements gives that

$$\frac{\partial}{\partial\theta}u_r(s,t) = u_\theta - Rw(s,t) = u_\theta + R\sin(\theta - \theta_0 + \omega t), \tag{2.42}$$

to second order, from (2.8). The quantity $u_\theta/R$ is second order by the same argument as outlined before (2.6). Therefore, to leading order

$$H = -R\int_{\theta_a(t)}^{\theta_b(t)} d\theta\, p(s,t)\sin(\theta - \theta_0 + \omega t) \approx -R\int_{\theta_a(t)}^{\theta_b(t)} d\theta\, p(s,t)(\theta - \theta_0 + \omega t). \tag{2.43}$$

This expression also results from quite a different approach, using (2.10) and (2.37). It follows from (2.43) that $H$ is of order $\epsilon W$, where $\epsilon$ is a typical strain.

We wish to write down a general expression for the displacement, in particular, on the surface of the cylinder. Our starting point is the third equation of (2.15), which gives

$$\begin{aligned} D(\underline{r},t) &= \frac{1}{2}\int_{s_0(t)}^{s} ds'\, \left\{\gamma * \left[\kappa\phi^+ + \phi^-\right]\right\}(s',t) \\ &= \int_{s_0(t)}^{s} ds'\, \left\{k * \left[\phi^+ + \phi^- + c_1\left(\phi^+ - \phi^-\right)\right]\right\}(s',t) \\ &= \int_{s_0(t)}^{s} ds'\, \{k * [iv - g - c_1 p]\}(s',t), \end{aligned} \tag{2.44}$$

where

$$k(t) = (1-\nu)\gamma(t), \qquad c_1 = \frac{1-2\nu}{2(1-\nu)}. \tag{2.45}$$

The quantity $k(t)$ is the inverse of $l(t)$ defined by (2.26) and $\gamma(t)$ is the inverse of $\mu(t)$, under the convolution product [3]. As noted after (2.4), the displacement is taken to be zero at $s_0(t)$.

One may check that the displacement is single-valued by showing that this expression gives zero around a complete contour. This follows from the fact that the integral of $\phi^+(s,t)$ around the circumference picks out the pole term in the first relation of (2.16) while the integral of $\phi^-(s,t)$ picks out the pole term in the second relation of (2.16).

For general $s$, we define (see (2.25))

$$w(s,t) = [k * v](s,t). \tag{2.46}$$

When $s \in \mathcal{C}(t)$, $w(s,t)$ is given by (2.8) or (2.9). For all $s$, we have

$$\begin{aligned} D(\underline{r},t) &= i\int_{s_0(t)}^{s} ds'\, w(s',t) - \Omega(s,s_0(t),t) - c_1\int_{s_0(t)}^{s} ds'[k * p](s',t), \\ \Omega(s,s_0(t),t) &= \int_{s_0(t)}^{s} ds'\,[k * g](s',t), \end{aligned} \tag{2.47}$$

where $g(s,t)$ is given by (2.22). We can write

$$\Omega(s,s_0(t),t) = A(t)\hat{k}(-\omega)\log\frac{s}{s_0(t)} + \frac{\bar{A}(t)\hat{k}(\omega)}{2R^2}\left[s^2 - s_0^2(t)\right]. \tag{2.48}$$

Equation (2.47) must retrieve the known form of $w(s,t)$ on the contact region. Differentiating with respect to $s$ and taking the imaginary part, we obtain an identity, remembering that $p(s,t)$ and $g(s,t)$ are real. It is shown in Section 5 that the normal displacement in the contact region is given by (2.5). Relation (2.47) is analyzed in depth in that section. Note that $k * p$ is of the order of a typical strain.

As mentioned earlier, the quantity $v(s,t)$, $s \in \mathcal{C}(t)$, is fundamental to our considerations. We will decompose this quantity in a manner that will be useful for developments in the next section. Let us put

$$\begin{aligned} v(s,t) &= q(s,t) + \Delta(s,t), \\ q(s,t) &= \int_{-\infty}^{t} dt'\, l\,(t-t')\, w_c\,(s,t')\,, \\ \Delta(s,t) &= \int_{-\infty}^{t} dt'\, l\,(t-t')\,[w\,(s,t') - w_c\,(s,t')]\,, \end{aligned} \tag{2.49}$$

where $w_c(s,t)$ is the known form of $w(s,t)$ in the contact region, as given by (2.9). The form of $q(s,t)$ is

$$\begin{aligned} q(s,t) &= Q_1(t)s + \frac{Q_2(t)}{s}, \\ Q_1(t) &= -\frac{\hat{l}(\omega)}{2is_0(t)}, \\ Q_2(t) &= \frac{\hat{l}(-\omega)}{2i} s_0(t) = R^2\bar{Q}_1(t). \end{aligned} \tag{2.50}$$

By definition, the quantity $w(s,t)-w_c(s,t)$ is zero in the contact region. The decomposition (2.49) separates $v(s,t)$ into a known polynomial part, of the same form as, though not identical to, the elastic limit of $v(s,t)$ and an unknown part resulting from the viscoelasticity of the cylinder. Note that $q(s,t)$ and $\Delta(s,t)$ are in general of order $l(t)$ but $v(s,t)$ is the same order as the pressure, i.e., $\epsilon l(t)$. For the slow rolling, short memory case, $q(s,t)$ and $\Delta(s,t)$ tend towards order $\epsilon l(t)$.

From (2.28) and (B.3), we deduce that

$$\begin{aligned} \phi(z,t) &= \frac{X(z,t)}{2\pi} \int_{a(t)}^{b(t)} ds\, \frac{\Delta(s,t)}{X^+(s,t)(s-z)} - \frac{1}{2}g(z,t) + \frac{i}{2}q(z,t) + \frac{i}{2}H(z,t), \\ H(z,t) &= -iX(z,t)\left[h_1(t) + \frac{h_2(t)}{z}\right], \\ h_1(t) &= -iQ_1(t) - 3\frac{\bar{A}(t)}{R^2}, \\ h_2(t) &= \frac{-iQ_2(t) + 3A(t)}{\rho}. \end{aligned} \tag{2.51}$$

We can write $H(z,t)$ in the form

$$\begin{aligned} H(z,t) &= X(z,t)\left[\frac{H_1 s_0(t)}{\rho z} + \frac{\bar{H}_1}{s_0(t)}\right], \\ H_1 &= \frac{i}{2}\hat{l}(-\omega) + \frac{3W_1}{R} = \frac{i}{2}\bar{\Lambda}(\omega), \\ \Lambda(\omega) &= [1 + 6\Lambda_1(\omega)]\hat{l}(\omega), \\ \Lambda_1(\omega) &= i\frac{\bar{W}_1}{R\hat{l}(\omega)}, \end{aligned} \tag{2.52}$$

where $W_1$ is defined by (2.14). The quantity $\Lambda_1(\omega)$ is generally much less than unity. However, terms proportional to $\Lambda_1(\omega)$ cannot always be neglected, in particular when calculating displacements.

From (2.24) we see that

$$\Delta(s,t)=\frac{X(s,t)}{i\pi}\int_{a(t)}^{b(t)}ds'\,\frac{\Delta\left(s',t\right)}{\left(s'-s\right)X^{+}\left(s',t\right)}+H(s,t),\qquad s\notin\mathcal{C}(t),\tag{2.53}$$

which also follows from (2.39). The quantities $v(s,t)$ and $q(s,t)$ are real, so that $\Delta(s,t)$ must also be real.

## 3. INTEGRAL EQUATION

We now derive an integral equation for $\Delta(s,t)$, following the general technique developed in [3]. The basic difference between the half-plane problem and the cylinder problem, namely that in the latter case, points re-enter the contact regions an infinite number of times, now becomes important.

We decompose $\Delta(s,t)$ according to the fundamental decomposition of hereditary integrals [3]. One term drops out because of the fact that $w(s,t')-w_c(s,t')$ is zero in the contact interval. This gives that

$$\Delta(s,t)=\int_{W_\sigma(s,t)}dt'\,\Pi_\sigma\left(t,t';s\right)\Delta\left(s,t'\right),\qquad s\in\mathcal{C}(t),\tag{3.1}$$

where $W_\sigma(s,t)$ is the set of all those times $t'\le t$ such that $s\notin\mathcal{C}(t')$. The quantity $\Pi_\sigma$ is given as follows [3]:

$$\Pi_\sigma\left(t,t';s\right)=T_1\left(t,t';s\right)R\left(t';t_2(s),t_1(s)\right)+T_3\left(t,t';s\right)R\left(t';t_4(s),t_3(s)\right)+\cdots,\tag{3.2}$$

where

$$R\left(t',t_2,t_1\right)=\begin{cases}1, & t\in[t_2,t_1],\\ 0, & t\notin[t_2,t_1],\end{cases}\tag{3.3}$$

and

$$\begin{aligned}T_0\left(t,t'\right)&=l\left(t-t'\right),\\ T_r\left(t,t';s\right)&=\begin{cases}\displaystyle\int_{t'}^{t_r(s)}dt''\,T_{r-1}\left(t,t'';s\right)l\left(t''-t'\right), & r\text{ even},\\ \displaystyle\int_{t'}^{t_r(s)}dt''\,T_{r-1}\left(t,t'';s\right)k\left(t''-t'\right), & r\text{ odd},\end{cases}\end{aligned}\tag{3.4}$$

the quantities $t_r(s)$ being the times of transition of $s$ between the contact region $\mathcal{C}(t)$ and its complement on the cylinder surface. Combining (3.1) and (2.53) gives the following integral equation for $\Delta(s,t)$, $s\in\mathcal{C}(t)$:

$$\begin{aligned}\Delta(s,t)&=\int_{W_\sigma(s,t)}dt'\int_{a(t')}^{b(t')}ds'\,K\left(s,s';t,t'\right)\Delta\left(s',t'\right)+N(s,t),\\ N(s,t)&=\int_{W_\sigma(s,t)}dt'\,\Pi_\sigma\left(t,t';s\right)H\left(s,t'\right),\\ K\left(s,s';t,t'\right)&=\frac{X\left(s,t'\right)\Pi_\sigma\left(t,t';s\right)}{i\pi X^{+}\left(s',t'\right)\left(s'-s\right)}.\end{aligned}\tag{3.5}$$

Recalling that $\Delta(s,t)$ is real, (3.5) must be a real integral equation. This may be checked with the aid of (2.29) and (2.30).

Explicit application of the steady-state assumption allows simplification of (3.5). All quantities are periodic. Let $s$ be in $\mathcal{C}(t)$ at time $t$ and let it have entered at time $t_1(s)$ and left after the previous cycle at $t_2(s)$. Then, its entry time in the previous cycle was

$$t_3(s)=t_1(s)-T,\qquad T=\frac{2\pi}{\omega}.\tag{3.6}$$

Similarly, for all earlier transition times

$$t_{r+2}(s) = t_r(s) - T, \qquad r = 1, 2, \ldots. \tag{3.7}$$

Using periodicity, we can write (3.5) in the form

$$\begin{aligned} \Delta(s,t) &= \int_{t_2(s)}^{t_1(s)} dt' \int_{a(t')}^{b(t')} ds'\, K_p\left(s,s';t,t'\right) \Delta\left(s',t'\right) + N(s,t), \\ N(s,t) &= \int_{t_2(s)}^{t_1(s)} dt'\, \Pi_{\sigma p}\left(t,t';s\right) H\left(s,t'\right), \\ K_p\left(s,s';t,t'\right) &= \frac{X\left(s,t'\right) \Pi_{\sigma p}\left(t,t';s\right)}{i\pi X^{+}\left(s',t'\right)\left(s'-s\right)}, \end{aligned} \tag{3.8}$$

where

$$\Pi_{\sigma p}\left(t,t';s\right) = \sum_{k=0}^{\infty} T_{2k+1}\left(t,t'-kT;s\right). \tag{3.9}$$

The quantity $\Pi_{\sigma p}$ is an infinite sum in the present problem, while it reduces to a simple closed expression for the half-plane problem. However, for discrete spectrum materials, it is possible to carry out these summations to obtain closed expressions [3,15–19]. This is a particularly straightforward procedure for a standard linear solid [15,16]. We will confine our attention to the latter case, for reasons of algebraic simplicity. The more general discrete problem can also be treated in a similar manner, though the algebra is considerably more complicated.

The case of the standard linear solid, and in fact the general discrete case, is susceptible to special treatment in another sense also. The general integral equation which will be derived in this section can be simplified to a degree that is not possible for more general materials. This will be shown in Section 4 where in fact the starting point is taken to be (3.8) and a different, though equivalent path is followed to that described in the remainder of this section. Of course, the general equation derived in this section is more widely applicable.

We now proceed to change to coordinates fixed in space. First, we must consider the dependence of the various quantities on space and time parameters. It is argued in Appendix A that $\phi(z,t)$ depends only on the combination $ze^{i\omega t}$. One deduces from (2.24) that $v_1(s,t)$ depends only on $se^{i\omega t}$. By virtue of (2.22), $g(s,t)$ has this property so that $v(s,t)$ has it also. It may be checked that this is consistent with (2.28). Finally, examination of (2.50) yields that $q(s,t)$ and therefore $\Delta(s,t)$ have the required property. If $s = Re^{i\theta}$, then all these quantities depend only on $(\theta + \omega t)$.

