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LORENTZ INVARIANT SPACELIKE SURFACES WITH CONSTANT MEAN CURVATURE IN ANTI-DE SITTER 3-SPACE

by

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ABSTRACT

LORENTZ INVARIANT SPACELIKE SURFACES OF CONSTANT MEAN CURVATURE IN ANTI-DE SITTER 3-SPACE

by Jamie Patrick Lambert

August 2015

In this thesis, I studied Lorentz invariant spacelike surfaces with constant mean curvature $H = c$ in the anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ of constant curvature $-c^2$. In particular, I construct Lorentz invariant spacelike surfaces of constant mean curvature c and maximal Lorentz invariant spacelike surfaces in $\mathbb{H}^3_1(-c^2)$. I also studied the limit behavior of those constant mean curvature c surfaces in $\mathbb{H}^3_1(-c^2)$. It turns out that they approach a maximal catenoid in Minkowski 3-space $\mathbb{E}^3_1$ as $c \to 0$. The limit maximal catenoid is Lorentz invariant in $\mathbb{E}^3_1$.

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This is to thank all of those who have assisted me in this effort. I want to also thank all of my friends and family for their support in my pursuit for higher education, and my fellow classmates and professors for a great learning experience. Lastly, I am forever indebted to my advisor for his infinite wisdom, and I hope to be half as great a teacher as he is.

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CMC - Constant Mean Curvature

NOTATION AND GLOSSARY

General Usage and Terminology

The notation used in this text represents fairly standard mathematical and computational usage. In many cases these fields tend to use different preferred notation to indicate the same concept, and these have been reconciled to the furthest extent possible, given the interdisciplinary nature of the material. In particular, the notation for partial derivatives varies extensively, and the notation used is chosen for stylistic convenience based on the application. While it would be convenient to utilize a standard nomenclature for this important symbol, the many alternatives currently in the published literature will continue to be utilized. The blackboard fonts are used to denote manifolds or standard sets of numbers, e.g., $\mathbb{R}$ for the field of real numbers, $\mathbb{C}$ for the complex field, $\mathbb{Z}$ for the integers, and $\mathbb{Q}$ for the rationals. When the blackboard fonts are used with subscripts and superscripts then it is to denote spaces, an example of which is $\mathbb{H}_1^3$ for the anti-de Sitter 3-space.The capital letters, $A, B, \cdots$ are used to denote domains of surfaces, including lowercase greek letters, e.g., φ for parametric surfaces. Vectors are in bold font, e.g., $\mathbf{v}$, and matrices are typeset in parentheses, e.g., $(\cdot)$. In general the norms are typeset using double pairs of lines, e.g., $||\cdot||$, and the absolute value of numbers is denoted using a single pair of lines, e.g., $|\cdot|$. A single pair of lines around matrices indicates the determinant of the matrix.

Chapter 1

Introduction

Surfaces of revolution in Euclidean 3-space have been an interesting topic in geometry due to their geometric beauty which reflects the symmetry of Euclidean 3-space. Examples of such surfaces include spheres, tori, the catenoid (the minimal surface of revolution), unduloids (surfaces of nonzero constant mean curvature obtained by rolling an ellipse along a fixed line, tracing the focus, and revolving the resulting curve around the line), and nodoids (surfaces of nonzero constant mean curvature obtained by rolling a hyperbola along a fixed line, tracing the focus, and revolving the resulting curve around the line).

One would also naturally be interested in studying surfaces of revolution in Minkowski 3-space. While there is no distinction between axes of rotation in Euclidean 3-space, in Minkowski 3-space there are three distinct types of axes of rotation, namely spacelike axes, timelike axis, and lightlike axes. Rotations about spacelike axes and timelike axis form a group of the symmetries of Minkowski 3-space, called the Lorentz group. In relativity, such rotations are called Lorentz transformations. In [2], Hano and Nomizu classified the surfaces of revolution in Minkowski 3-space by studying profile curves. Those profile curves are obtained by rolling quadratic curves along a spacelike axis or the timelike axis analogously to the Euclidean 3-space case. In [4] and [5], the authors noted that the surfaces of revolution in Minkowski 3-space studied by Hano and Nomizu are surface area minimizing while holding a volume. They set up an area functional with volume constraint and used calculus of variations to obtain the differential equations of profile curves for such surfaces of revolution about a spacelike axis or the timelike axis. They also had an interesting discussion on how to obtain such surfaces of revolution about lightlike axes.

One may wonder if we can consider surfaces of revolution in certain curved spaces; for example, the well-known hyperbolic 3-space. It turns out that we can, but there is limited rotational symmetry, namely $\mathrm{SO}(2)$ symmetry. Due to this limited rotational symmetry, we cannot study profile curves by rolling quadratic curves along the axis of rotation as done in Euclidean or Minkowski 3-space. In [6], Lee and Zarske considered the flat chart model of hyperbolic 3-space of constant negative curvature $-c^2$ and obtained the differential equation of the profile curve. They constructed the surface of revolution with constant mean curvature $H = c$ in hyperbolic 3-space of constant negative curvature $-c^2$ and noted that it converges to the catenoid, the minimal surface of revolution in Euclidean 3-space as $c \to 0$. In Euclidean 3-space, minimal surfaces are characterized by mean curvature, namely they are zero mean curvature surfaces. However, in hyperbolic 3-space there is no connection between minimal surfaces and mean curvature. In [6] the authors considered the area functional for surfaces of revolution and found the minimal surface of revolution as a critical point of the area functional. This minimal surface too converge to the catenoid in Euclidean 3-space as $c \to 0$.

There may be another curved space where surfaces of revolution can be considered. In fact, there is one: anti-de Sitter 3-space. Due its resemblance with hyperbolic 3-space, it is also called Lorentzian hyperbolic space. There are actually no rotational symmetries in anti-de Sitter space but it has limited space-time symmetry, namely $\mathrm{SO}(1,1)$ symmetry. However, we may consider Lorentz invariant ($\mathrm{SO}(1,1)$-invariant) surfaces in anti-de Sitter 3-space analogously to surfaces of revolution in hyperbolic 3-space. Motivated by the research done in [6], I studied Lorentz invariant spacelike surfaces with constant mean curvature in anti-de Sitter 3-space for my master's thesis. This thesis is organized as follows: In chapter 2, I discuss some basic differential geometry of surfaces as preliminaries. In chapter 3, I discuss the main results of my research.

Chapter 2

Preliminaries

In this chapter, I discuss some basic differential geometry of surfaces in $\mathbb{R}^3$ that I need to study the main results in chapter 3. Throughout this chapter, $\mathbb{R}^3$ is the space of ordered triples

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_i \in \mathbb{R},\ i = 1,2,3\}.$$

It is regarded as a 3-dimensional differentiable manifold (in fact a 3-dimensional Riemannian or pseudo-Riemannian manifold) but not necessarily as Euclidean 3-space.

2.1 Directional Derivatives and Covariant Derivatives

In order to do differential geometry, one should be able to differentiate functions and vector fields. I begin with the notion of *directional derivative* of a function f in the direction of a tangent vector v to $\mathbb{R}^3$ at $p \in \mathbb{R}$.

