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Chromatic Thresholds of Regular Graphs with Small Cliques

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CHROMATIC THRESHOLDS OF REGULAR GRAPHS WITH SMALL CLIQUES

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ABSTRACT

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The chromatic threshold of a class of graphs is the value θ such that any graph in this class with a minimum degree greater than θn has a bounded chromatic number. Several important results related to the chromatic threshold of triangle-free graphs have been reached in the last 13 years, culminating in a result by Brandt and Thomassé stating that any triangle-free graph on n vertices with minimum degree exceeding $\frac{1}{3}n$ has chromatic number at most 4. In this paper, the researcher examines the class of triangle-free graphs that are additionally regular. The researcher finds that any triangle-free graph on n vertices that is regular of degree $(\frac{1}{4}+\alpha)n$ with $\alpha > 0$ has chromatic number bounded by $f(\alpha)$, a function of α independent of the order of the graph n. After obtaining this result, the researcher generalizes this method to graphs that are free of larger cliques in order to limit the possible values of the chromatic threshold for regular K_r-free graphs.

ACKNOWLEDGMENTS

This is to thank everyone who has assisted me in this effort. In particular, I would like to thank Dr. Sungwook Lee for encouraging me to continue my study of mathematics, Dr. John Perry and Dr. Karen Kohl for their insightful questions and comments, and my advisor Dr. Jeremy Lyle for introducing me to extremal graph theory, for helping me find and understand the tools needed to obtain our results, and for his patient support throughout the project.

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General Usage and Terminology

The notation used in this paper for basic graph theory concepts is generally taken from [5]. In particular,

- G refers to a graph
- $V(G)$ and $E(G)$ refer to the vertex set and edge set of G, respectively
- $N(v)$ refers to the neighborhood of the vertex v, that is, $\{u \in V(G) : (u,v) \in E(G)\}$.
- n generally refers to the order of a graph, that is $|V(G)|$
- δ refers to the minimum degree of the graph
- K_r refers to the complete graph on r vertices
- $\chi(G)$ refers to the chromatic number of G
- $KG(n,k)$ refers to the Kneser graph related to k-subsets of a ground set of size n
- d refers to density
- $e(A,B)$ refers to the number of edges with one end-vertex in each of A and B

Additionally, when working with the Szemerédi regularity lemma,

- G' refers to the subgraph of G with an ε-regular partition
- G'' refers to the pure graph obtained by deleting the exceptional set V_0 from G'
- R refers to the reduced graph associated with the pure graph G''
- Capital letters such as V_i refer to sets of vertices as in the pure graph
- Lower-case letters such as v_i refer to a single vertex as in the reduced graph
- V_i is the cluster in the pure graph associated with v_i in the reduced graph

Chapter 1
BACKGROUND

1.1 Introduction

Triangle-free graphs of order n with sufficiently large minimum degree satisfy some strong structural properties. By restricting a triangle-free graph to be regular, the structure is limited further. In this paper, the researcher finds a constant C, more restrictive than the constant C obtained from Turán's Theorem, such that for a triangle-free graph that is regular of degree $\delta > (C+\alpha)n$, the chromatic number is bounded by $f(\alpha)$, a function of α independent of the order of the graph.

1.2 Terminology and Notation

For notation and definitions, this paper will follow that of [5].

A *graph* G is a finite nonempty set of objects called *vertices* (or *nodes*) together with a (possibly empty) set of unordered pairs of distinct vertices of G called *edges*. The *vertex set* of G is denoted by $V(G)$, while the *edge set* of G is denoted by $E(G)$. The *order* of the graph is $|V(G)|$, and the researcher often uses n to represent this order. The *size* of the graph is $|E(G)|$. This paper is concerned only with simple graphs, i.e., graphs with undirected edges, no loops, and no multiple edges.

The edge $e = (u, v)$ is said to *join* the vertices u and v. If $e = (u, v)$ is an edge of a graph G, then u and v are said to be *adjacent*, while u and e are said to be *incident*, as are v and e. u and v are said to be the *end-vertices* (singular: *end-vertex*) of e. The *neighborhood* of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v.

A *complete* graph is a graph wherein every pair of vertices is adjacent. A complete graph on r vertices is denoted K_r. K_3 is also called a *triangle*.

A *cycle* is a sequence of vertices $v_1, v_2, \ldots, v_k, v_1$ such that each $v_i \in V(G), i = 1, \ldots, k$ is distinct, and each pair of consecutive vertices is adjacent. A cycle containing k distinct vertices is called a *k-cycle* and denoted C_k. An *odd cycle* is a k-cycle with k odd.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *H-free graph* is a graph which does not contain H as a subgraph. If K_r is a subgraph of a graph, G, K_r is called a *clique*. Because K_r is a subgraph of K_s for $r < s$, any K_r-free graph is additionally K_s-free.

The *degree* of a vertex v in a graph G, denoted $d_G(v)$, is the number of edges in G incident to v. The minimum degree of a graph G is denoted $\delta(G)$. A graph G is said to be *regular* if each vertex in $V(G)$ has the same degree. If each vertex has degree δ, it is said to be *regular of degree* δ.

A *proper coloring* of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned different colors (it is often convenient to use numbers instead of colors as in Figure 1.2). A graph is said to be *k-colorable* if there exists a coloring of the graph with k or fewer colors. The *chromatic number* of a graph G is the minimum integer k for which G is k-colorable. The chromatic number is denoted $\chi(G)$. If $\chi(G) = k$, it is said that G is *k-chromatic*.

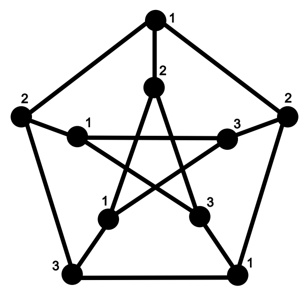

Figure 1.1: The Petersen graph, pictured here, is regular of degree 3, triangle-free, and 3-chromatic.

Two vertices that are not adjacent in a graph G are said to be *independent*. A set $S \subseteq V(G)$ is said to be *independent* if every two vertices of S are independent. Put another way, for all $v, w \in S$, $(v,w) \notin E(G)$. An independent set of vertices, therefore, can all be assigned the same color in a coloring, and so finding independent sets gives us a method of coloring a graph. In fact, one can define $\chi(G)$ as the minimum number of parts needed to partition $V(G)$ into independent sets.

A *bipartite graph* is a graph that can be partitioned into two independent sets. That is, any edges in a bipartite graph must have one end-vertex in one of the partitions and the other end-vertex in the other. A graph is bipartite if and only if it contains no odd cycles. Because a bipartite graph is made up of two independent sets, the chromatic number of a bipartite graph is 2.

A *complete bipartite graph* is a bipartite graph with all possible edges. That is, G is a complete bipartite graph if G has the property that $V(G)$ can be partitioned into disjoint, independent sets A and B with the property that for all $a \in A, b \in B$, $(a,b) \in E(G)$. The complete bipartite graph with partition sizes r and s is denoted $K_{r,s}$.

For a graph G, let $A, B \subset V(G)$. Denote by $e(A,B)$ the number of edges with an end-vertex in A and one end-vertex in B. When it is unclear, denote by $e_H(A,B)$ the number of edges in $E(H)$ with one end-vertex in A and one end-vertex in B.

Denote by $G-H$ the graph G with the subgraph H removed, that is, $V(G-H) = V(G)-V(H)$ and the edge set $E(G-H)$ consists of only the edges with both end-vertices in $V(G-H)$.

1.3 Background

1.3.1 Extremal Graph Theory

Finding the chromatic threshold of a class of graphs is a problem in the field of extremal graph theory. Extremal graph theory is a branch of graph theory studying extremal (that is, maximal or minimal) graphs with respect to some property. It is said to have started with a 1907 result from Mantel.

Theorem 1. *[14] The maximum number of edges in a triangle-free (K_3-free graph) of order n is $\frac{1}{4}n^2$.*

A complete bipartite graph is maximally triangle-free, that is, adding one edge to a complete bipartite graph forces a triangle.