Let $a_0$, $b_0$ in (2.3) have the form

$$a_0 = Re^{i\theta_a}, \qquad b_0 = Re^{i\theta_b}, \tag{3.10}$$

so that $\theta_a$ and $\theta_b$ will be used henceforth to denote $\theta_a(0)$, $\theta_b(0)$ in terms of the quantities introduced in (2.40). Then, we can write $X(s,t)$, $s \notin C(t)$, in the form

$$\begin{aligned} X(s,t) &= Re^{-i\omega t} X(\theta + \omega t), \\ X(u) &= \left[\left(e^{iu} - e^{i\theta_a}\right)\left(e^{iu} - e^{i\theta_b}\right)\right]^{1/2}. \end{aligned} \tag{3.11}$$

We put $X(u)$ in the form

$$X(u) = \sqrt{2}\,|[1-\cos(u-\theta_a)][1-\cos(u-\theta_b)]|^{1/4} \exp\frac{i(\theta_{ua}+\theta_{ub})}{2}, \tag{3.12}$$

where

$$\begin{aligned} \theta_{ua} &= \arg\left(e^{iu} - e^{i\theta_a}\right), \\ \theta_{ub} &= \arg\left(e^{iu} - e^{i\theta_b}\right), \qquad u \notin [\theta_a, \theta_b]. \end{aligned} \tag{3.13}$$

We take $u$ in the range $[\theta_b - 2\pi, \theta_b]$ as required later. Also, $\theta_{ua}$ and $\theta_{ub}$ are taken to be in this range. One can show that

$$\theta_{ua} = \frac{u+\theta_a}{2} - \frac{\pi}{2}, \tag{3.14}$$

$$\theta_{ub} = \frac{u+\theta_b}{2} - \frac{\pi}{2}, \qquad u \notin [\theta_a, \theta_b], \tag{3.15}$$

so that

$$\begin{aligned} X(u) &= \sqrt{2}\,|[1-\cos(u-\theta_a)][1-\cos(u-\theta_b)]|^{1/4} e^{(i/2)(u+\theta_c-\pi)}, \\ \theta_c &= \frac{\theta_a+\theta_b}{2}, \qquad u \notin [\theta_a, \theta_b]. \end{aligned} \tag{3.16}$$

We put

$$u = \psi + \theta_c, \qquad \psi_0 = \frac{\theta_b - \theta_a}{2}, \tag{3.17}$$

so that

$$\begin{aligned} X(\psi) &= -2ie^{i(\psi/2+\theta_c)}\sqrt{\sin\left(\frac{\psi+\psi_0}{2}\right)\sin\left(\frac{\psi-\psi_0}{2}\right)} \\ &= -\sqrt{2}\,ie^{i(\psi/2+\theta_c)}\sqrt{\cos\psi_0 - \cos\psi}. \end{aligned} \tag{3.18}$$

The sign ascribed in (3.18) will be justified below. Now, consider $X^+(s,t)$ in the contact interval. We have

$$X^+(s,t) = Re^{-i\omega t}X^+(\theta+\omega t), \tag{3.19}$$

where $X^+(u)$ has a form similar to (3.12) but where

$$\theta_{ua} = \frac{u+\theta_a}{2} - \frac{3\pi}{2}, \quad \theta_{ub} = \frac{u+\theta_b}{2} - \frac{\pi}{2}, \qquad u \in [\theta_a, \theta_b], \tag{3.20}$$

giving

$$X(u) \equiv X^+(\psi) = -\sqrt{2}\,e^{i(\psi/2+\theta_c)}\sqrt{\cos\psi - \cos\psi_0} \approx -e^{i(\psi/2+\theta_c)}\left(\psi_0^2 - \psi^2\right)^{1/2}, \tag{3.21}$$

to second order in $\psi_0$, which quantity is of the order of a typical strain. The sign on the left-hand side of (3.21) is conveniently fixed with the aid of the second relation of (B.5) for example. The quantity $X^-(s,t)$ is similarly connected to $X^-(\psi) = -X^+(\psi)$. The sign of (3.18) can be confirmed with the aid of (B.48). The quantity under the square root sign in the second form of (3.18) is approximately $\sqrt{2}\,|\sin(\psi/2)| = -\sqrt{2}\sin(\psi/2)$, away from the contact region for $\psi \in [\psi_0 - 2\pi, -\psi_0]$ which, as will emerge below, is the region of interest. We can express $H(s,t)$, given by (2.52), in the form

$$H(s,t) = H(\theta+\omega t), \qquad H(u) = 2\,\mathrm{Re}\left[H_1\bar{X}(u)e^{i\theta_0}\right]. \tag{3.22}$$

Note that

$$a(t_1(s)) = s, \qquad b(t_2(s)) = s, \tag{3.23}$$

which gives that

$$t_1(s) = \frac{\theta_a - \theta}{\omega}, \qquad t_2(s) = \frac{\theta_b - 2\pi - \theta}{\omega}. \tag{3.24}$$

Putting $u = \theta + \omega t$, $v = \theta + \omega t'$, we can write the equation of $\Delta(s,t)$ in (2.49) as

$$\begin{aligned} \Delta(s,t) &= \Delta(u) = \int_{-\infty}^{\theta_a} \frac{dv}{\omega} l\left(\frac{u-v}{\omega}\right)\{w(v) - w_c(v)\}, \\ w(v) &\equiv w(s,t'), \end{aligned} \tag{3.25}$$

where the equality of $w(v)$ and $w_c(v)$ in the contact region has been used.

The integration over $s'$ in (3.8) can be expressed as an integration over its argument $\theta'$. We change variables according to

$$\begin{aligned} \theta &\to \psi = u - \theta_c, & u &= \theta + \omega t, \\ \theta' &\to \psi' = u' - \theta_c, & u' &= \theta' + \omega t', \\ t' &\to \phi = v - \theta_c, & v &= \theta + \omega t', \end{aligned} \tag{3.26}$$

and (3.8) becomes

$$\Delta(\psi) = \int_{-\psi_0}^{\psi_0} d\psi' \, K(\psi, \psi') \, \Delta(\psi') + N(\psi), \qquad \Delta(\psi) \equiv \Delta(u), \tag{3.27}$$

where

$$\begin{aligned} K(\psi, \psi') &= \frac{1}{\pi} \int_{\psi_0 - 2\pi}^{-\psi_0} \frac{d\phi}{\omega} \frac{X(\phi) \Pi(\psi, \phi)}{X^+(\psi') \left(1 - e^{i(\phi - \psi')}\right)}, \\ N(\psi) &= \int_{\psi_0 - 2\pi}^{-\psi_0} \frac{d\phi}{\omega} \Pi(\psi, \phi) H(\phi), \\ H(\phi) &\equiv H(v), \qquad \Pi(\psi, \phi) = \Pi_{\sigma p}(t, t'; s). \end{aligned} \tag{3.28}$$

It is straightforward to show that the first term of $\Pi_{\sigma p}(t, t'; s)$ is a function only of $\psi$, $\phi$ in the steady-state limit. To demonstrate this for the other terms, one must make use of recurrence relations derived in [17–19]. The angular range $[\psi_0 - 2\pi, \psi_0]$ covered by the integration in (3.27) and (3.28) corresponds to $v$, given by (3.17), in the range $[\theta_b - 2\pi, \theta_b]$ as stated after (3.13).

Equation (3.27) is the stationary form of (3.8), for a general material. We now specialize to the case of a standard linear solid.

## 4. AN ALTERNATIVE APPROACH FOR THE STANDARD LINEAR SOLID

Let us first characterize the material. Under the proportionality assumption [3], the quantities $l(t)$ and $k(t)$ are given by

$$\begin{aligned} l(t) &= l_0 \delta(t) + l_1 e^{-\alpha t} = h\mu(t), \\ k(t) &= k_0 \delta(t) + k_1 e^{-\beta t} = \frac{\gamma(t)}{h}, \\ l_0 k_0 &= 1, \qquad k_1 = -\frac{l_1}{l_0^2}, \\ h &= \frac{1}{1 - \nu}, \qquad \beta = \alpha - \frac{k_1}{k_0}. \end{aligned} \tag{4.1}$$

The summation in (3.9) may be carried out as in a similar case described in [3, p. 131], to give

$$\begin{aligned} \Pi_{\sigma p}(t, t'; s) &= \frac{l_1 k_0}{1 - E} \exp\left\{-\alpha[t - t_1(s)] + \beta[t' - t_1(s)]\right\}, \\ E &= \exp\{[t_2(s) - t_1(s)](\beta - \alpha) - T\alpha\}, \end{aligned} \tag{4.2}$$

where $T$ is the period, defined by (3.6). In stationary coordinates, we have, with the aid of (3.26), that

$$\begin{aligned} \Pi(\psi, \phi) &= \frac{l_1 k_0}{1 - E} \exp\left[-\frac{\alpha}{\omega}(\psi + \psi_0) + \frac{\beta}{\omega}(\phi + \psi_0)\right], \\ E &= \exp\left[2\psi_0 \left(\frac{\beta - \alpha}{\omega}\right) - T\beta\right]. \end{aligned} \tag{4.3}$$

Also, from (3.25) and (3.26)

$$\begin{aligned} \Delta(\psi) &= Ce^{-(\alpha/\omega)\psi}, \\ C &= l_1 \int_{-\infty}^{-\psi_0} \frac{d\phi}{\omega} e^{(\alpha/\omega)\phi}[w(\phi) - w_c(\phi)], \\ w(\phi) &\equiv w(v), \qquad w_c(\phi) \equiv w_c(v). \end{aligned} \tag{4.4}$$

We now temporarily go back to the time-dependent system. Certain general manipulations have a greater transparency in this frame. Let us put

$$\begin{aligned} \Delta(\psi) &= \Delta(s,t) = C_1 e^{-\alpha t} s^p, \qquad s \in \mathcal{C}(t), \\ p &= i\frac{\alpha}{\omega}, \qquad C_1 = \frac{C}{R^p} e^{(\alpha/\omega)\theta_c}, \end{aligned} \tag{4.5}$$

and with the aid of (B.23), deduce the following expressions for $\phi(z,t)$ as given by (2.51):

$$\begin{aligned} \phi(z,t) &= \frac{i}{2} q(z,t) - \frac{1}{2} g(z,t) + \frac{i}{2} H(z,t) \\ &+ \frac{i}{2} C_1 e^{-\alpha t} \left[ D_{p-1} X(z,t) - (p-1) z^p \int_0^z \frac{dz' \, (z')^{-p} K_1(z',t)}{X(z',t)} \right], \end{aligned} \tag{4.6}$$

where $K_1(z,t)$ is given by (B.23), $D_q$ in $K_1(z,t)$ by (B.13), $g(z,t)$ by (2.22), $q(z,t)$ by (2.50), and $H(z,t)$ by (2.51). Using (B.32), we can also show that

$$\begin{aligned} \phi(z,t) &= \frac{i}{2} q(z,t) - \frac{1}{2} g(z,t) + \frac{i}{2} H(z,t) \\ &- \frac{i}{2} C_1 e^{-\alpha t} \left\{ \frac{D_p}{z} X(z,t) + (p+1) z^p \int_z^\infty \frac{dz' \, (z')^{-p-2} [z' D_{p+1} - ab D_p]}{X(z',t)} \right\}. \end{aligned} \tag{4.7}$$

The identity of (4.6) and (4.7) allows one to confirm that (2.22) holds, as may be seen with the aid of (B.29). Combining (2.24) and (2.49) with (4.6), we obtain that

$$\Delta(s,t) = -C_1 e^{-\alpha t} s^p (p-1) \int_0^{a(t)} \frac{dz' \, (z')^{-p} K_1(z',t)}{X(z',t)} = C_1 e^{-\alpha t} s^p, \qquad s \in \mathcal{C}(t), \tag{4.8}$$

by virtue of (B.34). One determines $\phi^-(s,t)$ by going from the origin to $s$ and then around the cut, through $a(t)$. Thus, the correct form for $\Delta(s,t)$ is recovered on $\mathcal{C}(t)$.

