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Conjugacy Classes and Finite P-Groups

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CONJUGACY CLASSES AND FINITE p-GROUPS

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Abstract. Let G be a finite p-group, where p is a prime number and $a \in G$. Denote by $\mathrm{Cl}(a) = \{gag^{-1} \mid g \in G\}$ the conjugacy class of a in G. Assume that $|\mathrm{Cl}(a)| = p^n$. Then $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \{xy \mid x \in \mathrm{Cl}(a), y \in \mathrm{Cl}(a^{-1})\}$ is the union of at least $n(p-1)+1$ distinct conjugacy classes of G.

1. Introduction

Let G be a finite group. Denote by $\mathrm{Cl}(a) = \{gag^{-1} \mid g \in G\}$ the conjugacy class of a in G, and by $|\mathrm{Cl}(a)|$ the size of $\mathrm{Cl}(a)$. If the subset X of G is G-invariant, i.e $X^g = \{x^g \mid x \in X\} = X$ for all $g \in G$, then X is the union of m distinct conjugacy classes of G, for some integer m. Set $\eta(X) = m$.

Given any conjugacy classes $\mathrm{Cl}(a)$ and $\mathrm{Cl}(b)$, we can check that the product $\mathrm{Cl}(a)\,\mathrm{Cl}(b) = \{xy \mid x \in \mathrm{Cl}(a), y \in \mathrm{Cl}(b)\}$ is a G-invariant set. In this note, we will explore the relation between $|\mathrm{Cl}(a)|$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}))$. Those results are the equivalent in conjugacy classes as some of the ones in irreducible characters in [1] and [2].

In Theorem A of [2], it is proved that if G is a p-group, χ is an irreducible character with degree p^n, then the product $\chi\overline{\chi}$ of χ with its complex conjugate $\overline{\chi}$ is the sum of at least $2n(p-1)+1$ distinct irreducible characters. The following is the equivalent for conjugacy classes

Theorem A. *Let G be a finite p-group and $a \in G$. Assume that $|\mathrm{Cl}(a)| = p^n$. Then the product $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ of the conjugacy class of a in G and the conjugacy class of the inverse of a in G, is the union of at least $n(p-1)+1$ distinct conjugacy classes of G, i.e. $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) \geq n(p-1)+1$.*

In Proposition 5.3, it is shown that for every prime p and every integer $n \geq 0$, there exist a p-group G and a conjugacy class $\mathrm{Cl}(a)$ of G such that $|\mathrm{Cl}(a)| = p^n$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = n(p-1)+1$. Thus the bound in Theorem A is optimal.

An application of Theorem A is the following

Theorem B. *Let n be a positive integer. Then there exists a finite set S_n of positive integers such that for any nilpotent group G and any conjugacy class $\mathrm{Cl}(a)$ of G with $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) \leq n$, we have that*

$$|\mathrm{Cl}(a)| \in S_n.$$

In Proposition 5.5, we prove that given any prime p, there exist a supersolvable group and a conjugacy class $\mathrm{Cl}(a)$ of G with $|\mathrm{Cl}(a)| = p$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = 2$. Thus the previous result does not remain true assuming the weaker hypothesis that

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the groups are supersovable. Theorem A is the equivalent in conjugacy classes of Theorem B of [1].

Theorem C. *Let p be a prime number. Let G be a finite p-group and $\mathrm{Cl}(a)$ be a conjugacy class of G. Then one of the following holds:*
i) $|\mathrm{Cl}(a)| = 1$ *and* $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = 1$.
ii) $|\mathrm{Cl}(a)| = p$ *and* $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = p$.
iii) $|\mathrm{Cl}(a)| \geq p^2$ *and* $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) \geq 2p-1$.

Given a fix prime $p > 2$, observe that Theorem C implies that there are "gaps" among the possible values that $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}))$ can take for any finite p-group and any conjugacy class $\mathrm{Cl}(a)$ in G. The previous result is the equivalent in conjugacy classes of Theorem B of [2].

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2. Proof of Theorem A

Notation. Let G be a finite p-group and N be a normal subgroup of G. Denote by $\overline{a}$ the element in G/N that contains a. Thus $\mathrm{Cl}(\overline{a})$ is the conjugacy class of $\overline{a}$ in G/N.

