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Abstract

Analogous to the fixed point property for ordered sets, a graph has the fixed vertex property iff each of its endomorphisms has a fixed vertex. The fixed point theory for ordered sets can be embedded into the fixed vertex theory for graphs. Therefore, the potential for cross-fertilization should be explored.

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This paper introduces the fixed vertex property for graphs as a graph-theoretical generalization of the fixed point property for ordered sets in Section 1. Section 2 provides examples of small graphs with the fixed vertex property. Sections 3 and 4 show that the whole fixed point theory for ordered sets (consider [17] or [20] for an overview), including questions about products, can be translated into analogous, more general, questions on the fixed vertex property for graphs.

*The bulk of the work on this paper was done when the author was still at Louisiana Tech University

1 Endomorphisms and Fixed Vertices

Throughout this paper, we assume that **graphs** are simple graphs without loops. The central notions for this paper are homomorphisms and fixed vertices.

Definition 1.1 *(See, for example, [5].) Let $G = (V, E)$ and $H = (W, F)$ be graphs. A function $f : V \to W$ is called a* **homomorphism** *iff, for all $v, w \in V$, if $\{v, w\} \in E$, then $\{f(v), f(w)\} \in F$. Consistent with standard terminology, an* **endomorphism** *is a homomorphism from G to G, an* **isomorphism** *is a bijective homomorphism whose inverse is a homomorphism, too, and an* **automorphism** *is an isomorphism from G to G.*

Definition 1.2 *Let $G = (V, E)$ be a graph. Then G is said to have the* **fixed vertex property** *iff every endomorphism f of G has a fixed vertex $v = f(v)$.*

Most graphs satisfy the fixed vertex property, and in impressive fashion to boot: Recall that a graph is called **rigid** iff the identity is the graph's only endomorphism. Asymptotically, almost every graph is rigid (see, for example [5], Theorem 4.7) and therefore has the fixed vertex property. This situation is fundamentally different from the situation for ordered sets: Most ordered sets do not have the fixed point property, because they can be retracted onto a 4-crown tower $\{a, b < c, d < e, f\}$.

This asymptotic behavior for the fixed vertex property is a natural consequence of the fact that we do not allow loops at the vertices. Indeed, we specifically must avoid loops, because a graph $G = (V, E)$ that has loops (edges whose initial and terminal vertex are the same) at two adjacent vertices $a \neq b$ cannot have the fixed vertex property: Mapping $V \setminus \{a\}$ to a and a to b is an endomorphism without a fixed vertex.[1] Moreover, without

[1]This type of observation may well be the reason why the fixed point property for ordered sets was not translated to graphs until now. Order-preserving maps f allow for points $p < q$ to have the same image $f(p) = f(q)$. Hence, from the point of view of ordered sets, the natural target for a translation of the fixed point property would be functions on graphs that map edges to edges and which are also allowed to collapse an edge into a single vertex. These functions do not even allow a fixed vertex property for graphs that have just a single edge $\{a, b\}$, because we could map all vertices, except a, to a and map a to b. For such functions, the fixed clique property (see, for example, [8] or Section 6.3 in [18]) is analogous to the fixed point property, and the functions can be translated into certain order-preserving functions on truncated lattices.

further conditions, multiple edges between distinct vertices do not affect our analysis. Hence we focus our attention on simple graphs without loops.

As is common for fixed point properties for endomorphisms of any kind, the fixed vertex property is inherited by retracts. For a set of vertices $S \subseteq V$ in a graph $G = (V, E)$, we denote the **induced subgraph** on S by $G[S]$.

Definition 1.3 *Let $G = (V, E)$ be a graph. An idempotent endomorphism r from G to G is called a* **retraction**. *For a retraction r, the induced subgraph $G[r[V]]$ is also called a* **retract** *of G.*

Proposition 1.4 *Let $G = (V, E)$ be a graph with the fixed vertex property and let r be a retraction from G to G. Then $G[r[V]]$ has the fixed vertex property.*

Proof. Let f be an endomorphism of $G[r[V]]$. Then $f \circ r$ is an endomorphism of G and hence there is a $v \in V$ so that $f \circ r(v) = v$. In particular, because v is the image of an element of $r[V]$ under f, we infer that $v \in r[V]$. But then $r(v) = v$ and $f(v) = f \circ r(v) = v$. Hence $G[r[V]]$ has the fixed vertex property. ■

Unlike for ordered sets, where finite chains have the fixed point property, complete graphs K_n with $n \geq 2$ vertices do not have the fixed vertex property: Any fixed vertex free permutation of the vertices of K_n is an endomorphism of K_n. Consequently, by Proposition 1.4, perfect graphs do not have the fixed vertex property: If G is perfect, we can pick a $\chi(G)$-clique C and retract each color class to the vertex in C of that color. In particular, nontrivial bipartite graphs do not have the fixed vertex property, which is different from the situation for ordered sets, where (see [10]) bipartite ordered sets have the fixed point property iff they are dismantlable. Moreover, comparability graphs, being perfect, do not have the fixed vertex property, which may make the results of Sections 3 and 4 a bit more surprising.

Even with stronger hypotheses, the Abian-Brown Theorem (see [1]), is not valid for graph endomorphisms: Wheels with an even number of spokes are perfect graphs. Hence, even a *unique* **universal vertex** (a vertex adjacent to all other vertices) does not guarantee the fixed vertex property.

A positive contrast to the fixed point property for ordered sets is that disconnected graphs *can* have the fixed vertex property, because it is possible for two graphs to have no homomorphisms between them.

Proposition 1.5 *Let $G = (V, E)$ be a disconnected graph. Then G has the fixed vertex property iff G has a component which has the fixed vertex property and which does not have a homomorphism into any of the other components.*

Proof. For "$\Rightarrow$," note that if every component $G[C]$ either has an endomorphism $f_C : C \to C$ without a fixed vertex or a homomorphism $f_C : C \to V$ into a different component of G, then the function f that is equal to f_C on each component of G is a fixed vertex free endomorphism.

For "$\Leftarrow$," let $G[C]$ be a component which has the fixed vertex property and which does not have a homomorphism into any of the other components, and let f be an endomorphism of G. Then $f[C] \subseteq C$ and, because $G[C]$ has the fixed vertex property, f has a fixed vertex. ■

Finally, some results from the fixed point theory for ordered sets translate to the fixed vertex property, albeit not in their full strength.

