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MARKOV CHAINS AND DYNAMIC GEOMETRY OF POLYGONS

JIU DING, L. RICHARD HITT, AND XIN-MIN ZHANG

ABSTRACT. In this paper we construct sequences of polygons from a given n-sided cyclic polygon by iterated procedures and study the limiting behaviors of these sequences in terms of nonnegative matrices and Markov chains.

1. INTRODUCTION

In classical geometry there are many transformations that change one geometric object into another. For instance, from a given n-sided plane polygon P_0, one may construct a sequence of polygons via an iterated procedure. Perhaps the simplest example of this type of construction uses the so-called *Kasner polygons*. In this construction one forms a second polygon P_1 whose vertices are the mid-points of the edges of P_0. Then a third polygon, P_2, is formed whose vertices are the midpoints of the edges of P_1. Continuing this process one obtains a sequence of n-sided polygons. It is natural to ask "What can we say about the limit of this sequence?" While the size of the polygons gets smaller rapidly, what can we say about the change of their shapes?

Consider the cases beginning with $n = 3$. Given any triangle, joining the midpoints on each side yields a similar triangle. Consequently, all triangles in the resulting sequence are similar.

Given any quadrilateral, joining the midpoints on each of the four sides produces a parallelogram. Consequently, all subsequent quadrilaterals will be parallelograms. If the original quadrilateral is non-square, then the resulting sequence simply alternates between two different similarity classes.

However, for a given n-sided plane polygon P_0 with $n \geq 5$, the sequence of polygons produced by this midpoint construction can have shapes that vary in complicated ways. Some conclusions within affine geometry have been obtained. From [BGS65], for example, when normalized in size the even descendants $P_2, P_4, P_6, \cdots$ approach a fixed polygon P and the odd descendants approach another fixed polygon P'. The polygons P and P' are affine images of regular polygons and are inscribed in the same ellipse. But very little is known in general within Euclidean geometry.

This kind of question has been investigated by several others including E. Kasner, B. H. Neuman, J. Douglas, I. Schoenberg, P. Davis, B. Grunbaum, and G. C. Shephard [BGS65, Cla79, Dav79, Dou40, Oss78]. Many powerful techniques have been developed and used to deal with this problem, such as finite Fourier series and circulant matrix theory [Sch82, Dav79]. Even if one

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is concerned with only triangles and quadrilaterals, and the points are chosen from each side of a triangle or quadrilateral in some systematic manner, the iteration of such a geometric construction can still generate an unpredictable sequence.

In this paper, we continue the discussion on the dynamic geometry of polygons that was initiated in the article [HZ01]. By using some fundamental results in nonnegative matrices and Markov chains, we are able to add new findings and reveal the connections between the geometry and linear algebra of cyclic polygons.

The contents of this paper are arranged as follows. In Section 2, we provide the necessary definitions and theorems that will be used in the next two sections. In Section 3, we introduce different constructions of sequences of polygons and discuss their convergence properties. To explain the motivations behind these geometric constructions and to see how complex a simple geometric problem can become, in Section 4 we deal with some special cases of sequences of cyclic polygons, namely, sequences of triangles. Finally, Section 5 contains a few concluding historical remarks.

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2. Stochastic Matrices and Markov Chains

Let $T = [t_{ij}]$ be an $n \times n$ nonnegative matrix over $\mathbb{R}$ and denote its i^{th} row sum and j^{th} column sum by r_i and c_j, respectively. That is,

$$\begin{aligned} r_i &= \sum_{k=1}^{n} t_{ik}, \qquad i = 1, 2, \cdots, n, \qquad \text{and} \\ c_j &= \sum_{k=1}^{n} t_{kj}, \qquad j = 1, 2, \cdots, n. \end{aligned}$$

The detailed and systematic study of eigenvalues of nonnegative matrices is referred as the *Perron-Frobenius theory*. For such nonnegative matrices T, let r be the maximal eigenvalue of T.

In this section, we collect some results in matrix theory that guarantee some convergence results in the dynamic geometry we study in the next section. We begin with a well-known result in the theory of nonnegative matrices about their maximal eigenvalues.

Let us recall that an $n \times n$ matrix $T = [t_{ij}]$ over $\mathbb{R}$ is called *row quasi-stochastic* (respectively, *column quasi-stochastic*) if $r_i = 1$ (respectively, $c_j = 1$) for all $1 \leq i \leq n$ (respectively, $1 \leq j \leq n$). T is called *row stochastic* (respectively, *column stochastic*) if T is row (respectively, column) quasi-stochastic and is also a nonnegative matrix. T is called *doubly stochastic* if it is both row stochastic and column stochastic. It is easy to see from [Min88, p. 24] that the maximal eigenvalue of every row or column stochastic matrix is 1.

