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Relativistic and Non-Relativistic Proton-Nucleus Scattering

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by

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Abstract

Calculations for proton-nucleus scattering often rely on transition amplitudes. We implement new transition amplitudes [7] with the relativistic equations. We can find the matrix elements of the operators between the usual Dirac spinor basis or the helicity spinor basis. The operators can also be written as a linear combination of non-relativistic spin operators. To transform from one basis to another, we need to find a transformation matrix. We must establish what one of the factors that appears in the transformed expression means in order to correctly complete our transformation matrix. Once this is resolved, our transformation matrix will be complete.

Key Terms: Nuclear physics, nuclear scattering, transition amplitudes

Dedication

My parents, Kirk and Kaycee Zarske, have been instrumental to my success. They have always encouraged intellectual growth. I also owe the writing of this thesis to my high school physics teacher, Mr. Scott Pfaff, who introduced me to physics in the first place. Additionally, many people here at USM have helped me through the years. Dr. Sung Lee, Dr. Lawrence Mead, Dr. James Lambers, Dr. Bill Ford, and Dr. Khin Maung Maung are a few of them. Further thanks go to my husband who has supported me through the completion of this thesis.

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Chapter 1: Introduction

One of the most useful experiments in nuclear physics is proton-nucleus scattering. Scattering observables such as the differential cross-section, polarization, and spin rotation functions can be measured during these experiments. There are wide ranges of experiments, with projectile energies ranging from a few MeV to a few GeV and target nuclei ranging from light nuclei such as deuterons to heavy nuclei such as lead. Together, all this makes for an extremely rich field of study, with many opportunities to learn more about the structure and inner workings of nuclei.

Traditionally, the non-relativistic Schrödinger equation,

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + U\right]\psi(r,\theta,\phi) = E\psi(r,\theta,\phi), \tag{1.1}$$

where μ is reduced mass, U is optical potential, E is energy, r, θ, ϕ, are in the center of mass frame, was used to predict nucleon-nucleus scattering observables, such as the differential cross-section and polarization. The optical potential (U) can be expressed in an infinite series called the multiple scattering series, and the usual approximation is to take the single-scattering term of the series. In this first order optical potential, the main ingredients are the nucleon-nucleon (NN) scattering t-matrix and the target nucleus density function. The NN scattering t-matrix is usually calculated by fitting parameters to the experimental NN data or from some theory. The target density usually comes from electron scattering experiments. Non-relativistic calculations were sufficiently accurate for small to medium energies (less than about 200 MeV), but even at the higher end of these medium energies, they left some to be desired. The natural next step was to add relativity.

Paul Adrian Maurice Dirac wrote the first relativistic equation for a spin 1/2 particle, and it is now known as the Dirac equation. A momentum space representation of the equation is

$$(E - V - mc^2\beta - c\alpha \cdot \mathbf{p})\psi(\mathbf{p}) = -\mathbf{e}\int \mathbf{d^3k}[\psi(-\mathbf{k}) - \alpha \cdot \mathbf{A}(-\mathbf{k})]\psi(\mathbf{p}+\mathbf{k}) \quad (1.2)$$

Spin and negative energy states (anti-particles) have a natural explanation through the Dirac equation. Using Dirac equations for scattering then requires relativistic input for the NN amplitudes. The typical literature has many relativistic proton-nucleus scattering calculations, all of which use NN amplitudes.[2, 3, 4, 7, 6] Unfortunately, none of them fit past more than a couple of GeV. There is a latest set of NN amplitudes that were fit to experimental NN data using Regge theory, allowing an energy range up to tens of GeV.[6, 7] Our research is the first time that anyone is going to attempt to use the new amplitudes in a proton-nucleus calculation with energies going up to tens of GeV. To use these amplitudes, we need to find a transformation matrix to the amplitudes required for p-n scattering.

Chapter 2: Literature Review

While there are many similar projects, as seen in the literature, we are the first to integrate these new amplitudes [7] with the relativistic equation, taking into account spin and other factors.

Scattering Analysis Interactive Dial-in (SAID) is a repository of experimental data and an interactive analysis facility that allows one to compare and extract data and partial wave solutions for elastic nucleon-nucleon, pion-nucleon, kaon-nucleon, and pion-deuteron scatterings.[1]

Arndt et al. used the SAID information with partial wave analysis of nucleon-nucleon scattering up to 3 GeV, which is far past the previous best of 1 GeV.[2] This method and use of the data can be very useful to us.

Rather than fitting phase shifts, like [2], Ford and Van Orden used the fitting of nucleon-nucleon (NN) amplitudes.[6, 7] The Regge model was used to fit these amplitudes, and the energy goes up to around 20 GeV. They fit these NN amplitudes for Mandelstam $s > 6$ GeV^2, and up to $s \approx 4000$ GeV^2.

Arnold et al. found a phenomenological fit to the optical potential for proton-nucleus scattering using the Dirac equation.[4] Arnold and Clark used a similar approach in a previous paper, [3].