Lemma 2.1. *Let G be a finite p-group and N be a normal subgroup of G. Let a and b be elements of G. Then*

i) $\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{b})$ *is a G-invariant set. If* $\mathrm{Cl}(\overline{a}) \cap \mathrm{Cl}(\overline{b}) = \emptyset$ *then* $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) = \emptyset$. *Thus* $\eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{b})) \leq \eta(\mathrm{Cl}(a)\,\mathrm{Cl}(b))$.

ii) If, in addition, $|N| = p$, *then either* $|\mathrm{Cl}(\overline{a})| = |\mathrm{Cl}(a)|$ *or* $|\mathrm{Cl}(\overline{a})| = \frac{|\mathrm{Cl}(a)|}{p}$. *Furthermore, if* $|\mathrm{Cl}(\overline{a})| = \frac{|\mathrm{Cl}(a)|}{p}$, *then* $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) \geq \eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{a}^{-1})) + (p-1)$.

Proof. **i)** Clearly if $a = gbg^{-1}$, then $\overline{a} = \overline{g}\overline{b}(\overline{g})^{-1}$. Thus if $\mathrm{Cl}(\overline{a}) \cap \mathrm{Cl}(\overline{b}) = \emptyset$ then $\mathrm{Cl}(a) \cap \mathrm{Cl}(b) = \emptyset$. Therefore $\eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{b})) \leq \eta(\mathrm{Cl}(a)\,\mathrm{Cl}(b))$.

ii) Since N is normal, $|N| = p$ and G is a p-group, then N is contained in the center $\mathbf{Z}(G)$ of G. Thus given any $n \in N$, $\mathrm{Cl}(n) = \{n\}$.

Suppose that $|\mathrm{Cl}(\overline{a})| \neq |\mathrm{Cl}(a)|$. Since $|N| = p$, we have that $|\mathrm{Cl}(a)| \leq p|\mathrm{Cl}(\overline{a})|$. Therefore $\frac{|\mathrm{Cl}(a)|}{p} \leq |\mathrm{Cl}(\overline{a})| \leq |\mathrm{Cl}(a)|$. Thus $|\mathrm{Cl}(\overline{a})| = \frac{|\mathrm{Cl}(a)|}{p}$ since G is a p-group and $|\mathrm{Cl}(\overline{a})|$ divides $|G/N|$.

If $|\mathrm{Cl}(\overline{a})| = \frac{|\mathrm{Cl}(a)|}{p}$, then given any $x \in \mathrm{Cl}(a)$ and any $n \in N$, we have that $nx \in \mathrm{Cl}(a)$. Thus $n = nx(x^{-1}) \in \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ for any $n \in N$. Therefore $N \leq \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ and ii) follows. □

Proof of Theorem A. We are going to use induction on the order of G. Let N be a normal subgroup of G of order p. Observe such group exists since G is a p-group. Let $|\mathrm{Cl}(\overline{a})| = p^m$. Since $|G/N| < |G|$, by induction we have that $\eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{a}^{-1})) \geq m(p-1)+1$. If $|\mathrm{Cl}(\overline{a})| = |\mathrm{Cl}(a)|$, i.e if $m = n$, then by Lemma 2.1 i) we have that

$$\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) \geq \eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{a}^{-1})) \geq m(p-1)+1 = n(p-1)+1.$$

We may assume then that $|\mathrm{Cl}(\overline{a})| \neq |\mathrm{Cl}(a)|$. By Lemma 2.1 ii), we have that $m = n-1$ and

$$\begin{aligned}\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) &\geq \eta(\mathrm{Cl}(\overline{a})\,\mathrm{Cl}(\overline{a}^{-1})) + (p-1)\\ &= (n-1)(p-1)+1+(p-1) = n(p-1)+1.\end{aligned}$$

□

3. Proof of Theorem B

Proof of Theorem B. Let

$$S_n = \{\Pi(p_i)^{t_i} \mid p_i \text{ is a prime number for all } i,\, t_i \geq 0 \text{ and } t_i(p_i-1)+1 \leq n\}.$$

Observe that the set S_n is a finite set of positive integers since $0 \leq t_i \leq n$ and if $t_i > 0$ then $p_i \leq n$.

