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Symmetric Groups and Conjugacy Classes

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SYMMETRIC GROUPS AND CONJUGACY CLASSES

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ABSTRACT. Let S_n be the symmetric group on n-letters. Fix $n > 5$. Given any nontrivial $\alpha, \beta \in S_n$, we prove that the product $\alpha^{S_n}\beta^{S_n}$ of the conjugacy classes α^{S_n} and β^{S_n} is never a conjugacy class. Furthermore, if n is not even and n is not a multiple of three, then $\alpha^{S_n}\beta^{S_n}$ is the union of at least three distinct conjugacy classes. We also describe the elements $\alpha, \beta \in S_n$ in the case when $\alpha^{S_n}\beta^{S_n}$ is the union of exactly two distinct conjugacy classes.

1. INTRODUCTION

Let G be a finite group, $a \in G$ and $a^G = \{a^g \mid g \in G\}$ be the conjugacy class of a in G. Let X be a G-invariant subset of G, i.e. $X^g = \{x^g \mid x \in X\} = X$ for all $g \in G$. Then X can be expressed as a union of n distinct conjugacy classes of G, for some integer $n > 0$. Set $\eta(X) = n$.

For any $a, b \in G$, the product $a^G b^G = \{xy \mid x \in a^G, y \in b^G\}$ is a G-invariant set. In this note we explore $\eta(a^G b^G)$ when G is the symmetric group S_n on n-letters, and a^G, b^G are conjugacy classes of S_n. We denote the identity of any group by e.

Arad and Herzog conjectured that the product of two nontrivial conjugacy classes in a finite simple nonabelian group is never a conjugacy class [AH], i.e., if $a, b \neq e$, then $\eta(a^G b^G) \neq 1$. This has been proved in some cases [AH], in particular, it has been proved for the alternating group A_n, i.e. if $n \geq 5$ and $\alpha, \beta \in A_n \setminus \{e\}$, then $\eta(\alpha^{A_n}\beta^{A_n}) \geq 2$.

In this note, we show that the symmetric group behaves similarly, and we give an explicit description of the minimum possible value of η.

Theorem A. *Let S_n be the symmetric group on n-letters, $n > 5$, and $\alpha, \beta \in S_n \setminus \{e\}$. Then $\eta(\alpha^{S_n}\beta^{S_n}) \geq 2$, and if $\eta(\alpha^{S_n}\beta^{S_n}) = 2$ then either α or β is a fixed point free permutation. Assume that α is fixed point free. Then one of the following holds*

i) n is even, α is the product of $\frac{n}{2}$ disjoint transpositions and β is either a transposition or a 3-cycle.

ii) n is a multiple of 3, α is the product of $\frac{n}{3}$ disjoint 3-cycles and β is a transposition.

Since for any group G and any $a, b \in G$, we have $a^G b^G = b^G a^G$ (see Lemma 3), Theorem A describes α and β when $\eta(\alpha^{S_n}\beta^{S_n}) = 2$.

Corollary A. *Fix $n > 5$. Let m be the least integer in $\{\eta(\alpha^{S_n}\beta^{S_n}) \mid \alpha, \beta \in S_n \setminus \{e\}\}$. Then*

i) $m = 2$ if n is divisible by 2 or 3,

ii) $m = 3$ otherwise.

Remark 1. For $n = 5$ we have $(1\ 2)^{S_5}(1\ 2\ 3\ 4\ 5)^{S_5} = (1\ 2\ 3\ 4)^{S_5} \cup ((1\ 2\ 3)(4\ 5))^{S_5}$. This describes, up to conjugation and ordering, the only pair of elements $\alpha, \beta \in S_5$ with $\eta(\alpha^{S_5}\beta^{S_5}) = 2$.