Proposition 1.6 *(Compare with Theorem 3.3 in [16].) Let $G = (V, E)$ be a graph so that, for some $a \in V$, $G[V \setminus \{a\}]$ is a retract of G. If $G[V \setminus \{a\}]$ and $G[N(a)]$ have the fixed vertex property, then G has the fixed vertex property.*

Proof. Let f be an endomorphism of G, let r be the retraction from G to $G[V \setminus \{a\}]$ and let $b := r(a)$. There is nothing to prove if $f(b) = b$, so we can assume $f(b) \neq b$. In case $f(b) \neq a$, let v be a fixed vertex of $r \circ f$. Because $f(b) \notin \{a, b\}$, we must have that $v \neq b$ and hence $r^{-1}(v) = \{v\}$, which means that $f(v) = v$. In case $f(b) = a$, because $r(a) = b$ implies that $N(a) \subseteq N(b)$, we have that f must map $N(a)$ to itself. Hence, f has a fixed vertex in this case, too. ■

Clearly, when $G[V \setminus \{a\}]$ is a retract of a graph G with the fixed vertex property, it follows from Proposition 1.4 that $G[V \setminus \{a\}]$ has the fixed vertex property. Unlike for ordered sets, it is not necessary for $G[N(a)]$ to have the fixed vertex property, as we will see in the next section once we have some examples.

2 Examples

By Proposition 1.6, if G is a connected graph with at least 3 vertices and p is a **pendant vertex** (a vertex with exactly one neighbor), then G has the fixed vertex property iff $G - a$ does. Hence we will focus on graphs that do not have pendant vertices. Although disconnected graphs can have the fixed vertex property, by Proposition 1.5 at least one component of a disconnected graph with the fixed vertex property must have the fixed vertex property. Hence we will focus on connected graphs.

To find examples of small graphs with the fixed vertex property, an argument similar to [15] and [16] is possible. However, 20 years after [15] and [16], lists of small graphs are readily available and it is now much more efficient to perform a computer search on these lists. The author processed the lists of adjacency matrices for connected graphs with up to 9 vertices from [9] with [21]. The first result is that there are no connected graphs with $2, 3, 4$ or 5 vertices and the fixed vertex property. (With an approach similar to those in [15] and [16], it can be shown that each of these graphs can be retracted onto a copy of K_2, K_3, K_4, K_5 or C_5.) This is different from the situation for ordered sets, where every finite totally ordered set has the fixed point property.

The only graph with 6 vertices and the fixed vertex property is the 5-wheel W_5 (see Figure 1). The 5-wheel shows that, for the fixed vertex property, there is no analogue of Rival's important theorem (see [10]) that the fixed point property is not affected by the adding or removal of an irreducible point: A vertex a of a graph $G = (V, E)$ is called **dominated** by the vertex $w \in V \setminus \{a\}$ iff $\{a, w\} \in E$ and, for all $x \in V$ with $\{a, x\} \in E$, we have $\{w, x\} \in E$. Dominated vertices are the graph theoretical analogues of irreducible points in ordered sets. Pick a vertex a on the periphery of the 5-wheel. Then a is dominated by the center of the 5-wheel and yet W_5 has the fixed vertex property and $W_5 - a$ does not. (This is not a contradiction to Proposition 1.4, because, unlike for order-preserving maps, there is no retraction from W_5 to $W_5 - a$.) On the other hand, let G be the graph obtained by adding to W_5 a vertex a that is adjacent to all vertices of W_5. Then G does not have the fixed vertex property, but $G - a$ does.

On 7 vertices, only the following 13 connected graphs without pendant vertices have the fixed vertex property. There are 7 connected graphs G without pendant vertices and the fixed vertex property that are obtained from W_5 by attaching a seventh vertex a with a neighborhood of $2, 3$ or 4 vertices. Each of these graphs is an example of a graph G and a vertex a so that $G[V \setminus \{a\}]$ has the fixed vertex property, G has the fixed vertex property and $G[N(a)]$ does not have the fixed vertex property. Hence, Proposition 1.6 cannot be turned into an equivalence like Theorem 3.3 in [16]. The neighborhoods of a for the 7 graphs are: The center and one vertex on the periphery, two non-consecutive vertices on the periphery, the center and two non-consecutive vertices on the periphery, two consecutive vertices on the periphery, three non-contiguous vertices on the periphery, three consecutive vertices on the periphery, and, four consecutive vertices on the periphery. Moreover, each of the 6 graphs with 7 vertices in Figure 1 has the fixed vertex property and none of them contains a copy of the 5-wheel. All of these graphs are 4-colorable.

On 8 vertices, there are 293 nonisomorphic connected graphs without pendant vertices that have the fixed vertex property. Of these graphs, 286 are 4-colorable and 7 are 5-colorable. Of these 7 graphs, 6 are obtained from the graphs with 7 vertices in Figure 1 by attaching a dominating vertex in each case; the seventh graph is obtained from the circulant graph $X(\mathbb{Z}_7, \{-3, -2, 2, 3\})$ by attaching a dominating vertex, see the graph with 8 vertices in Figure 1.

On 9 vertices, there are 11,613 nonisomorphic connected graphs without pendant vertices that have the fixed vertex property. Of these graphs, 4 are 3-colorable (so there are 3-colorable graphs with the fixed vertex property), 10,880 are 4-colorable, 723 are 5-colorable and 6 are 6-colorable. These 6 graphs are obtained from the graphs in Figure 1 that have 7 vertices by attaching two dominating vertices in each case.