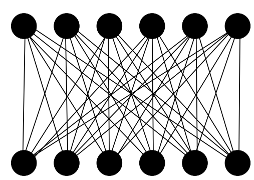

Figure 1.2: In the complete bipartite graph $K_{6,6}$, there are $36 = \frac{1}{4}(12)^2$ edges.

In 1941, Turán proved a significant result, generalizing Mantel's theorem for any K_r-free graph [18].

Definition 1. *The Turán graph $T(n,r)$ on n vertices is formed by partitioning the n vertices into r sets as evenly as possible and then connecting two vertices by an edge only if the two vertices belong to different sets. Note that by the pigeonhole principle, this graph contains no copy of K_{r+1}.*

Theorem 2. *(Turán [18]) The Turán graph $T(n,r)$ has the most edges among any K_{r+1}-free graph on n vertices. Put another way, a graph G with $|V(G)| = n$ vertices and $|E(G)| > |E(T(n,r))|$ must contain K_{r+1} as a subgraph.*

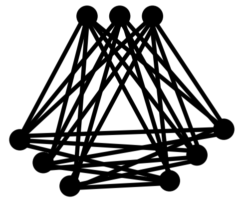

Figure 1.3: $T(9,3)$, the K_4-free graph on 9 vertices with the most possible edges.

Theorem 2 gives a bound on the number of edges in a K_{r+1}-free graph of order n. If n is divisible by r, each vertex of $T(n,r)$ has degree $\frac{r-1}{r}n$, so $T(n,r)$ has $\frac{n^2(r-1)}{2r}$ edges. Likewise, any graph G of order n with $|E(G)| > \frac{n^2(r-1)}{2r}$ must contain at least one copy of K_{r+1}.

In 1946, Erdős and Stone [7] generalized Turán's result to any subgraph H based on its chromatic number. The result below is a more restrictive minimum-degree version of their result.

Theorem 3. *(Erdős and Stone [7]) Let H be a graph such that $\chi(H) = r \geq 2$ and $\alpha > 0$. There is an integer $n_0 = n_0(r, \alpha)$ such that if $|V(G)| = n \geq n_0$ and*

$$\delta(G) \geq (\tfrac{r-2}{r-1} + \alpha)n$$

then G contains H as a subgraph.

This theorem implies that, given a graph H, a graph G with sufficiently large minimum degree and sufficiently large order cannot be H-free. (The above shows the case for $\chi(H) \geq 2$; if $\chi(H) < 2$ then H contains no edges, and thus as long as $|V(G)| \geq |V(H)|$, H is a subgraph of G.)

The result from Erdős and Stone motivated others to find results on H-free graphs, as well as to examine the structure of triangle-free graphs with a bound on the minimum degree.

1.3.2 The Chromatic Threshold Problem

In 1955, Mycielski [15] created a construction that preserves the property of being triangle-free but increases the chromatic number.

Definition 2. *The Mycielskian of a graph G, denoted $\mu(G)$, is constructed as follows:*

- *Let $v_1, \ldots, v_n$ be vertices in $V(G)$.*
- *Add a copy of G to $\mu(G)$.*
- *Add vertices $u_1, \ldots, u_n$ to $V(\mu(G))$.*

- *For each $(v_i, v_j) \in E(G)$, add the edges $(u_i, v_j), (v_i, u_j) \in E(\mu(G))$.*
- *Add a vertex $w \in V(\mu(G))$, and for each $u_i \in V(\mu(G))$, add the edge $(w, u_i) \in E(\mu(G))$.*

With $n = |V(G)|$ and $m = |E(G)|$, the resulting graph $\mu(G)$ has $2n+1$ vertices and $3m+n$ edges. The Mycielskian of a graph G preserves the property of being triangle-free but increases the chromatic number by 1. By repeated application of the construction, Mycielski showed that there exist triangle-free graphs with arbitrarily large chromatic number.

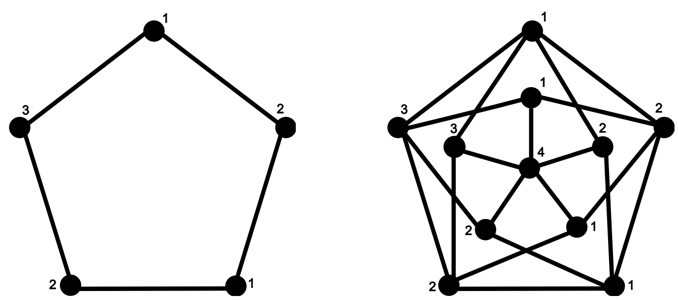

Figure 1.4: On the left, the 3-chromatic, triangle-free graph C_5.
On the right, the 4-chromatic, triangle-free graph $\mu(C_5)$, also known as the *Grötzsch graph*.

Also in 1955, Martin Kneser [9] began investigating a class of graphs that today bear his name.

Definition 3. *A Kneser graph, denoted $KG(n,k)$, is a graph whose vertices can be labeled as the k-element subsets of the ground set $\{1, \ldots, n\}$, with vertices adjacent if and only if their associated subsets are disjoint.*

A Kneser graph $KG(n,k)$ has $\binom{n}{k}$ vertices and $\frac{1}{2}\binom{n}{k}\binom{n-k}{k}$ edges.

By limiting n to be $2k+j$ for some $0 < j < k$, it is guaranteed that the Kneser graph is itself triangle-free; for there to be three disjoint k-element subsets, the ground set needs to contain at least $3k$ elements.

In 1978, Lovász proved a conjecture of Kneser regarding the chromatic number of the Kneser graph $KG(n,k)$.

Theorem 4. *(Lovász [11]) The chromatic number of the Kneser graph $KG(n,k)$ is $\chi(KG(n,k)) = n-2k+2$.*

As a corollary, for the Kneser graph $KG(2k+j,k)$, the chromatic number is

$$\chi(KG(2k+j,k)) = j+2.$$

The Kneser graph is another construction that is triangle-free and that can have arbitrarily large chromatic number.

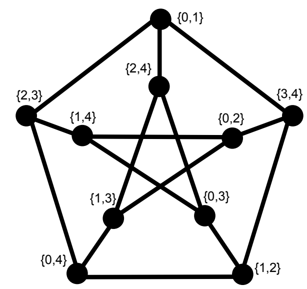

Figure 1.5: The Petersen graph is also the Kneser graph $KG(5,2)$.
Each vertex is assigned a 2-element subset of $\{1,\ldots,5\}$, with disjoint subsets adjacent.

1.3.3 Literature Review and Results

In 1973, Erdős and Simonovits [6] considered the problem of finding the minimum value $\psi = \psi(n, K_p, r)$ for which an r-chromatic graph G of order n and minimum degree ψ must contain K_p as a subgraph. The authors showed that

$$\psi(n, K_3, r) \geq (\tfrac{1}{3} + o(1))n,$$

where $o(1)$ refers to functions $f(n)$ such that $\lim_{n\to\infty} \frac{f(n)}{1} = 0$.

In the same paper, they conjectured that $\psi(n, K_3, r) \approx \frac{1}{3}n$. That is, they conjectured that a triangle-free graph on n vertices with minimum degree at least approximately $\frac{1}{3}n$ has chromatic number no greater than 3. This conjecture was based on a construction by Hajnal showing that $\frac{1}{3}$ is best possible, but it was later disproved. Hajnal's construction is described in more detail in Chapter 3.

In 1974, Andrásfai, Erdős, and Sós [2] proved the following theorem.

Theorem 5. *(Andrásfai et al. [2]) Let $r \geq 3$, and let G be a K_r-free graph on n vertices such that*

$$\delta(G) > \tfrac{3r-7}{3r-4}n.$$

Then $\chi(G) < r$.

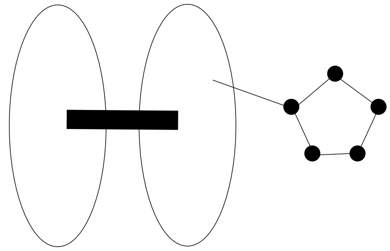

Figure 1.6: Here, each oval is an independent set of q vertices, with the thick edge indicating all possible edges between the two sets.