Let us agree that the cut for the power function is along the radius vector through $b(t)$, with $\arg(s)$ in the region $[\theta_b - 2\pi, \theta_b]$ as in the previous section. Off the contact region, we have

$$\begin{aligned} \Delta(s,t) = H(s,t) + C_1 e^{-\alpha t} \left[ D_{p-1} X(s,t) - (p-1) s^p \int_0^s \frac{dz' \, (z')^{-p} K_1(z',t)}{X(z',t)} \right], \\ s \notin \mathcal{C}(t). \end{aligned} \tag{4.9}$$

Now, the quantity $\Delta(s,t)$ is continuous at the ends of the contact interval. This follows from its definition, given by (2.49) and the continuity of $v(s,t)$. Therefore, we can use (B.34) in (4.9) to obtain

$$\begin{aligned} \Delta(s,t) &= H(s,t) + C_1 e^{-\alpha t} \Bigg[ D_{p-1} X(s,t) + s^p \\ &- (p-1) s^p \int_{a(t)}^s ds' \, \frac{(s')^{-p} K_1(s',t)}{X(s',t)} \Bigg], \qquad s \notin \mathcal{C}(t), \end{aligned} \tag{4.10}$$

where the modified integral sign is defined after (B.36). This clearly gives the correct behaviour as $s \to a(t)$. It can also be shown, with the aid of (B.36), that it gives the correct expression (i.e., continuity) as $s \to b(t)$. An alternative expression for $\Delta(s,t)$ can be derived, starting from (4.7). Comparison of this and (4.9) allows one to check that $\Delta(s,t)$ is real. In fact, (4.10) can be simplified and the reality made more apparent by combining the integral in (4.10) and the term proportional to $X(s,t)$, which together give a real quantity. This is done with the aid of the identity

$$X(s,t) = s^p \int_{a(t)}^{s} ds' \frac{d}{ds'} \left[ (s')^{-p} X(s',t) \right], \tag{4.11}$$

and (B.16). One finally obtains

$$\begin{aligned} \Delta(s,t) &= H(s,t) + C_1 e^{-\alpha t} \left[ s^p + p s^p \,⨍_{a(t)}^{s} ds' \frac{(s')^{-p-1} K_2(s',t)}{X(s',t)} \right], \\ K_2(s,t) &= s D_p - ab D_{p-1}, \qquad s \notin C(t). \end{aligned} \tag{4.12}$$

This form would also emerge on starting from (4.7) and using a similar trick. The integral term in (4.12) can be shown to be real with aid of (B.31). It also follows, by imposing the condition that $\Delta(s,t)$ is continuous at $b(t)$, that (cf. (B.36))

$$⨍_{a(t)}^{b(t)} ds' \frac{(s')^{-p-1} K_2(s',t)}{X(s',t)} = -\frac{\eta - 1}{\eta p}. \tag{4.13}$$

We now wish to write down a relationship for $C$. In the case of a general material, the corresponding relationship would be the integral equation derived in Section 3. The formalism leading to that equation was determined by the fact that, in the general case, $\Delta(s,t)$, $s \notin C(t)$, is given by (2.53). In the present case, the rather more explicit (4.12) is available. Therefore, instead of taking (3.27) as our starting point, we begin with the stationary form of (3.1), namely

$$\Delta(\psi) = \int_{\psi_0 - 2\pi}^{-\psi_0} \frac{d\phi}{\omega} \Pi(\psi,\phi) \Delta(\phi), \qquad \psi \in [-\psi_0, \psi_0], \tag{4.14}$$

where $\Pi(\psi,\phi)$ is given by (4.3). Continuing the strategy of temporarily going back from static coordinates to those moving with the cylinder, we write (4.14) in the form

$$\begin{aligned} \Delta(s,t) &= ⨍_{b(t)}^{a(t)} ds' \, \Pi(s,s') \, \Delta(s',t), \\ \Pi(s,s') &= \frac{l_1 k_0}{i\omega(1-E)s'} \left( \frac{s}{a(t)} \right)^p \left( \frac{s'}{a(t)} \right)^{-q}, \\ q &= \frac{i\beta}{\omega}, \qquad E = \frac{1}{\eta} \left( \frac{b'}{a} \right)^{p-q}, \qquad b'(t) = R e^{i(\theta_b - \omega t - 2\pi)}. \end{aligned} \tag{4.15}$$

The quantity $q$ is not to be confused with that given by (B.12). We have distinguished $b(t)$ approached clockwise in (4.15) as $b'(t)$. This will not generally be done explicitly, for notational economy. It is always clear from the context.

Using (4.13) and the fact (see (4.1)) that $\beta - \alpha = l_1 k_0$, we find that the expression resulting from the $s^p$ term on the right of (4.12) combined with one of the terms emerging from a partial integration of the integral term also on the right of (4.12) cancels with $\Delta(s,t)$ on the left of (4.15). This is similar to a cancellation that takes place for the problem of a rigid indentor passing over a discrete spectrum material [3]. The resulting equation simplifies to give

$$C = \frac{R^p Y}{Z} e^{(\omega t - \theta_c)\alpha/\omega}, \tag{4.16}$$

where, using (2.51) and (B.40),

$$
\begin{aligned}
Y &= \int_a^b ds\, s^{-q-1} H(s,t) \\
&= -i \int_a^b \frac{ds}{X(s,t)} \left\{ \frac{h_1(t)}{q} \left[ s^{-q+1} - cs^{-q} \right] + \frac{h_2(t)}{q+1} \left[ s^{-q} - cs^{-q-1} \right] \right\} \\
&= \pi \left\{ \frac{h_1(t)}{q} \left[ E_{q-1} - cE_q \right] + \frac{h_2(t)}{q+1} \left[ E_q - cE_{q+1} \right] \right\},
\end{aligned}
\tag{4.17}
$$

adopting the notation of (B.39), and

$$
Z = \frac{p}{p-q} \int_a^b ds \, \frac{s^{-q-1} K_2(s,t)}{X(s,t)} = \frac{i\pi p}{p-q} \left[ E_q D_p - abE_{q+1} D_{p-1} \right]. \tag{4.18}
$$

Applying (B.31),(B.40), and (B.41), we can give the following, explicitly real, expression for $C$:

$$
C = -ie^{(\omega t - \theta_c)\alpha/\omega} \frac{p-q}{qp} \frac{\operatorname{Re}\{R^q h_1(t)[E_{q-1} - cE_q]\}}{\operatorname{Re}[R^{q-p} D_p E_q]}. \tag{4.19}
$$

One can show that

$$
D_{p-r} = R^{p-r} e^{(\alpha/\omega)(\omega t - \theta_c)}\, e^{ir(\omega t - \theta_c)} d_r, \qquad r = 0, \pm 1, \pm 2, \ldots, \tag{4.20}
$$

where, for a small contact interval,

$$
\begin{aligned}
d_r &= -\frac{1}{\pi} \int_{-\psi_0}^{\psi_0} \frac{d\psi\, e^{-i(r-1/2)\psi} e^{-(\alpha/\omega)\psi}}{\left(\psi_0^2 - \psi^2\right)^{1/2}} = -I_0 \left[ \alpha' + i \left( r - \frac{1}{2} \right) \psi_0 \right], \\
\alpha' &= \frac{\alpha \psi_0}{\omega},
\end{aligned}
\tag{4.21}
$$

in terms of the modified Bessel functions with imaginary argument [20]. For $\alpha/\omega$ large, $\alpha'$ is not a small quantity and will not be treated as such in what follows, because the slow rolling case is of interest later. Also, for integer $r$,

$$
\begin{aligned}
E_{q+r} &= iR^{-q-r} e^{-(\beta/\omega)(\omega t - \theta_c)} e^{ir(\omega t - \theta_c)} e_r, \\
e_r &= \frac{1}{\sqrt{2}\pi} \int_{\psi_0 - 2\pi}^{-\psi_0} d\psi \, \frac{e^{(\beta/\omega)\psi} e^{-i(r-1/2)\psi}}{|\cos\psi_0 - \cos\psi|^{1/2}}.
\end{aligned}
\tag{4.22}
$$

Making transformation $u = \sin(\psi/2)$, $v = \cos(\psi/2)$, and using appropriate forms of the denominator in the integrand, one can show, after careful handling of the branches of the inverse trigonometrical function which arise, that

$$
\begin{aligned}
e_{00} &= \operatorname{Re}(e_0) = -\frac{2}{\pi} e^{-\beta\pi/\omega} \int_0^{y_0} d\theta \sinh \left[ \frac{2\beta}{\omega} f(\theta) - \frac{\beta\pi}{\omega} \right], \\
f(\theta) &= \sin^{-1}(s_0 \cosh\theta), \\
y_0 &= \cosh^{-1} \frac{1}{s_0} \approx -\log \frac{\psi_0}{4}, \\
e_{01} &= \operatorname{Im}(e_0) = -\frac{1}{\pi} \int_0^{\pi} d\theta \exp \left[ -\frac{2\beta}{\omega} \cos^{-1}(c_0 \cos\theta) \right], \\
s_0 &= \sin\frac{\psi_0}{2}, \qquad c_0 = \cos\frac{\psi_0}{2}.
\end{aligned}
\tag{4.23}
$$

Recalling the remark after (2.38), we see that the load terms in $H(s,t)$ given by (2.51) and (2.52) are small compared with the term proportional to the modulus of the material. We shall retain this term in the final expression for $C$ because it occurs later in the expression for displacements, which may be of order $\psi_0^2$. In the final expression for $p(s,t)$ evaluated on the cylinder surface, the load terms will be neglected.

One finally obtains for $C$

$$
\begin{aligned}
C &= -\frac{(\alpha-\beta)\omega}{2\alpha\beta}\,\frac{\mathrm{Re}[\Lambda(\omega)e^{i\theta_c}(e_{-1}-e_0\cos\psi_0)]}{\mathrm{Im}(d_0e_0)} \\
&= -\frac{l_0(\alpha-\beta)\omega}{2\alpha}\,\frac{\mathrm{Re}\{[1+6\Lambda_1(\omega)]e^{i\theta_c}(e_0\cos\psi_0-\bar{e}_0)/(\alpha+i\omega)\}}{\mathrm{Im}(d_0e_0)},
\end{aligned}
\tag{4.24}
$$

in the notation of (2.52), the last expression being more convenient for numerical evaluation using (4.23). It is obtained by combining (4.22) and (B.40) for $\delta = q$ and using the fact that $e_1 = \bar{e}_0$. One also requires the identity $\hat{l}(\omega)/l_0 = (\beta+i\omega)/(\alpha+i\omega)$.

Comparing (4.24) with the corresponding equations of Morland [2] for his coefficients $A_n$, in the case of one decay time, it may be seen that they are the same in the very short memory or slow rolling approximation. For such an approximation, the exponential in $e_r$, given by (4.22), becomes negligible for other than small values of $\psi$ so that the lower limit of the integral may be replaced by $(-\infty)$ and $\cos\psi$ in the denominator replaced by $1-\psi^2/2$. Thus, the integral is proportional to $K_0(\beta' - i(r-0.5)\psi_0)$ which, together with (4.24), gives essentially Morland's result. In such an approximation, the problem becomes similar in certain respects to that of a rigid punch on a viscoelastic half-plane [3].

Using (2.18),(2.51), and (4.6), we obtain an expression for $p(s,t)$ of the form

$$
\begin{aligned}
p(s,t) &= -X^+(s,t)\left[h_1(t)+\frac{h_2(t)}{s}\right] - iC_1e^{-\alpha t}D_{p-1}X^+(s,t) \\
&\quad + iC_1e^{-\alpha t}(p-1)s^p\int_{a(t)}^{s} ds'\,\frac{(s')^{-p}K_1(s',t)}{X^+(s',t)}.
\end{aligned}
\tag{4.25}
$$

It clearly vanishes at $s=a(t)$. It also vanishes at $s=b(t)$ by virtue of (B.24). The expression for the pressure must be real. This can be shown, in the general case, to be equivalent to (2.22), the latter property being essentially derived from this physical condition. For a standard linear solid, it follows, as for $\Delta(s,t)$, from the fact that (4.6) and (4.7) are equal. The first term in (4.24) is real and the last two terms combine to form a real quantity, though separately they are not. It is convenient to combine these terms using the same trick as for $\Delta(s,t)$ (see (4.11)). One finally obtains

$$
p(s,t) = -X^+(s,t)\left[h_1(t)+\frac{h_2(t)}{s}\right] - iC_1e^{-\alpha t}ps^p\int_{a(t)}^{s} ds'\,\frac{(s')^{-p-1}K_2(s',t)}{X^+(s',t)}, \tag{4.26}
$$

which can be shown to be real with the aid of (B.31). This form also follows from (4.7).