As for the maximum possible value of $\eta(a^G b^G)$, John Thompson conjectured that given any finite nonabelian simple group G, there exists a conjugacy class C such that $C^2 = G$ [see [KP]]. The conjecture has been proved for the alternating group A_n with $n \geq 5$ [see [H1]]. Since given any $\alpha, \beta \in S_n$, either $\alpha^{S_n}\beta^{S_n} \subseteq A_n$ or $\alpha^{S_n}\beta^{S_n} \subseteq S_n \setminus A_n$, it follows then that there exists a conjugacy class C in S_n such that $\eta(C^2)$ is the number of conjugacy classes of S_n in A_n and that is the largest possible value for $\eta(\alpha^{S_n}\beta^{S_n})$. See [AB1], [AB2], [AB3], [DY] for examples of recent developments in products of conjugacy classes. The products of conjugacy classes of symmetric groups have been studied extensively, for instance in [F1], [FH] [G1], [G2] and [J1].

The results of this paper were discovered by experimentation, using the computer algebra package MAGMA [BCP].

2. Notation

Our notation makes use of the following very well known result.

Lemma 1. *Let $\alpha \in S_n$. Then*
  a) α can be written as a product of disjoint cycles.
  b) α^{S_n} is the set of all permutations of S_n with the same cycle structure as α.

Corollary 2. *If $\alpha, \beta \in S_n$*
  a) have different cycle structures, then they are not conjugate.
  b) have different numbers of fixed points, then they are not conjugate.

Notation. We follow standard conventions, but repeat this for clarity.
- For a positive integer n, S_n denotes the symmetric group on n objects, which we identify with the set of bijective endomorphisms of the set $\{1, \dots, n\}$.
- For a group G and $a, b \in G$, we write $a \sim b$ if for some $g \in G$ we have $a^g = b$, where $a^g = g^{-1}ag$.
- For $\sigma \in S_n$, the cycle structure of σ is the multiset of the lengths of all the disjoint cycles comprising σ. We denote the cycle structure of σ by $\text{Type}(\sigma)$. For example, $\text{Type}((1\ 3)(2\ 5\ 6)(4)(7)) = \{1, 1, 2, 3\}$. By Lemma 1, $\text{Type}(\sigma)$ is well defined, and $\sigma \sim \tau \iff \text{Type}(\sigma) = \text{Type}(\sigma)$.

3. Proofs

Lemma 3. *Let G be a finite group, and let a^G and b^G be conjugacy classes of G. Then $a^G b^G = b^G a^G$. In particular, $a^G b^G = (ab)^G$ if and only if $b^G a^G = (ba)^G$.*

Proof. Observe that $ba^g = (a^g)^{b^{-1}}b$, so $b^G a^G \subseteq a^G b^G$. Similarly, $a^G b^G \subseteq b^G a^G$. $\square$

Lemma 4. *Let $\alpha, \beta \in S_r$ for $r < n$. We may consider α and β as elements of S_n by defining their action to be trivial on $r + 1, \dots, n$. Suppose that $\alpha_1, \beta_1 \in S_n$ fix $1, \dots, r$. Then $\eta(\alpha^{S_r}\beta^{S_r}) \leq \eta((\alpha\alpha_1)^{S_n}(\beta\beta_1)^{S_n})$.*

Proof. If $\eta(\alpha^{S_r}\beta^{S_r}) = k$, by Lemma 1 we have that there exist $\sigma_1, \ldots, \sigma_k \in S_r$ with $\text{Type}(\alpha^{\sigma_i}\beta) \neq \text{Type}(\alpha^{\sigma_j}\beta)$ for $i \neq j$. Since $\text{Type}(\alpha^{\sigma_i}\alpha_1\beta\beta_1) = \text{Type}(\alpha^{\sigma_i}\beta) \cup \text{Type}(\alpha_1\beta_1)$, $\alpha^{\sigma_i}\alpha_1\beta\beta_1$ are pairwise non-conjugate, and the result follows. $\qquad\square$

Lemma 5. *For $\alpha, \beta \in S_n$, $n \geq 5$, if α has r fixed points, and β has s fixed points, then provided $r + s \leq n$, there exists $\sigma \in S_n$ with $\alpha^\sigma(i) \neq \beta(i)$ for $i = 1, \ldots, n$.*

