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Particular Solutions of products of Helmholtz-type equations using Matern functions

Abstract

in progress ...

Keywords: Helmholtz-type equations, radial basis function Matern functions, fundamental solution, particular solution

1 Introduction

in progress ...

In this paper, we use the Matern functions as the basis function which means

$$\phi_\nu(r) = (cr)^\nu K_\nu(cr) \quad (1)$$

where K_ν is the modified Bessel function of the second kind of order $\nu > 0$ and shape parameter $c > 0$.

2 Particular solutions for Helmholtz-type equations using Matern function in 2D

First we consider to derive the particular solution for modified Helmholtz equation using Matern function $(cr)^n K_n(cr)$ where n is a positive integer as the basis function in 2D. In this section, our aim is to derive a close form particular solutions Φ^{MH} of the modified Helmholtz equations as follow:

$$(\Delta - \lambda^2)\Phi_n^{mh} = (cr)^n K_n(cr). \quad (2)$$

By the annihilator method [2, 5], we can apply a differential operator on the both sides of (2) and then convert the derivation of particular solution to fundamental solution which is well-known in the literature of boundary element methods.

Similar to the idea that $r^{2n} \ln$ is related to the fundamental solution of polyharmonic differential operator Δ^n , Matern function in (1) is also related to the fundamental solution of poly-modified Helmholtz operator. From [3], the fundamental solution of the following poly-modified Helmholtz equation

$$(\Delta - c^2)^m G_m(cr) = \delta(x, \xi), \quad m \geq 1, \quad (3)$$

where δ is a Dirac delta function, is given by

$$G_m(cr) = \frac{(-1)^m}{2^m \pi (m-1)!} \left(\frac{r}{c}\right)^{m-1} K_{m-1}(cr). \quad (4)$$

For $m = 1$, the fundamental solution of the modified Helmholtz equation in 2D is

$$G_1(cr) = \frac{-1}{2\pi} K_0(cr).$$

To illustrate how we convert the problem of finding the particular solution in (2) into the fundamental solution, we let $n = 2$ in (2). Repeatedly applying $(\Delta - c^2)$ on the both sides of (2) and using MAPLE for the symbolic computation, we have

$$\begin{aligned} (\Delta - c^2)(\Delta - \lambda^2)\Phi_2^{mh} &= (\Delta - c^2)(cr)^2 K_2(cr) = -4c^2(cr)K_1(cr), \\ (\Delta - c^2)^2(\Delta - \lambda^2)\Phi_2^{mh} &= (\Delta - c^2)(-4c^2(cr)K_1(cr)) = 8c^4 K_0(cr), \\ (\Delta - c^2)^3(\Delta - \lambda^2)\Phi_2^{mh} &= (\Delta - c^2)8c^4 K_0(cr) = -16\pi c^4 \delta(x, \xi). \end{aligned} \quad (5)$$

Note that (5) can be reformatted as follows:

$$(\Delta - c^2)^3(\Delta - \lambda^2) \left(\frac{\Phi_2^{mh}}{-16\pi c^4} \right) = \delta(x, \xi). \quad (6)$$

Let $\mathbb{G}(r; c, \lambda)$ be the fundamental solution of (6). Then, the relationship between the particular solution Φ_2^{mh} in (2) and the fundamental solution $\mathbb{G}(r; c, \lambda)$ of (6) can be expressed in the following way

$$\Phi_2^{mh} = -16\pi c^4 \mathbb{G}(r; c, \lambda). \quad (7)$$

As we shall see, we will turn our focus to finding the fundamental solution of the product of poly-modified Helmholtz equation. The above derivation is based on the help of symbolic software MAPLE. Next, we will show the general case for the Matern function $(rc)^n K_n(rc)$ in the following lemma.

Lemma 1 *Let n be a positive integer. In 2D, it can be shown that*

$$(\Delta - c^2)^{n+1}(cr)^n K_n(cr) = (-2)^{n+1} \pi c^{2n} n! \delta(x, \xi). \quad (8)$$

Proof. For the following derivation, we need the following two formulae [1]

$$\frac{d}{dr} (r^n K_n(r)) = -r^n K_{n-1}(r) \quad (9)$$

$$rK_{n+1}(r) = rK_{n-1}(r) + 2nK_n(r). \quad (10)$$

By direct differentiation, it follows that

$$\begin{aligned}
\Delta((cr)^n K_n(cr)) &= \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} (cr)^n K_n(cr) \right) \\
(\text{From (9)}) &= -\frac{1}{r} \frac{d}{dr} ((cr)^{n+1} K_{n-1}(cr)) \\
&= -\frac{1}{r} \frac{d}{dr} ((cr)^2 \cdot (cr)^{n-1} K_{n-1}(cr)) \\
(\text{Product rule}) &= -\frac{1}{r} (2rc^2(cr)^{n-1} K_{n-1}(cr) + (cr)^2 (-c(cr)^{n-1} K_{n-2}(cr))) \\
&= -c^2 (2(cr)^{n-1} K_{n-1}(cr) - (cr)^n K_{n-2}(cr)) \\
(\text{From (10)}) &= c^2 ((cr)^n K_n(cr) - 2n(cr)^{n-1} K_{n-1}(cr)).
\end{aligned}$$