Using (2.50) and (4.20) we obtain, neglecting load terms,

$$
\begin{aligned}
p(s,t) = p(u) &= \mathrm{Im}\left[\hat{l}(\omega)X^+(u)\right] + Ce^{-\alpha u/\omega}p\int_{\theta_a}^{u} dv\,\frac{e^{\alpha v/\omega}\left[d_0 - e^{-i(v-\theta_c)}d_1\right]e^{iv}}{X^+(v)}, \\
u &= \theta+\omega t,
\end{aligned}
\tag{4.27}
$$

where the relation

$$
\bar{X}^+(u) = \frac{s_0^2(t)}{\rho s}X^+(u) \tag{4.28}
$$

has been used in writing down the first term of (4.27). This follows from (2.30) and (3.19). Note that $d_1 = \bar{d}_0$. We put

$$
d_0 = d_{00} + id_{01}, \tag{4.29}
$$

where $d_{00}$ and $d_{01}$ are real. Then, relating only leading terms (i.e., of order $l_0\psi_0$ and $C$; the next contribution involving $C$ is order $C\psi_0^2$) we obtain

$$p(u) \equiv p(\psi) \doteq l_1(\omega)\psi_0\left(1-x^2\right)^{1/2} + Ce^{-\alpha' x}\alpha' \int_{-1}^{x} dy\, \frac{e^{\alpha' y}[2(d_{01}/\psi_0)+d_{00}y]}{\left(1-y^2\right)^{1/2}},$$
$$l_1(\omega) = \operatorname{Re}\left[\hat{l}(\omega)\right], \qquad x = \frac{\psi}{\psi_0}. \tag{4.30}$$

If $\alpha/\omega$ is of order unity or less, then $d_0 = -1$ to lowest order in $\psi_0$ and

$$p(\psi) \approx \left[l_1(\omega)\psi_0 + C\alpha'\right]\left(1-x^2\right)^{1/2}. \tag{4.31}$$

To second order in $\psi_0$, we have that

$$d_r = -\left\{I_0(\alpha') + i\left(r-\frac{1}{2}\right)\psi_0 I_1(\alpha') - \frac{1}{4}\left(r-\frac{1}{2}\right)^2 \psi_0^2\left[I_0(\alpha') + I_2(\alpha')\right]\right\}. \tag{4.32}$$

Neglecting terms of order $\psi_0^2$ and $C\psi_0^2$, the following expression for the pressure is then deduced:

$$p(\psi) = l_1(\omega)\psi_0\left(1-x^2\right)^{1/2} - Ce^{-\alpha' x}\alpha' \int_{-1}^{x} dy\, \frac{e^{\alpha' y}\left[yI_0(\alpha') - I_1(\alpha')\right]}{\left(1-y^2\right)^{1/2}}, \tag{4.33}$$

which reduces to $l_1(0)\psi_0(1-x^2)^{1/2}$ as $\omega \to 0$. Thus, a nonsymmetric pressure occurs only in the range where $\alpha/\omega$ is greater than unity but not very large. Equation (4.33) agrees with the result of Morland [2].

Let us now write down the constraint (2.34). Using (4.5), we obtain

$$\frac{(b-a)^2}{8}Q_1(t) + \frac{(\rho+c)}{\rho}Q_2(t) + C_1 e^{-\alpha t}\left[cD_p - D_{p+1}\right] = \frac{P(t)}{\pi i}, \tag{4.34}$$

with the aid of integrals in Appendix B. This reduces to, recalling that $\theta_0 = 3\pi/2$,

$$\frac{i}{2}\psi_0^2 e^{i\delta}\operatorname{Re}\left[\hat{l}(\omega)e^{i\delta}\right] + \psi_0 C e^{i\delta}\left[iI_1(\alpha') + \frac{1}{2}\psi_0 I_2(\alpha')\right] = \frac{H+iW}{\pi R}, \qquad \delta = \theta_c - \theta_0. \tag{4.35}$$

Only those terms are retained on the left of (4.35) that are required to give nontrivial relations for $W$ and $H$, which are of order $Rl_0\psi_0^2$ and $Rl_0\psi_0^3$, respectively. Splitting into real and imaginary parts and assuming that $\delta$ is of order $\psi_0$ (see Section 6), we obtain

$$\frac{1}{2}\psi_0^2 l_1(\omega) + C\psi_0 I_1(\alpha') = \frac{W}{\pi R}, \tag{4.36}$$

and

$$-\frac{1}{2}\psi_0^2\delta l_1(\omega) - C\psi_0\delta I_1(\alpha') + \frac{1}{2}C\psi_0^2 I_2(\alpha') = -\delta\frac{W}{\pi R} + \frac{1}{2}C\psi_0^2 I_2(\alpha') = \frac{H}{\pi R}, \tag{4.37}$$

where a term proportional to $C\psi_0^3$ was neglected in writing (4.36) and a term of order $C\psi_0^4$ in writing (4.37). The question of the order of magnitude of $C$ is discussed after (6.4). The constraint (2.35) takes the form

$$\operatorname{Im}\left\{i\left[\frac{Q_2}{\rho} - cQ_1 + C_1 e^{-\alpha t}D_p\right] - \frac{6A(t)}{\rho}\right\} = 0. \tag{4.38}$$

Keeping terms up to second order, we obtain

$$\left(1-\frac{\delta^2}{2}-\frac{\psi_0^2}{4}\right)l_2(\omega) + \delta l_1(\omega) - C\left\{I_0(\alpha') - \frac{\psi_0^2}{16}\left[I_0(\alpha') + I_2(\alpha')\right]\right\} = 0,$$
$$l_2(\omega) = \operatorname{Im}\left(\hat{l}(\omega)\right). \tag{4.39}$$

## 5. SURFACE DISPLACEMENTS

The surface displacements given by (2.44) will now be considered. Terms proportional to the applied loads will often be retained. In many cases, these are negligible compared with other contributions. They are always second order or higher. However, second order displacements in the contact patch are not neglected, so for consistency, such terms should be retained. It turns out also that terms involving loads play an important role in ensuring the continuity of displacements.

We write

$$\begin{aligned} D(\underline{r},t) &= \int_{s_0(t)}^{s} ds' \, \{k * [iv - c_1 p]\}(s',t) - \Omega(s, s_0(t), t) \\ &= D(\underline{a},t) + \int_{a(t)}^{s} ds' \, \{k * [i(q+\Delta) - c_1 p]\}(s',t) - \Omega(s, a(t), t), \end{aligned} \tag{5.1}$$

where $\underline{a}$ is the two-dimensional vector corresponding to $a(t)$ and where (2.47) and (2.49) have been used. Noting that $[k * q](s,t) = w_c(s,t)$, in the notation of (2.49), we write

$$D(\underline{r},t) = D(\underline{a},t) + iW_c(s,t) - \Omega(s,a(t),t) + D_1(\underline{r},t), \tag{5.2}$$

where

$$\begin{aligned} W_c(s,t) &= -\frac{1}{2i}\left[\frac{s^2 - a^2(t)}{2s_0(t)} - s_0(t)\log\frac{s}{a(t)}\right], \\ \Omega(s,a(t),t) &= A(t)\hat{k}(-\omega)\log\frac{s}{a(t)} + \frac{\bar{A}(t)\hat{k}(\omega)\left[s^2 - a^2(t)\right]}{2R^2}, \end{aligned} \tag{5.3}$$

on using the form of $\Omega$ given by (2.48). Also,

$$D_1(\underline{r},t) = \int_{a(t)}^{s} ds' \, \{k * [i\Delta - c_1 p]\}(s',t), \tag{5.4}$$

which is the nontrivial part of the displacement, the evaluation of which is the main topic of this section. We place, as before, the cut in the log function in $\Omega$ and $W_c$ through $s = b(t)$. Writing

$$\Gamma(s,t) = iW_c(s,t) - \Omega(s,a(t),t), \tag{5.5}$$

then

$$\Gamma_c(b,t) - \Gamma_a(b,t) = 2i\pi A(t)\hat{k}(-\omega) - i\pi s_0(t), \tag{5.6}$$

where $\Gamma_c$ denotes the limit, approaching the cut clockwise and $\Gamma_a$ approaching it counterclockwise.

We have from (4.5) and (4.12) that

$$\begin{aligned} \Delta(s,t) &= \Delta_0(s,t) + \Delta_1(s,t), \\ \Delta_0(s,t) &= C_1 e^{-\alpha t} s^p, \\ \Delta_1(s,t) &= \begin{cases} \phi_1(s,t), & s \notin \mathcal{C}(t), \\ 0, & s \in \mathcal{C}(t), \end{cases} \end{aligned} \tag{5.7}$$

where

$$\phi_1(z,t) = H(z,t) + C_1 e^{-\alpha t} p z^p \int_a^z dz' \, \frac{(z')^{-p-1} K_2(z',t)}{X(z',t)}. \tag{5.8}$$

Equation (5.7) defines $\Delta(s,t)$ around one cycle in the time (or space) variable. At earlier times, it is defined by periodicity. Note that

$$p(s,t) = -i\phi_1^+(s,t), \qquad s \in \mathcal{C}(t), \tag{5.9}$$

which follows from (4.26).

In the notation of (2.46) and (2.49), we have

$$w(s,t) = w_c(s,t) + [k * \Delta](s,t), \tag{5.10}$$

so that

$$[k * \Delta](s,t) = 0, \qquad s \in \mathcal{C}(t), \tag{5.11}$$

or

$$\begin{aligned} k_0\Delta_0(s,t) + k_1 e^{-\beta t} \Bigg\{ \int_{t_1(s)}^{t} dt'\, e^{\beta t'} \Delta_0(s,t') + \frac{1}{1-\xi} \Bigg[ \int_{t_2(s)}^{t_1(s)} dt'\, e^{\beta t'} \left(\Delta_0(s,t') + \Delta_1(s,t')\right) \\ + \frac{1}{\eta} \int_{t_1(s)-T}^{t_2(s)} dt'\, e^{\beta t'} \Delta_0(s,t') \Bigg] \Bigg\} = 0, \qquad \xi = e^{-\beta T}, \quad \eta = e^{\alpha T}. \end{aligned} \tag{5.12}$$

The factor $\eta$ (see also (B.37)) is included to enforce periodicity on the known form of $\Delta_0(s,t)$. The time $t_1(s)$ in (5.12) is the most recent time of entry of $s$ into the contact region, and $t_2(s)$ is the most recent time of leaving the contact region. Condition (5.12) reduces to

$$\int_{t_2(s)}^{t_1(s)} dt'\, e^{\beta t'} \Delta_1(s,t') = \frac{C_1(\eta-1)}{(\beta-\alpha)\eta} e^{(\beta-\alpha)t_2(s)} s^p, \qquad s \in \mathcal{C}(t), \tag{5.13}$$

where the relation $k_1/k_0 = \alpha - \beta$ has been used. It will emerge later that this is not a new condition, but is equivalent to (4.15), as of course it must be.

Outside of the contact region

$$\begin{aligned} [k * \Delta](s,t) = k_0\Delta(s,t) + k_1 \int_{t_1(s)}^{t} dt'\, e^{-\beta(t-t')} \Delta(s,t') \\ + \frac{k_1}{\eta(1-\xi)} \int_{t_2(s)}^{t_1(s)} dt'\, e^{-\beta(t-t')} \Delta_0(s,t') \\ + \frac{k_1}{(1-\xi)} \int_{t_1(s)-T}^{t_2(s)} dt'\, e^{-\beta(t-t')} \left[ \frac{1}{\eta}\Delta_0(s,t') + \Delta_1^{(p)}(s,t') \right], \qquad s \notin \mathcal{C}(t), \end{aligned} \tag{5.14}$$

where $t_1(s)$ is now the time of last exit of $s$ from the contact region. The quantity $\Delta_1^{(p)}(s,t')$, $t' \in [t_1(s)-T, t_2(s)]$ in the last integral of (5.14) is equal to $\Delta_1(s,t'+T)$, thus ensuring periodicity. Also,

$$\begin{aligned} [k * p](s,t) = k_0 p(s,t) + k_1 \int_{t_1(s)}^{t} dt'\, e^{-\beta(t-t')} p(s,t') \\ + \frac{k_1}{(1-\xi)} \int_{t_1(s)-T}^{t_2(s)} dt'\, e^{-\beta(t-t')} p(s,t'), \qquad s \in \mathcal{C}(t), \end{aligned} \tag{5.15}$$

where $t_1(s)$ is, in this case, the time of last entry of $s$ to the contact region, and so on. Outside of the contact region

$$[k * p](s,t) = \frac{k_1}{(1-\xi)} \int_{t_2(s)}^{t_1(s)} dt'\, e^{-\beta(t-t')} (s,t'), \qquad s \notin \mathcal{C}(t), \tag{5.16}$$

where $t_1(s)$ is the time of last exit of $s$ from the contact region.

We now re-express these in terms of variables fixed in space and use this form to revert to coordinates fixed in the cylinder, as was done in Section 4. The procedure will be illustrated in one case, namely (5.15). Using the angle variables $u, v$ defined by (3.26) (we could also use $\psi, \psi'$), equation (5.15) becomes, on recalling (3.24),

$$\begin{aligned}[k*p](s,t) = [k*p](u) = k_0 p(u) + \frac{k_1}{\omega}\int_{\theta_a}^{u} dv\, e^{-\beta(u-v)/\omega} p(v) \\ + \frac{k_1}{\omega(1-\xi)}\int_{\theta_a-2\pi}^{\theta_b-2\pi} dv\, e^{-\beta(u-v)/\omega} p(v),\end{aligned} \tag{5.17}$$

which then becomes, putting $s = Re^{i(u-\omega t)}$, $s' = Re^{i(v-\omega t)}$,

$$\begin{aligned}[k*p](s,t) = k_0 p(s,t) - \frac{ik_1}{\omega} s^q \int_{a(t)}^{s} ds'\, (s')^{-q-1} p(s',t) - \frac{ik_1\xi s^q}{\omega(1-\xi)} N(t), \\ N(t) = \int_{a(t)}^{b(t)} ds'\, (s')^{-q-1} p(s',t), \qquad s \in C(t).\end{aligned} \tag{5.18}$$

The extra factor $\xi$ in the numerator of the third term comes from shifting from $[\theta_a - 2\pi, \theta_b - 2\pi]$ to $[\theta_a, \theta_b]$ as the range of the last integral in (5.17). Similarly, (5.16) becomes

$$[k*p](s,t) = -\frac{ik_1\xi s^q}{\omega(1-\xi)} N(t), \qquad s \notin C(t). \tag{5.19}$$

Comparing (5.18) and (5.19), we see that $[k*p](s,t)$ is continuous. Let us now consider (5.13) and (5.14). Condition (5.13) becomes

$$\int_{a(t)}^{b(t)} ds'\, (s')^{-q-1} \Delta_1(s',t) = -\frac{C_1 b^{p-q}\xi(\eta-1)e^{-\alpha t}}{p-q}, \tag{5.20}$$

which can be seen, with the aid of (5.7), to be equivalent to (4.15), as mentioned earlier.