These small examples also show an important algorithmic difference between the fixed vertex property and the fixed point property for ordered sets. For a given graph $G = (V, E)$, for which we want to determine if there is a fixed vertex free endomorphism, consider the following graph CN: The vertex set is the set $\{(x, y) : x, y \in V, x \neq y\}$. Two vertices $(x_1, y_1), (x_2, y_2)$ with $x_1 \neq x_2$ are joined by an edge iff $\{(x_1, y_1), (x_2, y_2)\}$ is a (partial) endomorphism. Then G has a fixed vertex free endomorphism iff CN contains a $|V|$-clique. Moreover, any edge $(x_1, y_1), (x_2, y_2)$ so that there are distinct $u_1, \ldots, u_k \in V \setminus \{x_1, x_2\}$ so that there is no clique of the form $\{(x_1, y_1), (x_2, y_2), (u_1, v_1), \ldots, (u_k, v_k)\}$ is called **not** $(2, k)$**-consistent** (see [2], Chapter 2 in [14], or [22]). Because edges that are not $(2, k)$-consistent cannot be part of a $|V|$-clique, they can be removed from CN without removing a solution. Enforcing $(2, k)$-consistency means removing edges from CN until all edges in the remaining graph are $(2, k)$-consistent. Clearly, if this process terminates with an empty graph, then there is no fixed vertex free endomorphism. Note that, for fixed k, $(2, k)$-consistency can be enforced in polynomial time.

To date, the smallest known ordered set with the fixed point property for

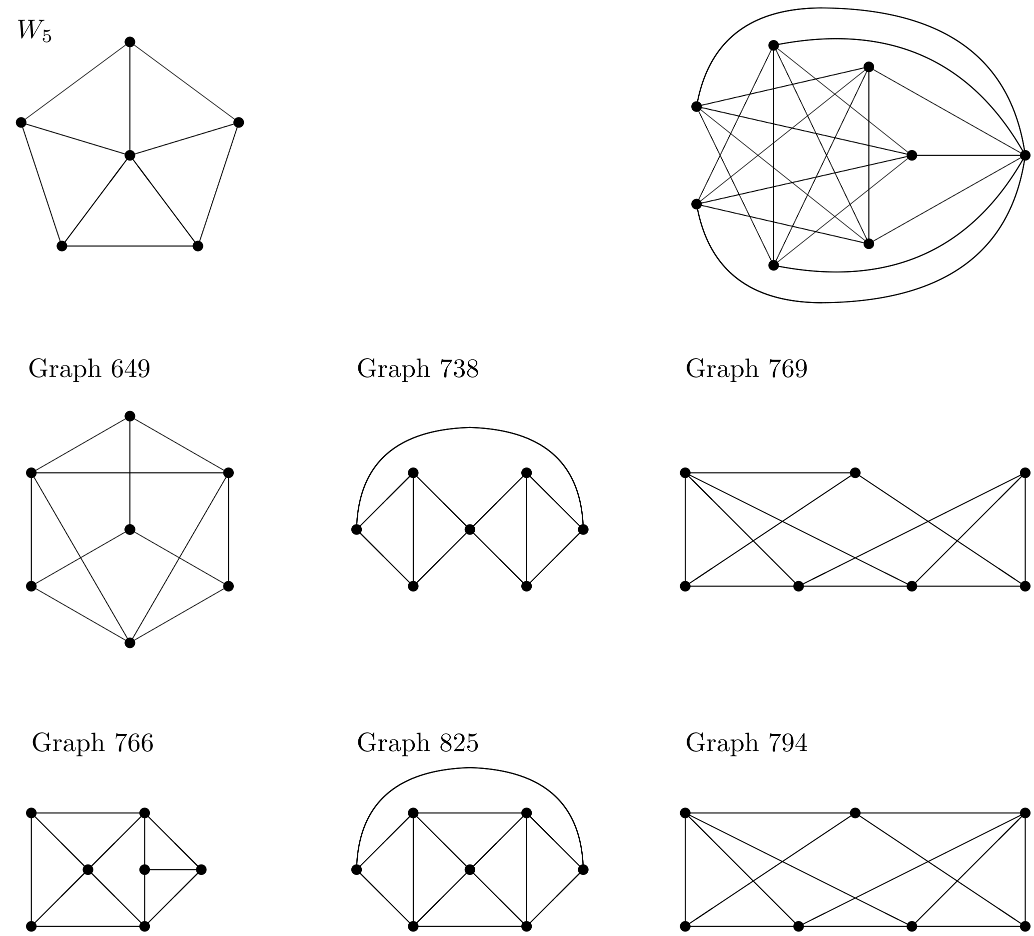

Figure 1: The 5-wheel W_5, the six nonisomorphic graphs with 7 vertices that have the fixed vertex property, no pendant vertex and do not contain a 5-wheel, and a 5-colorable graph with 8 vertices and the fixed vertex property. The numbers of the graphs with 7 vertices indicate where these graphs occur in the file graph8c.g6 available at [9].

which enforcing $(2,1)$-consistency on the corresponding constraint network (defined as above with order terminology replacing graph terminology) does not return an empty network has over 400 elements. (See Remark 11 on page 333 of [18].) Enforcing $(2,k)$-consistency is not nearly as successful for the fixed vertex property: For the 7 graphs with 8 vertices and no pendant vertices that are 5-colorable and have the fixed vertex property, even enforcing $(2,2)$-consistency on CN does not return an empty network. For the 6 graphs with 9 vertices and no pendant vertices that are 6-colorable and have the fixed vertex property, even enforcing $(2,3)$-consistency on CN does not return an empty network. The reason is that $(2,k)$-consistency enforcing algorithms rule out parts of potential solutions based on the non-existence of partial maps with small domains. If a graph contains a subgraph that is only a few edges short of being a large complete subgraph on m vertices, then it is likely that there are partial fixed vertex free maps on every subset with at most m vertices.

All graphs in Figure 1 are cores.

Definition 2.1 *A graph C is called a* **core** *iff all endomorphisms of C are bijective.*

For a graph $G=(V,E)$, if $W \subseteq V$ is so that $G[W]$ is a core and if there is an endomorphism from G to $G[W]$, then, for any endomorphism f of G so that $G[f[V]]$ is a core, we must have that $G[W]$ is isomorphic to $G[f[V]]$ and the isomorphism is $f|_W$. Hence, if G is a graph and $G[W]$ is a core so that there is an endomorphism from G to $G[W]$, we call $G[W]$ *the* core of G.

Because the core of a graph is a retract, if a graph has the fixed vertex property, then so does its core. So, because the core of an outerplanar graph is either a path with 2 vertices or an odd cycle, outerplanar graphs do not have the fixed vertex property. (The 5-wheel and the graphs with 7 vertices in Figure 1 show that planar graphs can have the fixed vertex property.)