Let us set

$$\mathbb{R}^n_+ = \{(x_1, x_2, \cdots, x_n)^t \in \mathbb{R}^n \mid x_i > 0, i = 1, 2, \cdots, n\}$$
$$H^n_{2\pi} = \Big\{\Theta = (\theta_1, \cdots, \theta_n)^t \in \mathbb{R}^n_+ \Bigm| \sum_{i=1}^n \theta_i = 2\pi\Big\}.$$

A nonnegative matrix $T = [t_{ij}]$ is called *primitive* if there exists a positive integer m such that all of the entries of T^m are positive. It is known that if r is the maximal eigenvalue of a primitive matrix T, then the eigenvector v of T associated with r is unique (up to a scalar multiple). It is clear that a row or column stochastic matrix $T = [t_{ij}]$ with all $t_{ij} > 0$ is primitive. Moreover, we have the following simple fact.

Lemma 2.1. *Every nonnegative matrix of the following form*

$$T = \begin{bmatrix} \lambda_1 & \beta_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

is primitive, where $\lambda_i, \beta_i > 0$, $i = 1, 2, \cdots, n$.

Proof. By induction, it is straightforward to show that all entries of T^{n-1} are positive. □

Recall that an $n \times n$ nonnegative matrix T is called a *Markov matrix* if it is a column stochastic matrix. The following theorem summarizes some standard results about a Markov matrix which can be found in many references such as [IM85, HJ85].

Theorem 2.2. *Let T be an $n \times n$ primitive Markov matrix. Then*

(i) *1 is an eigenvalue of T of multiplicity one;*
(ii) *every other eigenvalue r of T satisfies $|r| < 1$;*
(iii) *the eigenvalue 1 has an eigenvector $\mathbf{w}_1 = (w_1, w_2, \cdots, w_n)$ with $w_i > 0$ for $i = 1, 2, \cdots, n$;*
(iv) *let $\mathbf{v}_1 = \mathbf{w}_1/(\sum_{i=1}^n w_i)$, then for any positive probability vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^t$, we always have*

$$\lim_{m\to\infty} T^m \mathbf{x} = \mathbf{v}_1.$$

Corollary 2.3. *Let T be an $n \times n$ primitive Markov matrix. Then there is a vector $\Phi = (\phi_1, \phi_2, \dots, \phi_n)^t$ in $H^n_{2\pi}$ such that for any $\Theta \in H^n_{2\pi}$, $\lim_{m\to\infty} T^m\Theta = \Phi$ where Φ is the unique positive eigenvector of T associated with its maximal eigenvalue 1.*

Proof. Set $\mathbf{x} = \frac{1}{2\pi} \cdot \Theta$, then $\mathbf{x}$ is a positive probability vector and the corollary follows from Theorem 2.2. □

Corollary 2.4. *Let T be an $n \times n$ primitive doubly stochastic matrix, and $\Theta \in H^n_{2\pi}$. Then*

$$\lim_{m\to\infty} T^m\Theta = \left(\frac{2\pi}{n}, \frac{2\pi}{n}, \cdots, \frac{2\pi}{n}\right)^t.$$

Proof. Since T is doubly stochastic, its normalized positive eigenvector associated with the eigenvalue 1 is $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. □

Theorem 2.5. *Let $T_1, T_2, \cdots, T_k$ be primitive doubly stochastic matrices and let $\Theta \in H^n_{2\pi}$. Then*

$$\prod_{i=1}^{\infty} \tilde{T}_i \Theta = \left(\frac{2\pi}{n}, \frac{2\pi}{n}, \cdots, \frac{2\pi}{n}\right)^t$$

where $\tilde{T}_i$ is one of $\{T_1, T_2, \cdots, T_k\}$ for $i = 1, 2, \cdots$.

Proof. Refer to [IM85]. □

Theorem 2.6. *Let $T_0 = \left[t^{(0)}_{ij}\right]$ be a primitive Markov matrix and $T_1 = \left[t^{(1)}_{ij}\right]$, $T_2 = \left[t^{(2)}_{ij}\right], \cdots, T_m = \left[t^{(m)}_{ij}\right], \cdots$, be a sequence of Markov matrices such that*

$$\lim_{m\to\infty} t^{(m)}_{ij} = t^{(0)}_{ij} \quad \text{for all} \quad 1 \le i, j \le n.$$

Then there exists a vector $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)^t \in H^n_{2\pi}$ such that for any $\Theta \in H^n_{2\pi}$,

$$\prod_{i=1}^{\infty} T_i \Theta = \Phi$$

where Φ is the unique eigenvector of T_0 associated with its maximal eigenvalue 1.

Proof. See [IM85]. □

In general, if we view $\frac{1}{2\pi} \cdot \Theta$ as a probability vector for every $\Theta \in H^n_{2\pi}$, then iterations of a Markov matrix T acting on $\frac{1}{2\pi} \cdot \Theta$ form a Markov chain. Moreover, we call (after a normalization of Θ)

$$\{T^m \Theta\}_{m=0}^{\infty}$$

a *stationary Markov Chain*, and call

$$\left\{\prod_{i=0}^{m} T_i \Theta\right\}_{m=0}^{\infty}$$

a *non-stationary Markov Chain* if the Markov matrices T_i are (possibly) different.