In a triangle-free graph, if the minimum degree exceeds $\frac{2}{5}n$, the chromatic number of G is strictly less than 3; that is, G is bipartite. This bound is best possible due to C_5 (see Figure 1.4).

An upper bound given in terms of the minimum degree $\delta(G)$ of a graph G is often more restrictive than an upper bound given on the total number of edges in a graph G. For instance, consider a graph G that consists of a complete bipartite graph $K_{q,q}$ together with a 5-cycle C_5, with one vertex in $K_{q,q}$ adjacent to one vertex in C_5.

The complete bipartite graph $K_{q,q}$ has $\frac{1}{4}(2q)^2 = q^2$ edges, and C_5 contains 5 edges. Therefore, $|E(G)| = q^2 + 5 + 1 = q^2 + 6$, and this graph has no triangle.

On the other hand, generalizing Theorem 5, no more than $\frac{2}{5}$ of the $\frac{(2q+5)^2}{2}$ are in $|E(G)|$, that is, at most

$$\frac{2}{5}\left(\frac{(2q+5)^2}{2}\right) = \frac{1}{5}(2q+5)^2 < q^2 + 6.$$

for large enough q. As q grows very large, the upper bound given by the edge count is close to q^2, and the upper bound given by the minimum degree is close to $\frac{4}{5}q^2$, giving a more restrictive upper bound on the number of edges.

To give an idea of how Theorem 5 works, consider $r = 3$. The theorem states that a triangle-free graph with minimum degree $\delta > \frac{2}{5}$ is bipartite.

Proof. Let G be a non-bipartite, triangle-free graph with minimum degree δ. Because G is non-bipartite, G contains an odd cycle, and because G is triangle-free, the smallest possible such odd cycle is C_5.

Let C_{2k+1} be the smallest odd cycle in G, with $k \geq 2$. A vertex $v \in V(G - C_{2k+1})$ adjacent to 2 vertices $u_i, u_j \in C_{2k+1}$ will create two cycles: an even cycle and an odd cycle. If u_i and u_j share a neighbor, the cycles created are C_4 and C_{2k+1}; otherwise, a smaller odd cycle is created, a contradiction to the statement that C_{2k+1} is the smallest odd cycle in G. Because there are no three vertices in C_{2k+1} that are pairwise adjacent, each vertex in $G - C_{2k+1}$ is adjacent to at most 2 vertices in C_{2k+1}. Thus,

$$e(G - C_{2k+1}, C_{2k+1}) \leq (2)(n - (2k+1)).$$

Each vertex in C_{2k+1} is adjacent to exactly 2 vertices in C_{2k+1}, and each has a minimum degree of δ, so

$$e(C_{2k+1}, G - C_{2k+1}) \geq (2k+1)(\delta - 2).$$

These values must be equal, giving

$$\begin{gathered}(2k+1)(\delta-2) \leq e(C_{2k+1}, G - C_{2k+1}) \leq (n-(2k+1))(2) \\ (2k+1)\delta - 2(2k+1) \leq 2n - 2(2k+1) \\ (2k+1)\delta \leq 2n \\ \delta \leq \tfrac{2}{2k+1}n.\end{gathered}$$

The largest value of $\frac{2}{2k+1}$ with $k \geq 2$ is $\frac{2}{5}$. Consequently, for a non-bipartite triangle-free graph, the minimum degree cannot exceed $\frac{2}{5}n$. □

In 1982, the conjecture from Erdős and Simonivits was disproved by Häggkvist [8]. Häggkvist modified the Grötzsch graph (see Figure 1.4) by replacing the degree-5 vertex with a set of 4 vertices, replacing each degree-4 vertex with a set of 3 vertices, and replacing each degree-3 vertex with a set of 2 vertices. Two vertices in the new graph are adjacent only if the corresponding vertices are adjacent in the Grötzsch graph. The result is a 4-chromatic, triangle-free graph with minimum degree $\frac{10}{29}n$, providing a counterexample to the conjecture.

In 2001, Brandt [3] published the following theorem.

Theorem 6. *(Brandt [3]) Let G be a regular maximal triangle-free graph (that is, a regular triangle-free graph such that the addition of any edge produces a triangle) of order n with degree $\delta > \frac{1}{3}n$. Then G is 4-colorable.*

This result gives a very strict bound on the chromatic number of regular triangle-free graphs of degree greater than $\frac{1}{3}n$; this result is the motivation to examine the behavior of regular triangle-free graphs of smaller degree. In particular, the researcher is interested in a result on the *chromatic threshold* of such graphs.

The *chromatic threshold* θ of a family of graphs is the minimum value C such that for all H-free graphs G in the family of graphs and all $\alpha \geq 0$, if the minimum degree $\delta \geq (C+\alpha)n$, then $\chi(G)$ is bounded by $f(\alpha)$, a function of α independent of the order of the graph.

In 2002, Thomassen [17] showed that, for every $C > \frac{1}{3}$, the chromatic number of a triangle-free graph with minimum degree Cn is bounded. That is, the chromatic threshold of triangle-free graphs is no greater than $\frac{1}{3}$.

In 2006, Łuczak used graph homomorphisms in order to strengthen this result.

A *graph homomorphism* f from a graph G into a graph H is a map $f : V(G) \to V(H)$ such that if $(u,v) \in E(G)$, then $(f(u), f(v)) \in E(H)$. If there exists such a homomorphism, G is said to be *homomorphic* to H. For each $u \in V(G)$, $\{f(v) : (u,v) \in E(G)\} \subset N(f(u))$, that is, every adjacency in G has a corresponding adjacency in H, so

$$\chi(G) \leq \chi(H).$$

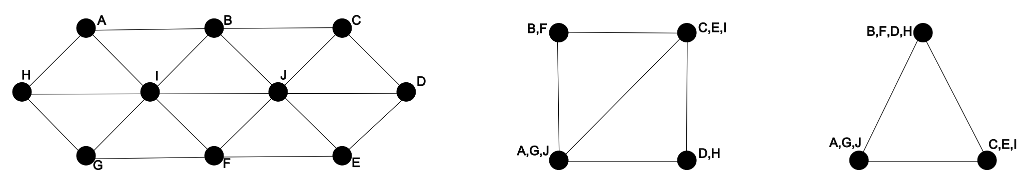

Figure 1.7: A set of independent vertices can be mapped to a single vertex in a homomorphism. The first graph is homomorphic to each graph on its right.

Łuczak [12] proved that a triangle-free graph G on n vertices with minimum degree exceeding $(\frac{1}{3} + \alpha)n$ is homomorphic to one of at most $M(\alpha)$ triangle-free graphs, where M is a function dependent only on α. Because the chromatic number of G is bounded by the chromatic number of a graph to which it is homomorphic, the maximum chromatic number among graphs to which a class of graphs is homomorphic serves as a bound, further strengthening Thomassen's result.

In 2010, Brandt and Thomassé [4] produced the strongest result on the chromatic threshold of triangle-free graphs thus far. Any triangle-free graph on n vertices with minimum degree exceeding $\frac{1}{3}n$ has chromatic number at most 4. This result by Brandt and Thomassé gives us a strong motivation to examine triangle-free graphs with additional restrictions.

In 2013, Allen et al. [1] produced an important result classifying the chromatic thresholds of H-free graphs by $\chi(H)$, the chromatic number of H. However, the result does not extend to additional restraints on a class of graphs, as in the case that the class of graphs is additionally regular.

The chromatic number is an important concept in graph theory in that it allows us to partition the vertices into independent sets, and it can be used to prevent conflicts in scheduling and in transmission of radio signals, among many others. Determining the chromatic threshold of a class of graphs can reduce the work considerably, and it can lead to stronger results about the class's chromatic number, as was the case with Brandt and Thomassé's result.

In this paper, the researcher that for triangle-free graphs that are additionally regular, the chromatic threshold can be reduced to $\frac{1}{4}$.