Every graph G with the fixed vertex property is homomorphically equivalent to a graph H (that is, there are homomorphisms from G to H and from H to G) that does not have the fixed vertex property and so that the core of H is isomorphic to the core of G: Simply construct H from two disjoint copies G and G' of G, in which corresponding vertices are denoted with primes, by choosing vertices $x \sim y$ and adding the adjacencies $x \sim y'$ and $x' \sim y$. In particular, the fixed vertex property is not invariant under homomorphic equivalence.

On the positive side, if a graph does not have the fixed vertex property, but its core does, then the core must be duplicated in some way.

Proposition 2.2 *Let $G = (V, E)$ be a graph with core $G[W]$ so that no other subgraph of G is isomorphic to $G[W]$. Then G has the fixed vertex property iff $G[W]$ has the fixed vertex property.*

Proof. The direction "$\Rightarrow$" follows from Proposition 1.4. For "$\Leftarrow$," let f be an endomorphism of G. Then f must be injective on W and $G[f[W]]$ must have a copy of $G[W]$ as a subgraph. Because G has no other copy of $G[W]$ as a subgraph, we infer that $f[W] = W$, and hence f will have a fixed vertex in W. ■

Proposition 2.2 shows that every core C with the fixed vertex property is the "heart" of an infinite family of graphs with the fixed vertex property: Simply add vertices and edges so that the resulting graph has C as its core and so that C is not duplicated, and the resulting graph will have the fixed vertex property. Moreover, if we attach a dominating vertex to *any* core that does not have a dominating vertex, we obtain a core with the fixed vertex property. On the other hand, Corollary 3.4 will provide a multitude of examples of graphs, with and without the fixed vertex property, that contain multiple copies of their core.

3 Connection to Ordered Sets

To be investigated as a property in its own right, the fixed vertex property for graphs requires new tools rather than a simple translation of results for the fixed point property for ordered sets. On the other hand, Corollaries 3.4 and 4.6 show that the fixed point theory for ordered sets (including the important product question) is embedded in the fixed vertex theory for graphs. Therefore, advances on the fixed vertex property for graphs will have impact on the fixed point property for ordered sets. Conversely, we can conclude here that the decision problem whether a graph has the fixed vertex property is co-NP-complete (see Corollary 3.5).

Directed graphs ("digraphs") are usually translated into graphs with the "arrow construction" or "replacement operation" (see Section 4.4 of [5]). Consequently, it may not be surprising that we can absorb order-theoretical fixed point theory into a fixed vertex theory for graphs. However, order relations are reflexive and we need the construction to be compatible with the product operation for ordered sets. Hence the requisite replacement graph (also called a "gadget") is not obvious. We will provide the details to assure that the gadget we will use (see Remark 3.2) has the right properties (see Lemma 4.3). Recall that a graph $G = (V, E)$ is called **triangle connected** iff, for any two vertices $x, y \in V$, there is a **triangle path** $x = p_1 \sim p_2 \sim \cdots \sim p_n = y$ so that, for all $i = 1, \ldots n-2$, we have $p_i \sim p_{i+2}$.

Definition 3.1 *Let $C = (V_C, E_C)$ be a rigid triangle connected core and let $b, t \in V_C$ be vertices so that the graph $C^{b=t}$ that is obtained by identifying b and t is a rigid triangle connected core, too. Let P be an ordered set, viewed as a directed graph $P = (V_P, E_P)$ with loops. The* **arrow construction** *for P using (C, b, t) yields a graph $P * (C, b, t)$ so that every directed edge of P is replaced by a copy of C. Formally,*

$$V(P * (C, b, t)) \quad = \quad V_P \cup (V_C \setminus \{b, t\}) \times E_P)$$

and

$$\begin{aligned} E(P * (C, b, t)) \quad = \quad & \bigcup_{e \in E_P} \{\{(x, e), (y, e)\} : \{x, y\} \in E_C, x, y \notin \{b, t\}\} \\ & \cup \{\{p_1, (y, e)\} : e = (p_1, p_2) \in E_P, \{b, y\} \in E_C\} \\ & \cup \{\{(x, e), p_2\} : e = (p_1, p_2) \in E_P, \{x, t\} \in E_C\}. \end{aligned}$$

For $p_1, p_2 \in P$ so that $p_1 \leq p_2$, we let C_{p_1,p_2} be the induced subgraph of $P * (C, b, t)$ on the vertices $\{p_1\} \cup (V_C \setminus \{b, t\}) \times \{(p_1, p_2)\} \cup \{p_2\}$. Note that, for $p_1 < p_2$, each C_{p_1,p_2} is isomorphic to C and C_{p_1,p_1} is isomorphic to $C^{b=t}$. We let i_{p_1,p_2} be the unique isomorphism from C or $C^{b=t}$ to C_{p_1,p_2}.

Remark 3.2 Cores as in Definition 3.1 exist. The graphs H_k from Section 4.4 in [5] are rigid and triangle connected, but these graphs are used in the replacement construction for digraphs *without loops* and identifying their vertices b and t into one does not produce a rigid core. Consider the graph C (see Figure 2) that is obtained as follows. Connect disjoint copies of H_1 and H_2 from Section 4.4 of [5] so that the last vertex of H_1 is joined with an edge to the first and second vertices of H_2, so that the second to last vertex of H_1 is joined to the first vertex of H_2 and let b be the first vertex of H_1 and t be the last vertex of H_2. (In Figure 2, H_1 is numbered in the opposite direction of the numbering in [5].) Moreover, a copy T_{10}^i of a triangle 10-cycle is attached to the start of H_1 and to the end of H_2 in the same fashion. (A **triangle n-cycle** is a cycle $c_1 \sim c_2 \sim \cdots \sim c_n \sim c_1$ so that, for all $i = 1, \ldots n$, we have $p_i \sim p_{i+2}$ modulo n.)

Although C is a rather large graph, H_1 and H_2 are needed to provide non-adjacent vertices b and t that can be identified in $C^{b=t}$. The two graphs T_{10}^i allow a relatively simple proof that various products are triangle connected (see Lemma 4.3), which will be needed for Theorem 4.5.