A different approach was taken by Tjon and Wallace. Instead of a phenomenological fit to the potential, they proposed a form for the potential, which is the product of the NN amplitude and the target density. The NN amplitudes are obtained from NN scattering data and the density was obtained from electron scattering. They solved the Dirac equation in position space using this.[10]

Hynes et al. solve the relativistic scattering problem by using the Dirac equation in momentum space, and the potential again assumes the form of the NN amplitude times the target density. A prescription for extending to negative energy states was given, and in this approach, the negative energy propagation of the projectile nucleon was explicitly taken into account. They conclusively showed that most of the success of relativistic calculation comes from a diagram called the Z-graph.[8]

Chapter 3: Strategy

The Dirac equation is relativistic, and in momentum-space, it contains propagators where the projectile nucleon propagates in a negative energy state. Since we are interested in the scattering amplitude in which the incoming and outgoing states of the projectile are positive energies, we are interested in finding T^{++}. T is the transition operator, and the plus signs refer to the initial and final energy states of the projectile particle.

In operator form, the coupled equation complete with positive and negative energy propagation of the projectile proton is

$$\begin{aligned} T^{++} &= U^{++} + U^{++}G^{+}T^{++} + U^{+-}G^{-}T^{-+} \\ T^{-+} &= U^{-+} + U^{-+}G^{+}T^{++} + U^{--}G^{-}T^{-+}. \end{aligned} \tag{3.1}$$

However, these equations are actually coupled 3-dimensional integral equations. In the above equations the plus signs and minus signs refer to the energy state of the projectile. The inputs to the integral equations are the components U^{++}, U^{-+} and U^{--} of the potential. This differs from the non-relativistic, in that the non-relativistic only has the U^{++} component. Here, G^{+} and G^{-} refer to the intermediate states in which the projectile propagates in a positive energy or negative energy state.

In order to solve the coupled equations, we do a partial wave decomposition to separate out the angular dependence and then rearrange the problem into a system of linear equations, using Gaussian quadrature for the necessary integrals. Gaussian quadrature is given by

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{N} f(x_i)w_i, \tag{3.2}$$

where x_i are the roots of the Legendre polynomial of order N, and w_i are weights calculated from the derivatives of the Legendre polynomial. To integrate over a different interval, a simple change of variables is used.

Any matrix elements of an operator Q can be expanded into partial waves as

$$\langle \mathbf{p}|Q|\mathbf{p}\rangle = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Q_l(p,p')Y_l^m(\hat{p}')Y_l^m(\hat{p}), \tag{3.3}$$

where Y_l^m are the spherical harmonics and Q_l are the lth partial wave component of Q. After the partial wave expansion and separation of the angular parts, and a lot of manipulation, the above coupled equations become

$$\begin{aligned} T_l^{++}(p',p) =& U_l^{++}(p',p) + \int_0^\infty U_l^{++}(p',p'')G^+(p'')T_l^{++}(p'',p)p''^2dp'' \\ &+ \int_0^\infty U_l^{+-}(p',p'')G^-(p'')T_l^{-+}(p'',p)p''^2dp'' \end{aligned} \tag{3.4}$$

$$\begin{aligned} T_l^{-+}(p',p) =& U_l^{-+}(p',p) + \int_0^\infty U_l^{-+}(p',p'')G^+(p'')T_l^{++}(p'',p)p''^2dp'' \\ &+ \int_0^\infty U_l^{--}(p',p'')G^-(p'')T_l^{-+}(p'',p)p''^2dp'' \end{aligned} \tag{3.5}$$

In this framework, the input potential for each component is given by

$$U_l^{\rho_1\rho_2} = \int_{-1}^{1} t^{\rho_1\rho_2}(q)\rho(q)P_l(x)dx \tag{3.6}$$

Here $\rho_1\rho_2$ are $\pm\pm$ components as appropriate and $\rho(q)$ is the target density. $t^{(\rho_1\rho_2)}(q)$ represents the NN amplitudes. For this work we will be employing the latest FVO model amplitudes. [6, 7]

Now, by using Gaussian quadrature, the coupled integral equation can be written in a matrix equation. While more involved, and potentially cumbersome, it is worth exploring solving it by directly solving the 3-D equation. It could be extremely useful for high energies, which is what we are trying to research. At these high energies, the partial wave method would produce a prohibitively large number of partial waves.

After using Gaussian quadrature for the integral, the first equation will be

$$T^{++}_{p'p} = U^{++}_{p'p} + \sum_{p''} U^{++}_{p'p''} G^{+}(p'') T^{++}_{p''p} w_{p''} + \sum_{p''} U^{+-}_{p'p''} G^{-}(p'') T^{-+}_{p''p} w_{p''} \tag{3.7}$$

The second equation is found in the same way. [8] These matrix equations will be solved numerically, using LAPACK routines. LAPACK is a collection of linear algebra routines for Fortran and other languages.