Chapter 2

TOOLS

2.1 Szemerédi Regularity Lemma

In 1975, Endre Szemerédi [16] proved a lemma that has had far-reaching consequences and has been useful in solving extremal graph theory problems. The Szemerédi regularity lemma allows us to construct a *reduced graph* from a large graph in such a way that it maintains strong regularity properties. Before the lemma is stated, some definitions are needed.

For $A, B \subset V(G)$, the *density* of the pair (A, B), denoted $d(A, B)$, is given by

$$d(A,B) = \tfrac{e(A,B)}{|A||B|},$$

that is, the number of edges between A and B divided by the number of possible edges between A and B. Because $0 \leq e(A,B) \leq |A||B|$, it follows that $0 \leq d(A,B) \leq 1$.

For $\varepsilon > 0$, (A,B) is said to be ε-*regular* if for all $X \subseteq A, Y \subseteq B$ where $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$,

$$|d(A,B) - d(X,Y)| < \varepsilon.$$

That is, any sufficiently large subsets of an ε-regular pair have approximately the same density as the ε-regular pair.

A partition of $V(G)$ into $k+1$ sets $V_0, \ldots, V_k$ is called an ε-*regular partition* if $||V_i| - |V_j|| \leq 1$ for all i, j, and all except εk^2 of the pairs (V_i, V_j), $i < j$ are ε-regular.

A graph with an ε-regular partition behaves somewhat like a *random graph*, that is, a graph on n vertices in which edges are placed randomly so that the probability of two vertices being adjacent is constant. In particular, subsets of two parts of an ε-regular partition have approximately the same density as the two parts themselves.

Lemma 1. *(Szemerédi regularity lemma [16]) For every ε such that $0 < \varepsilon < 1$, there exist integers $M = M(\varepsilon)$ and $N = N(\varepsilon)$ such that if a graph G has $n \geq N$ vertices, and $d \in [0,1]$ is a real number greater than ε, then there is an ε-regular partition of $V(G)$ into $k+1$ clusters with $\varepsilon^{-1} < k < M$. Denote these clusters $V_0, V_1, \ldots, V_k$. There is a subgraph $G' \subset G$ with $V(G') = V(G)$ such that the subgraph and the clusters have these properties:*

1. $1/\varepsilon < k < M$,

2. $|V_0| < \varepsilon n$,

3. *All clusters* V_i, $i \geq 1$, *are of the same size,* $|V_i| = m \leq \lceil \varepsilon n \rceil$,

4. $d_{G'}(v) > d_G(v) - (d+\varepsilon)n$ *for all* $v \in V(G)$,

5. *All pairs* (V_i, V_j), $i < j$ *in* G *are* ε*-regular with* $d(V_i, V_j) > d$ *or* $d(V_i, V_j) = 0$.

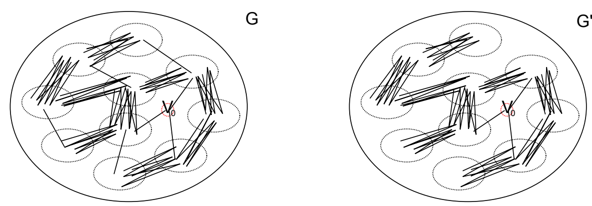

Figure 2.1: A small number of edges are deleted in order to obtain this ε-regular partition.

From this subgraph G', one can produce G'', called the *pure graph*, by deleting the exceptional set V_0. By the Szemerédi regularity lemma (2), $|V_0| < \varepsilon n$, so a vertex $v \in G$ is adjacent to at most εn vertices in V_0, so

$$d_{G''}(v) \geq d_{G'}(v) - \varepsilon n.$$

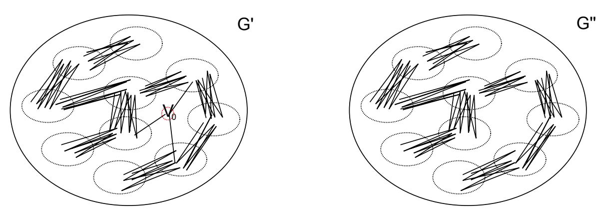

Figure 2.2: The pure graph G'' is obtained by deleting the exceptional set from G'.

Lemma 2. *Given* G *the original graph and an* ε*-regular partition that generates the pure graph* G'',

$$d_{G''}(v) > d_G(v) - (d+2\varepsilon)n.$$

Proof. This follows from $d_{G''}(v) \geq d_{G'}(v) - \varepsilon n$ together with the Szemerédi regularity lemma (4).

$$d_{G''}(v) \geq d_{G'}(v) - \varepsilon n > d_G(v) - ((d+\varepsilon) - \varepsilon)n.$$

□

Finally, from this pure graph G'', one can produce the *reduced graph*, denoted R. Each cluster $V_i \subset G''$ corresponds to a vertex $v_i \in V(R)$. Two vertices $v_i, v_j \in V(R)$ are adjacent if and only if $d(V_i, V_j) > d$. It is often convenient to associate $v_i \in V(R)$ with the corresponding cluster $V_i \subset V(G'')$.

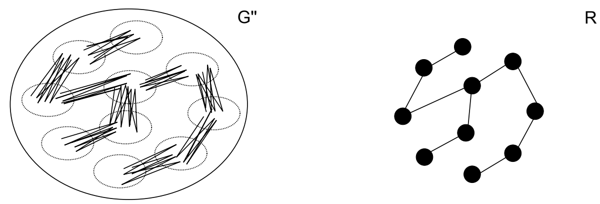

Figure 2.3: Each cluster in the pure graph G'' is associated with a vertex in R.

The Szemerédi regularity lemma is an important tool in theoretical graph theory, as it allows us to construct this *reduced graph* from a given graph of sufficiently large order. However, this order requirement is often incredibly large [16]. As a result, this lemma has little practical use when applied to a specific graph; instead, it is used to prove general results. Together with the following two lemmas, one is able to describe the structure of the reduced graph in relation to the original graph.

Lemma 3. *(Slicing [10]) Let ε, η such that $0 < \varepsilon < \eta < 1$ and d such that $1 - d \geq max\{2\varepsilon, \varepsilon/\eta\}$, and let (A,B) be an ε-regular pair with density d. Also let $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \eta|A|$, $|B'| \geq \eta|B|$. Then (A', B') is ε'-regular with $\varepsilon' = max\{2\varepsilon, \varepsilon/\eta\}$ and density in $[d - \varepsilon, d + \varepsilon]$.*

The slicing lemma states that if one considers a sufficiently large subset of each set in an ε-regular pair, the subsets are themselves ε'-regular, and they have approximately the same density as the original pair as long as $\varepsilon \ll d$.

Recall that a graph with an ε-regular partition behaves somewhat like a random graph, in which the probability of three vertices forming a triangle is the product of the probabilities that each pair of vertices is adjacent. The counting lemma [10] describes the conditions on the ε-regular graph that force a copy of K_r. It is described below for the triangle-free case.

Lemma 4. *(Counting [10]) Let G be a graph and let V_i, V_j, V_k be disjoint subsets of the vertices of G. Suppose that the partitions (V_i, V_j), (V_j, V_k), and (V_i, V_k) are ε-regular with densities d_1, d_2, d_3 respectively, and that $d_1, d_2, d_3 \leq 2\varepsilon$. Then the number of triples $(v_i, v_j, v_k) \in V_i \times V_j \times V_k$ that form triangles in G is at least*

$$(1 - 2\varepsilon)(d_1 - \varepsilon)(d_2 - \varepsilon)(d_3 - \varepsilon)|V_i||V_j||V_k|.$$

Note that if the reduced graph R, as constructed by the Szemerédi regularity lemma, has a triangle, then there are disjoint subsets $V_i, V_j, V_k \subset V(G)$ such that $d_1 = d(V_i, V_j) > d, d_2 = d(V_j, V_k) > d, d_3 = d(V_i, V_k) > d$. The above counting lemma then tells us that the number of triples of vertices that form triangles in G is at least

$$(1-2\varepsilon)(d_1-\varepsilon)(d_2-\varepsilon)(d_3-\varepsilon)m^3.$$

Each $d_i > \varepsilon$, so there are several triangles in the original graph.

