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A Comparison of Two Boundary Methods For Biharmonic Boundary Value Problems

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A COMPARISON OF TWO BOUNDARY METHODS FOR BIHARMONIC BOUNDARY VALUE PROBLEMS

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ABSTRACT

A COMPARISON OF TWO BOUNDARY METHODS FOR BIHARMONIC BOUNDARY VALUE PROBLEMS

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The purpose of this thesis is to solve biharmonic boundary value problems using two different boundary methods and compare their performances. The two boundary methods used are the method of fundamental solutions (MFS) and the method of approximate fundamental solutions (MAFS). The Delta-shaped basis function with the Abel regularization technique is used in the construction of the approximate fundamental solutions in MAFS. The MFS produces more accurate results but needs known fundamental solutions for the differential operator. The MAFS can provide comparable results, and is applicable to more general differential operators. The numerical results using both methods are presented.

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LIST OF ABBREVIATIONS

BEM	-	Boundary Element Method
FDM	-	Finite Difference Method
FEM	-	Finite Element Method
MAFS	-	Method of Approximate Fundamental Solutions
MFS	-	Method of Fundamental Solutions

NOTATION AND GLOSSARY

General Usage and Terminology

The notation used in this text represents fairly standard mathematical and computational usage. In many cases these fields tend to use different preferred notation to indicate the same concept, and these have been reconciled to the extent possible, given the interdisciplinary nature of the material. In particular, the notation for partial derivatives varies extensively, and the notation used is chosen for stylistic convenience based on the application. While it would be convenient to utilize a standard nomenclature for this important symbol, the many alternatives currently in the published literature will continue to be utilized.

The blackboard fonts are used to denote standard sets of numbers: $\mathbb{R}$ for the field of real numbers, $\mathbb{C}$ for the complex field, $\mathbb{Z}$ for the integers, and $\mathbb{Q}$ for the rationals. The capital letters, $A, B, \cdots$ are used to denote matrices, including capital greek letters, e.g., Λ for a diagnonal matrix. Functions which are denoted in boldface type typically represent vector valued functions, and real valued functions usually are set in lower case roman or greek letters. Lower case letters such as i, j, k, l, m, n and sometimes p and d are used to denote indices.

Vectors are typset in square brackets, e.g., $[\cdot]$, and matrices are typeset in parenthesese, e.g., $(\cdot)$. In general the norms are typeset using double pairs of lines, e.g., $||\cdot||$, and the abolute value of numbers is denoted using a single pairs of lines, e.g., $|\cdot|$.

Chapter 1

INTRODUCTION

The traditional numerical methods such as the finite element method (FEM) and the finite difference method (FDM) for solving partial differential equations are domain methods [3]. The domain methods require a mesh to the interior of a domain to approximate a solution. The boundary methods only require the numerical discretization on the boundary of a domain, with no mesh for the interior of a domain. They cost less than the domain methods in numerical modeling. They also have the benefits of easy refinement for the mesh and applicability to the moving boundaries.

The purpose of the thesis is to solve biharmonic boundary value problems using two boundary methods: the method of fundamental solutions (MFS) and the method of approximate fundamental solutions (MAFS) and to compare the numerical results obtained by these methods.

We consider the biharmonic equation

$$Lu \equiv \nabla^4 u = 0, \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a simply connected domain in $\mathbb{R}^2$. The biharmonic problem arises in Stokes fluid flow and the elasticity in engineering fields [5], also in structural and continuum mechanics with applications to thin beams and plates [4, 7, 9].
The boundary conditions are given by

$$u(x,y) = g_1(x,y), \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n}(x,y) = g_2(x,y), \quad \text{on } \partial\Omega, \tag{1.3}$$

where $g_1(x,y)$ and $g_2(x,y)$ are respectively the Dirichlet and Neumann data specified on $\partial\Omega$, and n denotes the outward normal of $\partial\Omega$.

We will use the methods MFS and MAFS for solving (1.1)-(1.3). In paper [5], several applications of MFS methods are reviewed. The MFS is shown to have the same benefits of the boundary element method (BEM) which has been adopted as a generic term in various numerical methods that use techniques of the boundary methods [3]. Besides, the MFS has advantages over the BEM since it requires relatively fewer boundary points and singularities for accurate results. The MFS has been used to solve PDEs [2, 5, 6, 7, 13, 15], and it is applicable when a fundamental solution of the differential operator is known. The adaptivity of the MFS has also been shown by allowing the singularities to move.

The MAFS is applicable to more general differential operators. It is coupled with appropriate regularization technique in the construction of approximate solutions. The MAFS has been considered for extending the applicability of the MFS because the fundamental solution of a given differential equation is not always available. In [10], a boundary method of Trefftz is suggested, and it follows the general scheme of the MFS, except the approximate solution can be obtained without knowing the fundamental solution of the differential operator. The MAFS for the heat conduction problem and for the ill-posed Cauchy problem can be found in [11, 13], and the MAFS are shown to provide satisfactory solutions.

We introduce the scheme of MFS method in Chapter 2. In Chapter 3, we introduce the MAFS incorporating the Delta-shaped basis functions with Abel regularization technique. In Chapter 4, we provide numerical results to compare the performances of the two numerical methods.

Chapter 2

THE METHOD OF FUNDAMENTAL SOLUTIONS

2.1 The Fundamental Solutions of Elliptic PDEs

We consider a boundary value problem governed by the equations of the form

$$Lu = 0, \quad \text{in } \Omega, \tag{2.1}$$

where L is an elliptic differential operator, and Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. A fundamental solution of the equation (2.1), $G(\mathbf{x}, \xi)$, is a function satisfying

$$LG(\mathbf{x}, \xi) = \delta(\mathbf{x} - \xi), \ \ \mathbf{x}, \xi \in \mathbb{R}^n, \ \ n = 2 \ \text{ or } \ 3, \tag{2.2}$$

where $\delta(\mathbf{x} - \xi)$ is the Dirac Delta function, and ξ is the singularity of the fundamental solution. The function $G(\mathbf{x}, \xi)$ is defined everywhere except at $\mathbf{x} = \xi$.