Then, we have

$$(\Delta - c^2)(cr)^n K_n(cr) = -2nc^2(cr)^{n-1} K_{n-1}(cr). \quad (11)$$

By recursively applying the operator $(\Delta - c^2)$ to (11), we can eventually reduce the Matern function to Dirac delta function as follows

$$(\Delta - c^2)^{n+1}(cr)^n K_n(cr) = (-2)^{n+1} \pi c^{2n} n! \delta(x, \xi).$$

■

Lemma 1 sets the stage of finding the particular solution Φ_n^{mh} in (2) through the use of fundamental solution. In Lemma 1, we show the Matern function can be annihilated by the poly-modified Helmholtz operator. Applying the operator $(\Delta - c^2)^{n+1}$ to (2), the particular solution Φ_n^{mh} for Metern basis function of order n can be reformulated in the following form:

$$\Phi_n^{mh} = (-2)^{n+1} \pi c^{2n} n! \mathbb{G}_{n+1}(r; c, \lambda) \quad (12)$$

where \mathbb{G}_{n+1} is the fundamental solution of the product of poly-modified Helmholtz operator $(\Delta - c^2)^{n+1}(\Delta - \lambda^2)$. In (12), we establish the relation between the particular solution and the fundamental solution of the product of poly-modified Helmholtz equation. The next important step is how to obtain $\mathbb{G}_{n+1}(r; c, \lambda)$ efficiently and systematically.

Next, we briefly describe how to find the fundamental solution of the following product of poly-modified Helmholtz operators

$$(\Delta - \lambda_1^2)^{k_1} (\Delta - \lambda_2^2)^{k_2} \cdots (\Delta - \lambda_n^2)^{k_n} \mathbb{G} = \delta(x, \xi) \quad (13)$$

where $k_i, i = 1, 2, \dots, n$ are nonnegative integers, $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$ are positive real numbers. In the literature of boundary element methods, there are various methods of construction of fundamental solutions of (13). A general expression of fundamental solution of (13) is given by [3]

$$\mathbb{G} = \sum_{i=1}^n \frac{1}{(k_i - 1)!} \frac{\partial^{k_i-1} c_i G_1(\lambda_i)}{\partial \lambda_i^{k_i-1}} \quad (14)$$

in which

$$c_i = \prod_{j=1, j \neq i}^n \frac{1}{(\lambda_i^2 - \lambda_j^2)^{k_j}} \quad (15)$$

and $G_1(\lambda)$ is the fundamental solution of the modified Helmholtz equation

$$(\Delta - \lambda) G_1 = \delta(x, \xi).$$

The derivation of \mathbb{G} in (14) through repeatedly taking derivative of G_1 is quite tedious. Another approach is using the algebraic procedure of partial fraction decomposition.

Lemma 2 *The following partial fraction decomposition is true [7]*

$$\prod_{i=1}^k \frac{1}{(\xi - \lambda_i^2)^{\alpha_i}} = \sum_{m=1}^k \sum_{l=0}^{\alpha_m-1} \frac{C_{m,l}}{(\xi - \lambda_m^2)^{\alpha_m-l}} \quad (16)$$

where $C_{m,l}$ can be obtained by partial fraction.

In general, we can regard the fundamental solution as a special case of particular solution. This means if the Dirac delta function $\delta(x, \xi)$ in the right hand side of (13) is replaced by a general function, then \mathbb{G} becomes the particular solution. The following theorem can be obtained in a similar way as shown in [9]. The only difference is replacing the particular solution by the fundamental solution in the statement of the theorem.

Theorem 3 *Let $x \in \mathbb{R}^d$, $d = 2, 3$. Assume*

$$(\Delta - \lambda_m^2)^{\alpha_m-l} G_{m, \alpha_m-l} = \delta(x, \xi), \quad m = 1, 2, \dots, \alpha_m-l, \quad (17)$$

then the fundamental solution for the following poly-modified Helmholtz equation

$$\left(\prod_{i=1}^k (\Delta - \lambda_i^2)^{\alpha_i} \right) \mathbb{G} = \delta(x, \xi), \quad (18)$$

where $\lambda_i \neq \lambda_j$ for all i and j , is given by

$$\mathbb{G} = \sum_{m=1}^k \sum_{l=0}^{\alpha_m-1} C_{m,l} G_{m, \alpha_m-l} \quad (19)$$

where $C_{m,l}$ is the partial fraction given in (16).

In the above theorem, the partial differential operator Δ was treated as an algebraic quantity in the partial fraction decomposition. We refer the reader to the book of Hörmander [6] for more details. The above theorem provides an easy and efficient way to find the fundamental solution of the product of poly-modified Helmholtz operators provided that G_{m, α_m-l} in (17) is known. On the other hand, the closed form of G_{m, α_m-l} is available in (4). All we need to do is to find $C_{m,l}$ by partial fraction decomposition which can be achieved using symbolic software such as MAPLE or MATHEMATICA.