As a corollary, if the reduced graph R has a triangle, then the original graph G necessarily also has a triangle. Thus if G is triangle-free, R is also triangle-free.

Likewise, if the reduced graph R has a copy of K_r, then the original graph G necessarily also has a copy of K_r. Thus if G is K_r-free, R is also K_r-free.

2.2 Cluster Neighborhoods

From here the researcher follows the methods and terminology in [12] and [13].

In the reduced graph, define the *cluster neighborhood* of $v \in V(G)$, denoted $N_d(v) \subseteq V(R)$, as

$$N_d(v) = \{v_i \in V(R) : |N_G(v) \cap V_i| > d \cdot m\},$$

where $N_G(v)$ is the set of vertices adjacent to v. That is, a vertex $v_i \in V(R)$ is in $N_d(v)$ if v is adjacent to at least $d \cdot m$ vertices in the corresponding cluster V_i in the partition formed from the Szemerédi regularity lemma.

The notion of cluster neighborhoods associates every $v \in V(G)$ with a subset $S_R \subset V(R)$. Because there is a maximum of M vertices in $V(R)$, there is a maximum of 2^M subsets of $V(R)$ and therefore a maximum of 2^M cluster neighborhoods.

For each cluster neighborhood $S_R \subset V(R)$, we define the inverse neighborhood $N_d^{-1}(S_R) \subseteq V(G)$ in the natural way:

$$N_d^{-1}(S_R) = \{v \in V(G) : N_d(v) = S_R\}.$$

By definition, because S_R is a neighborhood of some vertex $v \in G$, $N_d^{-1}(S_R)$ is nonempty.

As will be shown, if a triangle-free graph is regular of sufficiently high degree, the inverse neighborhoods are independent sets of vertices of $V(G)$, so identifying them will give us a method to color G.

Chapter 3

THE RESULT FOR REGULAR TRIANGLE-FREE GRAPHS

In order to find a chromatic threshold C for regular, triangle-free graphs, the researcher considers an upper bound and a lower bound on the chromatic threshold. First, the researcher considers a construction that is regular and triangle-free with unbounded chromatic number. The degree of such a construction is necessarily below the chromatic threshold, so it provides us with a lower bound for C. After that, the researcher uses the Szemerédi regularity lemma in order to generalize regular, triangle-free graphs and to give a lower bound on the degree that forces inverse neighborhoods to be independent and, consequently, a upper bound for the chromatic threshold.

3.1 A Regular, Triangle-Free Construction

A triangle-free construction by András Hajnal is obtained by taking a Kneser graph together with a complete bipartite graph. This construction was instrumental in work on the chromatic threshold done by Erdős and Simonovits as described in Chapter 1.

The original construction by Hajnal is formed as follows [6]. Consider the Kneser graph $KG(2\ell+k,\ell)$ together with the complete bipartite graph $K_{2m,m}$. Name the partitions of the complete bipartite graph A and B, with $|A| = 2m$ and $|B| = m$. Partition the vertices of A into $2\ell+k$ classes and label them $A_1, A_2, \ldots, A_{2\ell+k}$. Recall that each vertex in the $KG(2\ell+k,\ell)$ corresponds to an ℓ-subset of a $2\ell+k$-element ground set. A vertex in the Kneser graph is adjacent to all of the vertices in A_i if and only if its corresponding subset contains i as an element. Because no two vertices in the Kneser graph with i as an element in their corresponding subset are adjacent, adding these edges will not create a triangle. The Kneser graph and B have no edges between them. One is able to choose values of k, ℓ, m freely without affecting this structure. By restricting $k \ll \ell \ll m$, the order of the resulting graph is approximately $3m$. By Theorem 4, the chromatic number of this construction is $k+2$.

Theorem 3.1.1. *The chromatic threshold of a regular triangle-free graph is at least $\frac{1}{4}$.*

Proof. In order to obtain a result on regular triangle-free graphs, the researcher modifies Hajnal's construction in the following way. The Kneser graph $KG(2\ell+k,\ell)$ remains unchanged, but the sizes of the partitions $|A| = m_1, |B| = m_2$ are changed.

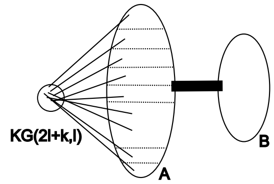

Figure 3.1: Hajnal's construction consists of a Kneser graph together with a complete bipartite graph, where the thick line implies all edges between these sets.

As described in the definition of the Kneser graph, $KG(2\ell+k,\ell)$ has $\binom{2\ell+k}{\ell}$ vertices and $\frac{1}{2}\binom{2\ell+k}{\ell}\binom{\ell+k}{\ell}$ edges. That is, for $v \in KG(2\ell+k,\ell)$, $e(v, KG(2\ell+k,\ell) = \frac{1}{2}\binom{\ell+k}{\ell}$. Let $p_{k,\ell} = \frac{1}{2}\binom{\ell+k}{\ell}$. For v, for each element i of its corresponding subset, it is adjacent to every vertex in A_i. $|A_i| = \frac{1}{2\ell+k}m_1$, and there are ℓ elements in the corresponding subsets. Therefore $e(v,A) = \frac{\ell}{2\ell+k}m_1$, and, for $v \in KG(2\ell+k,\ell)$,

$$d(v) = p_{k,\ell} + \frac{\ell}{2\ell+k}m_1.$$

A, B no longer form a complete bipartite graph, but the vertices are joined in a regular way. In particular, label the vertices of A $a_1, a_2, \ldots, a_{m_1}$ and B $b_1, b_2, \ldots, b_{m_2}$, and join each b_j to the d vertices $a_j, a_{j+1}, \ldots, a_{j+d-1}$, with each subscript evaluated in $\mathbb{Z}_{m_1}$ (that is, modulo m_1). In this construction, for each $b_j \in B$,

$$d(b_j) = d.$$

Likewise, because $e(A,B) = e(B,A)$, each $e(a_j,B) = d\frac{m_2}{m_1}$. Each a_j is in a partition of A. If $a_j \in A_i$, then a_j is adjacent to each vertex in the Kneser graph whose corresponding subset contains i, a total of $\binom{2\ell+k-1}{\ell-1}$ vertices. Let $q_{k,\ell} = \binom{2\ell+k-1}{\ell-1}$. Thus, for each $a_j \in A$,

$$d(a_j) = d\frac{m_2}{m_1} + q_{k,\ell}.$$

Because this is to be a regular graph, these degrees should be equal. For $d(b_j) = d(a_j)$,

$$d = d\frac{m_2}{m_1} + q_{k,\ell}$$
$$d = \frac{q_{k,\ell}}{m_1 - m_2}m_1.$$

Because our conjecture for the chromatic threshold is $\frac{1}{4}$, the construction should give a triangle-free graph of unbounded chromatic number that is regular of degree $\frac{1}{2}(1-\varepsilon)$ for any $\varepsilon > 0$. Then,

$$q_{k,\ell} = \tfrac{1}{2}(1-\varepsilon)(m_1 - m_2), \text{ and}$$
$$d = \tfrac{1}{2}(1-\varepsilon)m_1.$$

For $d(b_j) = d(v)$ with $v \in KG(2\ell+k,\ell)$,

$$d = p_{k,\ell} + \tfrac{\ell}{2\ell+k} m_1$$
$$\tfrac{1}{2}(1-\varepsilon)m_1 = p_{k,\ell} + \tfrac{\ell}{2\ell+k} m_1 = p_{k,\ell} + (\tfrac{1}{2} - \tfrac{k}{2(2\ell+k)})m_1$$
$$p_{k,\ell} = (\tfrac{k}{2(2\ell+k)} - \varepsilon)m_1.$$