Definition 1. *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a differential function on $\mathbb{R}_3$, and let $\mathbf{v} \in T_p\mathbb{R}^3$ where $T_p\mathbb{R}^3$ denotes the tangent space to $\mathbb{R}^3$ at $p \in \mathbb{R}^3$. Let $\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^3$ be a differentiable curve in $\mathbb{R}^3$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. Then the directional derivative $\nabla_{\mathbf{v}} f$ of f in $\mathbf{v}$ direction is defined by*

$$\nabla_{\mathbf{v}} f = \frac{d}{dt}(f(\alpha(t))|_{t=0}. \tag{2.1}$$

Example 1. Let $\mathbb{R}^3$ be Euclidean 3-space. Let $f = x^2yz$ with $p = (1,1,0)$ and $\mathbf{v} = (1,0,-3)$. In Euclidean 3-space, we may choose $\alpha(t)$ to be

$$\begin{aligned}\alpha(t) &= p + t\mathbf{v}\\ &= (1,1,0) + t(1,0,-3)\\ &= (1+t, 1, -3t).\end{aligned}$$

So the directional derivative $\nabla_{\mathbf{v}} f$ is computed to be

$$\begin{aligned}\nabla_{\mathbf{v}} f &= \frac{d}{dt}(f(p+tv))|_{t=0}\\ &= -3.\end{aligned}$$

Proposition 1. *The directional derivative* $\nabla_{\mathbf{v}} f$ *can be given by*

$$\nabla_{\mathbf{v}} f = \sum_{i=1}^{3} v_i \left(\frac{\partial f}{\partial x_i} \right)_p \tag{2.2}$$

where v_i *denotes the i-th component of* $\mathbf{v}$ *for each* $i = 1, 2, 3$.

Proof. It follows straightforwardly by the chain rule. □

Remark 1. If $\mathbb{R}^3$ is Euclidean 3-space, (2.2) can be written as

$$\nabla_{\mathbf{v}} f = \nabla f \cdot \mathbf{v}, \tag{2.3}$$

where ∇ on the right denotes the gradient operator in Euclidean 3-space $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$ and $\cdot$ denotes the standard dot product.

Example 2. Let $\mathbb{R}^3$ be Euclidean 3-space. Let $f = x^2 yz$ with $p = (1,1,0)$ and $\mathbf{v} = (1,0,-3)$. Then

$$\begin{aligned} &\frac{\partial f}{\partial x} = 2xyz, \ \frac{\partial f}{\partial y} = x^2 z, \\ &\frac{\partial f}{\partial z} = x^2 y \end{aligned}$$

and so

$$\begin{aligned} &\frac{\partial f}{\partial x}(p) = 0, \ \frac{\partial f}{\partial y}(p) = 0, \\ &\frac{\partial f}{\partial z}(p) = -1. \end{aligned}$$

Thus $\nabla_{\mathbf{v}} f$ is computed using (2.2) as $\nabla_{\mathbf{v}} f = -3$.

The directional derivative is linear on $\mathbf{v}$ and on f, and satisfies the Leibniz rule.

Proposition 2. *Let* $f, g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *be differentiable functions,* $\mathbf{v}, \mathbf{w} \in T_p\mathbb{R}^3$*, and* $a, b \in \mathbb{R}$. *Then*

$$\begin{aligned} \nabla_{a\mathbf{v}+b\mathbf{w}}(f) &= a\nabla_{\mathbf{v}} f + b\nabla_{\mathbf{w}} f \\ \nabla_{\mathbf{v}}(af + bg) &= a\nabla_{\mathbf{v}} f + b\nabla_{\mathbf{v}} g \\ \nabla_{\mathbf{v}}(fg) &= (\nabla_{\mathbf{v}} f)g(p) + f(p)(\nabla_{\mathbf{v}} g) \end{aligned} \tag{2.4}$$

.

In differential geometry, we not only need to differentiate functions but also vector fields. The notion of directional derivative can be naturally generalized to define a way to differentiate vector fields. The resulting derivative is called the *covariant derivative*. Let

W be a vector field on $\mathbb{R}^3$. Then the covariant derivative of W in the $\mathbf{v} \in T_p\mathbb{R}^3$ direction, denoted by $\nabla_{\mathbf{v}}W$, is defined to be the initial rate of change of $W(p)$ as p moves in the $\mathbf{v}$ direction. That is,

$$\nabla_{\mathbf{v}}W = \frac{d}{dt}W(\alpha(t))|_{t=0}, \tag{2.5}$$

where $\alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^3$ is a differentiable curve in $\mathbb{R}^3$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.

Example 3. Let $\mathbb{R}^3$ be Euclidean 3-space. Let W be a vector field on $\mathbb{R}^3$ defined by

$$W = x^2U_1 + yzU_3,$$

where U_1, U_2 and U_3 are vector fields on $\mathbb{R}^3$ given by

$$U_1(p) = (1,0,0)_p,\ U_2(p) = (0,1,0),$$
$$U_3(p) = (0,0,1)_p.$$

We calculate the covariant derivative $\nabla_{\mathbf{v}}W$ where $\mathbf{v} = (-1,0,2) \in T_p\mathbb{R}^3$ and $p = (2,1,0)$. As seen before, in $\mathbb{R}^3$ as Euclidean 3-space $\alpha(t)$ can be chosen to be $\alpha(t) = p + \mathbf{v}t$, so we have

$$W(p+tv) = (2-t)^2U_1 + 2tU_3$$

and

$$\nabla_{\mathbf{v}} = \frac{d}{dt}W(p+tv)|_{t=0} = -4U_1(p) + 2U_3(p).$$

Like directional derivatives, covariant derivatives also satisfy linearity (1 and 2) and Leibniz rules (3 and 4).

Proposition 3. *Let* $\mathbf{v}, \mathbf{w} \in T_p\mathbb{R}^3$ *and* Y, Z *vector field on* $\mathbb{R}^3$*. Then*

1. $\nabla_{a\mathbf{v}+b\mathbf{w}}Y = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{w}}Y$ *for all real numbers a and b.*

2. $\nabla_{\mathbf{v}}(aY + bZ) = a\nabla_{\mathbf{v}}Y + b\nabla_{\mathbf{v}}Z$ *for all real numbers a and b.*

3. $\nabla_{\mathbf{v}}(fY) = (\nabla_{\mathbf{v}}f)Y(p) + f(p)\nabla_{\mathbf{v}}Y$ *for all differentiable functions* f*.*

4. $\nabla_{\mathbf{v}}\langle Y, Z\rangle = \langle \nabla_{\mathbf{v}}Y, Z(p)\rangle + \langle Y(p), \nabla_{\mathbf{v}}Z\rangle$*, where* $\langle\ ,\ \rangle$ *denotes the inner product on* $T_p\mathbb{R}^3$*.*