For illustration, let us go back to (6) to complete the task of finding the particular solution Φ^{mh} . Based on Lemma 2, we decompose the following product into the partial fraction:

$$\begin{aligned} \frac{1}{(\xi - c^2)^3(\xi - \lambda^2)} = & -\frac{1}{(\lambda^2 - c^2)^3(\xi - c^2)} - \frac{1}{(\lambda^2 - c^2)^2(\xi - c^2)^2} \\ & - \frac{1}{(\lambda^2 - c^2)(\xi - c^2)^3} + \frac{1}{(\lambda^2 - c^2)^3(\xi - \lambda^2)}. \end{aligned} \quad (20)$$

Then, according to Theorem 3 and (20), the fundamental solution of (6) is given by

$$\mathbb{G}(r; c, \lambda) = \frac{G_1(\lambda r) - G_1(cr)}{(\lambda^2 - c^2)^3} - \frac{G_2(cr)}{(\lambda^2 - c^2)^2} - \frac{G_3(cr)}{(\lambda^2 - c^2)}$$

where $G_i(cr)$, $i = 1, 2, 3$, are given in (4). As a result, the particular solution Φ_2^{mh} in (7) can be obtained as follows

$$\Phi_2^{mh}(r; c, \lambda) = -16\pi c^4 \left(\frac{G_1(\lambda r) - G_1(cr)}{(\lambda^2 - c^2)^3} - \frac{G_2(cr)}{(\lambda^2 - c^2)^2} - \frac{G_3(cr)}{(\lambda^2 - c^2)} \right). \quad (21)$$

In the above expression, we observe that $G_2(cr)$ and $G_3(cr)$ are non-singular due to the fact that for positive integer n ,

$$\lim_{r \rightarrow 0} r^n K_n(r) = 2^{n-1}(n-1)!. \quad (22)$$

On the other hand, both $G_1(\lambda r)$ and $G_1(cr)$ are singular at $r = 0$. The particular solution Φ^{mh} is a smooth function since the singularities of $G_1(\lambda r)$ and $G_1(cr)$ are canceled out each other. Hence, it is crucial to know the behavior of Φ^{mh} as $r \rightarrow 0$. The series expansion of $K_0(cr)$ is known as follows

$$K_0(cr) = -\ln\left(\frac{c}{2}\right) - \ln r - \gamma + O(r^2)$$

where γ is the Euler number. It follows that

$$\lim_{r \rightarrow 0} (G_1(\lambda r) - G_1(cr)) = \frac{1}{2\pi} \ln\left(\frac{\lambda}{c}\right). \quad (23)$$

Furthermore, from (4) and (22), for positive integer $m \geq 2$, we have

$$\lim_{r \rightarrow 0} G_m(cr) = \lim_{r \rightarrow 0} \frac{(-1)^m}{2^m \pi (m-1)!} \left(\frac{r}{c}\right)^{m-1} K_{m-1}(cr) = \frac{(-1)^m}{4\pi (m-1) c^{m-1}}. \quad (24)$$

Hence, from (20) – (24), we have

$$\lim_{r \rightarrow 0} \Phi_2^{mh}(r; c, \lambda) = \frac{2(4c^4 \ln(c/\lambda) + 4c^2 \lambda^2 - 3c^4 - \lambda^4)}{(c^2 - \lambda^2)^3}. \quad (25)$$

For the general case, the following theorem can be obtained by the Mathematical Induction.

Theorem 4 *The particular solution of the following modified Helmholtz equation in 2D*

$$(\Delta - \lambda^2)\Phi_n^{mh} = (cr)^n K_n(cr)$$

is given by

$$\Phi_n^{mh}(r; c, \lambda) = (-2)^{n+1} c^{2n} \pi n! \left(\frac{G_1(\lambda r) - G_1(cr)}{(\lambda^2 - c^2)^{n+1}} - \sum_{i=2}^{n+1} \frac{G_i(cr)}{(\lambda^2 - c^2)^{n+2-i}} \right), \quad r \neq 0, \quad (26)$$

and

$$\lim_{r \rightarrow 0} \Phi_n^{mh}(r; c, \lambda) = (-2)^{n+1} c^{2n} n! \left(\frac{\ln(\lambda/c)}{2(\lambda^2 - c^2)^{n+1}} - \sum_{i=2}^{n+1} \frac{(-1)^i}{4(\lambda^2 - c^2)^{n+2-i} (i-1) c^{2(i-1)}} \right), \quad (27)$$

where

$$G_i(cr) = \frac{(-1)^i}{2^i \pi (i-1)!} \left(\frac{r}{c} \right)^{i-1} K_{i-1}(cr), \quad i = 1, 2, \dots, n+1. \quad (28)$$

The particular solution for the Helmholtz operator in 2D can be derived in a similar way as above. Similar to (8), we have

$$(\Delta - c^2)^{n+1} (\Delta + \lambda^2) \Phi_n^h = (-2)^{n+1} \pi c^{2n} n! \delta(x, \xi).$$

We can summarize the particular solution of the Helmholtz equation using Matern function in 2D in the following theorem.

Theorem 5 *The particular solution of the following Helmholtz equation in 2D*

$$(\Delta + \lambda^2)\Phi_n^h = (cr)^n K_n(cr) \quad (29)$$

is given by

$$\Phi_n^h(r; c, \lambda) = 2^{n+1} c^{2n} \pi n! \left(\frac{(Y_0(\lambda r)/4 - G_1(cr))}{(\lambda^2 + c^2)^{n+1}} + \sum_{i=2}^{n+1} \frac{(-1)^i G_i(cr)}{(\lambda^2 + c^2)^{n+2-i}} \right) \quad (30)$$

and

$$\lim_{r \rightarrow 0} \Phi_n^h(r; c, \lambda) = 2^n c^{2n} n! \left(\frac{\ln(\lambda/c)}{(\lambda^2 + c^2)^{n+1}} + \sum_{i=2}^{n+1} \frac{1}{2(\lambda^2 + c^2)^{n+2-i} (i-1) c^{2(i-1)}} \right) \quad (31)$$

where

$$G_i(cr) = \frac{(-1)^i}{2^i \pi (i-1)!} \left(\frac{r}{c} \right)^{i-1} K_{i-1}(cr)$$

in which K_ν is the modified Bessel function of the second kind of order ν .