In order to see the structure of how it can be solved, we can rewrite, for example, for the ++ component

$$\sum_{p''} \left[\delta_{p''p'} - U_{p'p''} G^{+} w_{p''}\right] T_{p''p} = U_{p'p} \tag{3.8}$$

$$\sum_{p''} \Lambda_{p'p''} T_{p''p} = U_{p'p} \tag{3.9}$$

To get the −+ and other components, we form a super-matrix with all the necessary components grouped into sub-matrices

$$\left[\begin{array}{c|c} \Lambda^{++} & \Lambda^{+-} \\ \hline \Lambda^{-+} & \Lambda^{--} \end{array}\right] \left[\begin{array}{c} T^{++} \\ \hline T^{-+} \end{array}\right] = \left[\begin{array}{c} U^{++} \\ \hline U^{-+} \end{array}\right]. \tag{3.10}$$

This allows us to have one giant matrix equation we will solve directly.

Chapter 4: Carrying out our mathematical manipulations

4.1 Preliminaries

The FVO model amplitudes use $\hat{\mathcal{F}}$, defined by,

$$\hat{\mathcal{F}} = \mathcal{F}_S I \cdot I + \mathcal{F}_V \gamma^{(1)} \cdot \gamma^{(2)} + \mathcal{F}_T \sigma^{(1)\mu\nu} \sigma^{(2)}_{\mu\nu} + \mathcal{F}_P \gamma^{(1)5} \gamma^{(2)5} + \mathcal{F}_A \gamma^{(1)5} \gamma^{(1)\mu} \gamma^{(2)5} \gamma^{(2)}_{\mu} \quad (4.1)$$

We wish to arrive at

$$M(E_{cm}, \theta) = A + B\vec{\sigma}_1 \cdot \hat{n} \vec{\sigma}_2 \cdot \hat{n} + C(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \hat{n} + E\vec{\sigma}_1 \cdot \hat{q} \vec{\sigma}_2 \cdot \hat{q} + F\vec{\sigma}_1 \cdot \hat{Q} \vec{\sigma}_2 \cdot \hat{Q}, \quad (4.2)$$

from Love and Franey [9], as this is the coordinate system used by the simulation we wish to use. Therefore, we must find a transformation matrix between the FVO invariant amplitudes and the amplitudes from Love and Franey. We know that

$$\bar{u}^{(1)}(\vec{p}')\bar{u}^{(2)}(\vec{p}')\hat{M} u^{(1)}(\vec{p}) u^{(2)}(\vec{p}) = \bar{u}^{(1)}(\vec{p}')\bar{u}^{(2)}(\vec{p}')\hat{F} u^{(1)}(\vec{p}) u^{(2)}(\vec{p}), \quad (4.3)$$

so we can find the matrix by comparing the result when we sandwich both sets of amplitudes with the same spin states.

The spin states we are sandwiching $\hat{\mathcal{F}}$ with are defined by

$$u(\mathbf{p}) = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \quad (4.4)$$

$$\bar{u}(\mathbf{p}) = \sqrt{\frac{E_p + m}{2m}} \begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix}, \quad (4.5)$$

where $\mathbf{p}$ is the momentum in the center of mass frame and $E_p = \sqrt{\mathbf{p}^2 + m^2}$.

4.2 Finding our sandwiched operator

We wish to find $\bar{u}^{(1)}(\mathbf{p}\ ')\bar{u}^{(2)}(\mathbf{p}\ ')\hat{\mathcal{F}}u^{(1)}(\mathbf{p})u^{(2)}(\mathbf{p})$, so we work it out term by term.

4.2.1 $I \cdot I$ term

$$\begin{aligned} t_S &= \bar{u}^{(1)}(\mathbf{p}')\bar{u}^{(2)}(\mathbf{p}')\mathcal{F}_S I^{(1)} \cdot I^{(2)} u^{(2)}(\mathbf{p})u^{(1)}(\mathbf{p}) \\ &= \mathcal{F}_S(\bar{u}^{(1)}(\mathbf{p}')I^{(1)}u^{(1)}(\mathbf{p}))(\bar{u}^{(2)}(\mathbf{p}')I^{(2)}u^{(2)}(\mathbf{p})) \end{aligned} \tag{4.6}$$

It is then simpler to evaluate the expression for one particle and then multiply the result with the appropriate spin operator in place.

$$\frac{E_p+m}{2m}\begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} = \frac{E_p+m}{2m}\left[1 - \frac{(\vec{\sigma}\cdot\mathbf{p}')(\vec{\sigma}\cdot\mathbf{p})}{(E_p+m)^2}\right] \tag{4.7}$$

From there, we get

$$t_S = \left(\frac{E_p+m}{2m}\right)^2\left[1 - \frac{(\sigma^{(1)}\cdot\mathbf{p}')(\sigma^{(1)}\cdot\mathbf{p})}{(E_p+m)^2}\right]\left[1 - \frac{(\sigma^{(2)}\cdot\mathbf{p}')(\sigma^{(2)}\cdot\mathbf{p})}{(E_p+m)^2}\right] \tag{4.8}$$