Proof. For a positive integer $k \leq n$, we inductively define $\sigma_k \in S_n$ such that $\alpha^{\sigma_k}(i) \neq \beta(i)$ for $1 \leq i \leq k$. We take $\sigma_0 = e$. Suppose σ_{k-1} has been defined for some $k > 0$. We define σ_k as follows:

case 1: If $\alpha^{\sigma_{k-1}}(k) \neq \beta(k)$ then set $\sigma_k = \sigma_{k-1}$.

case 2: Suppose $\alpha^{\sigma_{k-1}}(k) = \beta(k) = k$. Let A be the subset of $\{1, \ldots, n\}$ fixed by α^{α_k}, and let B be the subset fixed by β. Since $|A| + |B| = r + s \leq n$, and since $k \in A \cap B$, we must have some $t \in \{1, \ldots, n\} \setminus (A \cup B)$, and this t satisfies $\alpha^{\sigma_{k-1}}(t) \neq t$ and $\beta(t) \neq t$. Set $\sigma_k = \sigma_{k-1}(k\ t)$. This satisfies the required condition, since $\alpha^{\sigma_k}(i) = \alpha^{\sigma_{k-1}}(i)$ unless $i = t$ or $i = k$, and $\alpha^{\sigma_k}(t) = k \neq \beta(t)$ (because $\beta(i) = k$ only if $i = k$, and $k \neq t$) and $\alpha^{\sigma_k}(k) = \alpha^{\sigma_{k-1}}(t) \neq \beta(k) = k$ (because $\alpha^{\sigma_{k-1}}(k) = k$, so $\alpha^{\sigma_{k-1}}(t)$ has some other value).

case 3: $\alpha^{\sigma_{k-1}}(k) = \beta(k) \neq k$. In this case, since $n \geq 5$, there is some $t \in \{1, \ldots, n\}$ with $t \notin S := \{\alpha^{\sigma_{k-1}}\beta^{-1}(k), \beta(\alpha^{\sigma_{k-1}})^{-1}(k), \alpha^{\sigma_{k-1}}(k), k\}$ (these are labeled in the top row of the picture below). Set $\sigma_k = \sigma_{k-1}(k\ t)$. We have $\alpha^{\sigma_k}(i) = \alpha^{\sigma_{k-1}}(i)$ unless $i \in R := \{k, t, u := (\alpha^{\sigma_{k-1}})^{-1}(k), v := (\alpha^{\sigma_{k-1}})^{-1}(t)\}$. So, for $j \leq k$ not in R, we have $\alpha^{\sigma_k}(j) \neq \beta(j)$. We now show that for $i \in R$ we also have $\alpha^{\sigma_k}(i) \neq \beta(i)$. For $i = k$, we have $\alpha^{\sigma_k}(k) = \alpha^{\sigma_{k-1}}(t) \neq \beta(k)$, since otherwise $\alpha^{\sigma_{k-1}}(t) = \beta(k) = \alpha^{\sigma_{k-1}}(k)$ implies $k = t$, contradicting $t \notin S$. For $i = t$, from the assumption $t \neq \alpha^{\sigma_{k-1}}(k)$ it follows that $\alpha^{\sigma_k}(t) = \alpha^{\sigma_{k-1}}(k)$. We have $\alpha^{\sigma_{k-1}}(k) \neq \beta(t)$, since otherwise $t = \beta^{-1}\alpha^{\sigma_{k-1}}(k) = \beta^{-1}\beta(k) = k$, contradicting $t \neq k$. If $i = u$, then $\alpha^{\sigma_k}(u) = t \neq \beta(u)$ because otherwise $t = \beta(u) = \beta(\alpha^{\sigma_{k-1}})^{-1}(k)$, contradicting $t \notin S$. For $i = v$, we have $\alpha^{\sigma_k}(v) = k \neq \beta(v)$ because otherwise $k = \beta(v) = \beta(\alpha^{\sigma_{k-1}})^{-1}(t)$, so $t = \alpha^{\sigma_{k-1}}\beta^{-1}(k)$, contradicting $t \notin S$.