3 Particular solution for Helmholtz-type equations in 3D

For 3D case, the Laplacian Δ is defined as follows

$$\Delta = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr}. \quad (32)$$

In this case, the Matern function with positive integer order could not be annihilated by the differential operator $(\Delta - c^2)^{n+1}$ as shown in the 2D case and the annihilator method mentioned above could not be applied for the 3D case. We observe the Matern basis function in 2D is related to the fundamental solution G_n in (3). For the 3D case, the fundamental solution of (3) is given by

$$G_n(cr) = \frac{(-1)^n}{2^{n+1/2} \pi^{3/2} (n-1)!} \left(\frac{r}{c}\right)^{n-3/2} K_{n-3/2}(cr). \quad (33)$$

Hence, we will choose the Matern basis function with positive integer $n \geq 2$ in the form

$$\phi_{n-3/2}(r) = (cr)^{n-3/2} K_{n-3/2}(cr), \quad (34)$$

in the 3D case. In the literature, it is known that the Matern function with order $n-3/2$ can be simplified to the product of exponential basis function and radial power function. Some examples for various $n-3/2$ are:

$$\begin{aligned} n = 2, \quad \phi_{1/2}(r) &= \frac{\sqrt{2\pi}}{2} \exp(-cr), \\ n = 3, \quad \phi_{3/2}(r) &= \frac{\sqrt{2\pi}}{2} (1+cr) \exp(-cr), \\ n = 4, \quad \phi_{5/2}(r) &= \frac{\sqrt{2\pi}}{2} (3+3cr+c^2r^2) \exp(-cr). \end{aligned}$$

In this subsection, our goal is to derive the particular solution of the modified Helmholtz equation using $\phi_{n-3/2}(r)$, $n \geq 2$, in 3D:

$$(\Delta - \lambda^2)\Phi_n^{mh} = (cr)^{n-3/2} K_{n-3/2}(cr). \quad (35)$$

Similar to (11) in the proof of Lemma 1, we have

$$\begin{aligned} \Delta(cr)^{n-3/2} K_{n-3/2}(cr) &= \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} (cr)^{n-3/2} K_{n-3/2}(cr) \right) \\ &= \frac{1}{r^2} \frac{d}{dr} (-rc^2 (cr)^{n-3/2} K_{n-5/2}(cr)) \\ &= \frac{1}{r^2} \frac{d}{dr} (-c^2 r^3 \cdot (cr)^{n-5/2} K_{n-5/2}(cr)) \\ &= -3c^2 (cr)^{n-5/2} K_{n-5/2}(cr) + c^3 r (cr)^{n-5/2} K_{n-7/2}(cr) \quad (36) \end{aligned}$$

From (10), we have the following identity

$$(cr)K_{n-7/2}(cr) = (cr)K_{n-3/2}(cr) - (2n-5)K_{n-5/2}(cr). \quad (37)$$

From (36) and (37), after some algebraic manipulation, we have

$$(\Delta - c^2)(cr)^{n-3/2}K_{n-3/2}(cr) = -2(n-1)c^2(cr)^{n-5/2}K_{n-5/2}(cr). \quad (38)$$

It follows that

$$(\Delta - c^2)^n(cr)^{n-3/2}K_{n-3/2}(cr) = (-1)^n 2^n c^{2n-3} (n-1)! \pi \sqrt{2\pi} \delta(x, \xi) \quad (39)$$

which implies

$$\Phi_n^{mh} = (-1)^n 2^n c^{2n-3} (n-1)! \pi \sqrt{2\pi} \mathbb{G}_n(r; c, \lambda) \quad (40)$$

where $\mathbb{G}_n(r; c, \lambda)$ is the fundamental solution associated with the differential operator $(\Delta - c^2)^n(\Delta - \lambda^2)$.

Applying the Lemma 1 and Theorem 3, we have the following theorem.

Theorem 6 *Let n be the positive integer with $n \geq 2$. The particular solution Φ_n^{mh} of the following modified Helmholtz equation in 3D*

$$(\Delta - \lambda^2)\Phi_n^{mh} = (cr)^{n-3/2}K_{n-3/2}(cr)$$

is given by

$$\Phi_n^{mh} = (-1)^n 2^n c^{2n-3} (n-1)! \pi \sqrt{2\pi} \left(\frac{G_1(\lambda r) - G_1(cr)}{(\lambda^2 - c^2)^n} - \sum_{i=2}^n \frac{G_i(cr)}{(\lambda^2 - c^2)^{n+1-i}} \right) \quad (41)$$

where

$$G_i(cr) = \frac{(-1)^i}{2^{i+1/2} \pi^{3/2} (i-1)!} \left(\frac{r}{c} \right)^{i-3/2} K_{i-3/2}(cr) \quad (42)$$

and

$$\lim_{r \rightarrow 0} \Phi_n^{mh} = (-1)^n 2^n c^{2n-3} (n-1)! \sqrt{2\pi} \left(\frac{\lambda - c}{4(\lambda^2 - c^2)^n} - \sum_{i=2}^n \frac{(-1)^i c^{3-2i} \Gamma(i-3/2)}{8\sqrt{\pi} (i-1)! (\lambda^2 - c^2)^{n+1-i}} \right) \quad (43)$$

where $\Gamma(x)$ is the Gamma function.