A written proof that C and $C^{b=t}$ are rigid would require parity arguments similar to those in the proof of Lemma 4.3, so we only present a sketch. Let $f : C \to C$ be an endomorphism. (The argument for $C^{b=t}$ is similar.) We would first prove that a triangle 10-cycle is a core. (This requires care with parity, as, for example, a triangle 9-cycle is not a core.) Then we would prove that the only possible f-images of T_{10}^1 and T_{10}^2 are T_{10}^1 and T_{10}^2. After that, prove that T_{10}^1 and T_{10}^2 cannot be mapped to the same image set. Use the fact that the shortest triangle path from 10 to 34 is unique, together with the presence of the "extra edges" in $H_1 \cup H_2$ to show that $f[T_{10}^i] = T_{10}^i$ for $i = 1, 2$. Use the fact that the shortest triangle path from 10 to 34 is unique once more, to prove that f must be the identity.

Note that a computer search for endomorphisms (using, for example, [21]), even using a simple backtracking algorithm, reveals in seconds that C as well as $C^{b=t}$ indeed are rigid. Moreover, there is exactly one homomorphism from C to $C^{b=t}$ and, because C is a core, there is no homomorphism from $C^{b=t}$ to C. (Such a homomorphism, composed with the homomorphism from C to $C^{b=t}$ would induce a non-bijective endomorphism of C.) These facts about endomorphisms of and homomorphisms between C and $C^{b=t}$ are crucial for the proof of Theorem 3.3 below.

Theorem 3.3 *Let P, Q be ordered sets. Then every homomorphism f from $P * (C, b, t)$ to $Q * (C, b, t)$ must map V_P to V_Q in such a way that, if $p_1 \leq p_2$ in P, then $f(p_1) \leq f(p_2)$ in Q. Conversely, for every order-preserving map $F : P \to Q$, there is a homomorphism f from $P * (C, b, t)$ to $Q * (C, b, t)$ so that $f|_{V_P} = F$. If $P = Q$, then $f : P * (C, b, t) \to P * (C, b, t)$ has a fixed vertex iff $f|_{V_P} : P \to P$ has a fixed point.*

Proof. (The first paragraph is essentially the argument from [5], here provided to keep the presentation self-contained.) Let f be a homomorphism from $P * (C, b, t)$ to $Q * (C, b, t)$ and let $p_1 \leq p_2$ in P. Note that, for any

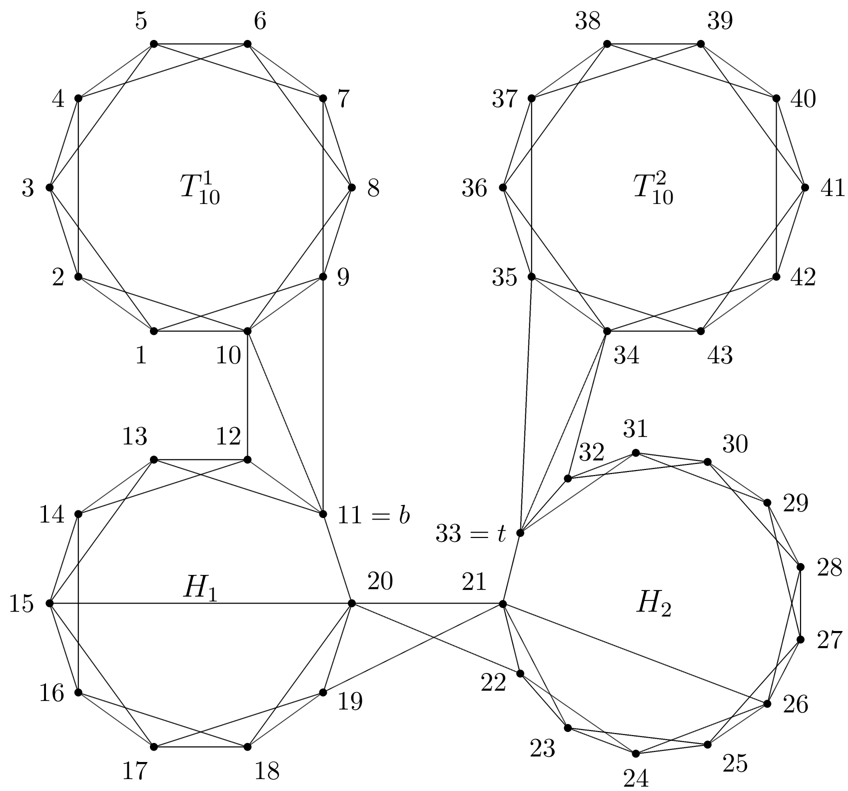

Figure 2: A rigid core C as needed in Definition 3.1 and in Theorem 4.5.

three elements q_1, q_2, q_3 of P so that $q_1 \neq q_3$ and so that q_1 is comparable to q_2 and q_2 is comparable to q_3, there is no triangle path from a vertex of $C_{q_1,q_2} - q_2$ to a vertex of $C_{q_2,q_3} - q_2$. Because C_{p_1,p_2} is triangle connected, this means there must be $q_1 \leq q_2 \in Q$ so that $f[C_{p_1,p_2}] \subseteq C_{q_1,q_2}$. Now the rigidity of C and of $C^{b=t}$ assures that $f(p_1) = q_1$ and $f(p_2) = q_2$, which means that $f(p_1) \leq f(p_2)$.

The homomorphism for the converse is the natural extension of F to $P * (C, b, t)$.

Regarding the fixed vertices when $P = Q$, note that, if f has a fixed vertex that is not in V_P, then f has a fixed vertex in some $C_{p_1,p_2} - p_1, p_2$. Hence f maps C_{p_1,p_2} to itself and thus f fixes p_1 and p_2. ■

Because C and $C^{b=t}$ are rigid cores, the natural extension of any order-preserving function $F : P \to Q$ to $P * (C, b, t)$ is unique. Therefore Theorem 3.3 defines an isomorphism between the category of ordered sets and a full subcategory of the category of graphs. Because fixed points of mappings are "preserved in both directions," Corollary 3.4 below is now a natural consequence. Moreover, the construction in Theorem 3.3 is polynomial in time and space, and by [3] the decision problem if a given finite ordered set has a fixed point free order-preserving self-map is NP-complete. Hence Corollary 3.5 is another natural consequence.