Proceeding in this way, we eventually obtain $\sigma = \sigma_n$ as required.
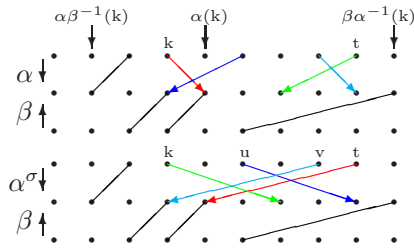
Picture corresponding to case 3 of Lemma 5. This shows a possible choice of t.

For simplicity we write α for $\alpha^{\sigma_{k-1}}$ and σ for $(k\ t)$.

The second picture shows the result of conjugating α by $(k\ t)$

$\qquad\square$

Remark 2. Lemma 5 also holds for $n = 4$, except (up to conjugacy and order) in the case $\alpha = (1\ 2\ 3)$, $\beta = (1\ 2)(3\ 4)$. One can check this case by case.

Corollary 6. *If $\alpha, \beta \in S_n$, $n \geq 4$, and α is a fixed point free permutation, then for some $\sigma \in S_n$, $\alpha^\sigma\beta$ is also fixed point free.*

Proof. By Lemma 5 and Remark 2, there exists σ with $(\alpha^{-1})^\sigma(i) \neq \beta(i)$ for $i = 1, \ldots, n$, and so $\alpha^\sigma\beta(i) \neq i$ for $i = 1, \ldots, n$, i.e., $\alpha^\sigma\beta$ has no fixed points. $\qquad\square$

3

Lemma 7. *Let $\alpha, \beta \in S_m$, $m \geq 4$, with α fixed point free and $\beta \neq e$. We may regard α and β as elements in S_n for any $n > m$ by defining $\alpha(i) = \beta(i) = i$ for any $m < i \leq n$. Then $\eta(\alpha^{S_m}\beta^{S_m}) < \eta(\alpha^{S_n}\beta^{S_n})$.*

Proof. All elements of $\alpha^{S_m}\beta^{S_m}$, considered as elements of S_n, fix at least $n - m$ points. Thus it suffices to show that some element of $\alpha^{S_n}\beta^{S_n}$ fixes fewer than $m - n$ points. By replacing α and β by conjugates if necessary, by Corollary 6, we may assume that $\alpha\beta$ does not fix $1, \ldots, m$, and that β does not fix m. Set $r = \beta^{-1}(m)$ and $s = \beta^{-1}\alpha^{-1}(m)$. Now $\alpha^{(m\ m+1)}\beta(i) = \alpha\beta(i)$ unless $i = r$, $i = s$, or $i = \beta^{-1}(m+1) = \beta^{-1}\alpha^{-1}(m+1) = m+1$. Because α is fixed point free, $\alpha(m) \neq m$, and so we have

$$\alpha^{(m\ m+1)}\beta(r) = \alpha^{(m\ m+1)}(m) = (m\ m+1)\alpha(m+1) = m \neq r,$$
$$\alpha^{(m\ m+1)}\beta(s) = \alpha^{(m\ m+1)}\beta(\beta^{-1}\alpha^{-1}(m)) = \alpha^{(m\ m+1)}\alpha^{-1}(m) = m+1 \neq s$$
$$\alpha^{(m\ m+1)}\beta(m+1) = \alpha^{(m\ m+1)}(m+1) = (m\ m+1)\alpha(m) = \alpha(m) \neq m+1,$$

and so $\alpha^{(m\ m+1)}\beta$ has $n - m - 1$ fixed points and the result follows. $\square$