2.2 Shape Operators and Applications

Recall that in calculus the curvature of a unit speed curve can be found by differentiating the normal vector field (acceleration) on the curve. One may also study the curvature of a surface M in $\mathbb{R}^3$ by differentiating the normal vector field on M. Let $p \in M$ and N denotes a unit normal vector field on a neighborhood of p in M. For each tangent vector $\mathbf{v} \in T_pM$, define

$$S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} N. \tag{2.6}$$

Since[1] $\langle N, N \rangle = \pm 1$, we obtain by differentiating

$$\begin{aligned} \langle \nabla_{\mathbf{v}} N, N \rangle + \langle N, \nabla_{\mathbf{v}} N \rangle &= 2 \langle \nabla_{\mathbf{v}} N, N \rangle \\ &= 0. \end{aligned}$$

This means that $\nabla_{\mathbf{v}} N \in T_pM$, therefore (2.6) defines a linear map $S_p : T_pM \longrightarrow T_pM$. This linear map S_p is called the *shape operator* of M at p. The shape operator is symmetric, i.e.

$$\langle S(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, S(\mathbf{w}) \rangle$$

for $\mathbf{v}$ and $\mathbf{w} \in T_pM$. So it has real eigenvalues. The eigenvalues of S is called the principal curvatures of M. As S is a 2×2 matrix, there can be at most two real eigenvalues κ_1, κ_2. The average $H = \frac{\kappa_1 + \kappa_2}{2}$ and the product $K = \kappa_1 \kappa_2$ are called, respectively, the *mean curvature* and the *Gaussian curvature* of M. Hence we have:

Proposition 4. *Let S be the shape operator of a surface M in $\mathbb{R}^3$. Then the mean curvature H and the Gaussian curvature K are given by*

$$H = \frac{1}{2} \mathrm{tr} S \tag{2.7}$$

$$K = \det S. \tag{2.8}$$

Let M be a parametric surface in $\mathbb{R}^3$, $\varphi : D(u, v) \longrightarrow \mathbb{R}^3$. Then the mean curvature H can be calculated by the Gauss' beautiful formula (see for instance [7])

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)}, \tag{2.9}$$

where N is a unit normal vector field on M and

$$\begin{aligned} &E := \langle \varphi_u, \varphi_u \rangle, \ F := \langle \varphi_u, \varphi_v \rangle, \ G := \langle \varphi_v, \varphi_v \rangle, \\ &\ell := \langle \varphi_{uu}, N \rangle, \ \mathfrak{m} = \langle \varphi_{uv}, N \rangle, \ \mathfrak{n} = \langle \varphi_{vv}, N \rangle. \end{aligned}$$

The classical proof of (2.9) given in [7] is no longer valid for parametric surfaces in a curved 3-space but (2.9) still holds for parametric surfaces in a curved 3-space. For a general proof see, for instance, [6] (Appendix A).

[1] As seen in the next chapter, a normal vector on a spacelike surface in $\mathbb{R}^3$ as a pseudo-Riemannian manifold has negative squared norm.

Chapter 3

Main Results

3.1 The Flat Chart Model of Anti-de Sitter 3-Space $\mathbb{H}^3_1(-c^2)$

Let $\mathbb{E}^4_2$ denote the semi-Euclidean 4-space $\mathbb{R}^4$ with coordinates $\hat{t}, \hat{u}, \hat{v}, \hat{w}$ and the metric

$$ds^2 = -(d\hat{t})^2 - (d\hat{u})^2 + (d\hat{v})^2 + (d\hat{w})^2.$$

The *anti-de Sitter 3-space* $\mathbb{H}^3_1(-c^2)$ is defined by the hyperquadric in $\mathbb{E}^4_2$

$$-\hat{t}^2 - \hat{u}^2 + \hat{v}^2 + \hat{w}^2 = -\frac{1}{c^2}.$$

The anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ has the constant negative sectional curvature $-c^2$. While the flat chart model of hyperbolic 3-space and de Sitter 3-space are well-known, the flat chart model of the anti-de Sitter 3-space was not known. Some even believed that the anti-de Sitter 3-space does not admit the flat chart model. However, my advisor Dr. Sungwook Lee showed in [3] that the anti-de Sitter 3-space indeed admits the flat chart model. So, one may study surfaces in the anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ analogously to surfaces in Euclidean 3-space. The flat chart model of $\mathbb{H}^3_1(-c^2)$ is obtained as follows ([3]): Let us consider an open chart

$$U = \{(\hat{t}, \hat{u}, \hat{v}, \hat{w}) \in \mathbb{H}^3_1(-c^2) : \hat{u} + \hat{v} > 0\},$$

and define the transformations:

$$\begin{aligned} t &= \frac{\hat{t}}{c(\hat{u}+\hat{v})}, \\ x &= \frac{\hat{w}}{c(\hat{u}+\hat{v})}, \\ y &= -\frac{1}{c}\log c(\hat{u}+\hat{v}). \end{aligned} \tag{3.1}$$

Then

$$ds^2 = e^{-2cy}\{-(dt)^2 + (dx)^2\} + (dy)^2. \tag{3.2}$$

$\mathbb{R}^3$ with coordinates (t, x, y) and the metric ds^2 in (3.2), or $(\mathbb{R}^3, ds^2)$ for short, is called the *flat chart model* of anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$. Hereafter, I mean $\mathbb{H}^3_1(-c^2)$ by the flat chart model $(\mathbb{R}^3, ds^2)$. As $c \to 0$, one can clearly see that $\mathbb{H}^3_1(-c^2)$ flattens out to Minkowski 3-space.

3.2 Parametric Spacelike Surfaces in $\mathbb{H}^3_1(-c^2)$

Let D be a domain[1] and $\varphi : D \longrightarrow \mathbb{H}^3_1(-c^2)$ which is an immersion. $\langle\ ,\ \rangle$ denotes the inner product on each tangent space $T_p\mathbb{H}^3_1(-c^2)$ induced by the pseudo-Riemannian metric (3.2).

Definition 2. *An immersed surface $\varphi : D \longrightarrow \mathbb{H}^3_1(-c^2)$ is said to be* spacelike *if both the tangent vectors $\frac{\partial\varphi}{\partial u}$ and $\frac{\partial\varphi}{\partial v}$ are spacelike vectors, i.e.*

$$\langle\frac{\partial\varphi}{\partial u},\frac{\partial\varphi}{\partial u}\rangle > 0,\ \langle\frac{\partial\varphi}{\partial v},\frac{\partial\varphi}{\partial v}\rangle > 0.$$

Using the inner product, one can speak of conformal surfaces in $\mathbb{H}^3_1(-c^2)$.

Definition 3. *$\varphi : D \longrightarrow \mathbb{H}^3_1(-c^2)$ is said to be* conformal *if*

$$\begin{aligned} \langle\varphi_u,\varphi_v\rangle &= 0,\\ |\varphi_u| = |\varphi_v| &= e^{\frac{\omega}{2}}, \end{aligned} \tag{3.3}$$

where (u,v) is a local coordinate system in D and $\omega : D \longrightarrow \mathbb{R}$ is a real-valued function in D.