Last, we consider (5.14). This becomes

$$\begin{aligned}[k*\Delta](s,t) = k_0\Delta(s,t) - \frac{ik_1 s^q}{\omega}\int_{b(t)}^{s} ds'\, (s')^{-q-1} \Delta(s',t) \\ - \frac{i\xi k_1 s^q}{\omega(1-\xi)}\left\{\int_{a(t)}^{b(t)} ds'\, (s')^{-q-1} \Delta_0(s',t)\right. \\ \left. - \int_{a(t)}^{b(t)} ds'\, (s')^{-q-1} \left[\Delta_0(s',t) + \Delta_1(s',t)\right]\right\}, \qquad s \notin C(t),\end{aligned} \tag{5.21}$$

where the last two integrals have been time-shifted before transformation resulting in the factor $\xi$ and the removal of the factor $1/\eta$ on $\Delta_0(s,t')$. The terms in brackets involving $\Delta_0(s',t)$ may be calculated explicitly, and the terms involving $\Delta_1(s,t)$ are given with the aid of (5.13) or (5.20). One obtains

$$\begin{aligned}[k*\Delta](s,t) = k_0\Delta(s,t) - \frac{ik_1 s^q}{\omega}\int_{b(t)}^{s} ds'\, (s')^{-q-1} \Delta(s',t) \\ - \frac{i\xi k_1 s^q b^{p-q}}{\omega(p-q)} C_1 s^q e^{-\alpha t}, \qquad s \notin C(t).\end{aligned} \tag{5.22}$$

It is convenient to shift the range of integration to $[a(t), s]$, which can be done by using (5.20) once more. Also, on integrating the $\Delta_0(s,t)$ term explicitly, certain cancellations occur and one finally obtains

$$[k*\Delta](s,t) = k_0\Delta_1(s,t) - \frac{ik_1 s^q}{\omega}\int_{a(t)}^{s} ds'\, (s')^{-q-1} \Delta_1(s',t), \qquad s \notin C(t). \tag{5.23}$$

In the contact region, $D_1(\underset{\sim}{r},t)$, defined by (5.4), has the form (recall (5.11))

$$D_1(\underset{\sim}{r},t) = -c_1\left\{\hat{k}(-\omega)\int_{a(t)}^{s} ds'\, p(s',t) - \frac{k_1}{\beta - i\omega}\left[s^{q+1}\int_{a(t)}^{s} ds'\,(s')^{-q-1}p(s',t) + \frac{\xi\left(s^{q+1}-a^{q+1}\right)N(t)}{1-\xi}\right]\right\}, \qquad s \in C(t), \tag{5.24}$$

while off the contact region

$$D_1(\underset{\sim}{r},t) = i\hat{k}(-\omega)\fint_{a(t)}^{s} ds'\,\Delta_1(s',t) - \frac{k_1}{\beta - i\omega}\left[is^{q+1}\fint_{a(t)}^{s} ds'\,(s')^{-q-1}\Delta_1(s',t) - \frac{c_1\xi\left(s^{q+1}-a^{q+1}\right)N(t)}{1-\xi}\right], \qquad s \notin C(t). \tag{5.25}$$

Comparing $D_1(\underset{\sim}{r},t)$ as $s \to b(t)$ from both directions and using (5.6), we find that the condition for continuity in the displacement is

$$\fint_{a(t)}^{b(t)} ds'\,\Delta_1(s',t) = -c_1P(t) - 2\pi A(t) + \pi\hat{l}(-\omega)s_0(t) - \frac{C_1(\eta-1)b^{p+1}}{p+1}e^{-\alpha t}, \tag{5.26}$$

where condition (5.20) has been invoked and also the relation

$$\frac{k_1}{(\beta - i\omega)\hat{k}(-\omega)} = \frac{\alpha-\beta}{\alpha - i\omega} = \frac{p-q}{p+1}. \tag{5.27}$$

Note that by virtue of (2.14),

$$c_1P(t) + 2\pi A(t) = P(t) + 6\pi A(t), \tag{5.28}$$

the right-hand side being the form used later.

We now proceed to write down a more detailed expression for $D_1(\underset{\sim}{r},t)$ in both regions. In preparation, we deduce from (5.8) that

$$\int_{a(t)}^{z} dz'\,\phi_1(z',t) = \int_{a(t)}^{z} dz'\,H(z',t) + \frac{C_1pe^{-\alpha t}}{p+1}\left[z^{p+1}\int_{a(t)}^{z} dz'\,\frac{(z')^{-p-1}K_2(z',t)}{X(z',t)} - \int_{a(t)}^{z} dz'\,\frac{K_2(z',t)}{X(z',t)}\right], \tag{5.29}$$

and, using the definition of $K_2(z,t)$ given by (4.12),

$$\int_{a(t)}^{z} dz'\,\frac{K_2(z',t)}{X(z',t)} = X(z,t)D_p + \left[cD_p - abD_{p-1}\right]\int_{a(t)}^{z} dz'\,\frac{1}{X(z',t)}. \tag{5.30}$$

Noting (B.42),(B.44), and (B.49), we find that

$$\begin{aligned}
&\int_{a(t)}^{z} dz'\,\phi_1(z',t) = F(z,t) + F_1(z,t),\\
&F(z,t) = -iX(z,t)\left[\frac{h_1(z-c)}{2} + h_2 - \frac{iD_pC_1pe^{-\alpha t}}{p+1}\right] + \frac{P(t)\log S(z,t)}{\pi i} - i\rho h_2\left[\log\left(-\frac{z}{p}\right) - \log(1+\lambda(z,t))\right],\\
&F_1(z,t) = \frac{C_1pz^{p+1}e^{-\alpha t}}{p+1}\int_{a(t)}^{z} dz'\,\frac{(z')^{-p-1}K_2(z',t)}{X(z',t)},
\end{aligned} \tag{5.31}$$

where $h_1(t)$, $h_2(t)$ are defined by (2.51) and where (4.34) and (B.16) have been used to obtain the coefficient of $\log S(z,t)$. Higher-order load terms have been neglected in doing this. The quantities $S(z,t)$, $\lambda(z,t)$ are defined by (B.42) and (B.47), respectively. Also

$$G(z,t) + G_1(z,t) = \int_{a(t)}^{z} dz' \, (z')^{-q-1} \phi_1(z',t),$$

$$G(z,t) = \int_{a(t)}^{z} dz' \, (z')^{-q-1} H(z',t) - \frac{C_1 p e^{-\alpha t}}{p-q} \int_{a(t)}^{z} dz' \, \frac{(z')^{-q-1} K_2(z',t)}{X(z',t)}, \tag{5.32}$$

$$G_1(z,t) = \frac{C_1 p z^{p-q} e^{-\alpha t}}{p-q} \int_{a(t)}^{z} dz' \, \frac{(z')^{-p-1} K_2(z',t)}{X(z',t)}.$$

For the combination required by (5.24) and (5.25) we see that, in the light of (5.27), the terms involving $F_1(z,t)$ and $G_1(z,t)$ cancel. Thus, we may write these equations in the form

$$D_1(\underline{r},t) = -c_1 \hat{k}(-\omega) \left\{ -iF_a^+(s,t) - \frac{p-q}{p+1}\left[ -is^{q+1} G_a^+(s,t) + \frac{\xi\left(s^{q+1} - a^{q+1}\right) N(t)}{1-\xi} \right] \right\}, \tag{5.33}$$
$$N(t) = -iG_a^+(b(t),t), \qquad s \in C(t),$$

and

$$D_1(\underline{r},t) = \hat{k}(-\omega) \left\{ iF_c(s,t) - \frac{p-q}{p+1}\left[ is^{q+1} G_c(s,t) - \frac{c_1 \xi\left(s^{q+1} - a^{q+1}\right) N(t)}{1-\xi} \right] \right\}, \qquad s \notin C(t), \tag{5.34}$$

where the subscripts "$a$", "$c$" indicate counterclockwise and clockwise contours in equations (5.31) and (5.32) while the "+" superscript means that one approaches $C(t)$ from inside the cylinder. Equations (5.33) and (5.34) are the most explicit forms attainable for the displacements.

Note that the relation for $N(t)$ in (5.33) follows from (5.18) and the fact that $G_{1a}^+(b(t),t)$ is zero (see (4.26)).

By considering $F_c(b(t),t) + F_{1c}(b(t),t)$, one can show that (5.26) holds, using (4.13),(5.28), the comment after (B.43) and (B.51). We can also make aspects of the continuity requirement more explicit. Note that by virtue of (4.16)–(4.18),

$$G_c(s,t) = T_1(s,t) - \frac{T_1(b(t),t)}{T_2(b(t),t)} T_2(s,t),$$
$$T_1(s,t) = \int_{a(t)}^{s} ds' \, (s')^{-q-1} H(s',t), \tag{5.35}$$
$$T_2(s,t) = \int_{a(t)}^{s} ds' \, \frac{(s')^{-q-1} K_2(s',t)}{X(s',t)},$$

where $b(t)$ is approached clockwise. Thus, $G_c(b(t),t)$ vanishes.

It remains to write down the complex displacements in terms of coordinates fixed in space. The following relations are helpful in this task:

$$\frac{s}{s_c} = e^{i\psi}, \qquad \frac{s_c}{s_0} = e^{i\delta}, \qquad s_c = Re^{i(\theta_c - \omega t)} = -\rho, \tag{5.36}$$

where $\delta$ is defined in (4.35). We start with the explicit term $\Gamma(s,t)$ given by (5.5)

$$\Gamma(s,t) = s_0(t)\left[\frac{1}{2} L_1 e^{2i\delta}\left(e^{2i\psi} - e^{-2i\psi_0}\right) + iL_2(\psi + \psi_0)\right],$$
$$L_1 = -\frac{1}{2} + \frac{i\bar{W}_1 \hat{k}(\omega)}{R}, \tag{5.37}$$
$$L_2 = \frac{1}{2} - \frac{iW_1 \hat{k}(-\omega)}{R} = \bar{L}_1 + 1,$$

in the notation of (2.14). In the contact region, we keep only second-order terms in $\psi_0$, obtaining

$$\begin{aligned} \Gamma(s,t) &= \frac{s_0(t)\psi_0^2}{2}\left[(x+D_0)^2-(1-D_0)^2\right], \\ x &= \frac{\psi}{\psi_0}, \qquad |x| \le 1, \\ D_0 &= \frac{\delta}{\psi_0}. \end{aligned} \tag{5.38}$$

Now, consider $D_1(\underset{\sim}{r},t)$, focusing first on (5.34),

$$\begin{aligned} F_c(s,t) = s_0(t)\Bigg\{ &-i\sqrt{2}\,e^{i(\psi/2+\delta)}\Sigma(\psi)\left[-\frac{C\alpha d_0}{\alpha-i\omega}+\frac{\bar{H}_1}{2}e^{i\delta}\left(e^{i\psi}-\cos\psi_0\right)-H_1e^{-i\delta}\right] \\ &+\frac{(W-iH)\log S(\psi)}{i\pi R}+H_1[i\psi-\log(1+\lambda(\psi))]\Bigg\}, \end{aligned} \tag{5.39}$$

where $H_1$ is given by (2.52) and

$$\begin{aligned} \Sigma(\psi) &= (\cos\psi_0-\cos\psi)^{1/2} = \sqrt{2}\left(\sin^2\frac{\psi}{2}-\sin^2\frac{\psi_0}{2}\right)^{1/2}, \\ S(\psi) &= \frac{1}{\sin\psi_0}\left[\sqrt{2}\,e^{i\psi/2}\Sigma(\psi)+i\left(e^{i\psi}-\cos\psi_0\right)\right] \\ &= \frac{e^{i\psi/2}}{\sin\psi_0}\left[\sqrt{2}\,\Sigma(\psi)-2\sin\frac{\psi}{2}+ie^{-i\psi/2}(1-\cos\psi_0)\right], \\ \lambda(\psi) &= \frac{2ie^{i\psi}\cos(\psi/2)}{1+\cos\psi_0}\left[\sqrt{2}\,\Sigma(\psi)+2\sin\frac{\psi}{2}-ie^{-i\psi/2}(1-\cos\psi_0)\right]. \end{aligned} \tag{5.40}$$

Also

$$\begin{aligned} is^{q+1}G_c(s,t) = -\sqrt{2}\,s_0(t)e^{i(\psi+\delta)}e^{-\beta\psi/\omega}\Bigg\{ &2\int_{-\psi_0}^{\psi} d\psi'\,e^{\beta\psi'/\omega}\Sigma(\psi')\,\mathrm{Im}\left[\bar{H}_1e^{i(\delta+\psi'/2)}\right] \\ &+\frac{C\alpha}{\alpha-\beta}\int_{-\psi_0}^{\psi} d\psi'\,\frac{\mathrm{Im}\left(d_0e^{i\psi'/2}\right)e^{\beta\psi'/\omega}}{\Sigma(\psi')}\Bigg\}, \end{aligned} \tag{5.41}$$

where, from (5.35), the value of $C$ is such as to render this term zero at $\psi=\psi_0-2\pi$. In fact, the value of $C$ deduced by imposing this condition on (5.41) can be shown to be the same as that given by the first expression of (4.24), after some manipulations similar to those implemented in deriving the second form of (4.24). It is convenient to use this property to write (5.41) in the form

$$\begin{aligned} is^{q+1}G_c(s,t) = -\sqrt{2}\,s_0(t)e^{i(\psi+\delta)}e^{-\beta\psi/\omega}\Bigg\{ &2\int_{\psi_0-2\pi}^{\psi} d\psi'\,e^{\beta\psi'/\omega}\Sigma(\psi')\,\mathrm{Im}\left[\bar{H}_1e^{i(\delta+\psi'/2)}\right] \\ &+\frac{C\alpha}{\alpha-\beta}\int_{\psi_0-2\pi}^{\psi} d\psi'\,\frac{\mathrm{Im}\left(d_0e^{i\psi'/2}\right)e^{\beta\psi'/\omega}}{\Sigma(\psi')}\Bigg\}, \end{aligned} \tag{5.42}$$

which avoid exponentials with positive arguments. These integrals can be transformed as in equations (4.23).