Let $\{p_1, \ldots, p_r\}$ be the set of distinct prime divisors of $|G|$. For $i = 1, \ldots, r$, let P_i be the Sylow p_i-subgroup of G. Observe that $a = \Pi_{i=1}^r a_i$, for some $a_i \in P_i$ for $i = 1, \ldots, r$. Since G is nilpotent, we have that $\mathrm{Cl}(a) = \Pi_{i=1}^r \mathrm{Cl}(a_i)$, where $\mathrm{Cl}(a_i)$ is the conjugacy class of a_i in P_i, for $i = 1, \ldots, r$. Let $m_i = \eta(\mathrm{Cl}(a_i)\,\mathrm{Cl}(a_i^{-1}))$. Observe that $m_i \leq n$ and

$$|\mathrm{Cl}(a)| = \Pi_{i=1}^r |\mathrm{Cl}(a_i)|.$$

We can check that

$$\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = \Pi_{i=1}^r \eta(\mathrm{Cl}(a_i)\,\mathrm{Cl}(a_i^{-1})).$$

For each i, let $|\mathrm{Cl}(a_i)| = p_i^{t_i}$. Since $\mathrm{Cl}(a_i)$ is the conjugacy class of a_i in the p_i-group P_i, by Theorem A we have that $m_i = \eta(\mathrm{Cl}(a_i)\,\mathrm{Cl}(a_i^{-1})) \geq t_i(p_i-1)+1$. Thus $n \geq t_i(p_i-1)+1$. Therefore $|\mathrm{Cl}(a)| = \Pi_{i=1}^r |\mathrm{Cl}(a_i)| = \Pi_{i=1}^r p_i^{t_i} \in S_n$. □

4. Proof of Theorem C

Lemma 4.1. *Let G be a finite p-group and $\mathrm{Cl}(a)$ be a conjugacy class of G with $|\mathrm{Cl}(a)| = p$. Then one of the following holds:*

i) $\mathrm{Cl}(a) = \{az | z \in Z\}$ for some subgroup Z of the center $\mathbf{Z}(G)$ of G. Therefore $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = Z$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = p$.

ii) $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ is the union of $p-1$ distinct conjugacy classes of size p and the class $\mathrm{Cl}(e) = \{e\}$. Therefore $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = p$.

Proof. Observe that if $z \in \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ and $|\mathrm{Cl}(z)| = 1$, then z is in the center $\mathbf{Z}(G)$ of G. Since $z = a^g a^{-1}$ for some $g \in G$ and $z \in \mathbf{Z}(G)$, $z^i \in \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ and $a^{g^i} = az^i$ for all integer i. Thus $< z > \leq \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$. Set $Z = < z >$.

i) If $z \neq e$, it follows that $|Z| \geq p$. Since $|\mathrm{Cl}(a)| = p$ and $a^{g^i} = az^i$ for all integer i, we have that $\mathrm{Cl}(a) = \{az | z \in Z\}$ and $|Z| = p$. Since Z is contained in $\mathbf{Z}(G)$, then $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = Z$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = p$.

ii) We may assume now that if $z \in \mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ and $\mathrm{Cl}(z) = 1$, then $z = e$. Thus all the conjugacy classes different from $\mathrm{Cl}(e)$ are of size p. Observe that $a^g(a^{-1})^g = e$ for all $g \in G$. Thus $|\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})| \leq p^2 - p + 1 = (p-1)p + 1$. Therefore by Theorem A it follows that $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ is the union of $p-1$ distinct conjugacy classes of size p and $\mathrm{Cl}(e)$. □

Remark. Let p be a prime number.

a) Let G be an extra special group of order p^3 and exponent p. We can check that given any $a \in G$, where a is not in the center of G, then $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \mathbf{Z}(G)$ and thus Lemma 4.1 i) occurs.

b) Let G be the wreath product of a cyclic group C_{p^2} of order p^2 by a cyclic group C_p of order p. Thus $|G| = p^{2p+1}$. Let $a = (c, e, \dots, e)$ in G, where $c \in C_{p^2}$ has order p^2. Observe that $a^{-1} \neq a$. Observe also that

$$\mathrm{Cl}(a) = \{(c, e, \dots, e), (e, c, \dots, e), \dots, (e, e, \dots, e, c)\}.$$