The induced metric on the spacelike surface φ is given by

$$ds^2_\varphi = \langle d\varphi, d\varphi\rangle = e^\omega\{(du)^2+(dv)^2\}. \tag{3.4}$$

If N denotes a unit normal vector field on a spacelike surface $\varphi : D \longrightarrow \mathbb{H}^3_1(-c^2)$, then we have:

$$\langle N,N\rangle = -1,\ \langle N,\varphi_u\rangle = \langle N,\varphi_v\rangle = 0.$$

While $(\mathbb{R}^3, ds^2)$ appears to look similar to Euclidean 3-space, it is not a vector space. But the cross product can be locally defined on each tangent space $T_p\mathbb{H}^3_1(-c^2)$. The tangent vectors $\mathbf{v},\mathbf{w} \in T_p\mathbb{H}^3_1(-c^2)$ can be represented as

$$\begin{aligned} \mathbf{v} &= v_0(\frac{\partial}{\partial t})_p + v_1(\frac{\partial}{\partial x})_p + v_2(\frac{\partial}{\partial y})_p,\\ \mathbf{w} &= w_0(\frac{\partial}{\partial t})_p + w_1(\frac{\partial}{\partial x})_p + w_2(\frac{\partial}{\partial y})_p, \end{aligned} \tag{3.5}$$

where $\{(\frac{\partial}{\partial t})_p, (\frac{\partial}{\partial x})_p, (\frac{\partial}{\partial y})_p\}$ is the canonical bases for $T_p\mathbb{H}^3_1(-c^2)$. Then the cross product $\mathbf{v}\times\mathbf{w}$ is defined by

$$\begin{aligned} \mathbf{v}\times\mathbf{w} &= \begin{vmatrix} -e^{2cy}\frac{\partial}{\partial t} & e^{2cy}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ v_0 & v_1 & v_2\\ w_0 & w_1 & w_2 \end{vmatrix}\\ &= e^{2cy}\{-(v_1w_2-v_2w_1)\frac{\partial}{\partial t} + (v_2w_0-v_0w_2)\frac{\partial}{\partial x}\} + (v_0w_1-v_1w_0)\frac{\partial}{\partial y} \end{aligned} \tag{3.6}$$

[1] A connected open 2-manifold.

where $p=(t,x,y)\in\mathbb{H}^3_1(-c^2)$. We can also define a triple scalar product $\langle\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle$ as a determinant

$$\langle\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle=\begin{vmatrix}-e^{2cy}u_0 & -e^{2cy}u_1 & u_2\\ v_0 & v_1 & v_2\\ w_0 & w_1 & w_2\end{vmatrix}. \tag{3.7}$$

Unlike the Euclidean case, the cross product and the inner product are not interchangeable.

$$\langle\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle\neq\langle\mathbf{u}\times\mathbf{v},\mathbf{w}\rangle.$$

Proposition 5. *Let $\varphi:M\longrightarrow\mathbb{H}^3_1(-c^2)$ be a spacelike parametric surface in $\mathbb{H}^3_1(-c^2)$. Then on each tangent plane $T_p\mathbb{H}^3_2(-c^2)$*

$$||\varphi_u\times\varphi_v||^2=e^{4cy}(F^2-EG) \tag{3.8}$$

where $p=(t(v,w),x(u,v),y(u,v))\in\mathbb{H}^3_1(-c^2)$.

Proof. It follows straightforwardly from a direct calculation. □

Remark 2. If $c\to 0$, (3.8) becomes the familiar formula in Lorentzian case [4]

$$||\varphi_u\times\varphi_v||^2=F^2-EG.$$

Remark 3. The normal vector field $\varphi_u\times\varphi_v$ is a timelike vector i.e. $F^2-EG<0$. Hence in geometry and physics the norm $||\varphi_u\times\varphi_v||$ is defined to be the *proper time*:

$$\begin{aligned}||\varphi_u\times\varphi_v||&:=\sqrt{-||\varphi_u\times\varphi_v||^2}\\&=e^{2cy(u,v)}\sqrt{EG-F^2}.\end{aligned} \tag{3.9}$$

The unit normal vector N of φ is then given by

$$N=\frac{\varphi_u\times\varphi_v}{e^{2cy(u,v)}\sqrt{EG-F^2}}. \tag{3.10}$$

In physics, the trajectory of a massive particle is a timelike path in spacetime. The physical meaning of proper time is that it is the actual time measured on a physical clock carried along the timelike path.

Let $\varphi:M\longrightarrow\mathbb{H}^3_1(-c^2)$ be a conformal spacelike surface and N a unit normal vector field of φ. Also let $S=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ be the shape operator of φ with respect to the orthogonal basis φ_u,φ_v of $T_p\varphi(M)$. Then

$$\begin{aligned}S(\varphi_u)=&a\varphi_u+b\varphi_v\,,\\ S(\varphi_v)=&c\varphi_u+d\varphi_v\,.\end{aligned}$$

So,

$$\begin{aligned}\langle S(\varphi_u),\varphi_u\rangle+\langle S(\varphi_v),\varphi_v\rangle=&e^w(a+d)\\=&e^wTrS\\=&2e^wH\quad.\end{aligned}$$

Let v_0 be fixed. Then $\varphi(u,v_0)$ a curve on the surface φ. Let N be restricted on this curve $\varphi(u,v_0)$. Then $S(\varphi_u)=-N_u$. Differentiating $\langle\varphi_u,N\rangle=0$ to the respect of u,

$$\begin{aligned}\langle\varphi_{uu},N\rangle=&-\langle\varphi_u,N_u\rangle\\=&\langle\varphi_u,S(\varphi_u)\rangle\quad.\end{aligned}$$

By similar calculation we obtain $\langle S(\varphi_v),\varphi_v\rangle=\langle\varphi_{vv},N\rangle$. Therefore the mean curvature H is calculated to be

$$\begin{aligned}H=&\frac{1}{2}e^{-w}(\langle\varphi_{uu},N\rangle+\langle\varphi_{vv},N\rangle)\\=&\frac{1}{2}e^{-w}\langle\triangle\varphi,N\rangle\end{aligned}$$

where $\triangle=\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}$.

Proposition 6. *Let $\varphi:M\longrightarrow\mathbb{H}^3_1(-c^2)$ be a conformal spacelike surface. Then the mean curvature H of φ is computed to be*

$$H=\frac{1}{2}e^{-w}\langle\triangle\varphi,N\rangle. \tag{3.11}$$

It can be readily seen that the formulas (2.9) and (3.11) coincide for conformal spacelike surfaces.