The last term $(s^{q+1} - a^{q+1})N(t)$ occurs also in (5.33) which we now address. Only the leading term will be retained and the load term will be dropped where they occur with clearly dominant contributions. We have

$$-is^{q+1}G_a^+(s,t) = is_0(t)e^{-\beta' x}\left\{\psi_0^2 l_1(\omega)\int_{-1}^{x} dy\, e^{\beta' y}\left(1-y^2\right)^{1/2}\right.$$
$$\left. - \frac{C\alpha\psi_0}{\alpha-\beta}\int_{-1}^{x} dy\,\frac{\left[yI_0(\alpha') - I_1(\alpha')\right]e^{\beta' y}}{\left(1-y^2\right)^{1/2}}\right\} = s_0(t)G_s(x), \qquad x = \frac{\psi}{\psi_0}. \tag{5.43}$$

We have, recalling (5.33),

$$G_s(1) = \frac{b^{q+1}N(t)}{s_0(t)} = i\pi\psi_0^2 l_1(\omega)\frac{e^{-\beta'}}{\beta'}I_1(\beta') - \frac{i\pi C\alpha\psi_0 e^{-\beta'}}{\alpha-\beta}$$
$$\times\left[I_0(\alpha')I_1(\beta') - I_0(\beta')I_1(\alpha')\right], \tag{5.44}$$

$$\beta' = \frac{\beta\psi_0}{\omega}, \tag{5.45}$$

so that

$$\left(s^{q+1} - a^{q+1}\right)N(t) = \begin{cases} s_0(t)G_s(1)e^{2\beta' - 2i\psi_0}\left[e^{-(\beta/\omega - i)(\psi+\psi_0)} - 1\right], & |\psi| > \psi_0, \\ s_0(t)G_s(1)e^{2\beta'}\left(e^{-\beta'(1+x)} - 1\right), & |\psi| \le \psi_0. \end{cases} \tag{5.46}$$

Finally, we consider $-iF_a^+(s,t)$. Two subsidiary results are required. To lowest order (since it is multiplied by the load)

$$\frac{1}{i\pi}\log S(\psi) = -\frac{1}{\pi}\left(\frac{\pi}{2} + \sin^{-1}x\right). \tag{5.47}$$

Also, to second order (see (B.50))

$$\lambda(\psi) - \frac{\lambda^2(\psi)}{2} = \psi_0\left[\left(1-x^2\right)^{1/2} + ix\right]. \tag{5.48}$$

It follows that

$$-iF_a^+(s,t) = is_0(t)\left\{\frac{1}{2}l_1(\omega)\psi_0^2 x\left(1-x^2\right)^{1/2}\right.$$
$$\left. + \frac{C\alpha\psi_0\left(1-x^2\right)^{1/2}}{\alpha - i\omega}I_0(\alpha') + \frac{\left(\pi/2 + \sin^{-1}x\right)W}{\pi R}\right\}, \tag{5.49}$$

where terms of order $C\psi_0^2$ have been neglected.

Complete expressions for the complex displacement are given by adding $\Gamma(s,t)$ given by (5.37) to (5.34) for $s \notin \mathcal{C}(t)$ and to (5.33) for $s \in \mathcal{C}(t)$ where the various components are set out in (5.39)–(5.49). The quantity $D(\underline{a},t)$ is deduced from the condition that $D(\underline{r},t)$ vanishes at $s = s_0(t)$ so that

$$D(\underline{a},t) = -\Gamma(s_0(t),t) - D_1(\underline{r}_0(t),t), \tag{5.50}$$

where $\underline{r}_0(t)$ is the vector corresponding to $s_0(t)$ and $D_1(\underline{r}_0(t),t)$ is found by using (5.33). The elastic limit is given by

$$D(\underline{r},t) = \begin{cases} s_0(t)\left\{\dfrac{\psi_0^2}{2}\left[x^2 - ic_1x\left(1-x^2\right)^{1/2}\right] - i\dfrac{c_1Wk_0}{\pi R}\sin^{-1}x\right\}, & |\psi| \le \psi_0, \\ s_0(t)\left\{\dfrac{L_0}{2}\left(e^{2i\psi} - e^{-2i\psi_0}\right) + i(L_0+1)(\psi+\psi_0)\right. & \\ \quad - iH_0\left[\sqrt{2}\,e^{i\psi/2}\Sigma(\psi)\left[\dfrac{1}{2}\left(e^{i\psi} - \cos\psi_0\right) + 1\right]\right. & \\ \quad \left.\left. + \psi + i\log[1+\lambda(\psi)]\right] + \dfrac{Wk_0}{\pi R}\log S(\psi)\right\} + D(\underline{a},t), & |\psi| > \psi_0, \end{cases} \tag{5.51}$$

where $S(\psi)$ and $\lambda(\psi)$ are defined by (5.40) and, recalling (2.14)

$$L_0 = -\frac{1}{2}[1 - V_0], \qquad H_0 = \frac{1}{2}[1 + 3V_0], \qquad V_0 = \frac{Wk_0}{4\pi(1-\nu)R}, \tag{5.52}$$

and the quantity $D(\underline{a}, t)$ is deduced from (5.51) to be

$$D(\underline{a}, t) = \frac{s_0(t)}{2}\left[\psi_0^2 + i\frac{c_1 W k_0}{R}\right]. \tag{5.53}$$

The quantities $\psi_0$ and $W$ are related by the elastic version of (4.36) which has the form

$$\frac{\pi}{2}\psi_0^2 l_0 R = W. \tag{5.54}$$

Equation (4.39) gives that the asymmetry parameter $\delta$ vanishes.

Continuity of the elastic displacement given by (5.51) can be demonstrated on neglecting terms of order $V_0\psi_0$.

## 6. NUMERICAL RESULTS

Using the relation for $l_2(\omega)$ defined in (4.39),

$$l_2(\omega) = \frac{l_0 r \alpha \omega}{\alpha^2 + \omega^2} = \frac{l_0 r \omega}{\alpha\,(1 + (\omega/\alpha)^2)}, \tag{6.1}$$

in terms of the parameter $r$, given by (see (4.1))

$$r = \frac{\alpha - \beta}{\alpha} = \frac{k_1}{\alpha k_0}, \tag{6.2}$$

we can write $C$, given by (4.24), in the form

$$C = -\frac{l_2(\omega)}{2\alpha}\,\frac{\operatorname{Im}\left[(1 + 6\Lambda_1(\omega))e^{i\delta}(\alpha - i\omega)(e_0\cos\psi_0 - \bar{e}_0)\right]}{\operatorname{Im}(d_0 e_0)}. \tag{6.3}$$

Also, from (4.39)

$$C = \frac{l_2(\omega)}{I_0(\alpha')}\left[1 - \frac{\delta^2}{2} + \delta r_l + \frac{\psi_0^2(1 + r_2)}{16}\right], \qquad r_2 = \frac{I_2(\alpha')}{I_0(\alpha')}, \qquad r_l = \frac{l_1(\omega)}{l_2(\omega)}. \tag{6.4}$$

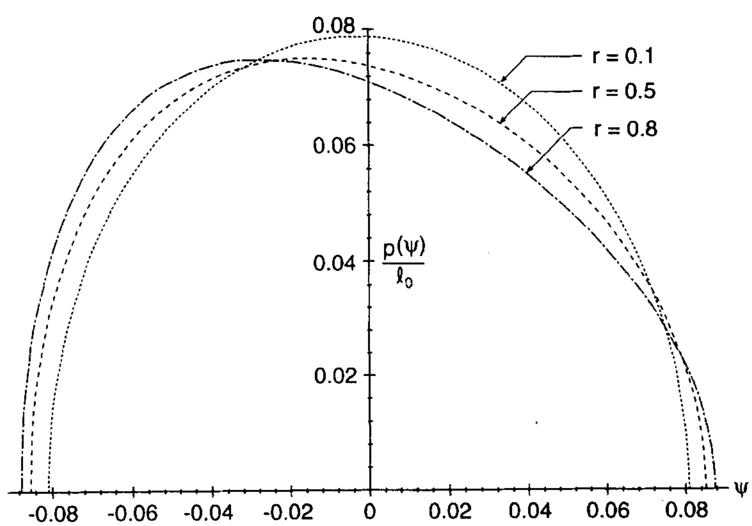


Figure 1. The pressure function $p(\psi)$, given by (4.33), for these values of r, with $\alpha/\omega = 10.0$ and $W/(l_0 R) = 0.01$.

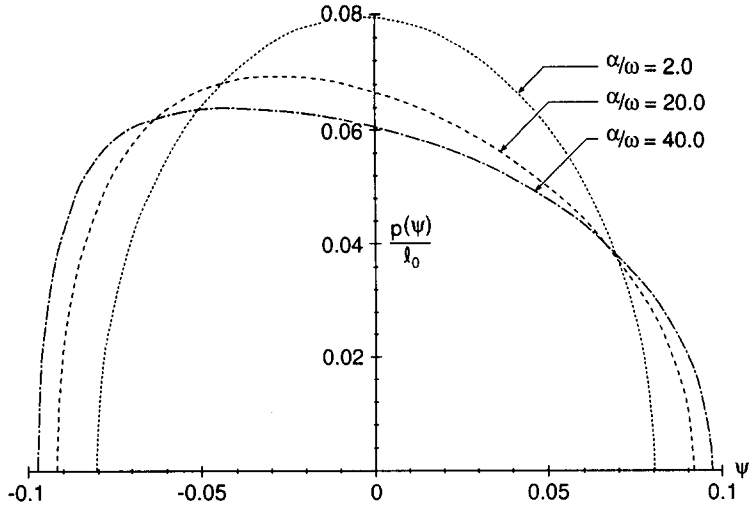


Figure 2. The pressure function $p(\psi)$, given by 4.33, for three values of $\alpha/\omega$, with $r = 0.5$ and $W/({}_0R) = 0.01$.

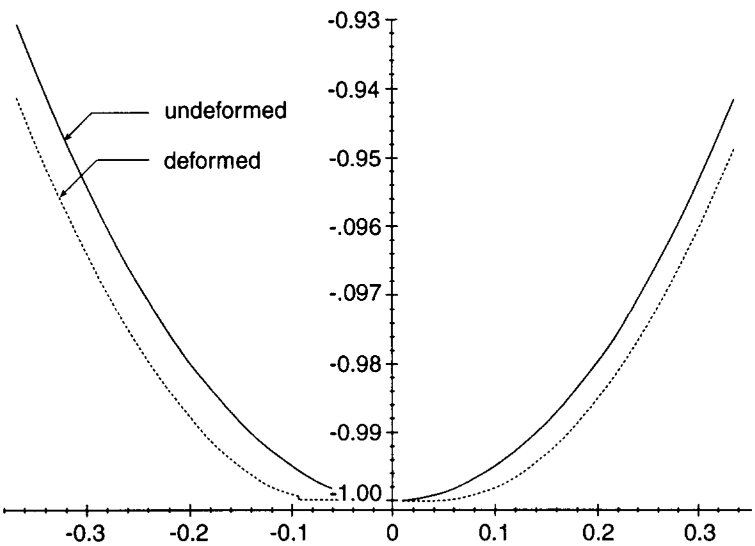


Figure 3. The deformed and undeformed cylinder in the vicinity of the contact region for $r = 0.5$, $\alpha/\omega = 0.5$, and $W/(l_0R) = 0.01$. The axes are not the same scale.