Corollary 3.4 *Let P be an ordered set. Then P has the fixed point property iff $P * (C, b, t)$ has the fixed vertex property.* ■

Corollary 3.5 *The decision problem*
Given. *A finite graph $G = (V, E)$.*
Question. *Does G have a fixed vertex free endomorphism?*
is NP-complete. ■

4 Products

The question whether the product of two ordered sets with the fixed point property has the fixed point property, too, gathered considerable attention in order theory until a positive answer was given in [11] for finite and certain kinds of infinite ordered sets. For each notion of a product in graph theory (see, for example, [5], p.74 for the cartesian product, p. 79 for the lexicographic and strong products), we can ask the question if the fixed vertex property is preserved by the product operation. However, the most natural product when considering homomorphisms, is the direct/categorical/tensor product. Corollary 4.6 below shows that the question if the product of two *arbitrary* (including infinite) connected ordered sets with the fixed point property again has the fixed point property can be embedded into the question whether the direct product of two connected graphs with the fixed vertex property has the fixed vertex property. On one hand, this provides examples of connected graphs whose product has the fixed vertex property. On the other hand, a graph theoretical investigation whether the fixed vertex property is "productive" for connected graphs could give new insights for the original question for ordered sets.

Definition 4.1 *Let $G = (V, E)$ and $H = (W, F)$ be two graphs. The* **direct product** *or* **categorical product** *or* **tensor product** *of G and H is the graph $G \times H$ whose vertices are the set $V \times W$ and for which there is an edge between (x, u) and (y, v) iff $\{x, y\} \in E$ and $\{u, v\} \in F$.*

Although the main focus of this section is again on ordered sets, we should note that there are "true" graph theoretical examples, too. Note that, in an n-fold direct product, the neighborhood of a vertex $c = (c_1, \ldots, c_n)$ satisfies $N(c_1, \ldots, c_n) = N(c_1) \times \cdots \times N(c_n)$.

Proposition 4.2 *Any direct product of a finite number of odd wheels has the fixed vertex property.*

Proof. Let c be the vertex of the product whose components are the centers of the wheels. Because the neighborhood of the vertex c is a product of odd cycles, it has chromatic number 3. For every vertex other than c, the neighborhood is a product of graphs among which there is at least one path, so the neighborhood's chromatic number is 2. Thus any endomorphism of the product of a finite number of odd wheels must fix c. ∎

In the proof of Theorem 4.5, we will need that products of the "arrows" that replace directed edges in Theorem 3.3 are triangle connected, too. The product $K_3 \times K_3$ shows that products of triangle connected graphs need not be triangle connected. The problem is that, if one triangle path ends at a vertex v and another starts at that vertex v, then their concatenation need not be a triangle path. The core in Figure 2 was chosen to make the proof of Lemma 4.3 as simple as possible.

Lemma 4.3 *The core C in Figure 2 is so that $C \times C$, $C \times C^{b=t}$ and $C^{b=t} \times C^{b=t}$ are triangle connected.*

Proof. Let $X, Y \in \left\{C, C^{b=t}\right\}$ and let (u,v) and (c,d) be vertices of $X \times Y$. Note that, if $u = x_0 \sim x_1 \sim \cdots \sim x_k = c$ and $v = y_0 \sim y_1 \sim \cdots \sim y_k = d$ are triangle walks of the same length $k \neq 0$ so that $(x_i, y_i) = (x_j, y_j)$ implies $i = j$, then $(u,v) = (x_0, y_0) \sim (x_1, y_1) \sim \cdots \sim (x_k, y_k) = (c,d)$ is a triangle path in $X \times Y$. To complete the proof, we will construct such triangle walks from u to c and from v to d.

First note that, for any vertices u and c in X, there is a triangle walk $u = x_0 \sim x_{-2} \sim x_{-1} \sim x_0 \sim x_1 \sim \cdots \sim x_n \sim x_{n+1} \sim x_{n+2} \sim x_n$ from u to c so that $\{x_{-2}, x_{-1}\} \cap \{x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}\} = \emptyset$ and $\{x_{n+1}, x_{n+2}\} \cap \{x_{-2}, x_{-1}, x_0, x_1, \ldots, x_n\} = \emptyset$ and so that n is as small as possible. (In case $u = c$, we have $n = 0$.)

In Y, there is a triangle walk from v to d of length m so that $m - n \equiv 0 \pmod 3$: Let "going straight" mean following a path $j, j+1, j+2, \ldots$ or a path $j, j-1, j-2, \ldots$ along with the labels in Y. Starting at v, go straight to the copy T_{10}^i of T_{10} that is farthest away from v in the consecutive order of the vertices, enter T_{10}^i in the order $12, 11, 10, 9$ or $32, 33, 34, 35$, go around T_{10}^i once, twice or thrice, as needed to get $m - n \equiv 0 \pmod 3$, exit T_{10}^i in the reverse order it was entered and then go straight to d.

Now extend the walk from u to c by adding copies of $x_0 \sim x_{-2} \sim x_{-1} (\sim x_0)$ at the front and adding copies of $(x_n \sim) x_{n+1} \sim x_{n+2} \sim x_n$ at the end so that the following hold. The repeated vertices x_{-2}, x_{-1}, x_0 are parallel to vertices of the path from v to d that occur on the way from v to T_{10}^i or before the last half lap around T_{10}^i. The repeated vertices x_n, x_{n+1}, x_{n+2} are parallel to vertices of the path from v to d that occur after the first half lap around T_{10}^i or on the way from T_{10}^i to d.

The repeated vertices in the X- and Y-coordinates will not cause any repeated vertices in $X \times Y$ when $u \neq c$: While cycling around T_{10}^i in Y, the parallel repeated vertices in X (if there are any) cycle with period 3, whereas the vertices in Y cycle with period 10, and any parallel repetition in X lasts for at most two and a half laps around T_{10}^i. (This avoids any complications with the entry and the exit from the cycle, too.) Moreover, as the walk in Y backtracks across vertices previously visited outside T_{10}^i note that, on the way to T_{10}^i, the repeated vertices were x_{-2}, x_{-1}, x_0, whereas on the way away from T_{10}^i the repeated vertices are x_n, x_{n+1}, x_{n+2}, which means there is no repetition as long as $c = x_n \neq x_0 = u$.