The idea of the formulation of the MFS method was first proposed by Kupradze and Aleksidze [5, 8]. The fundamental solution $G(\mathbf{x}, \xi)$ is used as a basis function in formulating the numerical solution. The expression of the approximate solution can be written as a linear combination of the fundamental solutions

$$\widetilde{u}(x) = \sum_{i=1}^{K} a_i G(x, \xi_i), \tag{2.3}$$

where ξ_i are the singular points of the fundamental solutions $G(x, \xi_i)$, and a_i are coefficients to be determined by boundary conditions.

The fundamental solutions of some elliptic operators, such as the Laplace operator, the Helmhotz operator, and the modified Helmholtz operator are given in the following for the cases $n = 2$ and $n = 3$ [2, 5, 6, 15].

- $L = \nabla^2, \; G(\mathbf{x}, \xi) = \begin{cases} \dfrac{1}{2\pi} \ln |\mathbf{x} - \xi|, & \text{if } n = 2, \\ -\dfrac{1}{4\pi|\mathbf{x} - \xi|}, & \text{if } n = 3, \end{cases}$ (2.4)

- $L = \nabla^2 + k^2, \; G(\mathbf{x}, \xi) = \begin{cases} \dfrac{i}{4} H_0^2(k|\mathbf{x} - \xi|), & \text{if } n = 2, \\ -\dfrac{e^{-ik|\mathbf{x}-\xi|}}{4\pi|\mathbf{x} - \xi|}, & \text{if } n = 3, \end{cases}$ (2.5)

- $L = \nabla^2 - k^2, \; G(\mathbf{x}, \xi) = \begin{cases} -\dfrac{1}{2\pi} K_0(k|\mathbf{x} - \xi|), & \text{if } n = 2, \\ -\dfrac{e^{-k|\mathbf{x}-\xi|}}{4\pi|\mathbf{x} - \xi|}, & \text{if } n = 3, \end{cases}$ (2.6)

where $i = \sqrt{-1}$, H_0^2 in (2.5) is the Hankel function of the second kind of order zero, and K_0 in (2.6) is the modified Bessel function of the second kind of order zero. The fundamental solution for the biharmonic operator $L = \nabla^4$ is

$$G(\mathbf{x}, \xi) = \begin{cases} \dfrac{1}{8\pi} |\mathbf{x} - \xi|^2 \ln |\mathbf{x} - \xi|, & \text{if } n = 2, \\ \dfrac{1}{8\pi} |\mathbf{x} - \xi|, & \text{if } n = 3. \end{cases} \tag{2.7}$$

2.2 Construction of the Approximate Solution

We can rewrite the biharmonic equation (1.1) as

$$\nabla^4 u = \nabla^2(\nabla^2 u) = 0, \tag{2.8}$$

which implies that a solution of the Laplace equation is also a solution of the biharmonic equation. Hence, we can approximate the solution of the biharmonic boundary value problem (1.1)-(1.3) by a linear combination of the fundamental solutions of both the biharmonic operator and the Laplace operator. We let the approximate solution of the problem (1.1)-(1.3) be expressed as

$$\widetilde{u}(x,y) = \sum_{i=1}^{K_1} c_i G_1(x,y;\xi_i,\eta_i) + \sum_{i=K_1+1}^{K_1+K_2} c_i G_2(x,y;\xi_i,\eta_i), \tag{2.9}$$

where $G_1(x,y;\xi_i,\eta_i)$ is a fundamental solution for the biharmonic operator with singularity at (ξ_i,η_i) and $G_2(x,y;\xi_i,\eta_i)$ is a fundamental solution for the Laplace operator with singularity at (ξ_i,η_i).

As in [7], we use the fundamental solutions for $n = 2$,

$$G_1(x_j,y_j;\xi_i,\eta_i) = {r_{ij}}^2 \log(r_{ij}), \tag{2.10}$$

$$G_2(x_j, y_j; \xi_i, \eta_i) = \log(r_{ij}), \tag{2.11}$$

where

$$r_{ij} = \sqrt{(x_j - \xi_i)^2 + (y_j - \eta_i)^2}. \tag{2.12}$$

We choose N collocation points $\{(x_j, y_j)\}_{j=1}^N$ on $\partial\Omega$, and $K = K_1 + K_2$ singular points $\{(\xi_i, \eta_i)\}_{i=1}^K$ outside of domain Ω. Requiring (2.9) to satisfy the boundary conditions at the boundary collocation points will lead to the linear system

$$\mathbf{A}_{N\times K} \cdot \mathbf{c}_{K\times 1} = \mathbf{b}_{K\times 1}, \tag{2.13}$$

where **A** and **b** are given by

$$\mathbf{A} = \begin{pmatrix} G_1 & G_2 \\ \dfrac{\partial G_1}{\partial n} & \dfrac{\partial G_2}{\partial n} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{2.14}$$

