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ON THE KAUFFMAN BRACKET SKEIN MODULE
OF THE QUATERNIONIC MANIFOLD

PATRICK M. GILMER AND JOHN M. HARRIS

ABSTRACT. We use recoupling theory to study the Kauffman bracket skein module of the quaternionic manifold over $\mathbb{Z}[A^{\pm 1}]$ localized by inverting all the cyclotomic polynomials. We prove that the skein module is spanned by five elements. Using the quantum invariants of these skein elements and the $\mathbb{Z}_2$-homology of the manifold, we determine that they are linearly independent.

1. INTRODUCTION

In [K], Kauffman presents an elegant construction of the Jones polynomial, an invariant of oriented links in S^3, by constructing a new invariant, the Kauffman bracket polynomial. The Kauffman bracket is an invariant of unoriented framed links in S^3, defined by the following skein relations:

$$(1) \quad \left\langle \diagup\!\!\!\!\diagdown \right\rangle = A \left\langle)(\right\rangle + A^{-1} \left\langle \asymp \right\rangle$$

$$(2) \quad < L \cup \text{unknot} > = (-A^{-2} - A^2) < L >$$

For the invariant to be well-defined, one also must normalize it by choosing a value for the empty link. $< \text{empty link} > = 1$, for instance.

Alternatively, we can use the skein relations to construct a module of equivalence classes of links in S^3, or, for that matter, in any oriented 3-manifold. See Przytycki ([P1]) and Turaev ([T]).

Definition 1. *Let N be an oriented 3-manifold, and let R be a commutative ring with identity, with a specified unit A. The Kauffman bracket skein module of N, denoted $S(N; R, A)$, or simply $S(N)$, is the free R-module generated by the framed isotopy classes of unoriented links in N, including the empty link, quotiented by the skein relations which define the Kauffman bracket.*

Since every crossing and unknot can be eliminated from a link in S^3 by the skein relations, $S(S^3)$ is generated by the empty link. Kauffman's argument that his bracket polynomial is well-defined shows that $S(S^3)$ is free on the empty link.

Hoste and Przytycki have, in fact, computed the skein modules of all of the closed, oriented manifolds of genus 1: $S(L(p, q))$, which is free on $\lfloor \frac{p}{2} \rfloor + 1$ generators ([HP1]), and $S(S^1 \times S^2) \cong \mathbb{Z}[A^{\pm 1}] \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4}))$, ([HP2]).

They have also computed the skein modules of I-bundles over surfaces ([HP1]).

Additionally, Bullock has found a presentation for the complement of a $(2, q)$ torus knot in [B1], and has determined whether or not the skein module of the
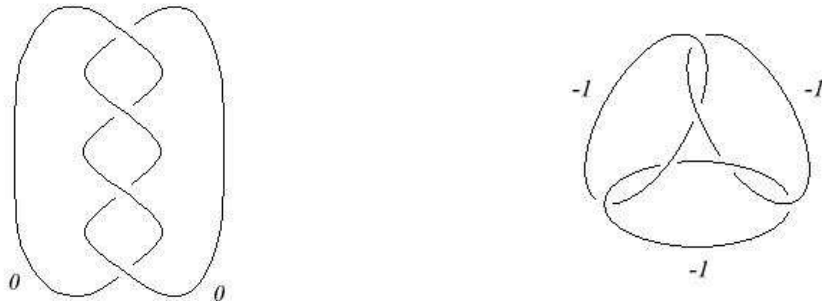
FIGURE 1. Surgery descriptions of the quaternionic manifold

result of integral surgery on a trefoil is finitely generated in [B2]. Bullock and LoFaro have computed the skein module of the exteriors of the twist knots in [BL].

All of the computations mentioned above have been carried out with $R = \mathbb{Z}[A^{\pm 1}]$.

Let M be the quaternionic manifold, S^3 quotiented by the action of the quaternionic group Q_8: M can be obtained by identifying opposite faces of a cube with one-quarter twists.

M is the three-fold cover of S^3 branched over the trefoil, and is an irreducible manifold of genus 2. Rolfsen gives surgery descriptions of this manifold in [Rol]. See Figure 1.