Lemma 8. *Let $n > 3$ be an integer. Then $\eta((1\ 2)^{S_n}(1\ 2)^{S_n}) = 3$.*

Proof. An element of $(1\ 2)^{S_n}(1\ 2)^{S_n}$ has the form $(i\ j)(k\ l)$, with $i, j, k, l \in \{1, \ldots, n\}$ and $i \neq j$, $k \neq l$. Depending on the size of the set $\{i, j\} \cap \{k, l\}$, $(i\ j)(k\ l)$ is conjugate to one of the following: $(1\ 2)(1\ 2) = e$, $(1\ 2)(1\ 3) = (1\ 3\ 2)$, $(1\ 2)(3\ 4)$. Thus the permutations in $(1\ 2)^{S_n}(1\ 2)^{S_n}$ are the identity, 3-cycles and the product of two disjoint transpositions. By Lemma 1 we have then that $\eta((1\ 2)^{S_n}(1\ 2)^{S_n}) = 3$. $\square$

Remark 3. The example given by the previous Lemma shows that the hypothesis that α is fixed point free can not be dropped from Lemma 7.

Lemma 9. *If $\alpha, \beta \in S_4 \setminus \{e\}$ and $\eta(\alpha^{S_4}\beta^{S_4}) = 1$, then up to conjugation $\{\alpha, \beta\} = \{(1\ 2\ 3), (1\ 2)(3\ 4)\}$.*

Proof. This can be checked by hand, or by computer e.g., MAGMA [BCP]. $\square$

Lemma 10. *If $\alpha, \beta \in S_n \setminus \{e\}$ then there exists a permutation $\gamma \in S_n$ such that γ fixes at least one point and $\gamma \in \alpha^{S_n}\beta^{S_n}$.*

Proof. Since $\alpha, \beta \in S_n \setminus \{e\}$, by Lemma 1 we may assume that $\alpha(1) = 2$ and $\beta(2) = 1$. Thus $\alpha\beta(1) = 1$ and the proof is complete. $\square$

Lemma 11. *Let $n \geq 5$ and $\alpha, \beta \in S_n \setminus \{e\}$. If $\eta(\alpha^{S_n}\beta^{S_n}) \leq 2$ then $\eta(\alpha^{S_n}\beta^{S_n}) = 2$, and at least one of α and β is a fixed point free permutation.*

Proof. Suppose $\alpha, \beta \in S_n$ both fix a point, and $\eta(\alpha^{S_5}\beta^{S_n}) \leq 2$. Because α and β fix some element, they can be considered as elements of S_{n-1}. Lemma 7 then implies that $\eta(\alpha^{S_{n-1}}\beta^{S_{n-1}}) = 1$. If $n \geq 6$, we can assume the result for $n - 1$ inductively, and this is a contradiction. If $n = 5$, by Lemma 9, up to conjugation, $\alpha = (1\ 2\ 3)$ and $\beta = (1\ 2)(3\ 4)$. But in S_5, we have $(1\ 2\ 3)(1\ 2)(3\ 4) = (1\ 3\ 4)(2)$, $(1\ 2\ 3)(1\ 2)(4\ 5) = (1\ 3)(4\ 5)$, $(1\ 2\ 3)(1\ 4)(2\ 5) = (1\ 4\ 2\ 5\ 3)$, which are all in different conjugacy classes, so $\eta(\alpha^{S_5}\beta^{S_5}) \geq 3$, a contradiction. Thus we may assume that at least one of α or β is a fixed point free permutation. By Corollary 6 and Lemma 10 we have that $\eta(\alpha^{S_n}\beta^{S_n}) \geq 2$ and the result follows. $\square$

Remark 4. Lemma 9 shows that the hypothesis $n \geq 5$ is necessary in Lemma 11.