The entry that is in j-th row and i-th column is determined as follows:

$$G_1(x_j, y_j; \xi_i, \eta_i), \quad \text{for } j = 1 \cdots N_1, i = 1 \cdots K_1,$$

$$G_2(x_j, y_j; \xi_i, \eta_i), \quad \text{for } j = 1 \cdots N_1, i = K_1 + 1 \cdots K_1 + K_2,$$

$$\frac{\partial G_1}{\partial n}(x_j, y_j; \xi_i, \eta_i), \quad \text{for } j = N_1 + 1 \cdots N_1 + N_2, i = 1 \cdots K_1,$$

$$\frac{\partial G_2}{\partial n}(x_j, y_j; \xi_i, \eta_i), \quad \text{for } j = N_1 + 1 \cdots N_1 + N_2, i = K_1 + 1 \cdots K_1 + K_2,$$

$$g_1(x_j, y_j), \quad \text{for } j = 1 \cdots N_1,$$

$$g_2(x_j, y_j), \quad \text{for } j = N_1 + 1 \cdots N_1 + N_2.$$

The derivative in the direction of the outward normal of the boundary is defined by

$$\frac{\partial F}{\partial n} = \nabla F \cdot \vec{N}, \tag{2.15}$$

where $\vec{N}$ is a normal vector at a given point $(f(t), g(t))$, and

$$\vec{N} = \frac{< \dot{g}, -\dot{f} >}{\sqrt{\dot{f}^2 + \dot{g}^2}}. \tag{2.16}$$

Chapter 3

THE METHOD OF APPROXIMATE FUNDAMENTAL SOLUTIONS

3.1 Delta-Shaped Basis Functions

In this section, we introduce the Delta-shaped basis functions [10, 11, 14] which is to be used for the construction of the approximate fundamental solutions. The eigenfunctions φ_n and eigenvalues λ_n of the Sturm-Liouville problem on the interval [-1,1]

$$-\varphi''(x) = \lambda^2\varphi(x), \tag{3.1}$$

$$\varphi(-1) = \varphi(1) = 0, \tag{3.2}$$

are given by

$$\varphi_n(x) = \sin(\lambda_n(x+1)),\ \lambda_n = \frac{n\pi}{2},\ n = 1,2,\cdots. \tag{3.3}$$

The eigenfunctions $\varphi_n(x)$ form an orthogonal system on [-1,1] with the scalar product

$$\int_{-1}^{1} \varphi_n(x)\varphi_m(x)\mathrm{d}x = \delta_{n,m} = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases}$$

Some regularization techniques and the formulas of the regularization factors are discussed in [11]. Here, we use the Abel regularization technique. For the heat equation,

$$\frac{\partial w(t,x,\xi)}{\partial t} = \frac{\partial^2 w(t,x,\xi)}{\partial x^2}, \tag{3.4}$$

with the initial distribution to be

$$w(0,x,\xi) = \delta(x-\xi) = \sum_{n=1}^{\infty} \varphi_n(\xi)\varphi_n(x). \tag{3.5}$$

We look for a solution that is in the same form of the series as in (3.5),

$$w(t,x,\xi) = \sum_{n=1}^{\infty} w_n(t)\varphi_n(\xi)\varphi_n(x). \tag{3.6}$$

Substitute (3.6) into (3.4), and we obtain

$$w_n(t) = e^{-\lambda_n{}^2 t}. \tag{3.7}$$

The regularizing coefficient is set as

$$r_n(\alpha) = e^{-\alpha \lambda_n^2}, \tag{3.8}$$

where α is the time moment.

The Delta-shaped basis function is the truncated form of (3.6), and it is a smooth function defined by

$$I(x,\xi) = \sum_{n=1}^{M} r_n(\alpha)\varphi_n(\xi)\varphi_n(x), \tag{3.9}$$

where M is a shape parameter since the support of the basis function decreases as M increases [14]. In [12, 14], the solutions of PDEs are approximated directly by Delta-shaped basis functions employing collocation points from both the interior and the boundary of the domain. The values of M and α are coupled parameters, and we choose any $\alpha \in [0.005, 0.01]$ for $M \leq 30$, $\alpha \in [0.001, 0.005]$ for $30 < M \leq 50$, and $\alpha \in [0.0012, 0.0015]$ for $50 < M \leq 100$ [13].

3.2 Construction of Approximate Fundamental Solutions

As we consider the two dimensional problem of (1.1)-(1.3), we extend one dimensional Delta-shaped basis functions to the two dimensional case, i.e.,

$$\begin{aligned} I(x,y;\xi,\eta) &= \left(\sum_{n=1}^{M} r_n\varphi_n(\xi)\varphi_n(x)\right)\left(\sum_{m=1}^{M} r_m\varphi_m(\eta)\varphi_m(y)\right) \\ &= \sum_{n,m=1}^{M} r_n r_m \varphi_n(\xi)\varphi_m(\eta)\varphi_n(x)\varphi_m(y) \\ &= \sum_{n,m=1}^{M} C_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y), \end{aligned} \tag{3.10}$$

where $C_{n,m}(\xi,\eta) = r_n r_m \varphi_n(\xi)\varphi_m(\eta)$.

We use the Delta-shaped basis functions as the forcing term to obtain the approximate fundamental solutions [11]. An approximate fundamental solution of a differential operator L is a function $R(x,y;\xi,\eta)$ satisfying

$$LR(x,y;\xi,\eta) = I(x,y;\xi,\eta). \tag{3.11}$$

Since the Delta-shaped basis functions are in the form of a finite sum of trigonometric functions, we assume that the approximate fundamental solution is written as a finite series of trigonometric functions as well [11]. We let

$$R(x,y;\xi,\eta) = \sum_{n,m=1}^{M} D_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y), \tag{3.12}$$

where the coefficients $D_{n,m}$ are to be determined.