3. Markov Chains of Cyclic Polygons

Let $\Theta = (\theta_1, \theta_2, \cdots, \theta_n)^t$ where $0 < \theta_i < \pi$, $i = 1, 2, \cdots, n$ and $\sum_{i=1}^n \theta_i = 2\pi$, and let $P_n = P(\Theta)$ be an n-side polygon that is inscribed in a circle of radius r with central angles $\theta_1, \theta_2, \cdots, \theta_n$. An n-sided plane polygon is called *cyclic* if it can be inscribed in a circle. Clearly, two n-sided cyclic plane polygons with the same n-tuple Θ of central angles must be similar. Thus from now on, we shall denote by $P(\Theta)$ a class of similar n-sided cyclic polygons. We set

$$\mathcal{P}_n = \text{the set of all classes of similar } n\text{-sided cyclic polygons},$$

and define a linear transformation

$$T = [t_{ij}] : \mathcal{P}_n \longrightarrow \mathcal{P}_n \qquad \text{by} \qquad T(P(\Theta)) = P(T\Theta).$$

Since the components in every n-tuple Θ of central angles for every cyclic polygon are constrained by the condition $\sum_{i=1}^{n} \theta_i = 2\pi$, a necessary condition for T to be well-defined on $\mathcal{P}_n$ is that $\sum_{i=1}^{n} t_{ij} = 1$ for $j = 1, 2, \cdots, n$. In other words, T has to be column quasi-stochastic. In general, the sequence generated by the iteration of such a matrix with a given Θ, $\{T^m(\Theta)\}$, may not be a Markov chain since T may not be a nonnegative transition matrix. Therefore the limiting behavior of the sequence $\{T^m(\Theta)\}$ can be very unpredictable as we shall see in later sections. However, by imposing a moderate restriction to our matrix T and by using some well-known results in Markov chain theory, we will arrive at interesting geometric conclusions.

In classical geometry, even for some simple geometric constructions, when one iterates them the results can become very complicated. In our approach to the dynamic geometry of cyclic polygons, we represent some geometric transformations and their iterations by certain special matrices so that we may predict in some cases the geometric outcomes and explain in other cases what causes the "chaotic" behavior.

Let P be an n-sided polygon with vertices $z_1, z_2, \cdots, z_n$ inscribed in a unit circle Γ centered at O. Joining O by a line segment to the midpoint on each side of P and extending the segments to meet the circle Γ at points $v_1, v_2, \cdots, v_n$, we form a second n-sided polygon inscribed in the same circle as P. Figure 1 illustrates this for a pentagon. Denote the second polygon by TP where T represents a transformation on the set of all n-sided polygons inscribed in Γ. We are interested in the sequence of polygons $\{P, TP, T^2P, \cdots\}$ and the limit of T^mP as $m \to \infty$. Since we have stretched every midpoint on the sides of P radially to Γ, we call the sequence of polygons $\{T^mP\}_{m=0}^{\infty}$ the *midpoint-stretching polygons* generated by P.

Theorem 3.1. *Every sequence of midpoint-stretching polygons converges to the regular polygon.*

Proof. Let P be a cyclic polygon inscribed in a circle Γ and let $a_i = z_i z_{i+1}$ be the i^{th} side of P, $i = 1, 2, \cdots, n$, where $z_{n+1} = z_1$. Also, let θ_i denote the central angle of Γ subtended by a_i for $1 \le i \le n$. Then $\sum_{i=1}^{n} \theta_i = 2\pi$. So P is a representative of the class $P(\Theta)$ where $\Theta = (\theta_1, \theta_2, \cdots, \theta_n)^t$. From the construction of TP whose vertices are, say, $v_1, v_2, \cdots, v_n$, we see that the central angles subtended by the sides of TP are $(\theta_1 + \theta_2)/2, (\theta_2 + \theta_3)/2, \cdots, (\theta_{n-1} + \theta_n)/2, (\theta_n + \theta_1)/2$. Set

$$T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We may represent TP by $P(T\Theta)$, and inductively, T^mP by $P(T^m\Theta)$. Because the matrix T is a doubly stochastic matrix and $\sum_{i=1}^{n} \theta_i = 2\pi$, from Lemma 2.1 and Corollary 2.4 we have

$$\lim_{m\to\infty} [T^m\Theta] = \left(\frac{2\pi}{n}, \frac{2\pi}{n}, \cdots, \frac{2\pi}{n}\right)^t.$$

Therefore, the sequence of midpoint-stretching polygons converges to the regular polygon. $\square$