Let $\mathcal{R}$ denote $\mathbb{Z}[A^{\pm 1}]$, localized by inverting the multiplicative set generated by the elements of the set $\{A^n - 1 | n \in \mathbb{Z}^+\}$. Equivalently, we could say that we are localizing by inverting all cyclotomic polynomials in A or all quantum integers (see below for the definition of a quantum integer).

Theorem 2. $S(M; \mathcal{R}, A)) \approx \mathcal{R}^5$.

Corollary 3. $S(M; \mathbb{Z}[A^{\pm 1}], A))$ *has at most cyclotomic torsion, i.e. every torsion element is annihilated by a product of cyclotomic polynomials.*

In fact, we conjecture that $S(M; \mathbb{Z}[A^{\pm 1}], A))$ is torsion-free. The quaternionic manifold is the first closed, irreducible, genus-two 3-manifold whose Kauffman bracket skein module (over a localization of $\mathbb{Z}[A^{\pm 1}]$) has been computed.

From now on, we consider all skeins to be over $\mathcal{R}$ and denote $S(N; \mathcal{R}, A)$ by $S(N)$.

Before we proceed, we note that the defining relations of the Kauffman bracket skein module respect $\mathbb{Z}_2$-homology. Since $H_1(M; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $S(M)$ is a direct sum of four submodules $S_1(M)$, $S_i(M)$, $S_j(M)$, and $S_k(M)$.

Furthermore, the permutation group on three elements acts on the manifold permuting the three non-trivial homology classes. Thus $S_i(M)$, $S_j(M)$, and $S_k(M)$ are isomorphic. We will see that $S_1(M)$ is isomorphic to $\mathcal{R}^2$, and that $S_i(M)$, $S_j(M)$, and $S_k(M)$ are isomorphic to $\mathcal{R}$.
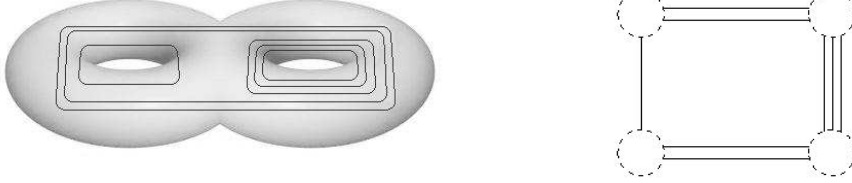
FIGURE 2. The link xy^2z^3 in the standard diagram and the cut-open diagram of H_2

where $\Delta_n =$

FIGURE 3. Definition of Jones-Wenzl idempotents

2. PRELIMINARY REMARKS

We draw H_2 as a ball before the handles are attached from top to bottom along the dashed lines. Equivalently, one can view the diagrams as pictures of handlebodies whose handles have been cut.

The standard basis B for the Kauffman bracket skein module of the solid handlebody H_g of genus g is the collection of all links without crossings in the standard diagram of H_g. For H_2, we use Bullock's algebraic notation $x^i y^j z^k$ ([B2]). See Figure 2.

Definition 4. *If $|a-b| \leq c \leq a+b$ and $a+b+c \equiv 0 \bmod 2$, then the triple $\{a, b, c\}$ is said to be admissible.*

An arc labelled n represents the nth Jones-Wenzl idempotent f_n: f_0 is an empty tangle, f_1 is a single arc, and f_n is a linear combination of (n, n)-tangles defined by the recursive relation in Figure 3.

Note that, for f_n to be defined, Δ_{k-1} must be invertible in $\mathcal{R}$ for all $k \leq n$. This holds, as $\Delta_{k-1} = (-1)^{k-1}[k]$ where the quantum integer $[k] = A^{2-2k} + A^{6-2k} + \cdots + A^{2n-k} + A^{2k-2}$. In Figure 3, the presence of a idempotent is indicated by a small rectangle. Hereafter, the rectangles will be dropped. Moreover, edges representing f_1 will be left unlabelled.

It is convenient to extend our view of skein modules to include banded trivalent graphs, whose edges are labelled so that at each vertex, the labels form an admissible
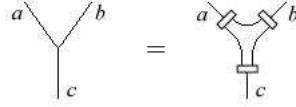
FIGURE 4. Trivalent graphs represent linear combinations of links

triple. See [BHMV2, 4.5] for the precise definition of a banded trivalent graph: all trivalent graphs we discuss will be assumed to have such a banding, coming from a regular neighborhood in the surface in which they are drawn. These graphs represent linear combinations of links, as expressed in Figure 4.