By following the scheme (3.10)-(3.12), the approximate fundamental solution of the biharmonic equation can be obtained. We let $R(x,y;\xi,\eta)$ denote an approximate fundamental solution for $L=\nabla^4$. Substitute (3.10) and (3.12) into (3.11) with $L=\nabla^4$ to determine $D_{n,m}(\xi,\eta)$, i.e.,

$$\begin{aligned}\nabla^4 R(x,y;\xi,\eta) &= \sum_{n,m=1}^{M} D_{n,m}(\xi,\eta)\nabla^4(\varphi_n(x)\varphi_m(y))\\ &= \sum_{n,m=1}^{M} D_{n,m}(\xi,\eta)(\lambda_n^4+2\lambda_n^2\lambda_m^2+\lambda_m^4)\varphi_n(x)\varphi_m(y)\\ &= \sum_{n,m=1}^{M} C_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y).\end{aligned} \tag{3.13}$$

Thus, we have

$$D_{n,m}(\xi,\eta)=\frac{C_{n,m}(\xi,\eta)}{(\lambda_n^2+\lambda_m^2)^2}. \tag{3.14}$$

Two types of fundamental solutions are used in the computation of an approximate solution in the MFS approach. For comparison purpose, we consider using the approximate fundamental solutions of both the biharmonic and the Laplace operators. The approximate fundamental solution for the Laplace operator can be obtained similarly,

$$\begin{aligned}\nabla^2 R(x,y;\xi,\eta) &= \sum_{n,m=1}^{M} D_{n,m}(\xi,\eta)\nabla^2(\varphi_n(x)\varphi_m(y))\\ &= \sum_{n,m=1}^{M} D_{n,m}(\xi,\eta)(-\lambda_n^2-\lambda_m^2)\varphi_n(x)\varphi_m(y)\\ &= \sum_{n,m=1}^{M} C_{n,m}(\xi,\eta)\varphi_n(x)\varphi_m(y),\end{aligned} \tag{3.15}$$

and

$$D_{n,m}(\xi,\eta)=-\frac{C_{n,m}(\xi,\eta)}{\lambda_n^2+\lambda_m^2}. \tag{3.16}$$

Let R_1 and R_2 denote respectively the approximate fundamental solutions for the biharmonic and the Laplace operators. As in (2.9), we let the approximate solutions of the problem (1.1)-(1.3) be expressed as a linear combination of R_1 and R_2 as follows,

$$\widetilde{u}(x,y)=\sum_{i=1}^{K_1} c_i R_1(x,y;\xi_i,\eta_i)+\sum_{i=K_1+1}^{K_1+K_2} c_i R_2(x,y;\xi_i,\eta_i), \tag{3.17}$$

where the coefficients c_i are to be determined by solving a linear system. The linear system is constructed in the same manner as the MFS such that

$$\mathbf{A}_{N\times K}\cdot\mathbf{c}_{K\times 1}=\mathbf{b}_{K\times 1}, \tag{3.18}$$

where **A** and **b** are given by

$$\mathbf{A} = \begin{pmatrix} R_1 & R_2 \\ \dfrac{\partial R_1}{\partial n} & \dfrac{\partial R_2}{\partial n} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{3.19}$$

The entry that is in j-th row and i-th column of the matrix **A** can be obtained by evaluating R_1, R_2, $\partial R_1/\partial n$, and $\partial R_2/\partial n$, which is similar to the structure of the linear system in the MFS approach discussed in Chapter 2.

Chapter 4

COMPARISON OF MFS AND MAFS

4.1 Solving Biharmonic Boundary Value Problems

In this chapter, we provide numerical examples by MFS and MAFS, which are the methods discussed in Chapter 2 and Chapter 3. Both approaches result in linear systems for determining the unknown coefficients. In our calculation, we let the number of boundary collocation points N be as twice as the number of source points K, i.e., the linear system from either method is overdetermined.

The test points $\{T_i\}_{i=1}^{N_t}$ are generated on the boundary and inside the domain to test the accuracy of the numerical results. The approximate solution is evaluated at the test points and is compared to the exact solution. We provide the mean square root error

$$E_1 = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(\widetilde{u}(T_i) - u(T_i))^2}.$$

To see how the error is relative to the magnitude of the solution, we also provide the relative mean square root error,

$$E_2 = \frac{\sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(\widetilde{u}(T_i) - u(T_i))^2}}{\sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(u(T_i))^2}}.$$

4.2 Numerical Examples

Let Ω be a circular domain of radius 0.5. For the MFS approach, we let the source points be distributed on a circle of a radius 1 for Example 1, and a circle of a radius 0.95 for Example 2. However, in the MAFS approach, the source points must be located inside the region $[-1,1]\times[-1,1]$ because the approximate fundamental solutions are constructed by eigenfunctions $\varphi_n(x)$ that vanishes at $x = \pm 1$; therefore, we choose the radius 0.95 for the location of source points for both examples.

We let $N_t = 100$ and these test points are distributed on $\partial\Omega$ and in Ω. The distribution of the test points are shown in Figure (4.1). Also, the distribution of source points for the examples are shown in Figures (4.2) and (4.3).

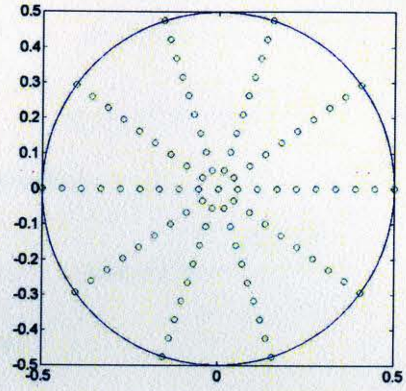

Figure 4.1: Distribution of test points

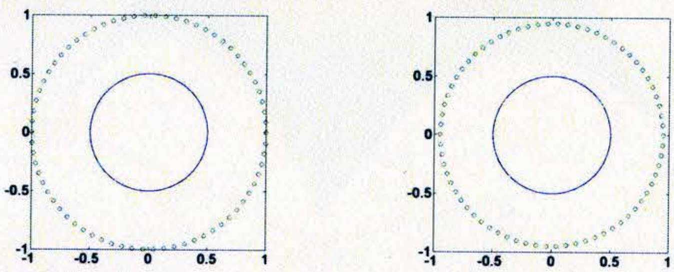

Figure 4.2: Distribution of the source points for Example 1 by MFS(Left) and MAFS(Right)