For the particular solution of the Helmholtz equation in 3D, we can obtain the following theorem in a similar way as the above theorem.

Theorem 7 *Let n be the positive integer with $n \geq 2$. The particular solution Φ_n^h of the following Helmholtz equation in 3D*

$$(\Delta + \lambda^2)\Phi_n^h = (cr)^{n-3/2}K_{n-3/2}(cr) \quad (44)$$

is given by

$$\Phi_n^h = (-2)^n c^{2n-3} (n-1)! \sqrt{2\pi^3} \left(\sum_{i=1}^n \frac{(-1)^{n+i} G_i(cr)}{(\lambda^2 + c^2)^{n+1-i}} + \frac{(-1)^{n+1} (\cos(\lambda r))}{4\pi r (\lambda^2 + c^2)^n} \right) \quad (45)$$

and

$$\lim_{r \rightarrow 0} \Phi_n^h = (-2)^n (n-1)! \sqrt{2} c^{2n-3} \left(\sum_{i=1}^n \frac{(-1)^n \Gamma(i-3/2) c^{3-2i}}{8(i-1)! (\lambda^2 + c^2)^{n+1-i}} \right) \quad (46)$$

where

$$G_i(cr) = \frac{(-1)^i}{2^{i+1/2} \pi^{3/2} (i-1)!} \left(\frac{r}{c} \right)^{i-3/2} K_{i-3/2}(cr), \quad (47)$$

in which K_ν is the modified Bessel function of the second kind of order ν and $\Gamma(x)$ is the Gamma function.

4 Particular solution for products of Helmholtz-type operators

In this section we will extend the technique we developed to the products of Helmholtz-type operator based on the particular solutions we derived in the previous section. Consider 2D case

$$(\Delta - \lambda_1^2)^{k_1} (\Delta - \lambda_2^2)^{k_2} \cdots (\Delta - \lambda_m^2)^{k_m} \Phi(r) = (cr)^n K_n(cr) \quad (48)$$

where $k_i, i = 1, 2, \dots, m$ are non-negative integers, $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_m$ are positive real number. To find the particular solution Φ in (60), we can apply the annihilator method using Lemma 1. Then we have

$$\begin{aligned} (\Delta - c^2)^{n+1} (\Delta - \lambda_1^2)^{k_1} (\Delta - \lambda_2^2)^{k_2} \cdots (\Delta - \lambda_n^2)^{k_n} \Phi(r) &= (\Delta - c^2)^{n+1} (cr)^n K_n(cr) \\ &= (-2)^{n+1} \pi c^{2n} n! \delta(x, \xi). \end{aligned}$$

We can then follow the same procedure as shown in Section 2 to find the particular solution $\dot{\Phi}$.

For example, let us consider the following equation

$$(\Delta - 4)^2 (\Delta + 9) \Phi(r) = (cr)^2 K_2(cr). \quad (49)$$

It follows that

$$(\Delta - c^2)^3 (\Delta - 4)^2 (\Delta + 9) \Phi(r) = -16\pi c^4 \delta(x, \xi).$$

By partial fraction, we have

$$\begin{aligned} \frac{1}{(\xi - c^2)^3 (\xi - 4)^2 (\xi + 9)} &= \frac{c^2 - 43}{\alpha^4} \frac{1}{(\xi - 4)} + \frac{6c^4 + 56c^2 + 187}{\alpha^4 \beta^3} \frac{1}{(\xi - c^2)} - \frac{1}{169\beta^3} \frac{1}{(\xi + 9)} \\ &\quad - \frac{1}{13\alpha^3} \frac{1}{(\xi - 4)^2} - \frac{3c^2 + 14}{\alpha^3 \beta^2} \frac{1}{(\xi - c^2)^2} + \frac{1}{\alpha^2 \beta^3} \frac{1}{(\xi - c^2)^3} \end{aligned}$$

where $\alpha = (c^2 - 4)$, $\beta = (c^2 + 9)$. From Theorem 3, we obtain

$$\begin{aligned} \Phi(r) = & -16\pi c^4 \left(\frac{c^2 - 43}{\alpha^4} G_1(2r) + \frac{6c^4 + 56c^2 + 187}{\alpha^4 \beta^3} G_1(cr) - \frac{1}{169\beta^3} \left(\frac{Y_0(3r)}{4} \right) \right. \\ & \left. - \frac{1}{13\alpha^3} G_2(2r) - \frac{3c^2 + 14}{\alpha^3 \beta^2} G_2(cr) + \frac{1}{\alpha^2 \beta^3} G_3(cr) \right) \end{aligned}$$

where Y_0 is the Bessel function of the second kind of order zero and $G_i, i = 1, 2, 3$, are defined in (28). However, the obtained Φ is quite lengthy and tedious. For the special case $k_1 = k_2 = \dots = k_m = 1$ in (48), we can adopt the technique introduced in Reference [9] to obtain the simple form of Φ . For example, consider

$$(\Delta - \lambda_1^2) (\Delta + \lambda_2^2) \Phi(r) = \phi(r) \quad (50)$$

where ϕ is a Matern function of order n . From Lemma 2, we decompose the following product by the partial fraction

$$\frac{1}{(\xi - \lambda_1^2)(\xi + \lambda_2^2)} = \frac{1}{(\lambda_1^2 + \lambda_2^2)} \left(\frac{1}{(\xi - \lambda_1^2)} - \frac{1}{(\xi + \lambda_2^2)} \right). \quad (51)$$