In [MV], Masbaum and Vogel describe an algorithm for reducing every trivalent graph in a 3-ball to a scalar multiple of the empty link, and trivalent graphs are often used to denote this scalar. In addition to the labelled unknot in Figure 3, two of these scalars appear frequently enough to merit symbols:

$$= \theta(a, b, c) \text{ and}$$

$$= Tet \begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}.$$

Additionally, we have the following useful local moves:

$$= (-A)^{a(a+2)}$$

,

$$= \lambda_c^{a\,b}$$

,

$$= \delta_d^c \, \frac{\theta(a, b, c)}{\Delta_c}$$

, and

$$
\text{(graph)} = \begin{cases} \dfrac{Tet\begin{bmatrix} a & b & e \\ c & d & f \end{bmatrix}}{\theta(a,d,e)} \quad \text{(graph)} & \text{, if } (a,d,e) \text{ admissible} \\ \\ 0 & \text{, otherwise.} \end{cases}
$$

Explicit formulas for these scalars in terms of A are given in [MV] and [KL]. The only skein-theoretic derivation of the formula for the tet is given in [MV]. Our notation is the same as in [KL].

Working with trivalent graphs is often much easier than working directly with links, thanks mainly to the following identity:

Theorem 5. *(Fusion Formula)*

$$
a \Big| \quad \Big| b \quad = \sum_i \frac{\Delta_i}{\theta(a,b,i)} \quad \text{(graph)} \quad ,
$$

where the sum is over all admissible labellings.

We also have the following well-known theorem. See, for example, [BFK].

Theorem 6. *(Sphere Lemma)*
If a sphere intersects a skein element in exactly 1 labelled arc , then

$$
\text{(graph with } a \text{)} = 0.
$$

The idea of using the fusion formula in conjunction with the sphere lemma is also well-known. See, for example, [Rob].

Theorem 7. *If a sphere intersects a skein element in exactly 2 labelled arcs, then*

$$
\text{(graph with } a, b \text{)} = \frac{\delta_b^a}{\Delta_a} \quad \text{(graph with } a \text{)} \quad .
$$

If a sphere intersects a skein element in exactly 3 labelled arcs, then

If a sphere intersects a skein element in exactly $n > 3$ arcs, then

where the sum is over all admissible labellings.

Proof. The proof proceeds by induction on n. The case $n = 1$ is the sphere lemma. Using the fusion formula, we can reduce the number of strands by 1 and then apply the inductve hypothesis. The factor resulting from fusion in each term can be absorbed into its coefficient, and for $n > 2$, the coefficient can be rewritten in the desired form with a bigon move. The result follows.   □

Hereafter, we will omit the cumbersome piecewise notation: when $\theta(a, b, c)$ appears in the denominator of an fraction, the fraction is to be taken as zero if (a, b, c) is not admissible. We also extend the Kronecker delta notation as follows: $\delta(a, b, c)$ is 1 when (a, b, c) is admissible and 0 otherwise.

For our work, we will use an alternate basis for $S(H_2)$: the admissible labellings of a trivalent graph which is itself a deformation retract of H_2. We use the notation (a, b, c): see Figure 5. Bullock, Frohman, and Kania-Bartoszynska describe this basis, for $R = \mathbb{C}$ and $|A| \neq 1$, in [BFK].

Note that, for (a, b, c) to be admissibly labelled, $b \leq 2a$, $b \leq 2c$, and b must be even.

Note that $S(S^1 \times S^2) \cong \mathcal{R}$. Przytycki has shown in [P2] that the skein module of the connected sum of two manifolds is the tensor product of their skein modules. Hence, $S((S^1 \times S^2)\#(S^1 \times S^2)) \cong \mathcal{R}$. In [G], the first author defined an $\mathcal{R}$-valued bracket evaluation of admissibly labelled trivalent graphs in a connected sum of copies of $S^1 \times S^2$ which essentially coincides with this isomorphism, using fusion as above for the evaluation.