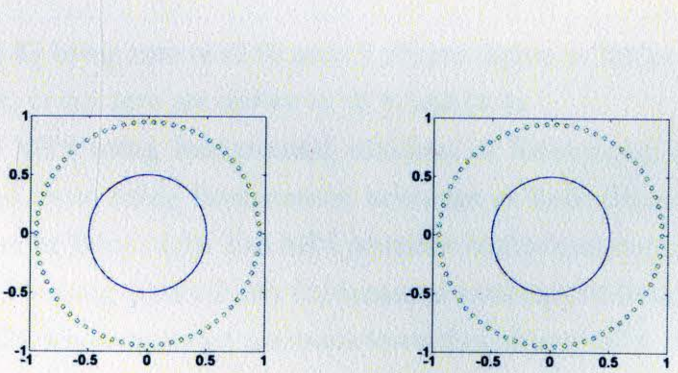

Figure 4.3: Distribution of the source points for Example 2 by MFS(Left) and MAFS(Right)

Example 1. We consider (1.1)-(1.3) with the boundary conditions g_1 and g_2 given by

$$g_1(x,y) = -\frac{x^2+y^2}{4},$$
$$g_2(x,y) = -\left(x^2+y^2\right).$$

The exact solution to the problem is

$$u(x,y) = -\frac{x^2+y^2}{4},$$

which is shown in Figure (4.4).

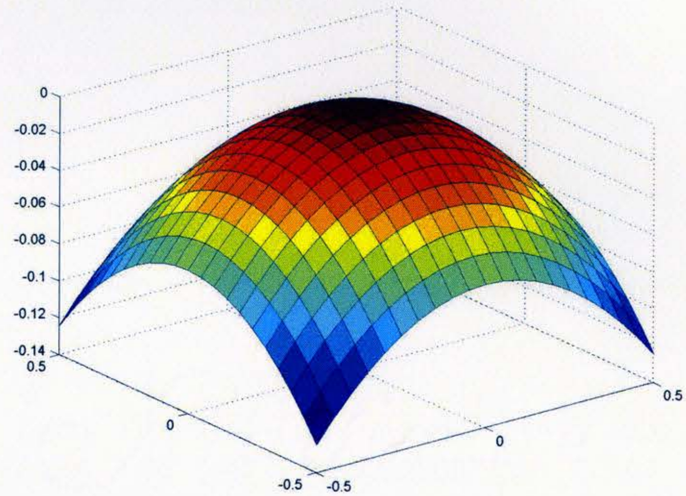

Figure 4.4: The graph of the exact solution for Example 1.

The results for K_2 being zero in (2.9) and (3.17) are shown in Tables (4.1) and (4.2), and the results when K_2 is not zero are shown in (4.3) and (4.4).

The results of MFS using fundamental solutions of Biharmonic operator are shown in Table (4.1), and those using fundamental solutions of both Biharmonic and Laplace operators are shown in Table (4.3). The MFS provides highly accurate results in either case. The results of MAFS using approximate fundamental solutions of Biharmonic operator are shown in Table (4.2), whose accuracy are much lower than that of MFS. However, the MAFS using both the approximate fundamental solutions of Biharmonic and Laplace operators show much better accuracy. The results by MAFS are excellent and comparable to those by MFS when the shape parameter $M = 100$ and the boundary collocation points are increased to $N = 301$ and $N = 401$.

Table 4.1: Example 1: Results of MFS when using fundamental solutions of Biharmonic operator (G_1).

N_1	N_2	K	E_1	E_2
71	70	70	$1.1108 \cdot 10^{-17}$	$3.6766 \cdot 10^{-16}$
101	100	100	$1.3567 \cdot 10^{-17}$	$4.4904 \cdot 10^{-16}$
151	150	150	$1.7830 \cdot 10^{-17}$	$5.9013 \cdot 10^{-16}$
201	200	200	$1.1888 \cdot 10^{-17}$	$3.9347 \cdot 10^{-16}$

Table 4.2: Example 1: Results of MAFS when using approximate fundamental solutions of Biharmonic operator (R_1).

M	α	N_1	N_2	K	E_1	E_2
30	0.007	71	70	70	$7.4390 \cdot 10^{-4}$	$2.4600 \cdot 10^{-2}$
50	0.005	71	70	70	$1.1000 \cdot 10^{-3}$	$3.7800 \cdot 10^{-2}$
70	0.0015	71	70	70	$2.3000 \cdot 10^{-3}$	$7.6800 \cdot 10^{-2}$
100	0.0015	71	70	70	$2.3000 \cdot 10^{-3}$	$7.6900 \cdot 10^{-2}$
30	0.007	101	100	100	$5.4958 \cdot 10^{-4}$	$1.8200 \cdot 10^{-2}$
50	0.005	101	100	100	$3.8234 \cdot 10^{-4}$	$1.2700 \cdot 10^{-2}$
70	0.0015	101	100	100	$2.5100 \cdot 10^{-4}$	$8.3000 \cdot 10^{-3}$
100	0.0015	101	100	100	$2.0186 \cdot 10^{-4}$	$6.7000 \cdot 10^{-3}$
30	0.007	151	150	150	$7.4476 \cdot 10^{-4}$	$2.4600 \cdot 10^{-2}$
50	0.005	151	150	150	$3.8306 \cdot 10^{-4}$	$1.2700 \cdot 10^{-2}$
70	0.0015	151	150	150	$2.4136 \cdot 10^{-4}$	$8.0000 \cdot 10^{-3}$
100	0.0015	151	150	150	$2.1146 \cdot 10^{-4}$	$7.0000 \cdot 10^{-2}$
30	0.007	201	200	200	$5.2320 \cdot 10^{-4}$	$1.7300 \cdot 10^{-2}$
50	0.005	201	200	200	$3.8464 \cdot 10^{-4}$	$1.2700 \cdot 10^{-2}$
70	0.0015	201	200	200	$2.7707 \cdot 10^{-4}$	$9.2000 \cdot 10^{-3}$
100	0.0015	201	200	200	$2.6925 \cdot 10^{-4}$	$8.9000 \cdot 10^{-3}$

Table 4.3: Example 1: Results of MFS when using fundamental solutions of both Biharmonic and Laplace operators ($G_1 + G_2$).