From Theorem 1 in [9], we have

$$\Phi(r) = \frac{\Phi_n^{mh}(r) - \Phi_n^h(r)}{(\lambda_1^2 + \lambda_2^2)} \quad (52)$$

where $\Phi_n^{mh}(r)$ and $\Phi_n^h(r)$ are given in Theorem 4 and Theorem 5 respectively.

5 Numerical results

To validate the particular solution we derived in the previous sections, we apply the method of particular solutions (MPS) using Matern function with various orders for solving modified Helmholtz equation in 2D and 3D, and product of Helmholtz and modified Helmholtz equation. The numerical accuracy is closely related to the selection of the shape parameter. In this section, we choose the algorithm of leave-one-out cross validation (LOOCV) [4, 8] for selecting a good shape parameter or so-called sub-optimal shape parameter. The MATLAB code on LOOCV can be found in [4]. We refer the interest reader to Reference [4] for further details. The optimal shape parameter of Matern function is denoted as C_{opt} and the sub-optimal shape parameter using LOOCV is denoted as C_{loocv} . Unlike C_{opt} which requires the exact solution in the selection process, a major advantage of LOOCV is that we do not need to know the exact solution for the determination of C_{loocv} . Since LOOCV is a statistical method, the more collocation points we have, the more accurate C_{loocv} we can obtain.

In this section, estimates of the numerical accuracy are based on the Root Mean Squared Error (RMSE)

$$RMSE = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} (\hat{u}(x_i, y_i) - u(x_i, y_i))^2}, \quad (53)$$

where u and \hat{u} are exact and approximate solution respectively, and N_t is the number of test points.

Example 1 We consider the following modified Helmholtz equation with Dirichlet boundary condition in 2D

$$(\Delta - \lambda^2) u(x, y) = f(x, y), \quad (x, y) \in \Omega, \quad (54)$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \quad (55)$$

where f and g are chosen according to the following exact solution:

$$u(x, y) = e^{x+y}.$$

The computational domain Ω is bounded by the curve defined by the following parametric equation:

$$\partial\Omega = \{(x, y) \mid x = \rho \cos \theta, y = \rho \sin \theta, 0 \leq \theta \leq 2\pi\},$$

where

$$\rho = \left(\cos(3\theta) + \sqrt{2 - \sin^2(3\theta)} \right)^{\frac{1}{3}}.$$

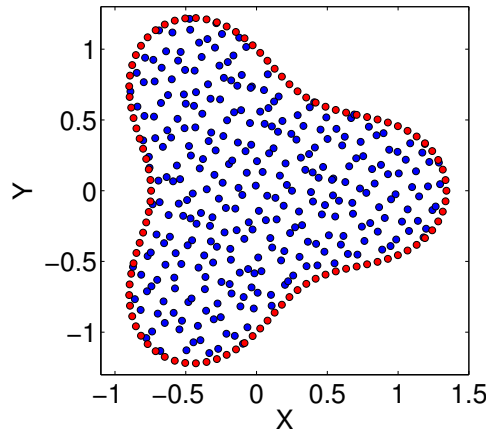


Figure 1: The profile of the computational domain.

We apply the method of particular solutions (MPS) to solve the above problem. For implementation, we require the particular solution derived in the previous section using Matern functions with various orders as the basis functions. Furthermore, we choose 300 quasi-random interior points, 100 equally distributed boundary points, 150 randomly distributed interior points as the test points, and $\lambda = 50$ in our numerical implementation. The profile of computational domain with above mentioned interior and boundary points is shown in Fig. 1. The overall profile of maximum absolute errors with various orders of Matern function versus the shape parameters is shown in Fig. 2. We observe that the lower order of Matern function is more stable but less accurate and vice versa.

This is due to the ill-conditioning of the higher order Matern function. As shown in Fig. 2, the curve of order 3 is much smoother than the curve of order 8. We also notice that the optimal shape parameter of higher order Matern function is larger than the one with lower order.

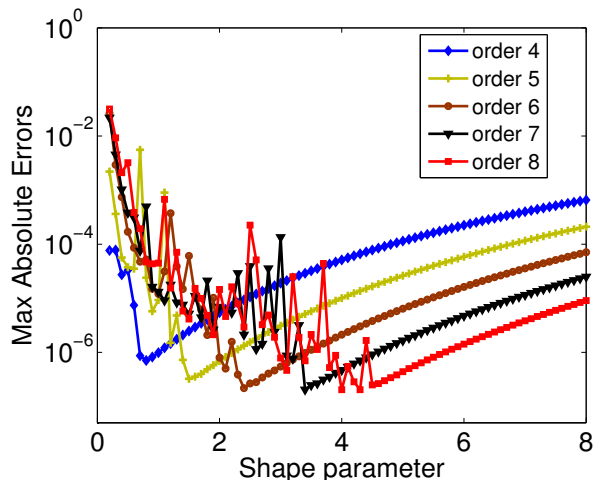


Figure 2: The profiles of error versus shape parameter for various orders of Matern function.