By gluing a copy of H_2 to itself with orientation reversed along the identity map on the boundaries, we obtain a Hermitian form $<, >: S(H_2) \times S(H_2) \longrightarrow \mathcal{R}$, and $\{(a, b, c)\}$ is orthogonal with respect to this product. (Here $\mathcal{R}$ is equipped with the involution which sends A to A^{-1}.) See Figure 6. In the figure, the graph in the outer, undrawn handlebody has been pushed into the inner handlebody. This form is closely related to the Yang-Mills measure on the skein algebra of a surface
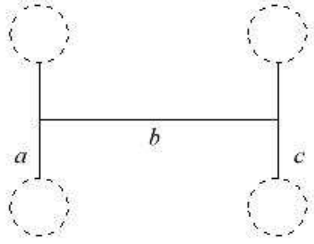
FIGURE 5. The trivalent graph (a, b, c) in H_2

$$= \quad \frac{\delta_x^a \, \delta_y^b \, \delta_z^c}{\Delta_x \, \Delta_y \, \Delta_z}$$

$$= \quad \frac{\delta_x^a \, \delta_y^b \, \delta_z^c \, \theta(x, x, y) \, \theta(y, z, z)}{\Delta_x \, \Delta_y \, \Delta_z}$$

FIGURE 6. Computation of $< (x, y, z), (a, b, c) >$

discussed by Bullock, Frohman, and Kania-Bartoszynska in [BFK]. The first author discussed the form $<, >$ and the orthogonality of our basis of trivalent graphs in a course on quantum topology in the fall of 2001.

We now prove that the alternate basis described above is indeed a basis, as there does not seem to be a proof in the literature. In the proof, we use the principle of well-founded induction:

A well-founded order on a set X is a partial order such that every nonempty subset of X has a minimal element. The principle of well-founded induction states that given a property p defined on a well-founded ordered set X, if p holds for every minimal element of X, and if, for every $y \in X$, p holds for y if p holds for every $x < y$ in X, then p holds for every element of X. See, for example, [Mo].

Proposition 8. $\{(a, b, c)\}$ *is a basis of* $S(H_2)$.

Proof. Let $\leq_B$ denote the partial ordering on the standard basis B defined by

$$x^{i_1} y^{j_1} z^{k_1} \leq_B x^{i_2} y^{j_2} z^{k_2} \Leftrightarrow i_1 \leq i_2, j_1 \leq j_2, \text{ and } k_1 \leq k_2.$$

Since $\leq_B$ is well-founded, we can show that $\{(a, b, c)\}$ spans $S(H_2; \mathcal{R}, A)$ by well-founded induction:

The empty link belongs to $\{(a, b, c)\}$, and so it can certainly be written as a linear combination of elements of $\{(a, b, c)\}$.

Suppose that every $x^{i_1} y^{j_1} z^{k_1} <_B x^i y^j z^k$ can be written as a linear combination of elements of $\{(a, b, c)\}$. $(i + j, 2j, k + j) = x^i y^j z^k +$ a linear combination of lesser terms, so $x^i y^j z^k$ can be expressed as a linear combination of elements in $\{(a, b, c)\}$. Hence, $\{(a, b, c)\}$ spans $S(H_2)$.

Now suppose that a linear combination $\sum C_{x,y,z}(a, b, c) = 0$. Then, for each (x, y, z),

$$C_{x,y,z} < (x, y, z), (x, y, z) >=< \sum C_{x,y,z}(a, b, c), (x, y, z) >= 0.$$

Hence, $C_{x,y,z} = 0$ for each (x, y, z), and so, $\{(a, b, c)\}$ is linearly independent.   □

By viewing a closed 3-manifold in terms of its handle decomposition, we can hope to analyze the structure of its skein module. A closed, oriented, genus-two 3-manifold is built by adding 2 solid cylinders, or 1-handles, to a ball, or 0-handle, and then by attaching 2 thickened disks, or 2-handles, and closed up by adding a 3-handle. Before the 2-handles are attached, the manifold is a solid handlebody. As each 2-handle is added, a set of relations is introduced among the skein elements, namely those obtained from sliding arcs over the newly attached thickened disk. The final 3-handle has no effect on the skein module.