4.2.2 $\gamma^{(1)} \cdot \gamma^{(2)}$ term

$$\begin{aligned} t_V &= \mathcal{F}_V(\bar{u}^{(1)}(\mathbf{p}')\gamma^{(1)}u^{(1)}(\mathbf{p}))(\bar{u}^{(2)}(\mathbf{p}')\gamma^{(2)}u^{(2)}(\mathbf{p})) \\ &= \bar{u}^{(1)}\bar{u}^{(2)}\mathcal{F}_V(\gamma_0^{(1)}\cdot\gamma_0^{(2)} - \bar{\gamma}_0^{(1)}\cdot\bar{\gamma}_0^{(2)})u^{(1)}u^{(2)} \\ &= \mathcal{F}_V(\bar{u}^{(1)}\gamma_0^{(1)}u^{(1)}\bar{u}^{(2)}\gamma_0^{(2)}u^{(2)} - \bar{u}^{(1)}\bar{\gamma}^{(1)}u^{(1)}\cdot\bar{u}^{(2)}\bar{\gamma}^{(2)}u^{(2)}) \end{aligned} \tag{4.9}$$

From there, we can find a general expression for the γ^0 and the $\bar{\gamma}$ parts and recombine the terms with the appropriate spin operators at the end.

γ^0 part

$$\begin{aligned}
\bar{u}\gamma^0 u &= \frac{E_p+m}{2m}\begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \\
&= \frac{E_p+m}{2m}\begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix}\begin{pmatrix} 1 \\ -\frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \\
&= \frac{E_p+m}{2m}\left(1+\frac{\vec{\sigma}\cdot\mathbf{p}'}{E_p+m}\frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m}\right) \qquad (4.10)
\end{aligned}$$

$\bar{\gamma}$ part

$$\begin{aligned}
\bar{u}\bar{\gamma} u &= \frac{E_p+m}{2m}\begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix}\begin{pmatrix} 0 & \bar{\sigma} \\ -\bar{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \\
&= \frac{E_p+m}{2m}\begin{pmatrix} 1 & \frac{-\vec{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix}\begin{pmatrix} \bar{\sigma}\frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} \\ -\bar{\sigma} \end{pmatrix} \\
&= \frac{E_p+m}{2m}\left(\bar{\sigma}\frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m}+\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m}\vec{\sigma}\right) \qquad (4.11)
\end{aligned}$$

We can then combine equations 4.10 and 4.11 for both particles, yielding

$$\begin{aligned}
t_V =&\mathcal{F}_V\left(\frac{E_p+m}{2m}\right)^2\left[\left(1+\frac{\vec{\sigma}^{(1)}\cdot\mathbf{p}'}{E_p+m}\right)\left(\frac{\vec{\sigma}^{(1)}\cdot\mathbf{p}}{E_p+m}\right)\left(1+\frac{\vec{\sigma}^{(2)}\cdot\mathbf{p}'}{E_p+m}\right)\left(\frac{\vec{\sigma}^{(2)}\cdot\mathbf{p}}{E_p+m}\right)\right. \\
&\left.-\left(\bar{\sigma}^{(1)}\frac{\bar{\sigma}^{(1)}\cdot\mathbf{p}}{E_p+m}+\frac{\bar{\sigma}^{(1)}\cdot\mathbf{p}'}{E_p+m}\right)\left(\bar{\sigma}^{(2)}\frac{\bar{\sigma}^{(2)}\cdot\mathbf{p}}{E_p+m}+\frac{\bar{\sigma}^{(2)}\cdot\mathbf{p}'}{E_p+m}\right)\right] \qquad (4.12)
\end{aligned}$$

4.2.3 $\sigma^{(1)\mu\nu}\sigma^{(2)}_{\mu\nu}$ term

From Appendix A of "Relativistic Quantum Fields, vol. 1", [5]

$$\sigma^{(1)\mu\nu}\sigma^{(2)}_{\mu\nu} = 2\bar{\alpha}^{(1)} \cdot \bar{\alpha}^{(2)} + 2\bar{\Sigma}^{(1)} \cdot \bar{\Sigma}^{(2)}, \tag{4.13}$$

where we have

$$\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \qquad \text{and} \qquad \bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix} \tag{4.14}$$

We will then use the same strategy from before of calculating the parts of the term and then recombining everything for both particles, using the appropriate spin operators.