Lemma 12. *Let $n \geq 7$ and $\alpha, \beta \in S_n \setminus \{e\}$. If at least one of α and β has a cycle of length at least three, and at least one of α, β is fixed point free, then for some $\sigma \in S_n$, $\alpha^\sigma \beta$ has exactly one fixed point.*

Proof. Since $\alpha, \beta \neq e$, after conjugation, we may assume $\alpha(2) = 1, \beta(1) = 2$. We have three cases: (i) α and β both contain cycles of length at least three; (ii) only α has a cycle length at least three; (iii) only β has a cycle length at least three. In these cases, illustrated in the diagram below, we may conjugate so that (i) $\alpha(3) = 2, \beta(2) = 4$, or (ii) $\alpha(1) = 3, \beta(2) = 1$, or (iii) $\alpha(1) = 2, \beta(3) = 1$ respectively.
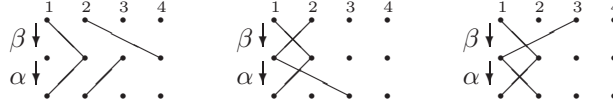

In all cases, $\alpha\beta(1) = 1$ and $\alpha\beta(2) \neq 2$. Now we proceed with the same inductive construction as in the proof of Lemma 5, starting at the step $k = 3$, since α has already been conjugated so that $\alpha\beta(1) = 1$ and $\alpha\beta(2) \neq 2$. Case 2 of the procedure never occurs, since by assumption one of α or β is fixed point free. When case 3 occurs, we must conjugate by $(\alpha(k)\ t)$ for some $t \notin S$, with S as in Lemma 5. Since $|S| \leq 4$, provided $n \geq 7$, we can pick $t \notin \{1, 2\} \cup S$. Then the property $\alpha\beta(1) = 1$ will be unaltered by replacing α by $\alpha^{(\alpha(k)\ t)}$. $\square$

Remark 5. Lemma 12 fails when $n = 6$ and $\alpha = (1\ 2\ 3)(4\ 5\ 6)$, $\beta = (1\ 2)(3\ 4)(5\ 6)$. Up to conjugation and change of order, this is the only pair of α, β in S_6 which satisfy the hypothesis, but fail the conclusion of Lemma 12. The possible cases may be checked by hand or computer.

Lemma 13. *Let $\alpha, \beta \in S_n \setminus \{e\}$ be permutations. Assume that at least one of α and β is fixed point free. If either (i) both α, β contain a cycle of length at least three, (ii) α, β have at least 4 non-fixed points, or (iii) both α, β contain a transposition, then there exists $\sigma \in S_n$ such that $\alpha^\sigma \beta$ has at least two fixed points.*

Proof. We may assume, after taking conjugates, that α, β act on 1, 2, 3, 4 as in the following diagram, where lines indicate conditions on the mapping, e.g., in all cases $\alpha(1) = 2$; if a line is not given, then no requirement is made.
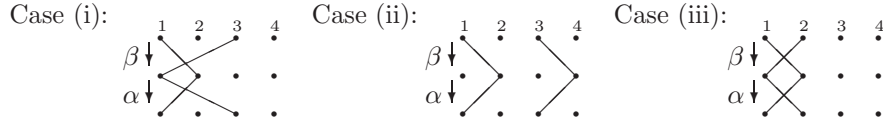
In cases (i) and (ii), 1 and 3 are fixed, and in case (iii), 1 and 2 are fixed. $\square$

Corollary 14. *If $n \geq 6$, $\alpha, \beta \in S_n$ and $\eta(\alpha^{S_n} \beta^{S_n}) = 2$, then up to change of order of α, β, α is fixed point free, and one of the following holds:*

(i) *α contains a cycle of length at least three, and β is a transposition.*
(ii) *α is a product of disjoint transpositions, and β is a three cycle.*
(iii) *Both α and β are products of disjoint transpositions.*