Finally, in case $c = x_n = x_0 = u$, use a similar argument using a triangle walk $x_0 \sim x_{-2} \sim x_{-1} \sim x_0 \sim x_1 \sim x_2 \sim x_3 \sim x_1 \sim x_2 \sim x_0$, adding cycles $x_0 \sim x_{-2} \sim x_{-1} (\sim x_0)$ at the front and adding cycles $(x_3 \sim) x_1 \sim x_2 \sim x_3 (\sim x_1)$ between x_3 and x_1. This adjustment avoids repetition of the vertex $x_0 = u = c$. ∎

Remark 4.4 For the graph C in Theorem 4.5 below, any graph C with vertices $b \neq t$ so that C and $C^{b=t}$ are rigid and triangle connected, so that C satisfies Lemma 4.3, and so that there is no homomorphism from $C^{b=t}$ to C will do. We refer to the graph in Figure 2 explicitly, because it is guaranteed to have the right properties.

Theorem 4.5 *Let P, Q, R and S be ordered sets viewed as directed graphs (V_P, E_P), (V_Q, E_Q), (V_R, E_R), (V_S, E_S) and let C be the graph in Figure 2. Every homomorphism f from $P * (C, b, t) \times Q * (C, b, t)$ to $R * (C, b, t) \times S * (C, b, t)$ must map $V_P \times V_Q$ to $V_R \times V_S$ in such a way that, if $(p_1, q_1) \leq (p_2, q_2)$ in $P \times Q$, then $f(p_1, q_1) \leq f(p_2, q_2)$ in $R \times S$. Conversely, for every order-preserving map $F : P \times Q \to R \times S$, there is a homomorphism f from $P * (C, b, t) \times Q * (C, b, t)$ to $R * (C, b, t) \times S * (C, b, t)$ so that $f|_{V_P \times V_Q} = F$. If $P \times Q = R \times S$, then a homomorphism f from $P * (C, b, t) \times Q * (C, b, t)$ to $R * (C, b, t) \times S * (C, b, t)$ has a fixed vertex iff $f|_{V_P \times V_Q}$ has a fixed point.*

Proof. Let $f : P * (C, b, t) \times Q * (C, b, t) \to R * (C, b, t) \times S * (C, b, t)$ be a homomorphism. For $(p_1, q_1), (p_2, q_2) \in P \times Q$ so that $(p_1, q_1) \leq (p_2, q_2)$ the diagonal

$$D_{(p_1,q_1),(p_2,q_2)} := C_{p_1,p_2} \times C_{q_1,q_2}[\{(i_{p_1,p_2}(x), i_{q_1,q_2}(x)) : x \in V_C \setminus \{b, t\}\} \cup \{(p_1, q_1), (p_2, q_2)\}]$$

of $C_{p_1,q_1} \times C_{p_2,q_2}$ is isomorphic to C when $(p_1, q_1) < (p_2, q_2)$ and it is isomorphic to $C^{b=t}$ when $(p_1, q_1) = (p_2, q_2)$.

In $R \times S$, consider three vertices $(r_1, s_1), (r_2, s_2), (r_3, s_3)$ so that $(r_1, s_1) \neq (r_3, s_3)$ and so that (r_1, s_1) is comparable to (r_2, s_2) and (r_2, s_2) is comparable to (r_3, s_3). We claim there is no triangle path from $C_{r_1,r_2} \times C_{s_1,s_2} - (r_2, s_2)$ to $C_{r_2,r_3} \times C_{s_2,s_3} - (r_2, s_2)$: If there was such a path, then there would be vertices (u, v) of $C_{r_1,r_2} \times C_{s_1,s_2} - (r_2, s_2)$, (c, d) of $C_{r_2,r_3} \times C_{s_2,s_3} - (r_2, s_2)$, and y so that $(u, v), (r_2, y), (c, d)$ are consecutive on the triangle path in this order. But then $u \sim c$ in $R * (C, b, t)$, which is not possible.

Therefore, just as in the proof of Theorem 3.3, every $D_{(p_1,q_1),(p_2,q_2)}$ must be mapped into a product $C_{r_1,r_2} \times C_{s_1,s_2}$. Moreover, if (p_1, q_1) does not get mapped to (r_1, s_1), then the composition of the appropriate projection onto the first or second component, the function f, and of the isomorphism from C (or $C^{b=t}$) to $D_{(p_1,q_1),(p_2,q_2)}$ would give a homomorphism from C or $C^{b=t}$ to C or $C^{b=t}$ that does not map b to b, which is not possible because of the uniqueness (or nonexistence in case of trying to map $C^{b=t}$ to C) of these homomorphisms. Similarly, (p_2, q_2) must be mapped to (r_2, s_2). Therefore f must map $V_P \times V_Q$ to $V_R \times V_S$ in such a way that, if $(p_1, q_1) \leq (p_2, q_2)$ in $P \times Q$, then $f(p_1, q_1) \leq f(p_2, q_2)$ in $R \times S$.

One possible homomorphism for the converse statement is the natural extension of F to $P * (C, b, t) \times Q * (C, b, t)$.

Regarding fixed vertices, note that, if f has a fixed vertex (u, v) in $C_{p_1,p_2} \times C_{q_1,q_2} - (p_1, q_1), (p_2, q_2)$, then, by Lemma 4.3, there is a triangle path from (u, v) to a vertex (x, x) of $D_{(p_1,q_1),(p_2,q_2)} - (p_1, q_1), (p_2, q_2)$. Thus $D_{(p_1,q_1),(p_2,q_2)}$ is mapped into $C_{p_1,p_2} \times C_{q_1,q_2}$, and hence f fixes (p_1, q_1) and (p_2, q_2). ∎

Unlike the proof of Theorem 3.3, the proof of Theorem 4.5 does not guarantee a bijective correspondence between order preserving maps $F : P \times Q \to R \times S$ and their counterparts $f : P * (C, b, t) \times Q * (C, b, t) \to R * (C, b, t) \times S * (C, b, t)$. In fact, any $C_{r_1,r_2} \times C_{s_1,s_2}$ can be mapped to $D_{(r_1,s_1),(r_2,s_2)}$ by projecting to the appropriate component and then using an isomorphism. Hence every order preserving map $F : P \times Q \to R \times S$ in fact has multiple counterparts. However, fixed points of mappings are "preserved in both directions," so that Corollary 4.6 below is a natural consequence.