$\bar{\alpha}$ term

$$\begin{aligned}
\bar{u}(\mathbf{p}')\bar{\alpha}u(\mathbf{p}) =& \frac{E_p + m}{2m} \begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \\
=& \frac{E_p + m}{2m} \begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix} \begin{pmatrix} \bar{\sigma}\frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} \\ \bar{\sigma} \end{pmatrix} \\
=& \frac{E_p + m}{2m} \left(\bar{\sigma}\frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} - \frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m}\bar{\sigma} \right)
\end{aligned} \tag{4.15}$$

$\bar{\Sigma}$ term

$$\begin{aligned}
\bar{u}(\mathbf{p}')\bar{\Sigma}u(\mathbf{p}) =& \frac{E_p + m}{2m} \begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix} \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix} \\
=& \frac{E_p + m}{2m} \begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m} \end{pmatrix} \begin{pmatrix} \bar{\sigma} \\ \bar{\sigma}\frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix}
\end{aligned}$$

$$=\frac{E_p+m}{2m}\left(\bar{\sigma}-\frac{(\bar{\sigma}\cdot\mathbf{p}')\bar{\sigma}(\bar{\sigma}\cdot\mathbf{p})}{(E_p+m)^2}\right) \tag{4.16}$$

Then, we recombine the terms given by equations 4.15 and 4.16 to obtain

$$\begin{aligned} t_T =&\mathcal{F}_T\left(\frac{E_p+m}{2m}\right)^2 2\left[\left(\bar{\sigma}^{(1)}\frac{\bar{\sigma}^{(1)}\cdot\mathbf{p}}{E_p+m}-\frac{\bar{\sigma}^{(1)}\cdot\mathbf{p}'}{E_p+m}\bar{\sigma}^{(1)}\right)\cdot\left(\bar{\sigma}^{(2)}\frac{\bar{\sigma}^{(2)}\cdot\mathbf{p}}{E_p+m}-\frac{\bar{\sigma}^{(2)}\cdot\mathbf{p}'}{E_p+m}\bar{\sigma}^{(2)}\right)\right. \\ &\left.+\left(\bar{\sigma}^{(1)}-\frac{(\bar{\sigma}^{(1)}\cdot\mathbf{p}')\bar{\sigma}^{(1)}(\bar{\sigma}^{(1)}\cdot\mathbf{p})}{(E_p+m)^2}\right)\left(\bar{\sigma}^{(2)}-\frac{(\bar{\sigma}^{(2)}\cdot\mathbf{p}')\bar{\sigma}^{(2)}(\bar{\sigma}^{(2)}\cdot\mathbf{p})}{(E_p+m)^2}\right)\right] \end{aligned} \tag{4.17}$$

4.2.4 $\gamma^{(1)5}\gamma^{(2)5}$ term

We use the same strategy as before, and evaluate $\bar{u}(\mathbf{p}')\gamma^5 u(\mathbf{p})$.

$$\begin{aligned} \bar{u}(\mathbf{p}')\gamma^5 u(\mathbf{p}) &= \frac{E_p+m}{2m}\begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m}\end{pmatrix} \\ &= \frac{E_p+m}{2m}\begin{pmatrix} 1 & -\frac{\bar{\sigma}\cdot\mathbf{p}'}{E_p+m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{\bar{\sigma}\cdot\mathbf{p}}{E_p+m}\end{pmatrix} \\ &= \frac{1}{2m}(\bar{\sigma}\cdot\mathbf{p}-\bar{\sigma}\cdot\mathbf{p}') \end{aligned} \tag{4.18}$$

From there, when we take both particles into account, we obtain

$$t_P = \frac{\mathcal{F}_P}{(2m)^2}(\vec{\sigma}^{(1)}\cdot\mathbf{p}-\vec{\sigma}^{(1)}\cdot\mathbf{p}')(\vec{\sigma}^{(2)}\cdot\mathbf{p}-\vec{\sigma}^{(2)}\cdot\mathbf{p}') \tag{4.19}$$

4.2.5 $\gamma^{(1)5}\gamma^{(1)\mu}\gamma^{(2)5}\gamma^{(2)}_{\mu}$ term

The term we wish to evaluate is expressed as

$$t_A =\bar{u}^{(1)}(\mathbf{p}')\bar{u}^{(2)}(\mathbf{p}')\gamma^{(1)5}\gamma^{(1)\mu}\gamma^{(2)5}\gamma^{(2)}_{\mu}u^{(2)}(\mathbf{p})u^{(1)}(\mathbf{p}) \tag{4.20}$$

$$\begin{aligned} &=\bar{u}^{(1)}(\mathbf{p}')\gamma^{(1)5}\gamma^{(1)0}u^{(1)}(\mathbf{p})\bar{u}^{(2)}(\mathbf{p}')\gamma^{(2)5}\gamma^{(2)0}u^{(2)}(\mathbf{p}) \\ &\quad-\bar{u}^{(1)}(\mathbf{p}')\gamma^{(1)5}\gamma^{(1)}u^{(1)}(\mathbf{p})\bar{u}^{(2)}(\mathbf{p}')\gamma^{(2)5}\gamma^{(2)}u^{(2)}(\mathbf{p}) \end{aligned} \tag{4.21}$$

For this term, we use the same strategy as before. We first evaluate $\bar{u}\gamma^5\gamma^0 u$, then we evaluate $\bar{u}\gamma^5\boldsymbol{\gamma} u$, then we reassemble the results.