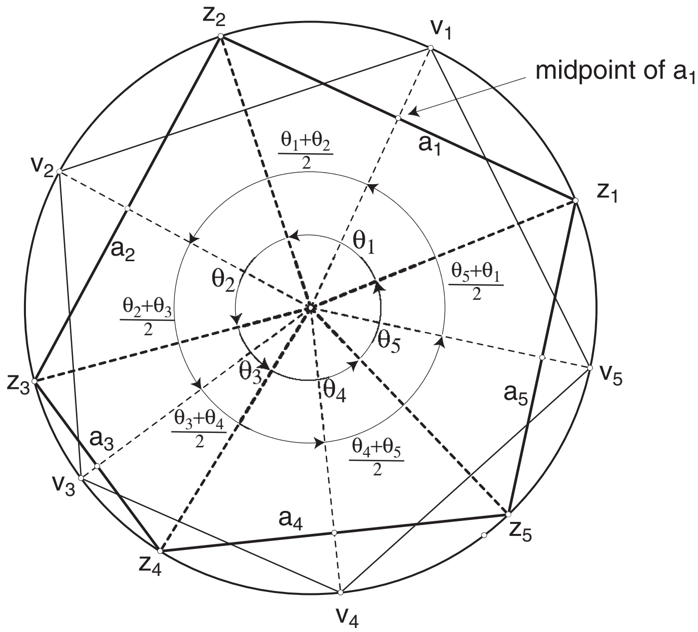

FIGURE 1. Midpoint stretching a pentagon

We can generalize this process by choosing an arbitrary point (rather than the midpoint) on the i^{th} side of P and stretching the line segment joining the center of Γ and this point to meet the circumference of Γ at a point v_i, $i = 1, 2, \cdots, n$. Denote the polygon with vertices $v_1, v_2, \cdots, v_n$ by TP, then we may characterize this polygon by the n-tuple of its central angles $\Phi = (\phi_1, \phi_2, \cdots, \phi_n)^t$, and rewrite TP as $P(\Phi) = P(T(\Theta))$, where the transformation T can be expressed as the following matrix:

$$T = \begin{bmatrix} 1-\lambda_1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1-\lambda_2 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & 0 & 0 & \cdots & 1-\lambda_n \end{bmatrix},$$

where the λ_i's $(i = 1, 2, \cdots, n)$ are real numbers between 0 and 1. An example is illustrated in Figure 2. It is clear that T is a doubly stochastic matrix if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Let us set $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, and call the sequence of polygons constructed by the iteration of T, $\{T^m P\}_{m=0}^{\infty}$, the Λ-*stretching polygons* generated by P under T.

Theorem 3.2. *The sequence of Λ-stretching polygons converges to a unique polygon $P(\Phi)$, where*

(i) *$P(\Phi)$ is regular if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = t$ where $0 < t < 1$;*
(ii) *$P(\Phi)$ is determined by the unique eigenvector of the Λ-stretching transformation associated with the eigenvalue 1.*

Proof.

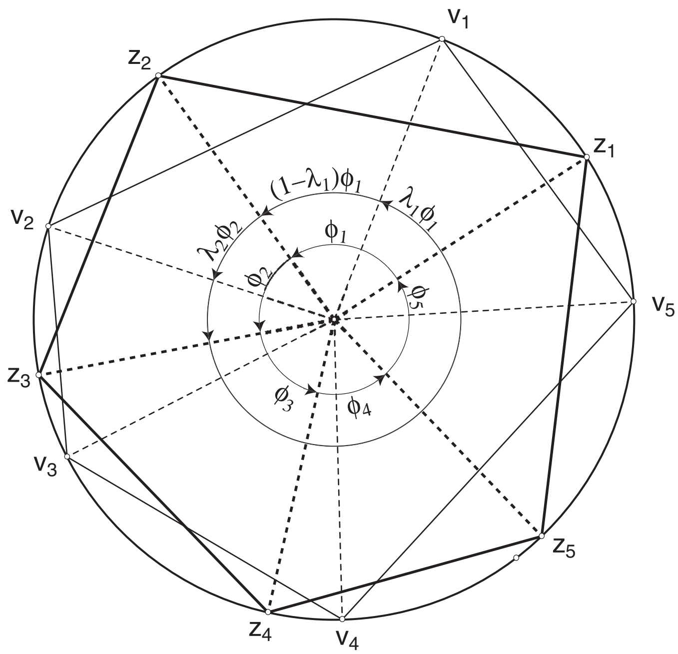

FIGURE 2. Λ-stretching a pentagon

(i) When all the components of Λ are equal, the matrix associated to the Λ-stretching is doubly stochastic. This theorem follows from Lemma 2.1 and Corollary 2.4. Note that Theorem 3.1 is a special case where $t = \frac{1}{2}$.
(ii) This follows from Lemma 2.1 and Theorem 2.2.

□

A picture that illustrates the convergence in case (i) in the above theorem can be found in [HZ01]. For an example that illustrates case (ii), let $n = 4$, $\Lambda = \left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{2}{3}\right)$. A direct calculation shows that the eigenvector of the Λ-stretching transformation matrix

$$T = \begin{bmatrix} 1-\lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 1-\lambda_2 & \lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 & \lambda_4 \\ \lambda_1 & 0 & 0 & 1-\lambda_4 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{2}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix},$$

associated with the maximal eigenvalue 1, is

$$\Phi = \left(\frac{36\pi}{47}, \frac{24\pi}{47}, \frac{16\pi}{47}, \frac{18\pi}{47}\right)^t.$$

Beginning with a square $P(\Theta)$ where $\Theta = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)^t$, Figure 3 shows the sequence of Λ-stretchings $\{\Theta, T\Theta, T^2\Theta, \cdots, T^m\Theta, \cdots\}$ out to $m = 11$ and demonstrates rapid approach to Φ. That is, the initial square becomes closer and closer to the quadrilateral $P(\Phi)$. These images were computed using a *Mathematica* program that implements the Λ-stretching method.