Thus $|\mathrm{Cl}(a)| = p$. Let $b_i = (c, e, \dots, c^{-1}, e, \dots, e)$, where c^{-1} is in the i-position for $i = 0, \dots, p-1$, i.e $b_0 = (cc^{-1}, \dots, e) = (e, e, \dots, e)$, $b_1 = (c, c^{-1}, e, \dots, e)$ and so for. Observe that $\mathrm{Cl}(b_0) = \mathrm{Cl}((e, e, \dots, e))$ has class size 1. We can check that $\mathrm{Cl}(b_i)$ has size p for $i = 1, \dots, p-1$. Since $c \neq c^{-1}$, then $\mathrm{Cl}(b_i) \cap \mathrm{Cl}(b_j) = \emptyset$ if $i \neq j$ and $i, j = 0, \dots, p-1$. Observe that $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \cup_{i=0}^{p-1} \mathrm{Cl}(b_i)$. Thus $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ is the union of a conjugacy class of size 1, namely $\mathrm{Cl}(b_0)$ and $p-1$ distinct conjugacy classes of size p, namely $\mathrm{Cl}(b_i)$ for $i = 1, \dots p-1$. We conclude that given any prime p, there exist some group G and some conjugacy class $\mathrm{Cl}(a)$ of G satisfying the condition in case ii) of Lemma 4.1.

Proof of Theorem C. If $|\mathrm{Cl}(a)| = \{a\}$, then $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \{e\}$ and so i) holds. Lemma 4.1 implies ii) and iii) follows from Theorem A. □

5. Examples

Lemma 5.1. *Let G_0 be a p-group and $\mathrm{Cl}(g_0)$ be the conjugacy class containing $g_0 \in G_0$. Assume that $\mathrm{Cl}(g_0) \neq \mathrm{Cl}(g_0^{-1})$. Let $N = G_0 \times G_0 \times \cdots \times G_0$ be the direct product of p-copies of G_0. Let $C = < c >$ be a cyclic group of order p. Observe that C acts on N by*

$$c : (n_0, n_1, \dots, n_{p-1}) \mapsto (n_{p-1}, n_0, \dots, n_{p-2}) \tag{5.2}$$

for any $(n_0, n_1, \dots, n_{p-1}) \in N$.

Let G be the semidirect product of N and C, i.e G is the wreath product of G_0 and C. Set $a = (g_0, e, \dots, e)$ in N, where e is the identity of G_0. Then $|\mathrm{Cl}(a)| = p|\mathrm{Cl}(g_0)|$, $\mathrm{Cl}(a) \neq \mathrm{Cl}(a^{-1})$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = \eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1})) + (p-1)$.

Proof. Observe that $\mathrm{Cl}(a) \neq \mathrm{Cl}(a^{-1})$ since $\mathrm{Cl}(g_0) \neq \mathrm{Cl}(g_0^{-1})$.

Let $\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1}) = C_1 \cup C_2 \cdots \cup C_m$, where $C_1, \dots, C_m$ are distinct conjugacy classes of G_0. Thus $m = \eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}(g_0^{-1}))$. We can check that the distinct conjugacy classes of $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ are of the following two types:

i) $\{(x, e, \dots, e)^c \mid x \in C_i, c \in C\}$ for $i = 1, \dots, m$.

ii)$\{(x, y, \dots, e, e)^c \mid x \in \mathrm{Cl}(g_0), y \in \mathrm{Cl}({g_0}^{-1}), c \in C\}$, $\{(x, e, y, \dots, e)^c \mid x \in \mathrm{Cl}(g_0), y \in \mathrm{Cl}({g_0}^{-1}), c \in C\}$ and $\{(x, e, \dots, e, y)^c \mid x \in \mathrm{Cl}(g_0), y \in \mathrm{Cl}({g_0}^{-1}), c \in C\}$.