$$\bar{u}\gamma^5\gamma^0 u = -\frac{1}{2m}(\vec{\sigma}\cdot(\mathbf{p}'+\mathbf{p})) \tag{4.22}$$

$$\bar{u}\gamma^5\boldsymbol{\gamma} u = \frac{E_p+m}{2m}\left(-\vec{\sigma} - \frac{(\vec{\sigma}\cdot\mathbf{p}')\vec{\sigma}(\vec{\sigma}\cdot\mathbf{p})}{(E_p+m)^2}\right) \tag{4.23}$$

$$\begin{aligned} t_A =& \mathcal{F}_A \frac{1}{(2m)^2}(\vec{\sigma}^{(1)}\cdot(\mathbf{p}'+\mathbf{p}))(\vec{\sigma}^{(2)}\cdot(\mathbf{p}'+\mathbf{p})) \\ &- \left(\frac{E_p+m}{2m}\right)^2\left(-\vec{\sigma}^{(1)} - \frac{(\vec{\sigma}^{(1)}\cdot\mathbf{p}')\vec{\sigma}^{(1)}(\vec{\sigma}^{(1)}\cdot\mathbf{p})}{(E_p+m)^2}\right)\cdot\left(-\vec{\sigma}^{(2)} - \frac{(\vec{\sigma}^{(2)}\cdot\mathbf{p}')\vec{\sigma}^{(2)}(\vec{\sigma}^{(2)}\cdot\mathbf{p})}{(E_p+m)^2}\right) \end{aligned} \tag{4.24}$$

4.2.6 Combining all the terms

We are then able to write the sandwiched operator by summing the terms we have already found.

$$\bar{u}^{(1)}(\mathbf{p}\,')\bar{u}^{(2)}(\mathbf{p}\,')\hat{\mathcal{F}}u^{(1)}(\mathbf{p})u^{(2)}(\mathbf{p}) = t_S + t_V + t_T + t_P + t_A \tag{4.25}$$

From before, t_S is given by equation 4.8, t_V is given by equation 4.12, t_T is given by equation 4.17, t_P is given by equation 4.19, and t_A is given by equation 4.24.

4.3 Changing Variables

We wish to change to coordinates defined by $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, $\mathbf{Q} = \mathbf{p} + \mathbf{p}'$, and $\hat{n} = \hat{q}\times\hat{Q}$. From there, we see that $\mathbf{p} = \frac{\mathbf{q}+\mathbf{Q}}{2}$, $\mathbf{p}' = \frac{\mathbf{Q}-\mathbf{q}}{2}$, and $\hat{n} = \frac{2\mathbf{p}\times\mathbf{p}'}{p^2}$. Additionally, we have the helpful vector relationship that $(\vec{\sigma}\cdot\mathbf{A})(\vec{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\vec{\sigma}\cdot(\mathbf{A}\times\mathbf{B})$

Love and Franey [9] give a structure of the N-N amplitudes as previously described in equation 4.2. Because the code we are using is in this coordinate system, we wish to change t_S, t_V, etc...to this coordinate system and simplify and rearrange things until it fits the form from the Love and Franey paper.

4.3.1 t_S Term:

In equation 4.8, we obtained an expression for t_S, if we apply vector relations to equation 4.8 and multiply out the expression, we obtain

$$t_s = \mathcal{F}_S \left(\frac{E_p+m}{2m}\right)^2 \Bigg[1 - \frac{1}{(E_p+m)^2}(2\mathbf{p}'\cdot\mathbf{p} + i\vec{\sigma}^{(1)}\cdot(\mathbf{p}'\times\mathbf{p}) + i\vec{\sigma}^{(2)}\cdot(\mathbf{p}'\times\mathbf{p}) \\ \frac{1}{(E_p+m)^4}\left((\mathbf{p}'\cdot\mathbf{p})^2 + i(\mathbf{p}'\cdot\mathbf{p})((\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})\cdot(\mathbf{p}\times\mathbf{p})) - \vec{\sigma}^{(1)}\cdot(\mathbf{p}'\times\mathbf{p})\vec{\sigma}^{(2)}\cdot(\mathbf{p}'\times\mathbf{p})\right)\Bigg] \tag{4.26}$$

At this point, we can convert to the other coordinate system, arriving, after some simplification, at

$$t_S = \mathcal{F}_S \left(\frac{E_p+m}{2m}\right)^2 \Bigg(1 - \frac{1}{(E_p+m)^2}(Q^2 - q^2 + iqQ(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})\cdot\hat{n} \\ + \frac{1}{(E_p+m)^4}\left[\frac{1}{4}(Q^2-q^2)^2 + \frac{1}{2}(Q^2-q^2)\left\{Qq(\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})\cdot\hat{n} - Q^2q^2(\vec{\sigma}^{(1)}\cdot\hat{n})(\vec{\sigma}^{(2)}\cdot\hat{n})\right\}\right]\Bigg). \tag{4.27}$$

The terms that appear in this expression can all be easily matched to those in equation 4.2.