To demonstrate the stability of the Matern function, we choose Matern function with order 6 and various initial search intervals to search the sub-optimal and optimal shape parameter using LOOCV and exact solution respectively. Note that in this paper the optimal shape parameter is obtained in the sense that the exact solution is being used and then the MATLAB code 'fminbnd' is used to search the minimal error in a search interval $[min, max]$ as shown in the column one of Table 1. As we can see in this table, the accuracy obtained using LOOCV is as good as the one produced by the known exact solution irrespective to the range of the search interval. This means our algorithm is numerically very stable. We obtain the similar results using various orders of Matern functions. Hence, we can comfortably choose a sufficiently large initial search interval for our forthcoming numerical computation.

The results in Table 2 are obtained using various orders of Matern functions with an initial search interval $[0, 8]$. The accuracy obtained using LOOCV is again as good as the one using exact solution. For higher order Matern function, the computational cost is higher since more terms of evaluating the particular solution in (26) are required.

Example 2 In this example, we consider the same modified Helmholtz equation (54) and Dirichlet boundary condition (55) in 3D. f and g are chosen according to the following exact solution:

$$u(x, y) = e^{x+y+z}.$$

The computational domain Ω is the unit sphere.

In the numerical implementation, we choose 895 uniformly distributed interior points, 500 uniformly distributed surface points, 300 randomly distributed interior points as

[min, max]	C_{loocv}	$RMSE_{loocv}$	C_{opt}	$RMSE_{opt}$
[0, 6]	2.338	$2.329E - 07$	2.427	$2.161E - 07$
[0, 8]	2.286	$2.242E - 07$	2.363	$2.135E - 07$
[0, 10]	2.380	$2.370E - 07$	2.435	$2.230E - 07$
[0, 12]	2.365	$2.363E - 07$	2.383	$2.378E - 07$
[0, 14]	2.300	$2.894E - 07$	2.622	$2.926E - 07$
[0, 16]	2.294	$2.146E - 07$	2.369	$2.215E - 07$
[0, 18]	2.311	$2.643E - 07$	2.428	$2.145E - 07$
[0, 20]	2.308	$2.204E - 07$	2.349	$2.088E - 07$

Table 1: Example 1: Sub-optimal shape parameters using LOOCV and the optimal shape parameter using exact solution for various initial search interval for Matern function with order 6.

order	C_{loocv}	$RMSE_{loocv}$	C_{opt}	$RMSE_{opt}$
4	0.680	$8.198E - 07$	0.753	$6.414E - 07$
5	1.399	$2.860E - 07$	1.503	$2.922E - 07$
6	2.286	$2.242E - 07$	2.363	$2.135E - 07$
7	3.316	$2.010E - 07$	3.448	$2.104E - 07$
8	4.474	$2.545E - 07$	4.452	$2.501E - 07$

Table 2: Example 1: Sub-optimal shape parameters using LOOCV and the optimal shape parameter using exact solution for various orders of Matern function with search interval [0, 8].

the test points, and $\lambda = 50$. The profiles of maximum absolute errors versus shape parameters for various orders of Matern functions are shown in Fig. 3. We observe the similar pattern between Fig. 2 and Fig. 3.

The results in Table 3 are obtained using various orders of Matern functions with an initial search interval [0, 10]. Similar to the 2D case in the last example, the results shown in Table 3 are as accurate as the results using exact solution.

Example 3 In this example, we consider the following plate vibration equation in 2D

$$\begin{cases} (\Delta^2 - \lambda^4)u = f(x, y), & (x, y) \in \Omega, \\ u = g_1(x, y), & (x, y) \in \partial\Omega, \\ \Delta u = g_2(x, y), & (x, y) \in \partial\Omega. \end{cases} \quad (56)$$

where f, g_1 and g_2 are chosen according to the following exact solution:

$$u(x, y) = \sin(\pi x) \cos\left(\frac{\pi}{2}y\right).$$

The boundary $\partial\Omega$ is defined by the following parametric equation:

$$\partial\Omega = \left\{ (x, y) \mid x = \rho(t) \cos\left(t + \frac{1}{2} \sin(5t)\right), y = \rho(t) \sin\left(t + \frac{1}{2} \sin(5t)\right), 0 \leq \theta < 2\pi \right\} \quad (57)$$

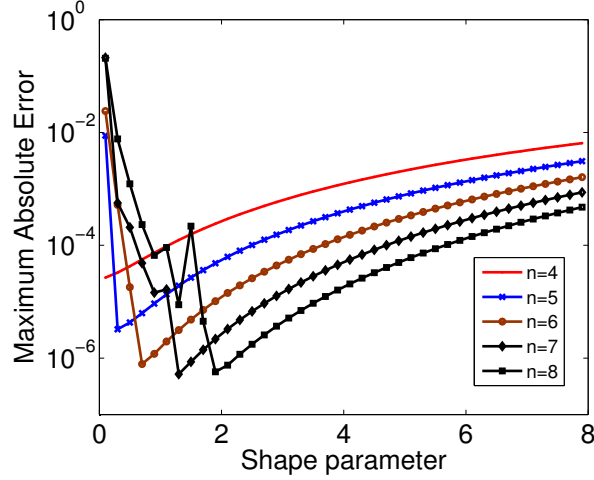


Figure 3: The profile of the computational domain in 3D.