3.3 Constructing Lorentz Invariant Spacelike Surfaces with CMC $H=c$ in $\mathbb{H}^3_1(-c^2)$

Note that the metric (3.2) is invariant under the Lorentz transformation (a hyperbolic rotation)

$$\begin{pmatrix}\cosh v & \sinh v & 0\\ \sinh v & \cosh v & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{3.12}$$

where $-\infty<v<\infty$ is a hyperbolic angle. So, $\mathbb{H}^3_1(-c^2)$ has SO$(1,1)$ symmetry. Let us consider the profile curve $(h(u),0,g(u))$ in the ty-plane. Rotating $(h(u),0,g(u))$ about the y-axis via the hyperbolic rotation (3.12), we obtain a surface

$$\varphi(u,v)=(h(u)\cosh v,h(u)\sinh v,g(u)). \tag{3.13}$$

Here we used the word rotation but only in a metaphorical sense. The hyperbolic rotation (3.12) is not really a rotation that we are familiar with in Euclidean 3-space. However the surface (3.13) shares similar properties to rotational surfaces in Euclidean 3-space. In particular, the induced metric ds^2_φ is given by

$$ds^2_\varphi = e^{-2cg(u)}[\{-(h'(u))^2 + (g'(u))^2\}du^2 + (h(u))^2 dv^2] \tag{3.14}$$

and this does not depend on the hyperbolic angle v analogously to rotational surfaces in Euclidean space. I call surfaces of the form (3.13) *Lorentz invariant surfaces*. If $g'(u) \neq 0$ for all u, (3.13) has a parametrization of the form

$$\varphi(w,v) = (f(w)\cosh v, f(w)\sinh v, w).$$

So without loss of generality we may assume that $g(u) = u$ in (3.13). From now on, I only consider Lorentz invariant surfaces of the form

$$\varphi(u,v) = (h(u)\cosh v, h(u)\sinh v, u). \tag{3.15}$$

I now calculate the mean curvature of the Lorentz invariant surface (3.15) using Gauss' formula (2.9). First, E, F, G are calculated to be

$$\begin{aligned}
E &= -e^{-2cu}h'(u)^2 + 1,\\
F &= 0,\\
G &= e^{-2cu}h(u)^2.
\end{aligned}$$

Since I want $\varphi(u,v)$ to be conformal, I require that $E = G$. Hence I obtain

$$(h(u))^2 = -(h'(u))^2 + e^{2cu}. \tag{3.16}$$

Also the quantities $\ell, \mathfrak{m}, \mathfrak{n}$ are calculated to be

$$\begin{aligned}
\ell &= -\frac{h''(u)h(u)}{\sqrt{-(h(u))^2[-e^{-2cu} + (h'(u))^2]}},\\
\mathfrak{m} &= 0,\\
\mathfrak{n} &= -\frac{(h(u))^2}{\sqrt{-(h(u))^2[-e^{-2cu} + (h'(u))^2]}}.
\end{aligned}$$

Hence, the mean curvature H is given by

$$H = -\frac{1}{2}\frac{h(u)h'(u) - h''(u)^2 + e^{2cu}}{[e^{-2cu}(h'(u))^2 - 1]\sqrt{-(h(u))^2[-e^{2cu} + (h'(u))^2]}}. \tag{3.17}$$

Using the conformality condition (3.16), (3.17) can be simplified as

$$H = \frac{h''(u) + h(u)}{-2e^{-2cu}(h(u))^3}. \tag{3.18}$$

Differentiating (3.16), I obtain

$$h'(u)(h''(u)+h(u)) = ce^{2cu}. \tag{3.19}$$

From (3.18) and (3.19) we see that if $H=0$ then $c=0$. Therefore, the following proposition holds.

Proposition 7. *There are no Lorentz invariant conformal spacelike surfaces of $H=0$ in $\mathbb{H}^3_1(-c^2)$.*

Those who are familiar with maximal spacelike surfaces in Minkowski 3-space $\mathbb{E}^3_1$ may think that proposition 7 implies that there are no Lorentz invariant maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$. In Minkowski 3-space $\mathbb{E}^3_1$, a conformal spacelike parametric surface φ is maximal if and only if $\triangle\varphi = 0$ if and only if $H=0$. However, this is no longer true in anti-de Sitter 3-space $\mathbb{H}^3_1(-c^2)$ as $\triangle\varphi = 0$ is not the harmonic map equation (see [3]) i.e. maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$ are not characterized by mean curvature. While there are no Lorentz invariant conformal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$, there are Lorentz invariant maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$. I will discuss this later.

Let $H=c$. Then from (3.18) I obtain the second order nonlinear differential equation

$$h''(u)+h(u)+2ce^{-2cu}h(u)^3 = 0. \tag{3.20}$$

This equation cannot be solved analytically. Hence I solve the equation numerically with the aid of MAPLE software. (Appendix contains details of computational procedure I performed.) The conformality condition (3.16) can be used to determine initial conditions. For all of the numerical solutions in this thesis, I used the initial conditions $h(0)=0$ and $h'(0)=1$.

If $c\to 0$, then the equation (3.20) becomes an equation of underdamped harmonic oscillator

$$h''(u)+h(u) = 0. \tag{3.21}$$

This equation has a general solution

$$h(u) = c_1\cos u + c_2\sin u.$$

This $h(u)$ results a Lorentz invariant maximal spacelike surface in $\mathbb{E}^3_1$ called a Lorentz invariant spacelike catenoid. Figure 3.1 shows a Lorentz invariant spacelike catenoid with $h(0)=0$ and $h'(0)=1$. It is now shown that Lorentz invariant conformal spacelike surfaces of constant mean curvature (CMC) $H=c$ in anti-de Sitter 3-space $\mathbb{H}^3_1$ approach the Lorentz invariant spacelike catenoid in Minkowski 3-space $\mathbb{E}^3_1$ as $c\to 0$. Figures 3.2 - 3.6 illustrate

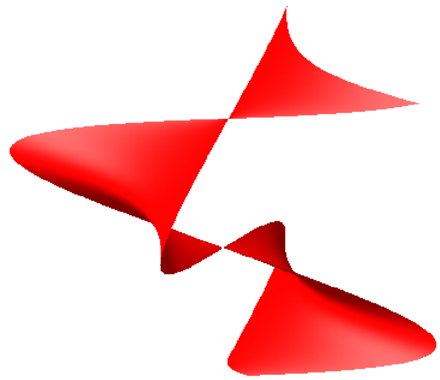
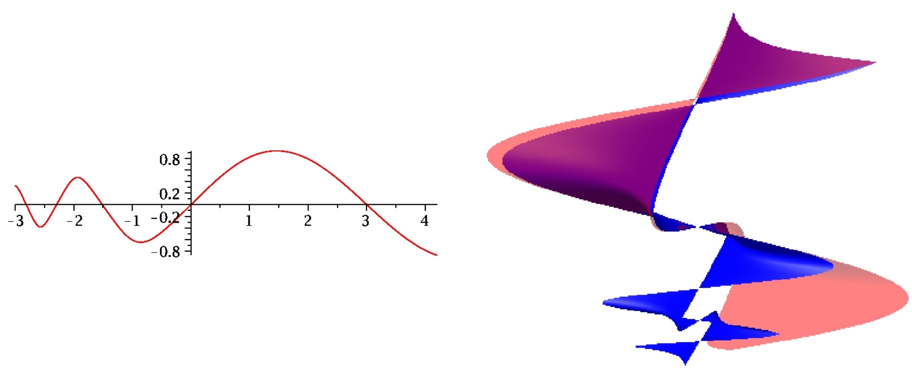


Figure 3.1: Spacelike Catenoid in $\mathbb{E}^3_1$

this limiting behavior. Each figure contains both Lorentz invariant spacelike surface of constant mean curvature $H = c$ in $\mathbb{H}^3_1(-c^2)$ (in blue) and the limit surface Lorentz invariant spacelike catenoid in $\mathbb{E}^3_1$ (in red) for visual comparison.