In this way, one obtains a presentation of the skein module. The generators are the basis elements of the solid handlebody, and the relations are given by the ways in which arcs may slide across the 2-handles. When applied to links in the manifolds, this is the most common method for generating a presentation of the module. See, for example, Hoste and Przytycki ([HP1],[HP2]), and Bullock ([B1], [B2].

As Masbaum has pointed out in [M], a good way to insure that one has found a complete set of relations is to use generators of the relative skein modules of the handlebody relative two points on the boundary along each of the attaching curves to index the relations. However, we do not take this route. Instead, we pick relatively simple relations which are not guaranteed to be complete. Then, we use a separate argument to show that the resulting generators are linearly independent.

We take this alternate route because the relations that are obtained following the first route are too difficult to analyze completely. However, the second author [H] did study these relations, and using Mathematica [Wo], found convincing evidence that the skein module of M over the field of rational functions in A should be five-dimensional.

Following the method of Rolfsen applied to the Poincare homology sphere in [Rol], we can construct a Heegaard splitting of the quaternionic manifold from its surgery description. See Figure 7. The curves mark the boundaries of the attached 2-handles.

3. Spanning

In this section, we show the skein module is generated by five elements. To this end, we present in Figures 8, 9, 10, and 11 six sets of slides over the attached 2-handles, yielding relations written in terms of our orthogonal basis.

Note that the diagrams of the second and fifth slides are just the diagrams of the first and fourth, respectively, rotated 180 degrees. Hence, the computations of
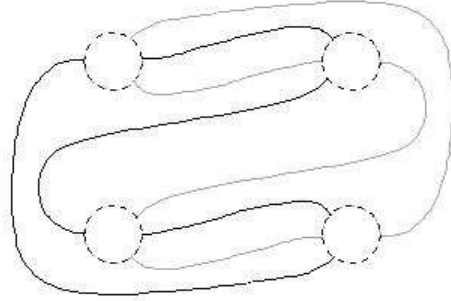
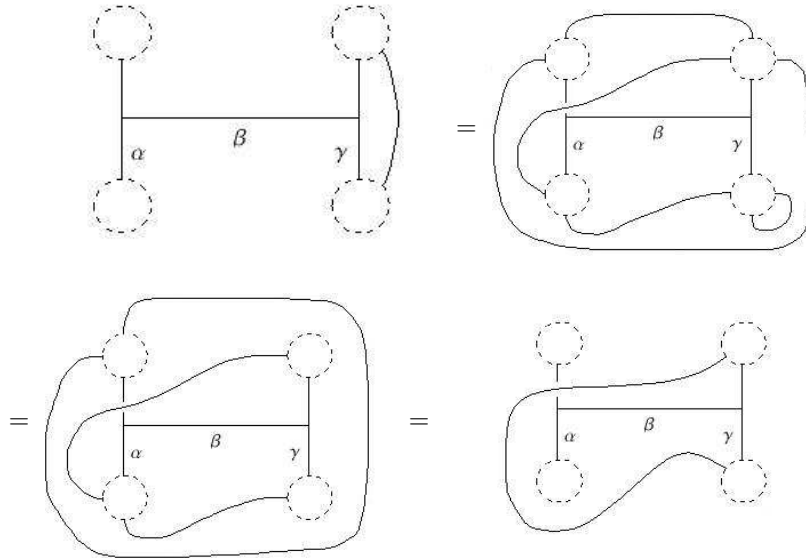FIGURE 7. A Heegaard splitting of the quaternionic manifold

FIGURE 8. Slide 1; Slide 2 is obtained by rotating figure 180º, leaving labels in place

the coefficients for the first and fourth relations yield the coefficients for the second and fifth as well.

In Figures 12 and 13, we demonstrate how the coefficients appearing in Relation 1 are computed using recoupling theory and orthogonality.

In Figures 14, 15, 16, and 17, we present the relations obtained from slides 1, 3, 4, and 6. In what follows, $<< a, b, c >>$ denotes $< (a, b, c), (a, b, c) >$ and terms in sums are taken to be zero if their denominators would be zero. See [H] for a detailed derivation of the later relations from the slides.