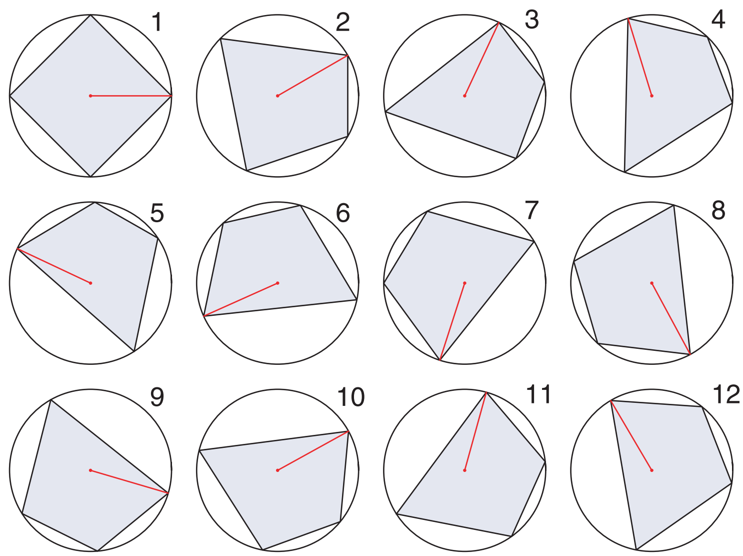

FIGURE 3. Λ-stretching a square with $\Lambda = \left(\frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{2}{3}\right)$.

Let $P_n = P_n(\Theta)$ be an n-sided cyclic polygon with the n-tuple of central angles $\Theta = (\theta_1, \theta_2, \cdots, \theta_n)^t$, $\sum_{i=1}^n \theta_i = 2\pi$. Let T_Λ be a Λ-stretching transformation where

$$\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n), \qquad 0 < \lambda_i < 1, \qquad i = 1, 2, \cdots, n.$$

If all λ_i's are equal, we call T_Λ an *even* Λ-stretching.

Theorem 3.3. *Let $P(\Theta)$ be a given n-sided cyclic polygon, and T_1, T_2, $\cdots$, T_m be a finite number of even Λ-stretchings. Then the sequence of polygons*

$$\left\{ \prod_{i=1}^{k} \tilde{T}_i \Theta \right\}_{k=1}^{\infty}$$

converges to the regular polygon where each $\tilde{T}_i$ $(1 \le i < \infty)$ is chosen from the set $\{T_1, T_2, \cdots, T_m\}$ at random.

Proof. Since every even Λ-stretching is a doubly stochastic matrix, this theorem follows from Theorem 2.5 in the previous section. □

For an example, let $\Lambda_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $\Lambda_2 = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$, $\Lambda_3 = \left(\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right)$, and $\Lambda_4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ be four different even Λ-stretchings. Let $\Theta = \left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{10\pi}{12}, \frac{7\pi}{12}\right)$. Figure 4 shows the first 12 steps in the non-stationary Markov chain that takes the initial polygon $P(\Theta)$ to the regular one based on the following choices of matrices:

$$\Theta,\ T_4\Theta,\ T_2T_4\Theta,\ T_1T_2T_4\Theta,\ T_3T_1T_2T_4\Theta,\ T_3^2T_1T_2T_4\Theta,\ \cdots,$$
$$T_3^7T_1T_2T_4\Theta,\ T_4T_3^7T_1T_2T_4\Theta.$$

These images were also created with a *Mathematica* program that implements this type of mixed Λ-stretching.

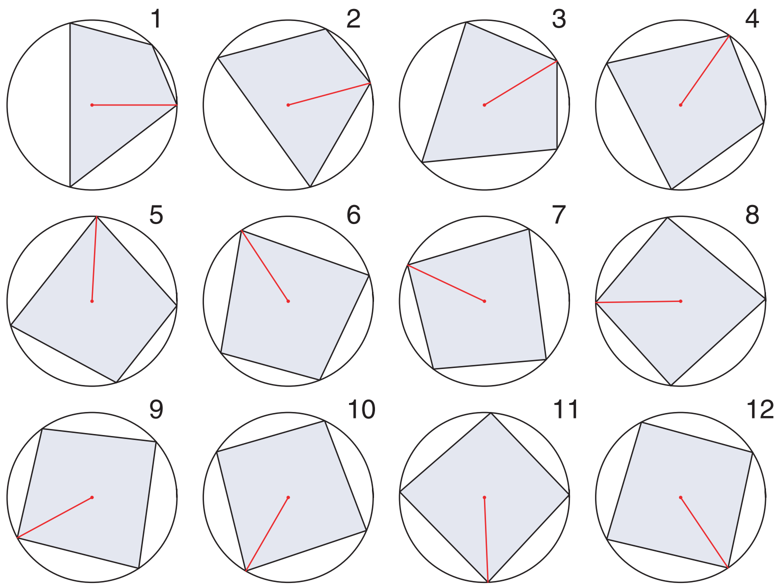

FIGURE 4. Λ-stretching with different Λs

Theorem 3.4. *Let $P(\Theta)$ be a given n-sided cyclic polygon, and $T_0 = \left[t_{ij}^{(0)}\right]$, $T_1 = \left[t_{ij}^{(1)}\right]$, $T_2 = \left[t_{ij}^{(2)}\right]$, $\cdots$, $T_m = \left[t_{ij}^{(m)}\right]$, $\cdots$ be a sequence of Λ-stretchings such that*

$$t_{ij}^{(m)} \longrightarrow t_{ij}^{(0)} \qquad \text{as} \qquad m \longrightarrow \infty \qquad \text{for all} \quad 1 \le i,j \le n.$$