Corollary 4.6 *Let P and Q be ordered sets. Then $P * (C, b, t) \times Q * (C, b, t)$ has the fixed vertex property iff $P \times Q$ has the fixed point property.* ∎

Neither Corollary 3.4 nor Corollary 4.6 assumes that the ordered sets should be finite. Therefore, generalizations of Roddy's Theorem (see [11]) show that if we start with two ordered sets P and Q with the fixed point property so that P is chain-complete with no infinite antichains, or so that P has width 3 (see [12]), then $P \times Q$ and hence $P * (C, b, t) \times Q * (C, b, t)$ has the fixed vertex property. So there is a substantial number of complex finite and infinite examples of pairs of graphs with the fixed vertex property whose product has the fixed vertex property, too. The question beckons if this is true in general. A positive answer would remove any additional conditions on P and Q from Roddy's Theorem and its generalizations. A negative answer might point the way towards a counterexample for ordered sets.

Corollary 4.7 *If the product of any two connected graphs with the fixed vertex property has the fixed vertex property, then the product of any two ordered sets with the fixed point property will have the fixed point property.*

In terms of necessary conditions for the product of graphs to have the fixed vertex property, note that, although the factors of a product need not be retracts of the product, the factors of a product with the fixed vertex property must have the fixed vertex property, too.

Proposition 4.8 *Let G_1 and G_2 be graphs so that $G_1 \times G_2$ has the fixed vertex property. Then G_1 and G_2 both have the fixed vertex property.*

Proof. Suppose for a contradiction that G_1 does not have the fixed vertex property and let f be a fixed vertex free endomorphism of G_1. Then $F(x, y) := (f(x), y)$ is a fixed vertex free endomorphism of $G_1 \times G_2$, a contradiction. ∎

5 Open Questions

Although the beauty of a particular property ultimately is in the eye of the beholder, Sections 3 and 4 show that, from the point of view of ordered sets, the fixed vertex property for graphs is a natural target of investigation. From the point of view of graph theory, after establishing existence or non-existence of homomorphisms, it is natural to now also analyze the behavior of the homomorphisms themselves. We conclude with open questions, some of which, the author hopes, will capture the reader's imagination.

1. If G and H are connected graphs with the fixed vertex property, does the product $G \times H$ have the fixed vertex property, too?

   Along these lines, note that the final example in [7] shows that products of disconnected graphs can have retracts R so that neither of the factors has a homomorphism into R. So it is not unthinkable that there may be a product of disconnected graphs with the fixed vertex property that has a retract that does not have the fixed vertex property.

2. What fixed vertex theorems can be derived for graphs G in which multiple subgraphs are isomorphic to the core of G?

With endomorphisms being injective on induced subgraphs that are isomorphic to the core, at least when these subgraphs do not overlap, it may be possible to use quotients to gain further insights.

3. What can be said about infinite products of graphs with the fixed vertex property?

   Because disconnected graphs can have the fixed vertex property, the fact that infinite products of nonempty graphs must be disconnected does not trivially dismiss this question. Results such as in [13], Section 10.3 of [18], and [19] do not translate easily.

   On the positive side, note that the proof of Proposition 4.2 does not refer to the product having finitely many factors. The key of the argument is that the neighborhood of the vertex c is the only neighborhood whose chromatic number is greater than 2. Thus an infinite product of a family of odd wheels of bounded size has the fixed vertex property. However, the product $\prod_{n\in\mathbb{N}} C_{2n+1}$ is bipartite. Hence the proof does not carry over to the product $\prod_{n\in\mathbb{N}} W_{2n+1}$, and it is unknown if this product has the fixed vertex property. The situation is similar to that of [13] and maybe similar arguments could resolve the question.

4. It is natural to also ask about the relation between the fixed vertex property and other notions of products in graphs, such as the cartesian product, the strong product or the lexicographic product.

   Related to the lexicographic product is the more general notion of a lexicographic sum. For lexicographic sums, a theorem similar to the one in [6] is not possible: The graph in Figure 3 is a lexicographic sum with index graph W_5 so that all maximal autonomous subgraphs are W_5 or singletons, and yet this graph does not have the fixed vertex property. The function that maps a to any vertex in V_a and that maps b and c to any two adjacent vertices in $V_{b,c}$ is a retraction onto a graph that does not have the fixed vertex property.

   The graph in Figure 3 is not a counterexample for lexicographic products, because, in a lexicographic product, *all* vertices of the first factor are replaced with autonomous isomorphic copies of the second one.

5. Let G and H be two graphs so that all automorphisms have a fixed vertex. Do all automorphisms of $G \times H$ have a fixed vertex?

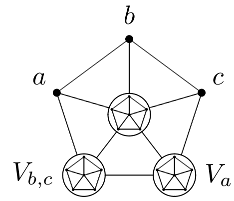

Figure 3: A decomposable graph $G = (V, E)$ that does not have the fixed vertex property, but for which all maximal autonomous subgraphs as well as the index graph have the fixed vertex property.

6. For what pairs of triangle connected graphs is the product triangle connected, too?

   The proof of Lemma 4.3 suggests that the answer may not be too hard. One could consider the same question for K_n-connected graphs or C-connected graphs, where a graph is C-connected iff, for any two vertices v and w there is a path $v = p_0, p_1, \ldots, p_n = w$ so that any subpath $p_{k+1}, \ldots, p_{k+|V(C)|}$ consisting of $|V(C)|$ consecutive vertices is contained in a subgraph that is isomorphic to C.

7. Is there a way to embed the study of the fixed vertex property for graphs into the study of the fixed point property for ordered sets that, at least for connected graphs, satisfies theorems similar to Corollaries 3.4 and 4.6?

   The author suspects that this is not possible, but a positive answer would solve the product problem for the fixed vertex property for finite connected graphs.

8. In Remark 3.2, the rigid core $C^{b=t}$ is 4-colorable and the rigid core C can be 4-colored in such a way that b and t can be assigned any desired color combination. Hence, all graphs $P * (C, b, t)$ as in Theorem 3.3 are 4-colorable. Is there a similar construction that produces 3-colorable graphs and satisfies Theorems 3.3 and 4.5, or is the class of 3-colorable graphs with the fixed vertex property a "genuinely graph theoretical class" with the fixed vertex property?

9. Are there sufficient conditions for the fixed vertex property for graphs that use iterated clique graphs as in [4]?

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