4.3.2 t_V Term:

To convert the vector term, I broke it down into $\bar{u}\gamma^0 u$ and $\bar{u}\gamma u$, converted those, and then recombined them into t_V.

To obtain $\bar{u}\gamma^0 u$, we make use of a few vector identities, and then convert to q and

Q, arriving at

$$\bar{u}\gamma^0 u = \frac{E_p + m}{2m}\left(1 + \frac{1}{(E_p+m)^2}\left[\frac{1}{2}(Q^2 - q^2) + i\vec{\sigma}\cdot\hat{n}Qq\right]\right). \tag{4.28}$$

To obtain $\bar{u}\gamma u$, we do the same, breaking the expression down into component notation and then reassembling it to something more useful as necessary. The result we arrive at is given by

$$\bar{u}\gamma u = \frac{1}{2m}(\mathbf{Q} + i\mathbf{q}\times\vec{\sigma}) \tag{4.29}$$

Then, we reassemble t_V in the same fashion as it was before, arriving, after some simplification, at

$$\begin{aligned} t_V =& \mathcal{F}_V \frac{1}{(2m)^2}\left[(E_p+m)^2\left(1 + \frac{1}{(E_p+m)^2}\left(\frac{1}{4}(Q^2-q^2)^2\right.\right.\right. \\ &+ \left\{\frac{i}{2}Qq(Q^2-q^2)\right\}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot\hat{n} - qQ(\vec{\sigma}^{(1)}\cdot\hat{n})(\vec{\sigma}^{(2)}\cdot\hat{n}) \\ &-(Q^2 - qQ(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot\hat{n} - q^2(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)} - \vec{\sigma}^{(1)}\cdot\hat{q}\vec{\sigma}^{(2)}\cdot\hat{q})))\Big] \end{aligned} \tag{4.30}$$

It is then apparent that with the exception of the terms of the form $\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}$, everything fits the form of equation 4.2. Terms like $\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}$ are handled elsewhere in the paper by Love and Franey, using the usual tensor operator.

4.3.3 t_T Term:

We use the same strategy as before, converting $\bar{u}\bar{\alpha}u$ and $\bar{u}\vec{\sigma}u$ separately and then combining the converted expressions back into t_V.

$$\bar{u}\bar{\alpha}u = \frac{1}{2m}(\mathbf{q} + i\mathbf{Q}\times\mathbf{q}) \tag{4.31}$$

This was calculated by breaking the vectors down into component notation and manipulating them until we found a more useful expression.

We also have a similar methodology to arrive at the conversion for the term with $\bar{\Sigma}$. For this one, we use the following identities

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \tag{4.32}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{4.33}$$

This gives us

$$\bar{u}\bar{\Sigma}u = \frac{E_p + m}{2m}\left(\left(1 + \frac{Q^2 - q^2}{(E_p + m)^2}\right)\vec{\sigma} - \frac{1}{(E_p + m)^2}\left(\frac{1}{2}\mathbf{Q}\vec{\sigma}\cdot\mathbf{Q} - \frac{1}{2}\mathbf{q}\vec{\sigma}\cdot\mathbf{q} + \frac{i}{2}\hat{n}\right)\right). \tag{4.34}$$

From there, when we reassemble t_T in the same manner we derived it earlier, we get

$$\begin{aligned} t_T =& 2\mathcal{F}_T\left\{\frac{1}{(2m)^2}\left(\left[q^2 + iQq(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot\hat{n} - Q^2(\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\right.\right.\right. \\ & \left.-(\vec{\sigma}^{(1)}\cdot\hat{Q})(\vec{\sigma}^{(2)}\cdot\hat{Q}))\right] + (E_p + m)^2\left[\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\left(1 + \frac{Q^2 - q^2}{(E_p + m)^2}\right)^2\right. \\ & -(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})\cdot\hat{n}\left[\left(\frac{iqQ}{(E_p + m)^2}\right)\left(1 + \frac{Q^2 - q^2}{(E_p + m)^2}\right) - \frac{1}{4}\right] \\ & + (\vec{\sigma}^{(1)}\cdot\hat{Q})(\vec{\sigma}^{(2)}\cdot\hat{Q})\left[Q^2\left(1 + \frac{Q^2 - q^2}{(E_p + m)^2}\right)\frac{1}{(E_p + m)^2} - \frac{Q^4}{4(E_p + m)^4}\right] \\ & \left.\left.+(\vec{\sigma}^{(1)}\cdot\hat{q})(\vec{\sigma}^{(2)}\cdot\hat{q})\left[\left(\frac{q}{E_p + m}\right)^2\left(1 + \frac{Q^2 - q^2}{(E_p + m)^2}\right) + \frac{q^4}{4(E_p + m)^4}\right] - \frac{1}{4}\right]\right\} \end{aligned} \tag{4.35}$$

From there, everything will match to the terms in equation 4.2 except for terms of the form $\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}$. Those terms will have to be handled separately.