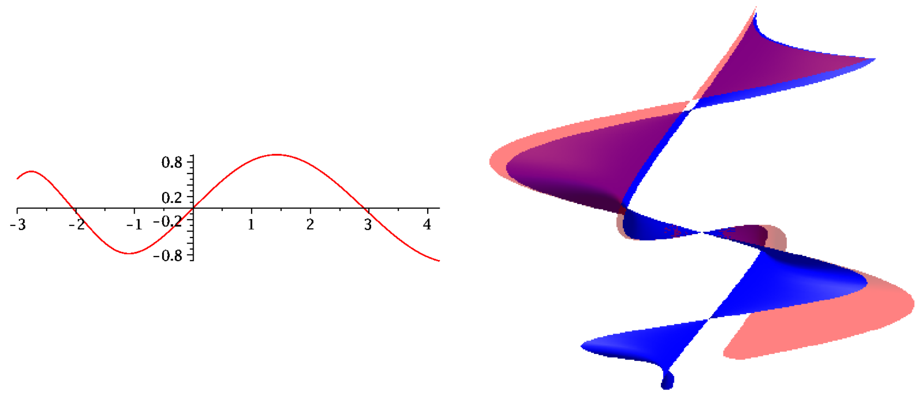


(a) (b)

Figure 3.2: CMC $H = 1$: (a) Profile Curve $h(u)$, $-3 \le u \le 4.2$, (b) Lorentz Invariant Spacelike Surface in $\mathbb{H}^3_1(-1)$

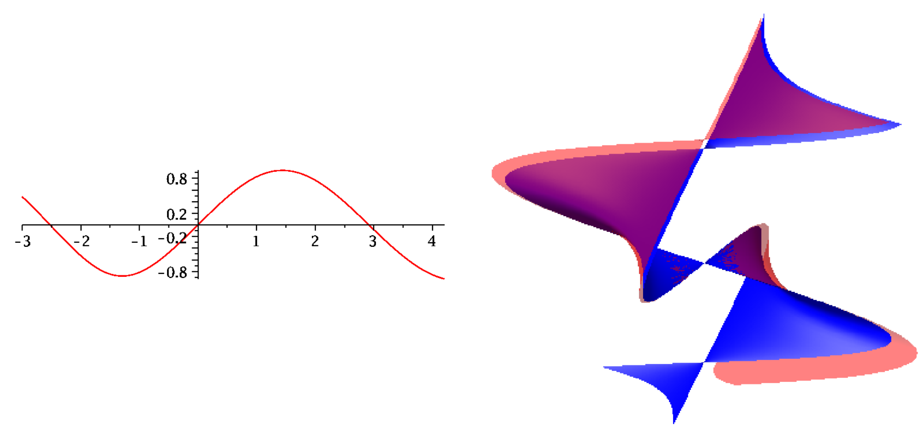
3.4 Lorentz Invariant Maximal Spacelike Surface in $\mathbb{H}^3_1(-c^2)$

I mentioned earlier that maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$ are not characterized by mean curvature. In this section, I find the Lorentz invariant maximal spacelike surface in $\mathbb{H}^3_1(-c^2)$ as a critical point of the area functional using the calculus of variations.



(a) (b)

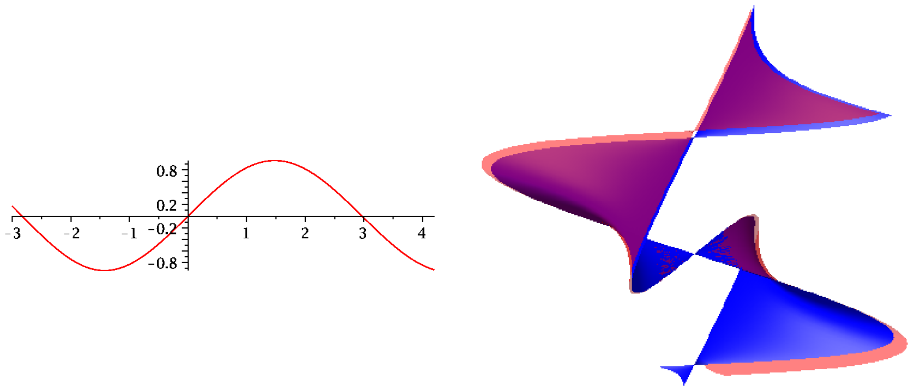
Figure 3.3: CMC $H = \frac{1}{2}$: (a) Profile Curve $h(u)$, $-3 \leq u \leq 4.2$, (b) Lorentz Invariant Spacelike Surface in $\mathbb{H}_1^3(-\frac{1}{4})$



(a) (b)

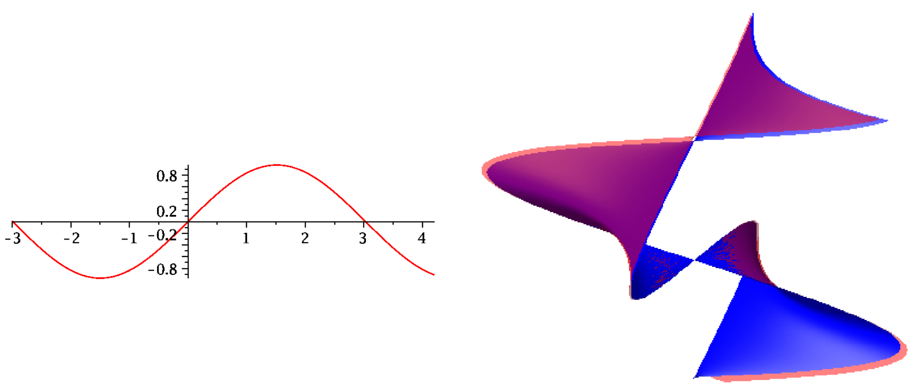
Figure 3.4: CMC $H = \frac{1}{4}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 5$, (b) Lorentz Invariant Spacelike Surface in $\mathbb{H}_1^3(-\frac{1}{16})$

By invoking a Euclidean picture, let us consider a rotational surface which is obtained by rotating a curve $t(y)$ in the ty-plane about the y-axis. I require the boundary conditions (y_1, t_1) and (y_2, t_2) on the curve $t(y)$ as seen in Figure 3.7. I want to find the curve $t(y)$ so that the area of the resulting rotational surface is a maximum. The area element dA from



(a) (b)

Figure 3.5: CMC $H = \frac{1}{8}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 5$, (b) Lorentz Invariant Spacelike Surface in $\mathbb{H}_1^3(-\frac{1}{64})$