From (6.3) and (4.32), we find that the zeroth order estimate (in $\psi_0$ and $\delta$) of $C$ is $l_2(\omega)/I_0(\alpha')$, in agreement with (6.4). Expanding (6.3) in powers of $\psi_0$ and $\delta$ comparing with (6.4), we find that $\delta$ is of order $\psi_0$ (including a $\psi_0 \log \psi_0$ term). After some algebra it may be determined up to order $\psi_0^2$.

The issue of the order of $C$ can now be settled. Using the zeroth-order term for $C$, i.e., $l_2(\omega)/I_0(\alpha')$ and the second form of $l_2(\omega)$ given by (6.1), we see that $C\alpha'$ is of order of $l_0\psi_0$. This is also true of $CI_1(\alpha')$ since $I_1(\alpha')/(\alpha' I_0(\alpha'))$ is of order unity or less in the range of interest of $\alpha'$. The quantity $CI_2(\alpha')$ is at least of order $l_0\psi_0$, since $r_2/\alpha'$ is less than order unity in the range of interest. These observations show that the terms proportional to $C$ in (4.33), (4.36), (4.37), and (5.43) are of the correct order.

Numerical results for pressure, displacements, and hysteretic friction are now presented. In all cases, the material is taken to have Poisson's ratio equal to 0.4.

The pressure function $p(\psi)$, given by (4.33), is plotted in Figures 1 and 2 for three values of $r$ with $\alpha/\omega$ fixed in the first case, and for three values of $\alpha/w$ with $r$ fixed in the second case. We

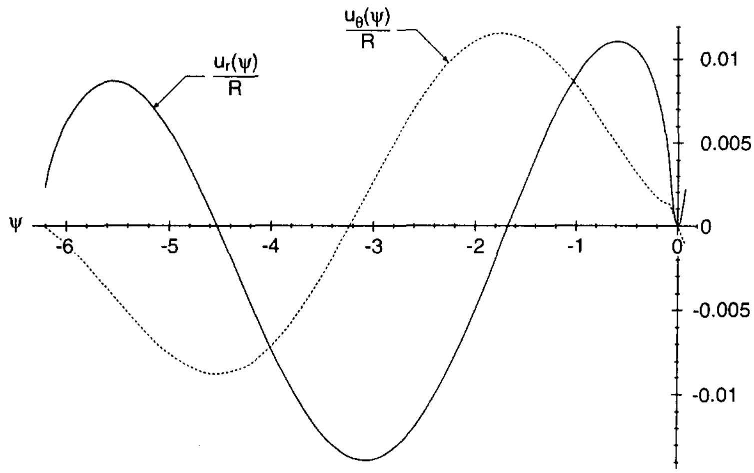


Figure 4. The radial and angular surface displacements for $r = 0.5$, $\alpha/\omega = 5.0$, and $W/(l_0 R) = 0.0.1$.

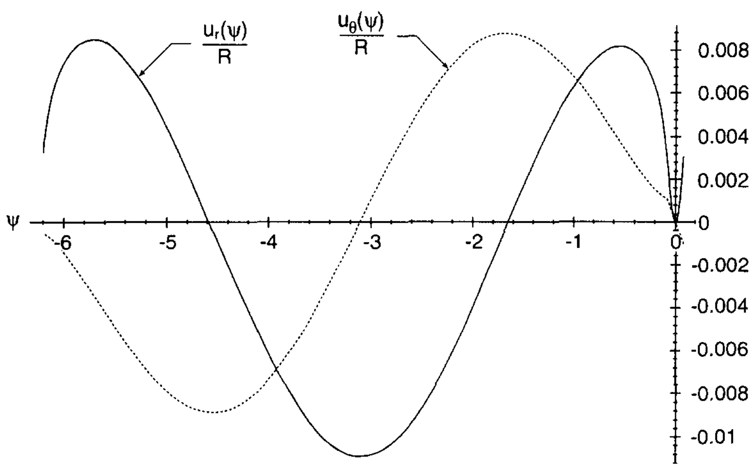


Figure 5. The radial and angular surface displacements for an elastic material, with $w/(l)0R) = 0.01$.

see that the asymmetry of the curve increases as $r$ increases for fixed $\alpha/\omega$ and as $\alpha/\omega$ increases for fixed $r$.

The deformed and nondeformed cylinder in the vicinity of the contact region is plotted on Figure 3, for a specified material, using the results of Section 5. The scales are chosen to be different on the two axes to magnify the separation of the curves. One sees that the flat contact region is reproduced quite well even through the formulae contain approximations, i.e., are correct to order $\psi_0^2$.

The radial and angular displacement at all points on the cylinder surface are plotted in Figure 4 for the same material. Comparable results for an elastic material with modulus $l_0$ are presented in Figure 5. For $u_r(\psi)$, the front positive and subsequent negative peaks are more pronounced in the viscoelastic case, while the rear positive peaks are similar in both cases. For $u_\theta(\psi)$, the front positive peak is higher in the viscoelastic case, while the two rear negative peaks are similar.

On these plots of displacements, discontinuities are detectable at the ends of the contact region, particularly for $u_\theta(\psi)$ in Figure 4. Also, discontinuities in $\frac{du_\theta(\psi)}{d\psi}$ are detectable at these endpoints, while this quantity should in fact be continuous. This is to be expected, given that the displacements are calculated only to order $\psi_0^2$, and terms of order $\psi_0^3$ ($\sim 0.0005$) are neglected. The discontinuities are of this order, and are detectable in the magnified presentation of Figure 4.

The hysteretic friction is plotted against a velocity related variable in Figure 6, for various values of $r$, which is a measure of viscous energy losses. The angle $\psi_e$ is the half contact angle

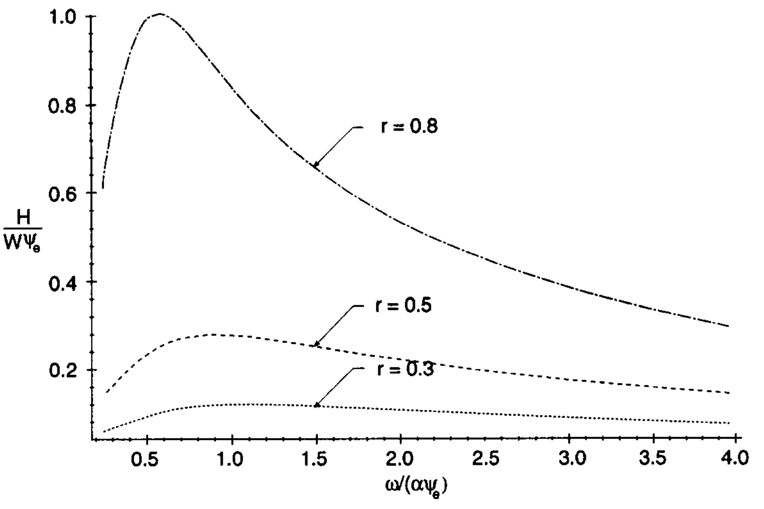


Figure 6. The hysteretic friction coefficient plotted against a velocity related variable for $w/(l_0 R) = 0.01$. The quantity $\psi_e = [2W/\pi l_0 R)]^1/2$.

for the instantaneous elastic limit. Note that the maximum occurs near $\alpha\psi_e/\omega \sim 1$ in each case. This variable is of course approximately equal to $\alpha'$ defined in (4.21).

In contrast to the problem of a rigid indentor on a viscoelastic half-space, the expression for the hysteretic friction in the present case depends not only on $\alpha'$, $\beta'$ but also on $e^{2\pi\beta/\omega}$—for example, through the expression for $e_0$ given by (4.23). This is interesting in that it indicates that the periodic return of each surface point to the contact region makes a contribution to the hysteretic friction—which is in agreement with physical intuition. This effect does not, however, show up in a dramatic way on the hysteretic friction curve; i.e., there would appear to be no, maximum, for example, around $\alpha/\omega \sim 1$. Such a feature would in fact occur outside of the range shown in Figure 6, but did not emerge even on plots over a more extensive range.

The dominant effect is the classic one, namely that related to pressure asymmetry in the contact patch, and the crucial variables are $\alpha'$, $\beta'$.

## APPENDIX A
## FORM OF STRESSES AND DISPLACEMENTS

In this appendix, we show that stresses and displacements in polar form, using coordinates fixed in the cylinder, depend only on $\theta + \omega t$ rather than on $\theta$, $t$, separately, if steady-state conditions prevail. Some consequences of this are also noted.

In nonrotating coordinates fixed at the centre of the cylinder, all stresses and displacements are independent of time. We use a prime to indicate these quantities. The relationship between primed and unprimed quantities will now be written down, under the assumption that the two sets of axes coincide at $t = 0$. We have for Cartesian components [11] but using polar coordinates fixed in the cylinder to indicate positions

$$\begin{aligned}
[\sigma_{xx} + \sigma_{yy}]\,(r,\theta,t) &= \left[\sigma'_{xx} + \sigma'_{yy}\right](r,\theta+\omega t),\\
[\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}]\,(r,\theta,t) &= e^{2i\omega t}\left[\sigma'_{yy} - \sigma'_{xx} + 2i\sigma'_{xy}\right](r,\theta+\omega t),\\
[u_x + iu_y]\,(r,\theta,t) &= e^{-i\omega t}\left[u'_x + iu'_y\right](r,\theta+\omega t).
\end{aligned} \tag{A.1}$$

Now, switching to polar components given by

$$\begin{aligned}
[\sigma_{rr} + \sigma_{\theta\theta}]\,(r,\theta,t) &= [\sigma_{xx} + \sigma_{yy}]\,(r,\theta,t),\\
[\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta}](r,\theta,t) &= e^{2i\theta}\left[\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}\right](r,\theta,t),\\
[u_r + iu_\theta](r,\theta,t) &= e^{-i\theta}\left[u_x + iu_y\right](r,\theta,t),
\end{aligned} \tag{A.2}$$

we see that the unprimed polar components of stress and displacement (but not the unprimed Cartesian components) are functions of $\theta + \omega t$ only. Inspections of (2.1) and (2.15) show that we can achieve this dependence by restricting $\phi(z,t)$ and $z^2\psi(z,t)$ to depend only on $ze^{i\omega t}$ rather than on $z, t$ separately. It follows that $[\mu * D'](\underline{r}, t)/iz$ depends only on $(r, \theta+\omega t)$. The convolution product does not influence this property, so that $\frac{\partial D}{(iz\partial\theta)}$ also has the same restricted dependence. This is consistent with (2.8).

# APPENDIX B
# INTEGRALS

We present or derive certain integrals in this appendix, which are required in the main text. Consider first integrals of the form

$$I_n(z,t) = \frac{1}{i\pi}\int_{a(t)}^{b(t)} \frac{ds\, s^n}{(s-z)X^+(s,t)}, \qquad \text{for } n = -1, 0, 1, 2. \tag{B.1}$$

This integral may be evaluated using general formulae of Gladwell [13]. However, we can confine ourselves here to those actually required. The basic relationship is

$$I_0(z,t) = \frac{1}{X(z,t)}, \tag{B.2}$$

which can be shown by using Cauchy's theorem with a contour around the cut and over the infinite boundary [13]. It follows by simple manipulation that

$$I_{-1}(z,t) = \frac{1}{zX(z,t)} - \frac{1}{z\rho}, \qquad I_1(z,t) = \frac{z}{X(z,t)} - 1, \qquad I_2(z,t) = \frac{z^2}{X(z,t)} - z - c, \tag{B.3}$$

where

$$c = \frac{a+b}{2}, \qquad \rho = X(0,t) = [a(t)b(t)]^{1/2}. \tag{B.4}$$

The relations

$$\begin{aligned}
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{1}{sX^+(s,t)} &= \frac{1}{\rho},\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{1}{X^+(s,t)} &= -1,\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{s}{X^+(s,t)} &= -c,\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{s^2}{X^+(s,t)} &= -\frac{1}{8}\left[3\left(a^2+b^2\right)+2ab\right],
\end{aligned} \tag{B.5}$$

follow by considering the limiting behaviour of the appropriate $I_n(z)$ as $z$ tends to zero and infinity. The first three of these relations have been used in deriving (B.3). The relation

$$\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{X^+(s,t)}{s-z} = [c(t) - z] + X(z,t) \tag{B.6}$$

is also required. This follows from (B.2) and (B.3). By considering the limit of (B.6), at large and small $z$, and its derivatives at small $z$ we see that

$$\begin{aligned}
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\, X^+(s,t) &= \frac{1}{8}(a-b)^2,\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{X^+(s,t)}{s} &= c+\rho,\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{X^+(s,t)}{s^2} &= -1-\frac{c}{\rho},\\
\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds\,\frac{X^+(s,t)}{s^3} &= -\frac{(a-b)^2}{8\rho^3}.
\end{aligned} \tag{B.7}$$