N_1	N_2	K	E_1	E_2
71	70	70	$4.2724 \cdot 10^{-17}$	$1.4141 \cdot 10^{-15}$
101	100	100	$2.6659 \cdot 10^{-17}$	$8.8235 \cdot 10^{-16}$
151	150	150	$1.3653 \cdot 10^{-16}$	$4.5187 \cdot 10^{-15}$
201	200	200	$1.8343 \cdot 10^{-16}$	$6.0709 \cdot 10^{-15}$

Table 4.4: Example 1: Results of MAFS when using approximate fundamental solutions of both Biharmonic and Laplace operators ($R_1 + R_2$).

M	α	N_1	N_2	K	E_1	E_2
30	0.007	71	70	70	$4.1073 \cdot 10^{-5}$	$1.4000 \cdot 10^{-3}$
50	0.005	71	70	70	$2.0719 \cdot 10^{-6}$	$6.8572 \cdot 10^{-5}$
70	0.0015	71	70	70	$1.7520 \cdot 10^{-9}$	$5.7988 \cdot 10^{-8}$
100	0.0015	71	70	70	$1.7536 \cdot 10^{-9}$	$5.8040 \cdot 10^{-8}$
30	0.007	101	100	100	$4.1074 \cdot 10^{-5}$	$1.4000 \cdot 10^{-3}$
50	0.005	101	100	100	$2.0719 \cdot 10^{-6}$	$6.8575 \cdot 10^{-5}$
70	0.0015	101	100	100	$1.5217 \cdot 10^{-11}$	$5.0365 \cdot 10^{-10}$
100	0.0015	101	100	100	$3.6143 \cdot 10^{-12}$	$1.1962 \cdot 10^{-10}$
30	0.007	151	150	150	$4.1074 \cdot 10^{-5}$	$1.4000 \cdot 10^{-3}$
50	0.005	151	150	150	$2.0719 \cdot 10^{-6}$	$6.8575 \cdot 10^{-5}$
70	0.0015	151	150	150	$1.3344 \cdot 10^{-11}$	$4.4165 \cdot 10^{-10}$
100	0.0015	151	150	150	$2.3270 \cdot 10^{-16}$	$7.7018 \cdot 10^{-15}$
30	0.007	201	200	200	$4.1074 \cdot 10^{-5}$	$1.4000 \cdot 10^{-3}$
50	0.005	201	200	200	$2.0719 \cdot 10^{-6}$	$6.8575 \cdot 10^{-5}$
70	0.0015	201	200	200	$1.0433 \cdot 10^{-11}$	$3.4529 \cdot 10^{-10}$
100	0.0015	201	200	200	$1.5389 \cdot 10^{-15}$	$5.0935 \cdot 10^{-14}$

Example 2. We consider (1.1)-(1.3) with the boundary conditions g_1 and g_2 given by

$$g_1(x,y) = -e^{2y}\sin(2x) - e^x\cos(y),$$

$$g_2(x,y) = -2xe^{2y}\cos(2x) - 2ye^{2y}\sin(2x) - xe^x\cos(y) + ye^x\sin(y).$$

The exact solution is

$$u(x,y) = -e^{2y}\sin(2x) - e^x\cos(y),$$

which is shown in Figure (4.5).

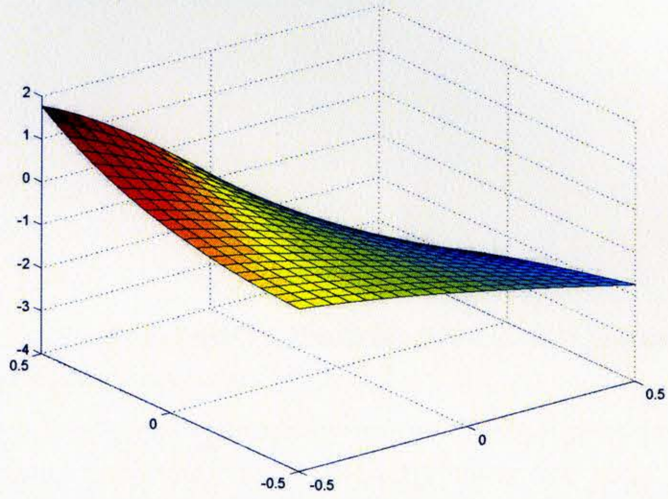

Figure 4.5: The graph of the exact solution for Example 2.

We notice that it is necessary to use fundamental solutions or approximate fundamental solutions of two types of differential operators in MFS/MAFS approach. In Example 2, we compare results by MFS and MAFS using two types of fundamental solutions or approximate fundamental solutions. In Table (4.5), the MFS approach achieves an accuracy of order 10^{-16} at large number of collocation points. In Table (4.6), the MAFS achieves better and better results as M and the number of boundary collocation points increase. The results by MAFS with $(M,\alpha) = (100, 0.0015)$ are excellent and comparable to those by MFS when the boundary collocation points $N = 301$ and $N = 401$.

Table 4.5: Example 2: Results of MFS when using fundamental solutions of both Biharmonic and Laplace operators ($G_1 + G_2$).