order	C_{loocv}	$RMSE_{loocv}$	C_{opt}	$RMSE_{opt}$
4	0.053	$2.785E - 05$	0.083	$2.631E - 05$
5	0.268	$4.071E - 06$	0.323	$3.187E - 06$
6	0.704	$7.822E - 07$	0.700	$7.761E - 07$
7	1.251	$5.242E - 07$	1.214	$4.611E - 07$
8	1.857	$4.130E - 07$	1.748	$3.821E - 07$

Table 3: Example 2: Sub-optimal shape parameters using LOOCV and the optimal shape parameter using exact solution for various orders of Matern function with search interval $[0, 10]$.

where

$$\rho(t) = \frac{1}{2} \left(2 + \frac{1}{2} \sin(5t) \right). \quad (58)$$

The computational domain Ω is a gear-shape domain as shown in Figure 4.

First, we examine the particular solution Φ of the following equation

$$(\Delta^2 - \lambda^4)\Phi = (cr)^n K_n(cr). \quad (59)$$

The left hand side of the above equation can be factored into the product of Helmholtz and modified Helmholtz operators

$$(\Delta - \lambda^2)(\Delta + \lambda^2)\Phi(r) = (cr)^n K_n(cr). \quad (60)$$

From (50), let $\lambda_1 = \lambda_2 = \lambda$, we have

$$\Phi(r) = \frac{1}{2\lambda^2} (\Phi_n^{mh}(r) - \Phi_n^h(r)) \quad (61)$$

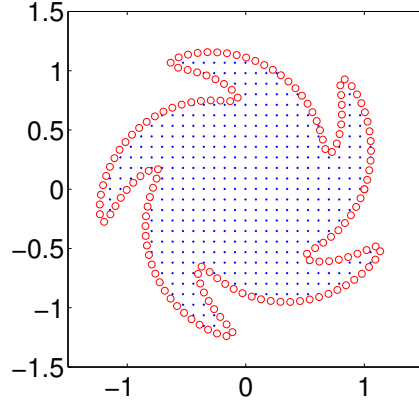


Figure 4: The profile of the gear-shape domain.

where $\Phi_n^{mh}(r)$ and $\Phi_n^h(r)$ are available from (26) and (30) respectively. In the implementation of Laplace boundary condition in (56), we need to find

$$\Delta\Phi = \frac{1}{2\lambda^2} (\Delta\Phi_n^{mh}(r) - \Delta\Phi_n^h(r)). \quad (62)$$

The direct derivation of $\Delta\Phi_n^{mh}(r)$ and $\Delta\Phi_n^h(r)$ using (26) and (30) respectively are very tedious. Instead, we observe that

$$(\Delta - \lambda^2)\Phi_n^{mh}(r) = (cr)^n K_n(cr)$$

which implies

$$\Delta\Phi_n^{mh}(r) = \lambda^2\Phi_n^{mh}(r) + (cr)^n K_n(cr). \quad (63)$$

Similarly,

$$\Delta\Phi_n^h(r) = -\lambda^2\Phi_n^h(r) + (cr)^n K_n(cr). \quad (64)$$

From (62) – (64), $\Delta\Phi_n^h(r)$ can be obtained effortlessly as follows

$$\Delta\Phi = \frac{1}{2} (\Phi_n^{mh}(r) + \Phi_n^h(r)). \quad (65)$$

In the numerical implementation, we choose $\lambda = 50$, 412 interior points, 150 boundary points, both uniformly distributed, as shown in Fig. 4. We measure the accuracy on 119 quasi-randomly distributed test points using LOOCV. In Table 4, we show that the numerical results for various order of Matern function using two initial search intervals: $[0,10]$ and $[0,20]$. Note that we modified LOOCV slightly by using only the interior points only in the search of the sub-optimal shape parameter. In this way, the resultant matrix is square and in the code of LOOCV we can replace the MATLAB code 'pinv' by 'inv'. To our experience, the generalized matrix solver 'pinv' in LOOCV normally overestimates the shape parameter than using 'inv'.

order	[min,max]	C_{loocv}	$RMSE_{loocv}$
4	[0,10]	0.658	$7.755E - 05$
	[0,20]	0.702	$5.140E - 05$
5	[0,10]	1.418	$2.457E - 05$
	[0,20]	1.435	$2.168E - 05$
6	[0,10]	2.382	$7.148E - 06$
	[0,20]	1.967	$7.456E - 06$
7	[0,10]	2.360	$3.740E - 06$
	[0,20]	3.221	$6.638E - 06$
8	[0,10]	3.819	$5.206E - 06$
	[0,20]	4.102	$4.170E - 06$
9	[0,10]	4.974	$3.046E - 06$
	[0,20]	4.609	$3.075E - 06$
10	[0,10]	6.316	$2.286E - 06$
	[0,20]	5.759	$2.352E - 06$

Table 4: Example 3: Sub-optimal shape parameters using LOOCV for various orders of Matern function using two different initial search intervals.

6 Conclusion

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