Let $\gamma = \frac{k}{2(2\ell+k)} - \varepsilon$, so $p_{k,\ell} = \gamma m_1$, and

$$m_1 = \tfrac{p_{k,\ell}}{\gamma}.$$

By choosing k, ℓ such that $\frac{k}{2(2\ell+k)} \to \varepsilon$ (i.e., $\gamma \to 0$), $m_1 \gg p_{k,\ell}$.

$$m_2 = \tfrac{1}{\gamma} p_{k,\ell} - \tfrac{1}{\frac{1}{2}(1-\varepsilon)} q_{k,\ell},$$
$$d = \tfrac{1}{2}(1-\varepsilon)m_1.$$

By choosing

$$m_2 = \tfrac{1}{2}\tfrac{p_{k,\ell}}{\gamma},$$
$$m_1 = m_2 + \tfrac{2}{1-\varepsilon} q_{k,\ell},$$

and setting $d = (\frac{1}{2} - \varepsilon)m_1$, for $b_j \in B$,

$$d(b_j) = (\tfrac{1}{2} - \varepsilon)m_1 = \tfrac{1}{2}(1-\varepsilon)\tfrac{p_{k,\ell}}{\gamma},$$

for $a_j \in A$,

$$d(a_j) = d\tfrac{m_2}{m_1} + q_{k,\ell}$$
$$d(a_j) = \tfrac{1}{2}(1-\varepsilon)(\tfrac{p_{k,\ell}}{\gamma} - \tfrac{2}{1-\varepsilon} q_{k,\ell}) + q_{k,\ell}$$
$$d(a_j) = \tfrac{1}{2}(1-\varepsilon)\tfrac{p_{k,\ell}}{\gamma},$$

and for $v \in KG(2\ell+k, \ell)$,

$$d(v) = p_{k,\ell} + \tfrac{\ell}{2\ell+k}(\tfrac{p_{k,\ell}}{\gamma} + \tfrac{2}{1-\varepsilon} q_{k,\ell}$$
$$d(v) = \tfrac{1}{2}(1-\varepsilon)\tfrac{p_{k,\ell}}{\gamma},$$

giving a triangle-free regular graph of degree $(\frac{1}{2} - \varepsilon)(\frac{\binom{\ell+k}{\ell}}{\frac{k}{2\ell+k} - 2\varepsilon})$. The sizes of A and B depend on the choices of k and ℓ, which can be chosen based on ε.

Hence, for regular triangle-free graphs, the chromatic threshold $C \geq \frac{1}{4}$. $\square$

3.2 Coloring via Inverse Neighborhoods

Let G be an arbitrary, regular triangle-free graph of large order n; if there is a bound on the order, then the chromatic number is trivially bounded by n. Using the Szemerédi regularity lemma, one can construct an ε-regular partition, and from that, a reduced graph R. By the counting lemma, a triangle in the reduced graph forces a triangle in the original graph, so by hypothesis, R is triangle-free. Łuczak gives us the following proposition, and the researcher uses an approach similar to the one outlined in that paper.

Proposition 1. *[12] For a triangle-free graph G, each neighborhood $S_R \subset V(R)$ is independent in R.*

Proof. By way of contradiction, suppose there exists a neighborhood $S_R \subset V(R)$ that is not independent in R. By definition of independent, this means there exist $v_i, v_j \in S_R$ such that an edge $e = (v_i, v_j) \in E(R)$. This edge in the reduced graph implies that $d(V_i, V_j) > d$.

Let $v \in N_d^{-1}(S_R)$. Define $V_i' \subset V_i$ to be $N(v) \cap V_i$. By definition of a cluster neighborhood, this means that

$$|V_i'| = |N(v) \cap V_i| > d \cdot |V_i|$$

and

$$|V_j'| = |N(v) \cap V_j| > d \cdot |V_j|.$$

By the slicing lemma, choosing $\alpha = d$, $d(V_i', V_j')$ is nonzero because $\varepsilon < d$, implying that there exists at least one edge between the two sets. Because v is adjacent to all the vertices in each of V_i' and V_j', an edge between V_i and V_j implies a triangle in G. This triangle in G is a contradiction to the premises (G is triangle-free). Therefore, for a triangle-free graph G, each neighborhood is independent in R. □

It follows that the bound on the size of an independent set in R is also a bound on the size of a neighborhood.

In order to determine a bound on the size of an independent set in R, consider a related graph, R_W, with weighted edges. While the reduced graph of a regular graph may not be regular, there is a strong restriction on the sum of the weights of the edges adjacent to a vertex in R_W, and this bounds the size of an independent set.

Definition 4. *The weighted reduced graph R_W is defined in the same way as the reduced graph, but with each edge (v_i, v_j) assigned a weight equal to the number of edges in the pure graph with one end-vertex in V_i and one end-vertex in V_j, that is, $e_{G''}(V_i, V_j)$. Define* *the weighted degree of a vertex v_i to be the sum of the weights of the edges incident to v_i, and denote it $d_W(v_i)$.*

Lemma 5. *Let G be a graph that is regular of degree Cn, with an ε-regular partition that produces a pure graph G'' with clusters of size m. The weighted degree $d_W(v)$ of each vertex in the reduced graph R_W has the property*

$$m(C-(d+2\varepsilon))n < d_W(v_i) \leq m(Cn).$$

Proof. Let G be a graph of order n and regular of degree Cn. The pure graph G'' as described in the Regularity Lemma is formed by constructing an ε-regular partition into $k+1$ clusters and then deleting the exceptional set V_0. By Lemma 2, each vertex $v \in G''$ has the property

$$Cn-(d+2\varepsilon)n < d_G(v) \leq Cn.$$

Each cluster $V_i \subset V(G)$ has m vertices, and each cluster is an independent set in the pure graph. Therefore, each cluster V_i has at least $m(C-(d+2\varepsilon))n$ and no more than $m(Cn)$ edges with (exactly) one end-vertex in V_i. Each edge with an end-vertex in V_i contributes to the total weight of edges incident to v_i in R_W, so it follows that the weighted degree of a vertex $v_i \in R_W$ has the property

$$m(C-(d+2\varepsilon))n < d_W(v_i) \leq m(Cn).$$

□

For a regular, connected graph G, the size of the largest possible independent set of vertices is known to be $\frac{1}{2}n$, which is the case when G is bipartite with two equal partitions. The researcher wants to show that if G is approximately regular in this way, the bound on the size of an independent set is still approximately $\frac{1}{2}n$.

Lemma 6. *In a weighted graph G on n vertices, if, for every $v_i \in G$, there is a fixed $C \leq 1$ and $\beta \geq 0$ such that*

$$(C-\beta)n < d_W(v_i) \leq Cn,$$

then the size of the largest independent set S is no larger than $(\frac{1}{2}+\frac{\beta}{2(2C-\beta)})n$.

Proof. Let G be a weighted graph of order n with the property that, for every vertex $v \in V(G)$, $(C-\beta)n < d_W(v) \leq Cn$.

Let $S \subset V(G)$ be an independent set. $e(S,S)=0$, so it follows that

$$e(S,G)=e(S,S)+e(S,G-S)=e(S,G-S).$$

Because of the restriction on the degree of each vertex in S, it also follows that

$$
\begin{gathered}
|S|(C-\beta)n < e(S,G-S) \leq Cn(n-|S|) \\
|S|(C-\beta) \leq Cn - C|S| \\
|S|(2C-\beta) \leq Cn \\
|S| \leq \frac{C}{2C-\beta}n = (\frac{1}{2}+\frac{\beta}{2(2C-\beta)})n.
\end{gathered}
$$

□

Note that for $\beta \ll C$, this is approximately

$$|S| \leq \frac{1}{2}n.$$

Lemma 7. *Let G be a triangle-free graph that is regular of degree Cn, together with a reduced graph R formed from an ε-regular partition of G in which $(v_i,v_j) \in E(R)$ iff $d_{G''}(V_i,V_j) > d$. If u_i,u_j form an edge in $N_d^{-1}(S_R)$, then $|S_R| \geq 2(Cn-(2d+2\varepsilon))n$.*

Proof. Let G be a regular, triangle-free graph such that $u_i,u_j \in N_d^{-1}(S_R)$, with $S_R \subset R$, and suppose by way of contradiction there is an edge $(u_i,u_j) \in E(G)$. In the pure graph G'',

$$|N(u_i)| = d_{G''}(u_i) \geq Cn-(d+2\varepsilon)n,$$

and

$$|N(u_i) \cap N_d(u_i)| \geq Cn-(d+2\varepsilon)n-dm \geq Cn-(2d+2\varepsilon)n.$$

Because the original graph is triangle-free, the total number of vertices in the clusters in G'' in the cluster neighborhood must be at least $2(Cn-(2d+2\varepsilon)n)$. □

Lemma 6 and Lemma 7 give a bound on the size of an independent set in the reduced graph which, in turn, gives a bound on the size of a cluster neighborhood and a bound on the size of an inverse neighborhood.