N_1	N_2	K	E_1	E_2
71	70	70	$9.7404 \cdot 10^{-11}$	$8.1482 \cdot 10^{-11}$
101	100	100	$4.3775 \cdot 10^{-15}$	$3.6619 \cdot 10^{-15}$
151	150	150	$6.7508 \cdot 10^{-16}$	$5.6473 \cdot 10^{-16}$
201	200	200	$7.8799 \cdot 10^{-16}$	$6.5918 \cdot 10^{-16}$

Table 4.6: Example 2: Results of MAFS when using approximate fundamental solutions of both Biharmonic and Laplace operators ($R_1 + R_2$).

M	α	N_1	N_2	K	E_1	E_2
30	0.007	71	70	70	$1.8264 \cdot 10^{-4}$	$1.5279 \cdot 10^{-4}$
50	0.005	71	70	70	$9.2951 \cdot 10^{-6}$	$7.7757 \cdot 10^{-6}$
70	0.0015	71	70	70	$8.0598 \cdot 10^{-8}$	$6.7423 \cdot 10^{-8}$
100	0.0015	71	70	70	$8.0641 \cdot 10^{-8}$	$6.7459 \cdot 10^{-8}$
30	0.007	101	100	100	$1.8265 \cdot 10^{-4}$	$1.5279 \cdot 10^{-4}$
50	0.005	101	100	100	$9.2948 \cdot 10^{-6}$	$7.7755 \cdot 10^{-6}$
70	0.0015	101	100	100	$7.1508 \cdot 10^{-11}$	$5.9819 \cdot 10^{-11}$
100	0.0015	101	100	100	$3.5055 \cdot 10^{-11}$	$2.9325 \cdot 10^{-11}$
30	0.007	151	150	150	$1.8265 \cdot 10^{-4}$	$1.5279 \cdot 10^{-4}$
50	0.005	151	150	150	$9.2949 \cdot 10^{-6}$	$7.7755 \cdot 10^{-6}$
70	0.0015	151	150	150	$4.6697 \cdot 10^{-11}$	$3.9064 \cdot 10^{-11}$
100	0.0015	151	150	150	$1.5459 \cdot 10^{-15}$	$1.2932 \cdot 10^{-15}$
30	0.007	201	200	200	$1.8265 \cdot 10^{-4}$	$1.5279 \cdot 10^{-4}$
50	0.005	201	200	200	$9.2949 \cdot 10^{-6}$	$7.7755 \cdot 10^{-6}$
70	0.0015	201	200	200	$4.8189 \cdot 10^{-11}$	$4.0312 \cdot 10^{-11}$
100	0.0015	201	200	200	$1.5187 \cdot 10^{-15}$	$1.2704 \cdot 10^{-15}$

We use a square domain $[-0.5,0.5]\times[-0.5,0.5]$ for Example 3 and let the source points be distributed on a circle of radius 0.95 for both the MFS and the MAFS approaches. The distribution of source points and the test points for Example 3 are shown in Figures (4.6)-(4.7).

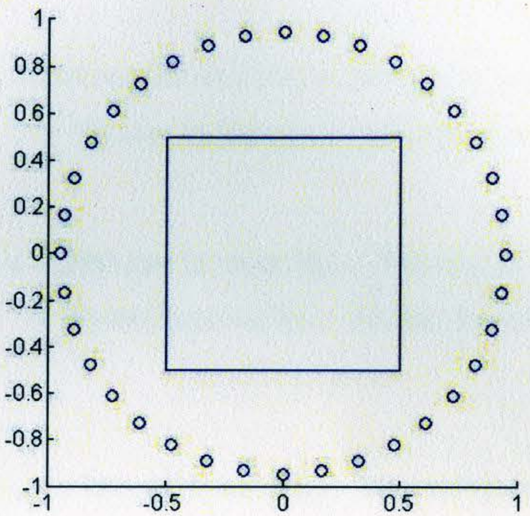

Figure 4.6: Distribution of the source points for Example 3.

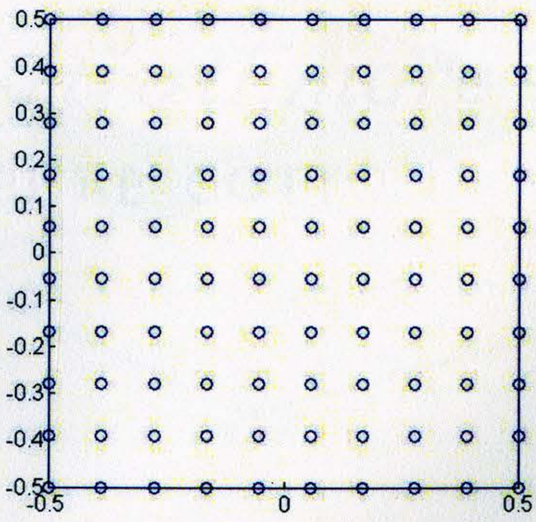

Figure 4.7: Distribution of test points for Example 3.

Example 3. We consider (1.1)-(1.3) with the boundary conditions g_1 and g_2, where g_1 is given by

$$\begin{aligned} g_1(x,y) =& -98\cos(3x)\sinh(3y) - 98\cos(3x)\cosh(3y) \\ & -10\sin(3x)\sinh(3y) - 10\sin(3x)\cosh(3y), \end{aligned}$$

and g_2 is given by

$$\begin{aligned} g_2(x,y) =& 294x\sin(3x)\sinh(3y) + 294x\sin(3x)\cosh(3y) \\ & -30x\cos(3x)\sinh(3y) - 30x\cos(3x)\cosh(3y), \end{aligned}$$

at $(x,0)$ on $\partial\Omega$,

$$\begin{aligned} g_2(x,y) =& -294y\cos(3x)\cosh(3y) - 294y\cos(3x)\sinh(3y) \\ & -30y\sin(3x)\cosh(3y) - 30y\sin(3x)\sinh(3y), \end{aligned}$$

at $(0,y)$ on $\partial\Omega$.
The exact solution is

$$\begin{aligned} u(x,y) =& -98\cos(3x)\sinh(3y) - 98\cos(3x)\cosh(3y) \\ & -10\sin(3x)\sinh(3y) - 10\sin(3x)\cosh(3y), \end{aligned}$$

which is shown in Figure (4.8).