Then the sequence of polygons

$$\left\{ \prod_{i=1}^{k} T_i \Theta \right\}_{k=1}^{\infty}$$

converges to a fixed polygon which is determined by the eigenvector of T_0.

Proof. This is a direct consequence of Theorem 2.6. □

4. SEQUENCES OF TRIANGLES

Since every triangle is inscribed in a unique circle, triangles are special cyclic polygons. In this section we re-examine some well-known examples of sequences of triangles in terms of iterations of matrices.

Example 4.1. Take any scalene triangle $\triangle A_0B_0C_0$ and construct the inscribed circle. The points of tangency form a second triangle, $\triangle A_1B_1C_1$. Then construct the inscribed circle for $\triangle A_1B_1C_1$. The points of tangency on the three sides of $\triangle A_1B_1C_1$ form a third triangle $\triangle A_2B_2C_2$. Continuing this process one gets a sequence of triangles $\{\triangle A_nB_nC_n\}_{n=0}^{\infty}$. See Figure 5. What does the shape of $\triangle A_nB_nC_n$ look like as n increases? The answer is that $\triangle A_nB_nC_n$ will approach an equilateral triangle. (Of course, if $\triangle A_0B_0C_0$ is equilateral,

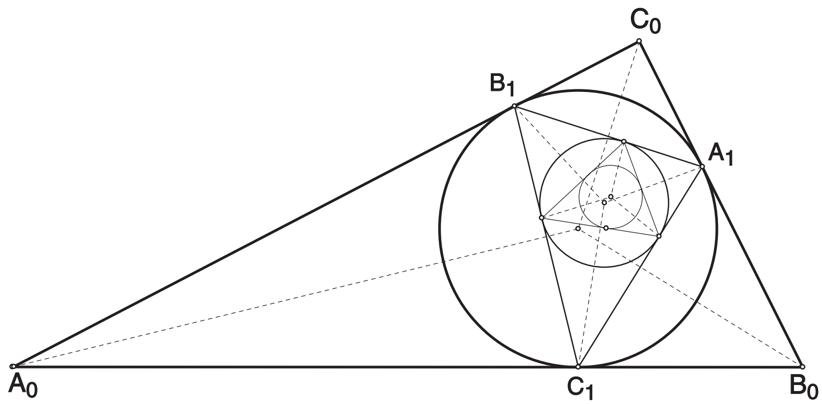

FIGURE 5. Example 4.1

then every subsequent $\triangle A_nB_nC_n$, $n \geq 1$, will be equilateral.) To confirm the answer, forming a simple geometric sketch one can see that

$$A_1 = \frac{\pi - A_0}{2}, \quad A_n = \pi \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{2^k} + (-1)^n \frac{A_0}{2^n}, \quad \text{and} \quad \lim_{n\to\infty} A_n = \frac{\pi}{3}.$$

By a similar calculation, we also have $\lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \frac{\pi}{3}$.

Example 4.2. For a variation of the first example, let $T_0 = \triangle A_0B_0C_0$ be any scalene triangle circumscribing a circle Γ_0 with center O. The line segments AO, BO, and CO (the angle bisectors of T_0) intersect Γ_0 at points A_1, B_1 and C_1. Form a second triangle $T_1 = \triangle A_1B_1C_1$ that circumscribes the circle Γ_1 with center O_1. Construct a third triangle from T_1 in the same manner, and so on. See Figure 6. We have a new sequence of triangles that are nested in a coherent manner. The triangle T_n in this sequence also approaches the equilateral one as n increases. To see this, notice that

$$A_n = \sum_{k=1}^{n} \frac{\pi}{4^k} + \frac{A}{4^n}, \quad \text{for} \quad n \geq 1, \quad \text{so} \lim_{n\to\infty} A_n = \frac{\pi}{3}.$$

Remark 4.3.

(i) There are other ways to construct nesting triangles that converge to the equilateral one. For instance, consider triangles formed by the intersection points of the three angle bisectors with the opposite sides of a triangle [CG67].