Theorem 7. *Let G be a triangle-free graph that is regular of degree Cn, together with the reduced graph R formed by an ε-regular partition and $(v_i,v_j) \in E(R)$ iff $d(V_i,V_j) > d$. Then either each inverse neighborhood is independent, or*

$$C \leq \frac{(10d+12\varepsilon+1)+\sqrt{36d^2+48d\varepsilon+16\varepsilon^2+20d+24\varepsilon+1}}{8}.$$

Proof. By Proposition 1, Lemma 6, and Lemma 7, the researcher produces the inequality

$$(\tfrac{1}{2}+\tfrac{d+2\varepsilon}{2(2C-(d+2\varepsilon))})n \geq m\cdot|S_R| \geq 2(C-(2d+2\varepsilon))n$$
$$\tfrac{1}{2}+\tfrac{d+2\varepsilon}{2(2C-(d+2\varepsilon))} \geq 2(C-(2d+2\varepsilon)$$
$$2C-(d+2\varepsilon)+(d+2\varepsilon) \geq 2(C-(2d+2\varepsilon))2(2C-(d+2\varepsilon))$$
$$2C \geq 4(2C^2-C(d+2\varepsilon)-2C(2d+2\varepsilon)+(d+2\varepsilon)(2d+2\varepsilon))$$
$$C \geq 2(2C^2-Cd-2C\varepsilon-4Cd-4C\varepsilon+2d^2+2d\varepsilon+4d\varepsilon+4\varepsilon^2)$$
$$C \geq 4C^2-10Cd-12C\varepsilon+4d^2+12d\varepsilon+8\varepsilon^2$$
$$0 \geq 4C^2-(10d+12\varepsilon+1)C+(4d^2+12d\varepsilon+8\varepsilon^2)$$

For the right-hand side to equal 0, the solutions are given by the quadratic formula.

$$C=\frac{(10d+12\varepsilon+1)\pm\sqrt{(10d+12\varepsilon+1)^2-16(4d^2+12d\varepsilon+8\varepsilon^2)}}{8}.$$

In order for the right-hand side to be less than or equal to 0, either C must equal one of those solutions, or it must be between the two solutions, that is,

$$\frac{(10d+12\varepsilon+1)-\sqrt{36d^2+48d\varepsilon+16\varepsilon^2+20d+24\varepsilon+1}}{8} \leq C$$

and

$$C \leq \frac{(10d+12\varepsilon+1)+\sqrt{36d^2+48d\varepsilon+16\varepsilon^2+20d+24\varepsilon+1}}{8}.$$

Consequently, if $C > \frac{(10d+12\varepsilon+1)+\sqrt{36d^2+48d\varepsilon+16\varepsilon^2+20d+24\varepsilon+1}}{8}$, then the inverse neighborhoods are independent sets, and these independent sets give us a method to color G. □

3.3 Result and Example

Showing that the chromatic number is bounded for a triangle-free graph that is regular of degree Cn where $C > \frac{1}{4}$ can be achieved by choosing sufficiently small d and ε so that

$$C > \frac{(10d+12\varepsilon+1)+\sqrt{36d^2+48d\varepsilon+16\varepsilon^2+20d+24\varepsilon+1}}{8},$$

that is, by Theorem 7 the inverse neighborhoods are forcibly independent sets.

The researcher can use these sets as a partition of $V(G)$, that is, a coloring with at most 2^M colors, where M is the number of possible neighborhoods. For $C=\frac{1}{4}+\alpha$, ε, d can be chosen based on α, and then M depends only on the choice of ε.

For instance, consider the class of triangle-free graphs that is regular of degree $\delta = 0.27n = (\frac{1}{4}+\frac{1}{50})n$. By our result, this class of graphs has chromatic number bounded by $f(0.02)$, independent of n.

By choosing $d=0.001, \varepsilon=0.0001$, this bound becomes

$$\frac{(10(0.001)+12(0.0001)+1)+\sqrt{36(0.001)^2+48(0.001)(0.0001)+16(0.0001)^2+20(0.001)+24(0.0001)+1}}{8}$$
$$\frac{1.0112+\sqrt{0.000036+0.0000048+0.00000016+0.02+0.0024+1}}{8}$$
$$\frac{1.0112+\sqrt{1.02244096}}{8} \approx \frac{2.02236}{8} \approx 0.252795.$$

Because $0.27 > 0.252795$, there exists a $d, \varepsilon > 0$ for which the inverse neighborhoods will necessarily be independent sets, bounding the chromatic number of this class of graphs by the number of possible neighborhoods in the reduced graph, that is, no more than 2^M.

Chapter 4
CONCLUSIONS AND REGULAR K_r-FREE GRAPHS

In order to determine the chromatic threshold of graphs that are regular and K_r-free for $r \geq 4$, the researcher takes a similar approach: Finding a regular K_r-free construction with unbounded chromatic number in order to obtain a lower bound for the chromatic threshold, and considering a bound on the size on an independent set in an arbitrary regular K_r-free graph in order to find an upper bound.

Here, the researcher shows how to obtain a lower bound for the chromatic threshold for regular K_4-free graphs as well as a note on the upper bound, and then how to generalize this method for K_r-free graphs.

4.1 Regular K_4-free graphs

Every triangle-free graph is also K_4-free because K_3 is a subgraph of K_4. As shown before, the researcher used a modification of Hajnal's construction in order to find a triangle-free, regular graph of unbounded chromatic number. By adding additional parts to the construction described in the previous chapter, the researcher can construct a regular K_r-free graph of unbounded chromatic number for any $r \geq 3$.

4.1.1 Further modifying Hajnal's construction

Theorem 8. *The chromatic threshold of the class of regular K_4-free graphs is not less than $\frac{4}{7}$.*

Proof. To the construction H described in Chapter 3, a new set of independent vertices T is added such that each vertex $v \in T$ is adjacent to each vertex in H. In order for this new construction H' to be regular, for $v_T \in T$, $v_H \in H$, $d(v_T) = |H|$ must equal $d(v_H) = \frac{1}{2}(1-\varepsilon)\frac{p_{k,\ell}}{\gamma} + |T|$, so

$|T| = |H| - \frac{1}{2}(1-\varepsilon)\frac{p_{k,\ell}}{\gamma}$.

By a similar method to that used in Chapter 3, the researcher is able to calculate the minimum degree as a fraction of the total degree of H'. In particular for any $\varepsilon > 0$, one can produce a regular graph using this construction that is regular of degree $\frac{4}{7}(1-\varepsilon)|H'|$. □

In order to find an upper bound, consider a bound on the size of a triangle-free set of vertices in a regular K_4-free graph.

4.1.2 Coloring via inverse neighborhoods

The researcher again uses the Szemerédi regularity lemma in order to find an ε-regular partition and to produce a reduced graph. By the counting lemma, the reduced graph will contain a copy of K_4 as a subgraph only if there is a copy of K_4 in the original graph, so by hypothesis, the reduced graph will be K_4-free as well.

Proposition 2. *In a K_4-free graph G, each neighborhood $S_R \subset V(R)$ is triangle-free in R.*

Proof. By way of contradiction, suppose there exists a neighborhood $S_R \subset V(R)$ that is not triangle-free in R. Then there exist $v_i, v_j, v_k \in S_R$ that form a triangle in R, which implies that $d(V_i,V_j) > d, d(V_i,V_k) > d, d(V_j,V_k) > d$.