Proof. By Lemma 11, one of α and β is fixed point free, and by Lemma 3 without loss of generality, we may assume that α is fixed point free. By Corollary 6 and Lemmas 12, 13 and Remark 5, it follows that unless we are in cases (i), (ii), (iii), or in case $n = 6$ and $\{\alpha^{S_6}, \beta^{S_6}\} = \{((1\ 2\ 3)(4\ 5\ 6))^{S_6}, ((1\ 2)(3\ 4)(5\ 6))^{S_6}\}$,

then $\alpha^{S_n}\beta^{S_n}$ contains elements with no fixed points, with exactly one fixed point, and with at least two fixed points. These are in different conjugacy classes from each other, so the result follows, except for the case $n = 6$ and $\{\alpha^{S_6}, \beta^{S_6}\} = \{((1\ 2\ 3)(4\ 5\ 6))^{S_6}, ((1\ 2)(3\ 4)(5\ 6))^{S_6}\}$. In the remaining case, we can explicitly see that $\eta(\alpha^{S_6}, \beta^{S_6}) \geq 3$, since $(1\ 2)(3\ 4)(5\ 6)(1\ 2\ 3)(4\ 5\ 6) = (2\ 4\ 6\ 3)$, $(1\ 4)(2\ 5)(3\ 6)(1\ 2\ 3)(4\ 5\ 6) = (1\ 5\ 3\ 4\ 2\ 6)$ and $(1\ 4)(2\ 6)(3\ 5)(1\ 2\ 3)(4\ 5\ 6) = (1\ 6)(2\ 5)(3\ 4)$, and so the result also holds for this case. $\qquad\square$

Proof of Theorem A. By Lemma 11 the minimal value of η when α, β are non trivial is at least 2, and by Lemma 8, it is at most 3. Corollary 14 gives three cases when the minimal value is 2.

Case (i): β is a transposition, and α is fixed point free and contains a cycle of length at least three. Note that $(1\ 2\ \cdots\ r)(1\ 2) = (2)(1\ 3\ \cdots\ r)$ and that for $s > r$, $(1\ 2\ \cdots\ r)(r+1\ r+2\ \cdots s)(r\ r+1) = (1\ 2\ \cdots r\ r+2\ r+3\ \cdots s\ r+1)$. This implies that if $\text{Type}(\alpha) = \{a_1, \cdots, a_k\}$, then for $1 \leq i \neq j \leq k$, $\alpha^{S_n}\beta^{S_n}$ contains elements with cycle types $\text{Type}(\alpha) \setminus \{a_i\} \cup \{1, a_i - 1\}$ and $\text{Type}(\alpha) \setminus \{a_i, a_j\} \cup \{a_i + a_j\}$, (where these are all operations on multisets, not sets). If $r > 3$, observe that $(1\ 2\ \cdots\ r)(1\ 3) = (1\ 4\ \cdots\ r)(2\ 3)$. Thus $\alpha^{S_n}\beta^{S_n}$ contains an element with cycle type $\text{Type}(\alpha) \setminus \{a_i\} \cup \{a_i - 2, 2\}$ if $a_i > 3$ for some i. So, if $\eta(\alpha^{S_n}\beta^{S_n}) = 2$, we must have that $a_i = a_j = 3$ for all j and so we must be in case (ii) of the theorem.

Case (ii): This is the second possibility of case (i) of the theorem.

Case (iii): Suppose β consists of at least two disjoint transpositions. Suppose $\alpha = (1\ 2)\alpha_1$, and $\beta = (1\ 2)\beta_1$ where α_1, β_1 fix 1 and 2. As elements of S_{n-2}, α_1 is fixed point free, and β_1 contains a transposition, so by Corollary 6 and Lemma 13 $\alpha_1^{S_{n-1}}\beta_1^{S_{n-1}}$ contain elements with no fixed points, and elements with at least two fixed points. Composing these elements with $(1\ 2)$ gives elements in $\alpha^{S_n}\beta^{S_n}$ with exactly 2 fixed points, and with at least 4 fixed points. On the other hand, by Corollary 6, $\alpha^{S_n}\beta^{S_n}$ contains elements with no fixed points. Since all of these elements are in different conjugacy classes, we have $\eta(\alpha^{S_n}\beta^{S_n}) \geq 3$. So, for $\eta = 2$, we must be in the first possibility of case (i) of the theorem.

Finally, it is easy to check that in the cases of the theorem, we do indeed have $\eta(\alpha^{S_n}\beta^{S_n}) = 2$. $\qquad\square$

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