4.3.4 t_P Term:

In equation 4.19, we obtained an expression for t_P. If we take part of that term and put the changed coordinates in, we get

$$\frac{1}{2m}(\vec{\sigma}\cdot\mathbf{p}-\vec{\sigma}\cdot\mathbf{p}') = \frac{1}{2m}\left(\vec{\sigma}\cdot\left(\frac{\mathbf{q}+\mathbf{Q}}{2}\right)-\vec{\sigma}\cdot\left(\frac{\mathbf{Q}-\mathbf{q}}{2}\right)\right) = \frac{1}{2m}(\vec{\sigma}\cdot\mathbf{q}). \tag{4.36}$$

From there, we reassemble the term, obtaining

$$t_P = \mathcal{F}_P\left(\frac{1}{2m}\right)^2(\vec{\sigma}^{(1)}\cdot\mathbf{q})(\vec{\sigma}^{(2)}\cdot\mathbf{q}). \tag{4.37}$$

Since $\mathbf{q} = q\hat{q}$, we arrive at

$$t_P = \mathcal{F}_P\left(\frac{q}{2m}\right)^2(\vec{\sigma}^{(1)}\cdot\hat{q})(\vec{\sigma}^{(2)}\cdot\hat{q}). \tag{4.38}$$

This lines up with the term whose coefficient is E in equation 4.2.

4.3.5 t_A Term:

This term is handled exactly like t_T was, yielding

$$\begin{aligned} t_A =& \mathcal{F}_A\frac{1}{(2m)^2}\Big(Q^2(\vec{\sigma}^{(1)}\cdot\hat{Q})(\vec{\sigma}^{(2)}\cdot\hat{Q}) \\ &-(E_p+m)^2\left[\vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)}\left(1-\frac{Q^2-q^2}{(E_p+m)^2}\right)^2\right. \\ &-\ (\vec{\sigma}^{(1)}+\vec{\sigma}^{(2)})\cdot\hat{n}\left(\frac{iqQ}{(E_p+m)^2}\left(1-\frac{Q^2-q^2}{(E_p+m)^2}\right)\right) \\ &+\ (\vec{\sigma}^{(1)}\cdot\hat{Q})(\vec{\sigma}^{(2)}\cdot\hat{Q})\left(\left(\frac{Q}{E_p+m}\right)^2\left(1-\frac{Q^2-q^2}{(E_p+m)^2}\right)-\frac{Q^4}{4(E_p+m)^4}\right) \\ &\left.+(\vec{\sigma}^{(1)}\cdot\hat{q})(\vec{\sigma}^{(2)}\cdot\hat{q})\left(\left(\frac{q}{E_p+m}\right)^2\left(1-\frac{Q^2-q^2}{(E_p+m)^2}\right)-\frac{q^4}{4(E_p+m)^4}\right)-\frac{1}{4}\right]\Big) \end{aligned} \tag{4.39}$$

This term has the same problems as t_T, so when those are solved, it will immediately be applicable to this term.

Chapter 5: Conclusions

We can take the transformed terms and construct a transformation matrix in order to use an existing, proven numerical simulation. To do this, however, we must first take care of the terms mentioned above as not fitting the equation (4.2) we are using from Love and Franey's 1981 paper.

Since we must have

$$\bar{u}^{(1)}(\vec{p}')\bar{u}^{(2)}(\vec{p}')\hat{M}u^{(1)}(\vec{p})u^{(2)}(\vec{p}) = \bar{u}^{(1)}(\vec{p}')\bar{u}^{(2)}(\vec{p}')\hat{F}u^{(1)}(\vec{p})u^{(2)}(\vec{p}), \tag{5.1}$$

where $\hat{M}$ is given by equation 4.2 and $\hat{F}$ is given by equation 4.1, we get the following transformation matrix

$$\begin{bmatrix} A \\ B \\ C \\ E \\ F \end{bmatrix} = \begin{bmatrix} A_S & A_V & A_T & A_P & A_A \\ B_S & B_V & B_T & B_P & B_A \\ C_S & C_V & C_T & C_P & C_A \\ E_S & E_V & E_T & E_P & E_A \\ F_S & F_V & F_T & F_P & F_A \end{bmatrix} \begin{bmatrix} \mathcal{F}_S \\ \mathcal{F}_V \\ \mathcal{F}_T \\ \mathcal{F}_P \\ \mathcal{F}_A \end{bmatrix}. \tag{5.2}$$

The $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}$ that are not accounted for from each of the previously calculated t_V, t_T, etc... must sort into one of the terms in the transformation matrix given above.

While it is tempting to use what we have in the calculation, it is not yet known what contribution $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}$ will have to the entries in the transformation matrix. Therefore, we cannot yet test these amplitudes in an actual calculation. This extra term will be handled in the future so that we can perform the transformation and use the FVO Regge amplitudes in an actual calculation.

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