Also, using the Plemelj formulae, we obtain

$$\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds' \, \frac{X^+(s',t)}{s'-s} = c - s, \qquad s \in [a(t), b(t)], \tag{B.8}$$

where the principal value of the integral is understood. Finally, we have

$$\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds \, \frac{sX^+(s,t)}{s-z} = \frac{1}{8}(a-b)^2 + z(c-z) + zX(z,t), \tag{B.9}$$

giving

$$\frac{1}{i\pi}\int_{a(t)}^{b(t)} ds \, sX^+(s,t) = \frac{c(a-b)^2}{8}. \tag{B.10}$$

We also need to consider values of $n$ that are nonintegral, in fact complex. Consider the integral

$$I_p(z,t) = \frac{1}{i\pi}\int_{a(t)}^{b(t)} ds \, \frac{s^p}{(s-z)X^+(s,t)}, \tag{B.11}$$

where $p$ is given by (4.5). We will evaluate this by first considering $I_q(z,t)$ where, in this appendix,

$$q = p - 1, \tag{B.12}$$

though not in the main text. One can easily show that

$$I_p = zI_q + D_q, \qquad D_q = \frac{1}{i\pi}\int_{a(t)}^{b(t)} ds \, \frac{s^q}{X^+(s,t)}. \tag{B.13}$$

It is convenient to start by considering the related integral

$$\begin{aligned} J_q(z,t) &= \frac{1}{i\pi}\int_{a(t)}^{b(t)} ds \, \frac{s^q X^+(s,t)}{s-z} = S(z,t)I_q(z,t) + D_{q+1} + (z-(a+b))D_q, \\ S(z,t) &= (z-a(t))(z-b(t)) = X^2(z,t). \end{aligned} \tag{B.14}$$

Let us define

$$\begin{aligned} J_0(z,t) &= J_q(z,t) - J_q(0,t) = J_q(z,t) - D_{q+1} + (a+b)D_q - abD_{q-1} \\ &= J_q(z,t) + \frac{1}{q}\left[D_{q+1} - cD_q\right]. \end{aligned} \tag{B.15}$$

The latter form is proved by partial integration. Comparing the two ways of evaluating $J_q(0,t)$ leads to the identity

$$\frac{(q+1)}{q}D_{q+1} - \frac{c}{q}(2q+1)D_q + abD_{q-1} = 0. \tag{B.16}$$

We have

$$\frac{\partial}{\partial z}J_0(z,t) = \frac{q}{z}J_0(z,t) + \frac{1}{i\pi}\int_a^b ds \, \frac{s^q(s-c)}{X^+(s,t)(s-z)}, \tag{B.17}$$

so that, putting

$$U(z,t) = \frac{J_0(z,t)}{X(z,t)}, \tag{B.18}$$

we obtain after some algebra

$$\begin{aligned} \frac{\partial}{\partial z}U(z,t) &= \frac{q}{z}U(z,t) - \frac{zD_1 + D_2}{S(z,t)X(z,t)}, \\ D_1 &= \frac{q+1}{q}\left[D_{q+1} - cD_q\right], \\ D_2 &= -\frac{c(q+1)}{q}D_{q+1} - \left[ab - \frac{c^2(2q+1)}{q}\right]D_q. \end{aligned} \tag{B.19}$$

We can rewrite this equation as

$$
\begin{aligned}
\frac{\partial}{\partial z} U(z,t) &= \frac{q}{z} U(z,t) - \frac{\partial}{\partial z} \frac{zD_3 + D_4}{X(z,t)}, \\
D_3 &= -D_q, \\
D_4 &= -\frac{q+1}{q} D_{q+1} + \frac{c(2q+1)}{q} D_q = abD_{q-1},
\end{aligned}
\tag{B.20}
$$

the last relation following from (B.16). Solving the differential equation, we obtain

$$
J_q(z,t) = D_{q+1} + D_q[z - (a+b)] - qz^q X(z,t) \int_0^z dz' \, \frac{(z')^{-q-1} \left(abD_{q-1} - z'D_q\right)}{X(z',t)}, \tag{B.21}
$$

so that, from (B.14),

$$
I_q(z,t) = -\frac{qz^q}{X(z,t)} \int_0^z dz' \, \frac{(z')^{-q-1}(abD_{q-1} - z'D_q)}{X(z',t)}. \tag{B.22}
$$

One can show that this has the correct behaviour at large and small $z$. Finally, from (B.13) we deduce that

$$
\begin{aligned}
I_p(z,t) &= D_{p-1} - \frac{(p-1)z^p}{X(z,t)} \int_0^z dz' \, \frac{(z')^{-p} K_1(z',t)}{X(z',t)}, \\
K_1(z',t) &= abD_{p-2} - z'D_{p-1}.
\end{aligned}
\tag{B.23}
$$

The quantity $I_p(z,t)$ is analytic except on the cut over $[a,b]$. It cannot matter which direction we come around to approach the cut from outside the cylinder. Therefore, we must have

$$
\int_{a(t)}^{b(t)} ds \, \frac{s^{-p} K_1(s,t)}{X^+(s,t)} = 0. \tag{B.24}
$$

This can be shown directly by observing, with the help of (2.29) and (2.30), that for example

$$
D_{p-1} = \frac{R^{2p}}{i\pi\rho} \int_{\bar{a}(t)}^{\bar{b}(t)} ds \, \frac{s^{-p}}{\bar{X}^+(s,t)} = -\left(\frac{R^2}{n}\right)^p \frac{1}{\rho} D_{-p} = -(ab)^{p-1/2} D_{-p}, \tag{B.25}
$$

where

$$
\begin{aligned}
n(t) &= e^{2i(\omega t - \theta_c)} = \frac{R^2}{a(t)b(t)} = \frac{\bar{b}(t)}{a(t)} = \frac{\bar{a}(t)}{b(t)}, \\
\bar{X}^+(ns,t) &= nX^+(s,t).
\end{aligned}
\tag{B.26}
$$

Noting that (B.25) holds for arbitrary complex $p$, we obtain, replacing $p$ by $(p-1)$,

$$
D_{p-2} = -\left(\frac{R^2}{n}\right)^p \frac{1}{ab\rho} D_{-p+1} = -(ab)^{p-3/2} D_{-p+1}, \tag{B.27}
$$

giving that

$$
abD_{-p}D_{p-2} - D_{-p+1}D_{p-1} = 0, \tag{B.28}
$$

which is equivalent to (B.24). From (B.11), we have that

$$
\bar{I}_p\left(\frac{R^2}{z}, t\right) = \frac{z\rho}{R^{2(p+1)}} I_p(z,t), \tag{B.29}
$$

and from (B.23) one can show that

$$
\bar{I}_p\left(\frac{R^2}{z}, t\right) = -\frac{\rho}{R^{2(p+1)}} \left\{ D_p + \frac{(p+1)z^{p+1}}{X(z,t)} \int_z^\infty dz' \, \frac{(z')^{-p-2} \left(z'D_{p+1} - D_p ab\right)}{X(z',t)} \right\}, \tag{B.30}
$$

with the aid of the identities

$$\bar{D}_{p-\delta} = -\frac{\rho}{R^{2(p+\delta)}} D_{p+\delta-1}, \tag{B.31}$$

where $\delta$ is real. Thus, we have

$$I_p(z,t) = -\frac{D_p}{z} - \frac{(p+1)z^p}{X(z,t)} \int_z^\infty dz' \, \frac{(z')^{-p-2} \left[ z' D_{p+1} - D_p ab \right]}{X(z',t)}. \tag{B.32}$$

The Plemelj formulae [3] give that

$$I_p^+(s,t) - I_p^-(s,t) = \frac{2s^p}{X^+(s,t)}, \tag{B.33}$$

where $I_p^\pm(s,t)$ are the limits of $I_p(z,t)$ from inside and outside the cylinder, respectively. Combining (B.23) and (B.33) gives that

$$\int_0^{a(t)} dz' \, \frac{(z')^{-p} K_1(z',t)}{X(z',t)} = -\frac{1}{p-1}. \tag{B.34}$$

Taking a contour that is a sector of a circle gives that

$$\int_0^{b(t)} dz' \, \frac{(z')^{-p} K_1(z',t)}{X(z',t)} = -\frac{1}{p-1}. \tag{B.35}$$

As stated in the main text, we take the cut for the power function along the radius vector through $b(t)$ and $\arg(z)$ in the region $[\theta_b - 2\pi, \theta_b]$. The relation (B.35) is for $\arg(z) = \theta_b$. We consider a contour that goes around the cylinder, starting on one side of this cut and ending on the other. It then goes along the cut, around the origin, and back out along the other side of the cut. This gives that

$$⨍_{a(t)}^{b(t)} ds' \, \frac{(s')^{-p} K_1(s',t)}{X(s',t)} = \frac{\eta - 1}{\eta(p-1)}, \tag{B.36}$$

where the slashed integral sign indicates that the integration is over the complement of $\mathcal{C}(t)$ on the cylinder surface. The quantity $\eta$ is given by

$$\eta = e^{-i\pi p}. \tag{B.37}$$

We write (B.36) in the form

$$abE_p D_{p-2} - E_{p-1} D_{p-1} = \frac{\eta - 1}{i\pi\eta(p-1)}, \tag{B.38}$$

where

$$E_\delta = \frac{1}{i\pi} ⨍_{a(t)}^{b(t)} ds \, \frac{s^{-\delta}}{X(s,t)}, \tag{B.39}$$

for arbitrary complex $\delta$. The relationship corresponding to (B.16) for the $E_\delta$ has the form

$$\frac{\delta - 1}{\delta} E_{\delta-1} - \frac{c(2\delta - 1)}{\delta} E_\delta + abE_{\delta+1} = 0, \tag{B.40}$$

and that corresponding to (B.31) is

$$\bar{E}_{q+r} = \rho R^{2(q-r)} E_{q-r+1}, \tag{B.41}$$

where $r$ is real.

Finally, we give certain integrals [20] required in Section 5. First,

$$\int_{a(t)}^{z} \frac{dz}{X(z,t)} = \log S(z,t), \qquad S(z,t) = \frac{2[X(z,t)+z-c]}{a-b}. \tag{B.42}$$

We have

$$⨍_{a(t)}^{b(t)} \frac{ds}{X(s,t)} = \int_{a(t)}^{b(t)} \frac{ds}{X^{+}(s,t)} = -i\pi, \tag{B.43}$$

by virtue of (B.5). This fixes the branch of the log function. In fact, $\arg(S(s,t))$ varies from zero at $s = a(t)$ to $-\pi$ at $s = b(t)$. The modulus of $S(s,t)$ is unity at both limits. Also

$$\int_{a(t)}^{z} dz'\, X(z',t) = \frac{1}{2}\left[(z-c)X(z,t) - \frac{(a-b)^2}{4}\log S(z,t)\right] \tag{B.44}$$

and

$$\begin{aligned} \int_{a(t)}^{z} dz' \frac{X(z',t)}{z'} &= X(z,t) + \rho \log\left(\frac{z}{a}\right) - \rho \log T(z,t) - c \log S(s,t), \\ T(z,t) &= \frac{2[\rho X(z,t) - cz + ab]}{a(b-a)}. \end{aligned} \tag{B.45}$$

Following a similar argument to that used in (B.43), we have by virtue of (B.7) that

$$\arg(T(b,t)) = \arg\left(\frac{b}{a}\right) - \pi = 2\psi_0 - \pi. \tag{B.46}$$

Thus, $\arg(T(s,t))$ varies from zero at $s = a(t)$ to $2\psi_0 - \pi$ at $s = b(t)$. The modulus of $T(s,t)$ is unity at both limits. Note that

$$\begin{aligned} T(z,t) &= -\frac{\rho}{a} S(z,t)[1+\lambda(z,t)], \\ \lambda(z,t) &= \frac{(c+\rho)(\rho-z)}{\rho(X-z+c)} = \frac{(\rho-z)(X-z+c)}{\rho(\rho-c)}. \end{aligned} \tag{B.47}$$

On or near the contact patch $X+z-c$ is first order and $c+\rho$ is second order (see (2.33)) so that $\lambda(z,t)$ is first order. Away from the contact patch

$$X(z,t) \approx (z-c)\left[1 - \frac{(a-b)^2}{8(z-c)^2}\right], \tag{B.48}$$

on the relevant branch of $X(z,t)$ so that $X+z-c$ is not small and $\lambda(z,t)$ is second order. It is of course zero at $z = \rho$.

We can write (B.45) in the form

$$\int_{a(t)}^{z} dz' \frac{X(z',t)}{z'} = X(z,t) + \rho \log\left(-\frac{z}{\rho}\right) - (\rho+c)\log S(z,t) - \rho \log[1+\lambda(z,t)]. \tag{B.49}$$

The last term can be approximated according to

$$\log[1+\lambda(z,t)] \approx \lambda(z,t) - \frac{1}{2}\lambda^2(z,t) \tag{B.50}$$

to second order near the contact patch and even more accurately elsewhere. However, this expansion is practically useful only on the contact patch where it allows a simple form of the displacement to be derived. Elsewhere, no particular simplifications arise, nor is the process of numerical evaluation greatly helped by its use. Note that

$$\arg[1+\lambda(b,t)] = \psi_0, \tag{B.51}$$

and its modulus is unity.

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