Let $v \in N_d^{-1}(S_R)$. Define $V_i' \subset V_i$ to be $N(v) \cap V_i$. By definition of a cluster neighborhood, this means that

$$\begin{aligned}
|V_i'| &= |N(v) \cap V_i| > d \cdot m = d \cdot |V_i|,\\
|V_j'| &= |N(v) \cap V_j| > d \cdot m = d \cdot |V_j|,\\
|V_k'| &= |N(v) \cap V_k| > d \cdot m = d \cdot |V_k|.
\end{aligned}$$

By the slicing lemma, this set of inequalities implies the existence of a triangle with one vertex in each of V_i', V_j', V_k'. However, because v is adjacent to each of these vertices, this creates a copy of K_4, a contradiction to the premise that G is K_4-free. Hence, each neighborhood is triangle-free in R. □

It follows that the bound on the size of a triangle-free set $S \subset R$ is also a bound on the size of a cluster neighborhood. Theorem 1 (Mantel's theorem) gives that $e(S,S) \leq \frac{1}{4}|S|^2$.

Modifying Lemma 6, one can consider, for a triangle-free set $S \subset R$,

$$e(S,G) = e(S,S) + e(S,G-S) \leq \tfrac{1}{4}|S|^2 + e(S,G-S),$$

Also because of the restriction on the degree of each vertex in S given by Lemma 5, if G is regular, then R_W is approximately regular in terms of weighted degree as described in Chapter 3. In particular, if G is regular of degree Cn, then the weighted degree of each vertex $v_i \in V(R_W)$ has the property

$$m(C - (d + 2\varepsilon))n < d_W(v_i) \leq m(Cn).$$

As a result, it follows that

$$\begin{gathered}
|S|(C-\beta)n - 2(\tfrac{1}{4}|S|^2) \leq e(S,G-S) \leq (n - |S|)(Cn)\\
\tfrac{1}{2}|S|^2 + (\beta - C)n|S| - Cn|S| + Cn^2 \geq 0\\
\tfrac{1}{2}|S|^2 + (\beta - 2C)n|S| + Cn^2 \geq 0.
\end{gathered}$$

The values of $|S|$ for which the left-hand side equals 0 are given by the quadratic formula.

$$|S| = \frac{(2C-\beta)n \pm \sqrt{(\beta-2C)^2n^2-2Cn^2}}{1}$$
$$|S| = (2C-\beta)n \pm \sqrt{(\beta^2-2\beta C+4C^2)n^2-2Cn^2}$$
$$|S| = \left((2C-\beta) \pm \sqrt{\beta^2-2C(2\beta+1)+4C^2}\right)n.$$

For the inequality to hold, $|S|$ must be smaller than both of these values or larger than both these values; however, because the reseacher obtained from the previous section that $C \geq \frac{4}{7}n$,

$$|S| \geq \left((2C-\beta)+\sqrt{\beta^2-2C(2\beta+1)+4C^2}\right)n$$

has no solutions because

$$\left((2C-\beta)+\sqrt{\beta^2-2C(2\beta+1)+4C^2}\right)n > (2C-\beta)n > n$$

and $|S| \leq n$.

Hence,

$$|S| \leq ((2C-\beta)-\sqrt{\beta^2-2C(2\beta+1)+4C^2})n.$$

This bound on the size of a triangle-free subgraph provides a somewhat weak upper bound on the minimum degree for which the chromatic number is bounded; this result is initial progress toward a more restrictive bound for K_4-free graphs.

4.2 Regular K_r-free graphs

Extending this method to regular K_r-free graphs is done in much the same way: modifying Hajnal's construction in order to find a lower bound, and finding a bound on the size of an independent set in order to find an upper bound.

4.2.1 Modifying Hajnal's construction for regular K_r-free graphs

Theorem 9. *The chromatic threshold of the class of regular K_r-free graphs is not less than $\frac{3r-8}{3r-5}$.*

Proof. Again, Hajnal's construction is modified as it was in the K_4 case, now adding $r-3$ independent sets of order $|T|$, each adjacent to all vertices outside of its own set. In this case, given the new construction has n vertices, each vertex is adjacent to $\frac{3r-8}{3r-5}(1-\varepsilon)n$, where $\varepsilon > 0$ can be chosen arbitrarily small. The result is a regular, K_r-free graph of unbounded chromatic number. □

4.2.2 Coloring via inverse neighborhoods

In order to find an upper bound on the size of an independent set in the class of K_r-free graphs that are regular of degree δ, the researcher uses the Szemerédi regularity lemma in order to find an ε-regular partition and to produce a reduced graph. By the counting lemma, the reduced graph will contain a copy of K_r as a subgraph only if there is a copy of K_r in the original graph, so by hypothesis, the reduced graph will be K_r-free as well.

Proposition 3. *In a K_r-free graph G, each neighborhood $S_R \subset V(R)$ is K_{r-1}-free in R.*

Proof. As before, if R is not K_{r-1}-free, there exists a copy of K_{r-1} in $N(v)$, and hence a copy of K_r in G, a contradiction to the premises. □

It follows that the bound on the size of a K_{r-1}-free set $S \subset R$ is also a bound on the size of a neighborhood. Turán's theorem gives us $|e(S,S)| \leq \frac{|S|^2(r-1)}{2r}$.

In order to determine a bound on the size of a triangle-free set in R, the researcher considers the weighted reduced graph R_W. Considering the degree limitations of the reduced graph imposed by Lemma 5, it is possible to limit the size of a K_{r-1}-free set S by considering $e(S, G-S)$.

Because, for all $v_i \in R_W$, $(C-\beta)n \leq d_W(v_i) \leq Cn$, $e(S, G-S)$ is given by

$$e(G,S) - e(S,S) \geq |S|(C-\beta)n - 2(\tfrac{r-1}{2r}|S|^2).$$

Also, because G is regular, $e(S, G-S) \leq (n-|S|)(Cn)$.

$$\begin{gathered}
|S|(C-\beta)n - \tfrac{r-1}{r}|S|^2 \leq e(S,G-S) \leq (n-|S|)(Cn) \\
\tfrac{r-1}{r}|S|^2 + (\beta - C)n|S| - Cn|S| + Cn^2 \geq 0 \\
\tfrac{r-1}{r}|S|^2 + (\beta - 2C)n|S| + Cn^2 \geq 0 \\
(r-1)|S|^2 + (\beta - 2C)rn|S| + Crn^2 \geq 0
\end{gathered}$$

The values of $|S|$ for which the left-hand side equals 0 are given by the quadratic formula.

$$\begin{gathered}
|S| = \frac{(2C-\beta)rn \pm \sqrt{(\beta-2C)^2 r^2 n^2 - 4(r-1)(Crn^2)}}{2(r-1)} \\
|S| = \frac{(2C-\beta)rn \pm \sqrt{(\beta^2 - 4\beta C + 4C^2)r^2n^2 - 4Cr^2n^2 + 4Crn^2)}}{2(r-1)} \\
|S| = \frac{r}{r-1} \frac{\left((2C-\beta) \pm \sqrt{(\beta^2 - 4C(\beta+1) + 4C^2) + \frac{4C}{r}}\right)}{2} n
\end{gathered}$$

This bound on the size of a K_{r-1}-free subgraph provides a weak upper bound for the minimum degree for which the chromatic number is bounded. More work is necessary to obtain a more restrictive bound.

4.3 Conclusion and Future Work

In conclusion, the method described in this paper gives a sharp bound for the chromatic threshold of regular triangle-free graphs: $\frac{1}{4}$. It also provides a smaller set of possibilities for the chromatic thresholds of regular K_r-free graphs for $r \geq 4$.

The lower bound of the chromatic threshold of regular K_r-free graphs could potentially be raised by examining the properties of the K_{r-1}-free set beyond the number of internal edges it contains.

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