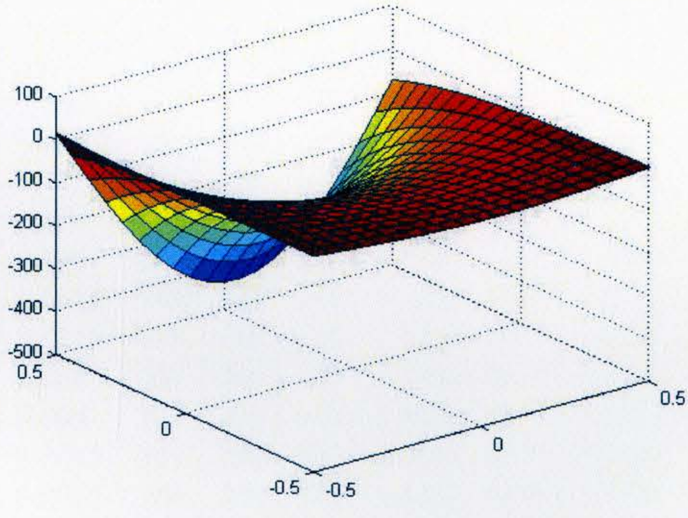

Figure 4.8: The graph of the exact solution for Example 2.

In Example 3, we compare results by MFS and MAFS on the square domain using two types of fundamental solutions or approximate fundamental solutions. In Table (4.7), the MFS provides accurate results of order 10^{-14} when the boundary collocation points $N = 400$. In Table (4.8), the MAFS when N is about 300 achieves an accuracy of order 10^{-8}, which is comparable to the result by the MFS at $N = 200$. Mostly, the MAFS provides better results with $(M,\alpha) = (70, 0.0015)$ and $(M,\alpha) = (100, 0.0015)$.

Table 4.7: Example 3: Results of MFS when using fundamental solutions of both Biharmonic and Laplace operators ($G_1 + G_2$).

N_1	N_2	K	E_1	E_2
68	68	68	$1.4000 \cdot 10^{-3}$	$1.0432 \cdot 10^{-5}$
100	100	100	$8.5373 \cdot 10^{-6}$	$6.2163 \cdot 10^{-8}$
148	148	148	$4.9637 \cdot 10^{-9}$	$3.6142 \cdot 10^{-11}$
200	200	200	$2.4342 \cdot 10^{-12}$	$1.7724 \cdot 10^{-14}$

Table 4.8: Example 2: Results of MAFS when using approximate fundamental solutions of both Biharmonic and Laplace operators ($R_1 + R_2$).

M	α	N_1	N_2	K	E_1	E_2
30	0.007	68	68	68	$1.1090 \cdot 10^{-1}$	$8.0715 \cdot 10^{-4}$
50	0.005	68	68	68	$3.0600 \cdot 10^{-2}$	$2.2281 \cdot 10^{-4}$
70	0.0015	68	68	68	$1.3000 \cdot 10^{-3}$	$9.1479 \cdot 10^{-6}$
100	0.0015	68	68	68	$1.3000 \cdot 10^{-3}$	$9.1479 \cdot 10^{-6}$
30	0.007	100	100	100	$7.0500 \cdot 10^{-2}$	$5.1366 \cdot 10^{-4}$
50	0.005	100	100	100	$2.7000 \cdot 10^{-2}$	$1.9674 \cdot 10^{-4}$
70	0.0015	100	100	100	$1.3384 \cdot 10^{-5}$	$9.7453 \cdot 10^{-8}$
100	0.0015	100	100	100	$1.3381 \cdot 10^{-5}$	$9.7434 \cdot 10^{-8}$
30	0.007	148	148	148	$8.9500 \cdot 10^{-2}$	$6.5155 \cdot 10^{-4}$
50	0.005	148	148	148	$1.7700 \cdot 10^{-2}$	$1.2857 \cdot 10^{-4}$
70	0.0015	148	148	148	$2.2685 \cdot 10^{-6}$	$1.6518 \cdot 10^{-8}$
100	0.0015	148	148	148	$2.2666 \cdot 10^{-6}$	$1.6503 \cdot 10^{-8}$
30	0.007	200	200	200	$7.2900 \cdot 10^{-2}$	$5.3116 \cdot 10^{-4}$
50	0.005	200	200	200	$1.5600 \cdot 10^{-2}$	$1.1373 \cdot 10^{-4}$
70	0.0015	200	200	200	$2.8088 \cdot 10^{-6}$	$2.0451 \cdot 10^{-8}$
100	0.0015	200	200	200	$2.4397 \cdot 10^{-6}$	$1.7764 \cdot 10^{-8}$

The MATLAB has been used for performing the above calculations. Although the results from using the MAFS are promising, much longer processing time for MAFS is noticed. In order to optimize the performance and make use of the features of MATLAB, the code should be further vectorized.

Chapter 5

CONCLUSION

We solve biharmonic boundary value problems using the MFS and the MAFS. In spite of the accurate results the MFS provides, it has limitations since it is only applicable to differential operators whose fundamental solutions are known. In general the results by MAFS is not as accurate as MFS. But the results of MAFS improve and are comparable to the those of MFS when the number of collocation points increase with appropriate values of shape parameter and regularizing coefficients. We also note that it is necessary to use two types of fundamental solutions or approximate fundamental solutions in MFS or MAFS. Considering the rapid convergence of the MFS, the MAFS has a disadvantage of computational costs since it requires more function evaluations. The merit of MAFS is its applicability to more general type differential operators. Both methods can be extended to solving higher-dimensinal PDEs of elliptic type.

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