(a) (b)

Figure 3.6: CMC $H = \frac{1}{16}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 5$, (b) Lorentz Invariant Spacelike Surface in $\mathbb{H}_1^3(-\frac{1}{256})$

Figure 3.7 is given by

$$dA = 2\pi t(y)ds = 2\pi t(y)\sqrt{1 - e^{-2cy}\left(\frac{dt}{dy}\right)^2}dy. \tag{3.22}$$

The area functional J is

$$J = \int_{y_1}^{y_2} 2\pi t(y)\sqrt{1 - e^{-2cy}\left(\frac{dt}{dy}\right)^2}dy. \tag{3.23}$$

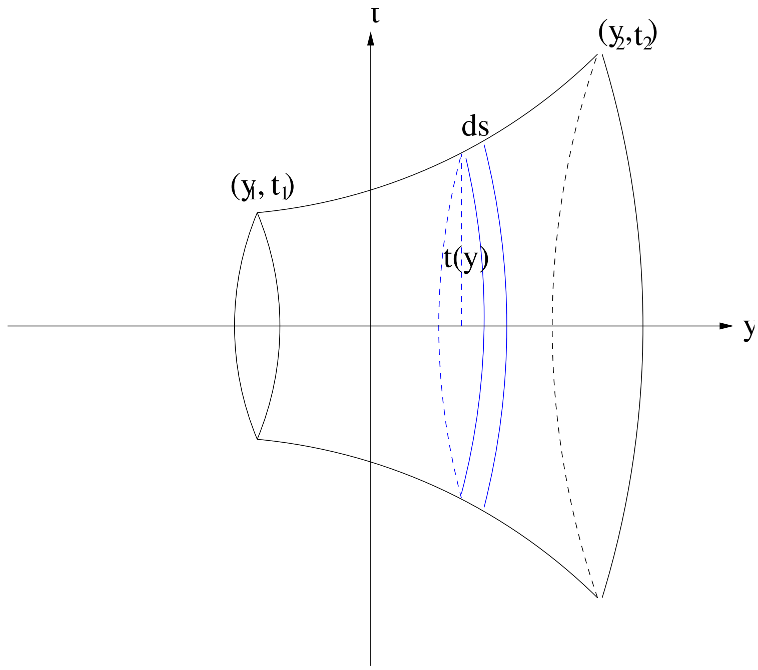


Figure 3.7: Surface of Revolution in $\mathbb{H}^3_1(-c^2)$

Remark 4. I obtained the area functional (3.23) from a Euclidean perspective of rotational surfaces. But there is no rotation in the ty-plane in Euclidean sense. So how can I justify the functional (3.23)? For one, the area functional (3.23) is obtained by mimicking a Euclidean rotation and we may as well accept (3.23) as the definition of area functional for spacelike surfaces in $\mathbb{H}^3_1(-c^2)$. On the other hand, the area of a parametric surface $\varphi(u,v) = (t(u,v), x(u,v), y(u,v))$ in $\mathbb{H}^3_1(-c^2)$ is given by

$$\int\int_D ||\varphi_u \times \varphi_v|| du dv = \int\int_D e^{cy(u,v)}(EG - F^2) du dv \tag{3.24}$$

which is consistent with the area of a parametric surface in Euclidean 3-space. (Recall the way $||\varphi_u \times \varphi_v||$ is defined in Remark 3.) So this may reasonably justify the way the area functional (3.23) is obtained.

Let[2]

$$f(t, t', y) = \sqrt{1 - e^{-2cy}(t')^2},$$

where $t' = \frac{dt}{dy}$. I find a critical point of the area functional (3.23) by solving the Euler-Lagrange equation (see, for instance, [1])

$$\frac{\partial f}{\partial t} - \frac{d}{dy}\frac{\partial f}{\partial t'} = 0. \tag{3.25}$$

[2]I neglect the constant 2π as it has no effect on the solution of the variational problem.

The Euler-Lagrange equation (3.25) is equivalent to

$$-(t')^2+e^{2cy}-ce^{-2cy}tt'(-(t')^2+e^{2cy})-ctt'+t''t=0. \quad (3.26)$$

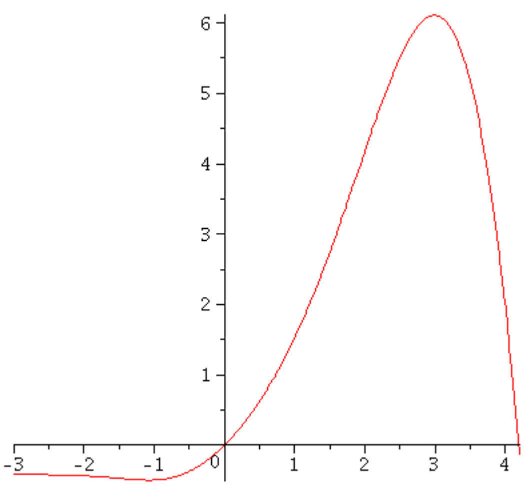
I require the conformality condition (3.16) i.e.

$$(t(y))^2=-(t'(y))^2+e^{2cy}. \quad (3.27)$$

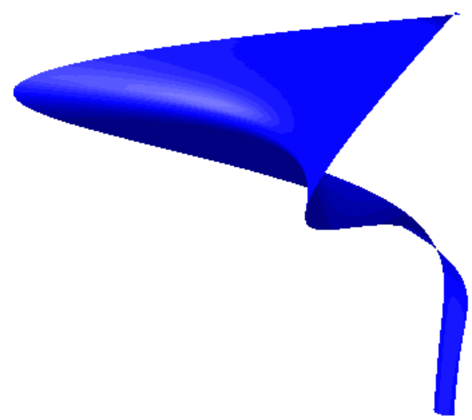
The resulting equation is

$$t''-c(1+e^{-2cy}t^2)t'+t=0. \quad (3.28)$$

The second order nonlinear equation (3.28) cannot be solved analytically so I solve it numerically. I use the same initial condition $t(0)=0$ and $t'(0)=1$ for the numerical solution. Figure 3.8 shows the profile curve $t(y)$ and the Lorentz invariant maximal spacelike surface in $\mathbb{H}^3_1(-1)$.