(ii) Since the sequences of triangles in Examples 4.1 and 4.2 are nested triangles with diameters that approach 0, their limits are actually just single points. Thus it might be misleading to talk about the shape of the limits of these sequences. However, since we are concerned with only the shapes of these triangles and not their sizes, we may re-scale the triangles by an appropriate proportion. For instance, notice that the initial triangle in Example 4.1, $\triangle A_0B_0C_0$, is always inscribed in a circle Γ of radius r, After the construction of the second triangle $\triangle A_1B_1C_1$ in terms of the "incircle" Γ_0 of $\triangle A_0B_0C_0$, we may re-scale the circle Γ_0

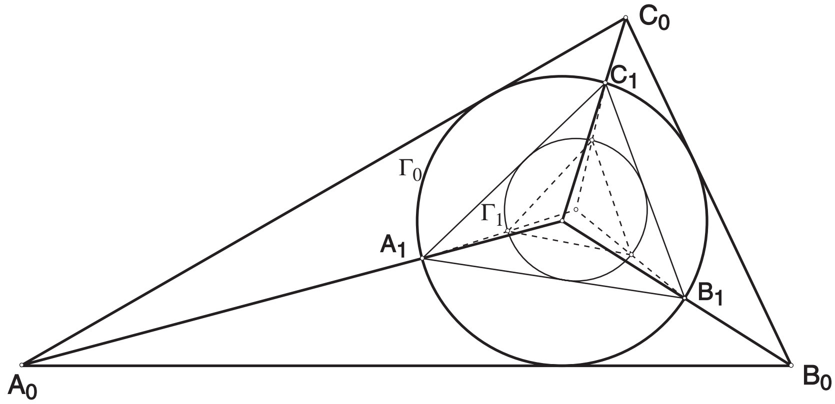

FIGURE 6. Example 4.2

to have radius r. Consequently, the triangle $\triangle A_1B_1C_1$ will be rescaled also. Thus, we can change the size of $\triangle A_1B_1C_1$ while preserving its shape. Continue this rescaling for subsequent triangles. The result is a sequence of triangles that are all inscribed in the same circle Γ. Applying this kind of magnification to other sequences of nested triangles allows us to use the theorems developed in the previous section to the study of limiting triangles.

First of all, let's look at the Example 4.1 again. From Figure 7, if we denote by θ_1, θ_2, θ_3, and ϕ_1, ϕ_2, ϕ_3 the three central angles of the initial triangle $\triangle A_0B_0C_0$, and the second triangle $\triangle A_1B_1C_1$, respectively, then it is clear that

$$A_0 = \frac{\theta_1}{2},\ B_0 = \frac{\theta_2}{2},\ C_0 = \frac{\theta_3}{2}, \qquad A_1 = \frac{\phi_1}{2},\ B_1 = \frac{\phi_2}{2},\ C_1 = \frac{\phi_3}{2}, \quad \text{and}$$

$$\phi_1 = 2A_1 = \pi - A_0 = \pi - \frac{\theta_1}{2} = \frac{\theta_2 + \theta_3}{2}, \quad \phi_2 = \frac{\theta_3 + \theta_1}{2}, \quad \phi_3 = \frac{\theta_1 + \theta_2}{2}.$$

That is, the triangles in Example 4.1 are actually midpoint-stretching triangles. By Theorem 3.1, they converge to the equilateral one.

Second, let's take another look at Example 4.2. Let $P(\Theta) = \triangle A_0B_0C_0$ be the initial triangle and $P(\Phi) = P(T\Theta) = \triangle A_1B_1C_1$ be the second triangle (up to a magnification) where $\Theta = (\theta_1, \theta_2, \theta_3)^t$ and $\Phi = (\phi_1, \phi_2, \phi_3)^t$ are the sets of central angles of the two triangles, respectively. A direct calculation shows that the matrix T is

$$T = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix},$$

which is a doubly stochastic matrix. Hence the limiting shape of the triangles in Example 4.2 must be an equilateral triangle by Corollary 2.4.

In classical geometry, there are several ways to associate equilateral triangles to a given triangle. Two of the best known examples are perhaps the so-called *Morley triangle* and the *Napoleon triangle* (refer to [CS97, CG67]). From a dynamic system point of view, there exists an iteration T acting on the set of all triangles $\{P(\Theta)\}$ such that $TP(\Theta) = P(T\Theta)$ is equilateral for any triangle $P(\Theta)$.

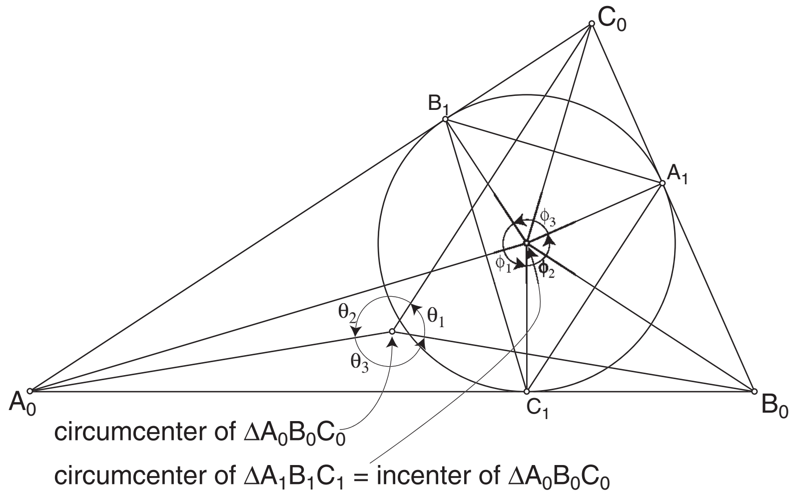

FIGURE 7. Another look at Example 4.1

It is clear that such a transformation is represented by the following special doubly stochastic matrix:

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

In the long history of classical geometry, there are many other elegant and interesting geometric transformations on triangles, quadrilaterals, and general polygons. Some of these transformations can be iterated and generate sequences of polygons. For more details, refer to the works by Coxeter [Cox89, CG67] and Yaglom [Yag62, Yag68, Yag73]. We hope that these transformations and their iterations can provide more inspiring models for the study of dynamical system in geometry.

However, simple geometric transformations on simple geometric figures by no means always create “simple” dynamical systems. A typical example of a sequence of triangles which could be “chaotic" is the so-called *sequence of pedal triangles*. That is, if $\triangle A_0B_0C_0$ is a given triangle, consider the three altitudes dropped from each vertex to the opposite side. The points of intersection of altitudes and the three sides of the triangles are called the feet of the altitudes, and they form a second triangle $\triangle A_1B_1C_1$ which is called the pedal triangle of $\triangle A_0B_0C_0$ (Coxeter [CG67, p. 9] calls it an *orthic triangle*). Then construct the second pedal triangle $\triangle A_2B_2C_2$ of $\triangle A_1B_1C_1$ in the same way. Continuing this process one gets a sequence of triangles $\{\triangle A_nB_nC_n\}_{n=0}^{\infty}$ which is called the “sequence of pedal triangles". This sequence was studied more than a century ago [Hob97]. In the late 1980’s, Kingston and Synge revisited this topic and discovered many surprising properties of such sequences and corrected some errors in the earlier literature [KS88]. The limiting shape of $\{\triangle A_nB_nC_n\}_{n=0}^{\infty}$ can be almost any triangular shape if one chooses an appropriate initial triangle

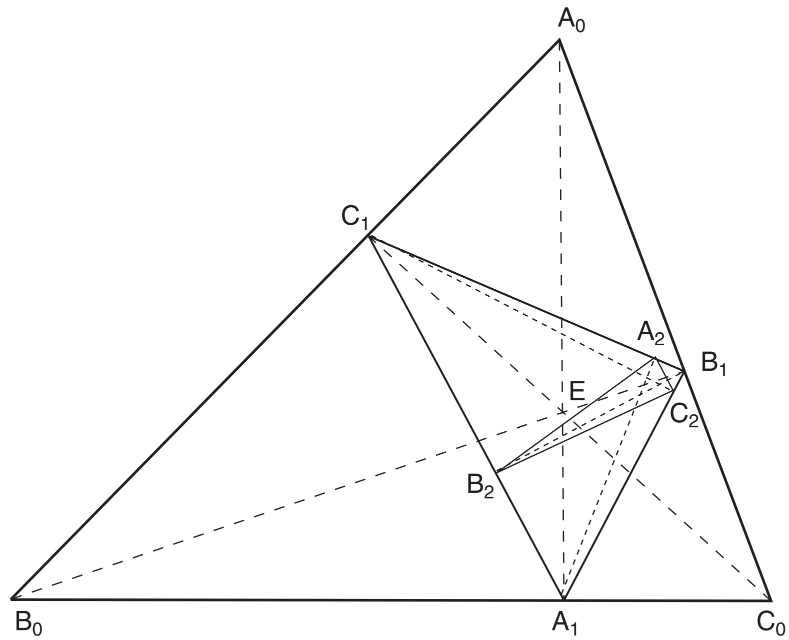

FIGURE 8. The Pedal triangle construction

$\triangle A_0B_0C_0$. Soon after their work, a number of articles made nice connections between the sequence of pedal triangles and symbolic dynamic systems and ergodic theory [Ale93, Lax90, Ung90].

5. Concluding Remarks

The special geometric transformations and their iterations, i.e., Λ-stretching, we introduced in Section 3 are only examples of how geometric transformations in classical geometry may generate interesting dynamical systems. The idea of the construction was first motivated by triangles that are always inscribed in circles and the special role played by cyclic polygons in isoperimetric problems (see [HZ01]). However, the idea of Λ-stretching can also be interpreted differently with some historical connections.

A very popular idea in the geometry of triangles has been to erect a regular (or even an irregular) geometric figure along each side of a triangle and then derive some nice properties about that triangle. It can be dated back to the proof of the Pythagorean theorem. Two famous examples of this kind of result are Napoleon's Theorem and its generalization to the Douglas-Neumann Theorem [Dav97]. Geometers are still discovering new results based on these kinds of simple constructions. Recent work of W. Schuster illustrates this [Sch98]. As a matter of fact, one may regard the geometric construction of the Λ-stretching as a variation of Napoleon-Douglas-Neumann construction. We hope that the utilization of computer graphic techniques and dynamical system theory can stimulate research in classical geometry leading to new questions and providing more intuitive background for some abstract theories of dynamic systems.

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