Observe that there are $\eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1}))$ distinct conjugacy classes of type i) and exactly $p-1$ distinct conjugacy classes of type ii). Thus $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = \eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1})) + (p-1)$. □

Proposition 5.3. *Given any prime p, and any integer $n \geq 0$, there exist a finite p-group G and a conjugacy class $\mathrm{Cl}(a)$ of G with $|\mathrm{Cl}(a)| = p^n$, $\mathrm{Cl}(a) \neq \mathrm{Cl}(a^{-1})$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = n(p-1) + 1$.*

Proof. Observe that if G is an abelian group and $a \in G$ has order p^2, then $|\mathrm{Cl}(a)| = 1$, $\mathrm{Cl}(a) \neq \mathrm{Cl}(a^{-1})$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = 1 = 0(p-1) + 1$. Thus the statement is true for $n = 0$. Assume by induction that the statement is true for $n-1$, i.e. there exist a finite p-group G_0 and a conjugacy class $\mathrm{Cl}(g_0)$ of G_0 with $|\mathrm{Cl}(g_0)| = p^{n-1}$,

$\mathrm{Cl}(g_0) \neq \mathrm{Cl}(g_0^{-1})$ and $\eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1})) = (n-1)(p-1)+1$. Using the notation of Lemma 5.1, we have that

$$\begin{aligned}\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) &= \eta(\mathrm{Cl}(g_0)\,\mathrm{Cl}({g_0}^{-1})) + (p-1)\\ &= (n-1)(p-1)+1+(p-1) = n(p-1)+1.\end{aligned}$$

Since $|\,\mathrm{Cl}(a)| = p|\,\mathrm{Cl}(g_0)| = p \times p^{n-1} = p^n$, the proof is complete. □

Hypothesis 5.4. *Fix a prime p and let $F = \{0, 1, \ldots, p-1\}$ be the finite field with p elements. Observe that F is also a vector space of dimension 1 over itself. Let $A = \mathrm{Aff}(F)$ be the affine group of F. Observe that the group A is a cyclic by cyclic group and thus it is supersolvable.*

Let C be a cyclic group of order p. Set $X = F$ and $K = C^X = \{f : X \to C\}$. Observe that K is a group via pointwise multiplication, and clearly A acts on this group (via its action on X).

Let G be the wreath product of C and A relative to X, i.e. $G = K \rtimes A$. We can check that G is a supersolvable group.

Proposition 5.5. *Assume Hypotheses 5.4. Set $a = (c, e, e, \ldots, , e)$ in K. Then $a \in G$, the conjugacy class $\mathrm{Cl}(a)$ of G has size p and $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \mathrm{Cl}((e, e, \ldots, e)) \cup \mathrm{Cl}((c, c^{-1}, e, \ldots, e))$. Thus $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = 2$.*

Therefore, given any prime p, there exist a supersolvable group G and a conjugacy class $\mathrm{Cl}(a)$ of G with $|\,\mathrm{Cl}(a)| = p$ and $\eta(\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})) = 2$.

Proof. Observe that $\mathrm{Cl}(a) = \{(c, e, \ldots, e), (e, c, e, \ldots, e), \ldots, (e, e, \ldots, e, c)\}$. Thus $\mathrm{Cl}(a)$ has p-elements. Observe that

$\{(c, c^{-1}, \ldots, e)^y \mid y \in F \backslash \{0\}\} = \{(c, c^{-1}, e, \ldots, e), (c, e, c^{-1}, \ldots, e), \ldots, (c, e, \ldots, e, c^{-1})\}$.

Observe also that

$\{(c, c^{-1}, e, \ldots, e)^x \mid x \in F\} = \{(c, c^{-1}, e, \ldots, e), \ldots, (e, e, \ldots, c, c^{-1}), (c^{-1}, e, \ldots, e, c)\}$.

Thus

$$\begin{aligned}\mathrm{Cl}((c, c^{-1}, e, \ldots, e)) =& \{(c, c^{-1}, e, \ldots, e)^x, (c, e, c^{-1}, \ldots, e)^x, \ldots,\\ & (c, e, \ldots, c^{-1}, e), (c, e, \ldots, e, c^{-1})^x \mid x \in F\}\end{aligned}$$

has $(p-1)p = p^2 - p$ elements. Since $a^g (a^{-1})^g = (e, \ldots, e)$, then $\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1})$ has at most $p^2 - p + 1$ elements. We conclude that

$$\mathrm{Cl}(a)\,\mathrm{Cl}(a^{-1}) = \mathrm{Cl}((e, e, \ldots, e)) \cup \mathrm{Cl}((c, c^{-1}, e, \ldots, e)).$$

□

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