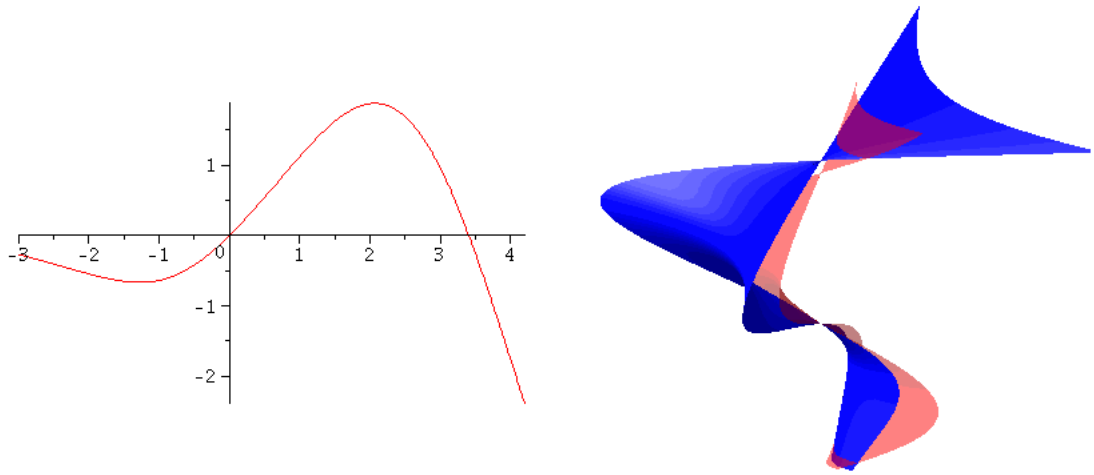
(a)



(b)

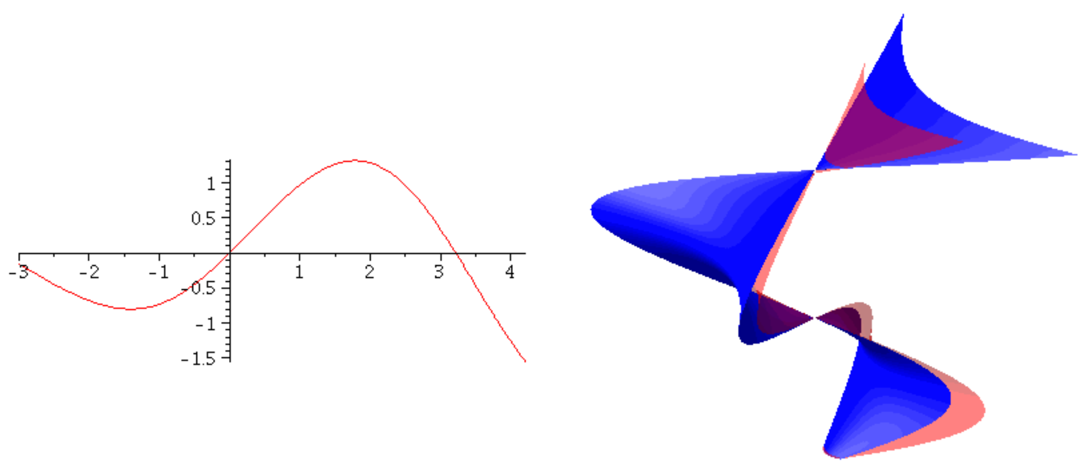
Figure 3.8: (a) Profile Curve $t(y)$, $-3 \le y \le 4.2$, (b) Maximal Lorentz Invariant Spacelike Surface in $\mathbb{H}^3_1(-1)$

As $c \to 0$, the equation (3.28) becomes the equation of underdamped simple harmonic oscillator (3.21). So, we see that Lorentz invariant maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$ also approach the Lorentz invariant spacelike catenoid in $\mathbb{E}^3_1$. Figures 3.9 - 3.11 illustrate this limiting behavior of Lorentz invariant maximal spacelike surfaces in $\mathbb{H}^3_1(-c^2)$. Each figure contains both Lorentz invariant maximal spacelike surface in $\mathbb{H}^3_1(-c^2)$ (in blue) and the limit surface Lorentz invariant spacelike catenoid in $\mathbb{E}^3_1$ (in red) for visual comparison.



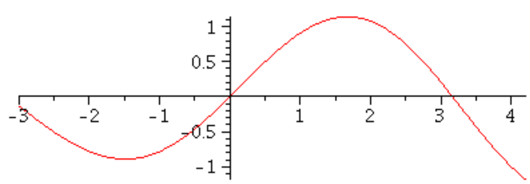
(a) (b)

Figure 3.9: (a) Profile Curve $t(y)$, $-3 \leq y \leq 4.2$, (b) Maximal Lorentz Invariant Spacelike Surface in $\mathbb{H}^3_1\left(-\frac{1}{4}\right)$

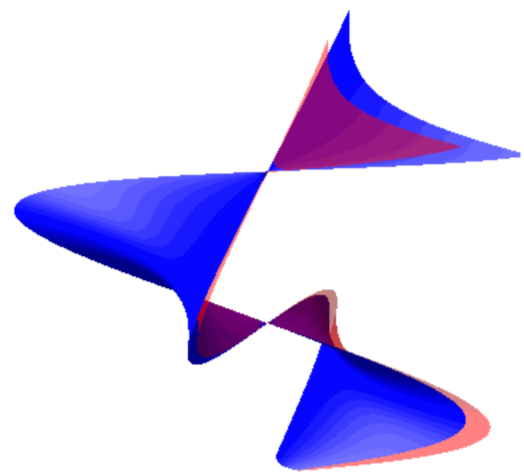


(a) (b)

Figure 3.10: (a) Profile Curve $t(y)$, $-3 \leq y \leq 4.2$, (b) Maximal Lorentz Invariant Spacelike Surface in $\mathbb{H}^3_1\left(-\frac{1}{16}\right)$



(a)



(b)

Figure 3.11: (a) Profile Curve $t(y)$, $-3 \leq y \leq 4.2$, (b) Maximal Lorentz Invariant Spacelike Surface in $\mathbb{H}^3_1\left(-\frac{1}{64}\right)$

Appendix A

COMPUTER RESULTS

A.1 The Numerical Solution of (3.20) with MAPLE

The numerical solution of the differential equation (3.20) was obtained with the aid of MAPLE software version 15. For the readers who want to try by themselves, here are the MAPLE commands that I used to obtain the numerical solutions and the graphics. The commands need to be run in the following order.

First we clear the memory.

```
restart:
```

In order to solve the equation numerically, we need a MAPLE package called DEtools.

```
with(DEtools):
```

Set the c value. In this example, we set $c = 1$.

```
c:=1;
```

Define the differential equation (3.20).

```
eq:=diff(h(u),u,u)+h(u)+2*c*exp(-2*c*u)*h(u)^3=0;
```

Define the initial conditions for the equation (3.20).

```
ic:=h(0)=0,D(h)(0)=1;
```

Get the numerical solution.

```
sol:=dsolve({eq,ic},numeric,output=listprocedure);
```

Define the numerical solution as a function Y.

```
Y:=subs(sol,h(u)):
```

For testing, we evaluate $Y(0.8)$.

```
Y(0.8);
```

The output is

```
0.707356122085521
```

Now, we are ready to plot the profile curve $h(u)$.

```
plot(Y,-3..4.2,scaling=constrained);
```

The output is Figure 3.2 (a).

Define the surface of revolution X.

```
X:=[Y(u)*cosh(v),Y(u)*sinh(v),u)];
```

Finally, we plot the Lorentz invariant spacelike surface X of CMC $H = 1$.

```
plot3d(X,u=-3..4.2,v=-1..1,grid=[85,85],style=patchnogrid,
shading=zhue,orientation=[62,64]);
```

The output is Figure 3.2 (b).

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