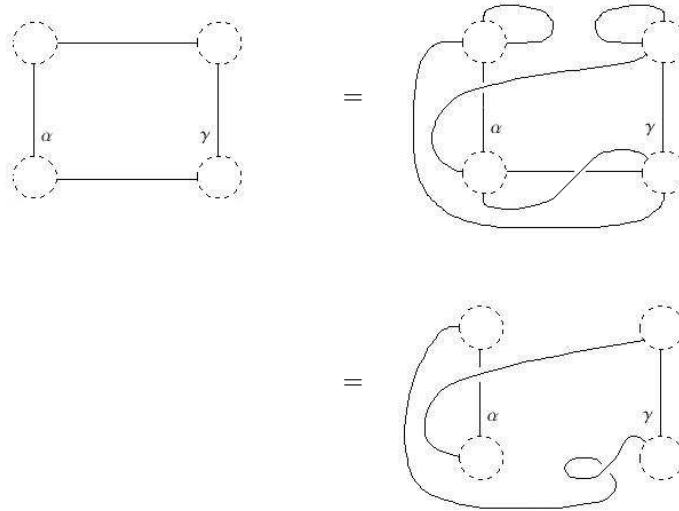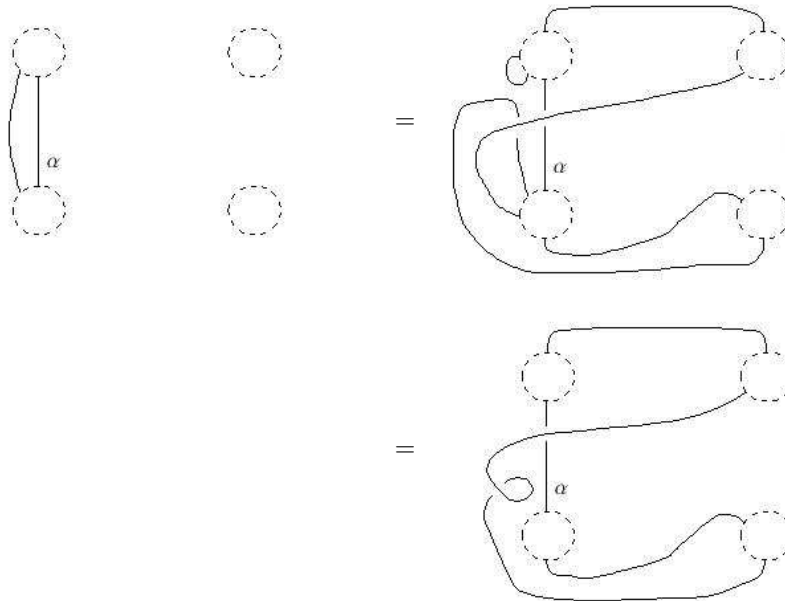Let $r_i = $ (right-hand side of relation i) - (left-hand-side of relation i).

FIGURE 9.   Slide 3

FIGURE 10.   Slide 4; Slide 5 is obtained by rotating figure 180º, leaving labels in place
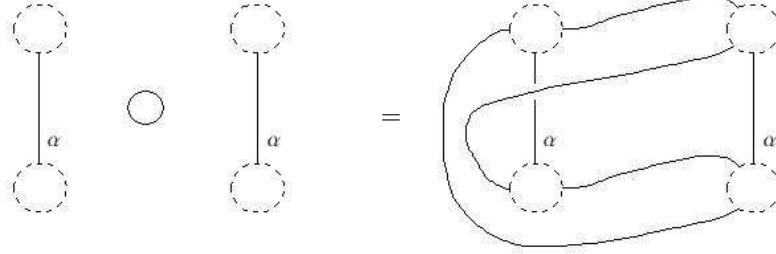
FIGURE 11. Slide 6

FIGURE 12. Computing the coefficients for the left side of Relation 1

Note that, due to admissibility conditions, (a, b, c) can only appear in r_1 if $a = \alpha, b \in \{\beta - 2, \beta, \beta + 2\}$, and $c = \gamma \pm 1$. See Figure 14.

(a, b, c) can only appear in r_2 if $a = \alpha \pm 1, b \in \{\beta - 2, \beta, \beta + 2\}$, and $c = \gamma$.
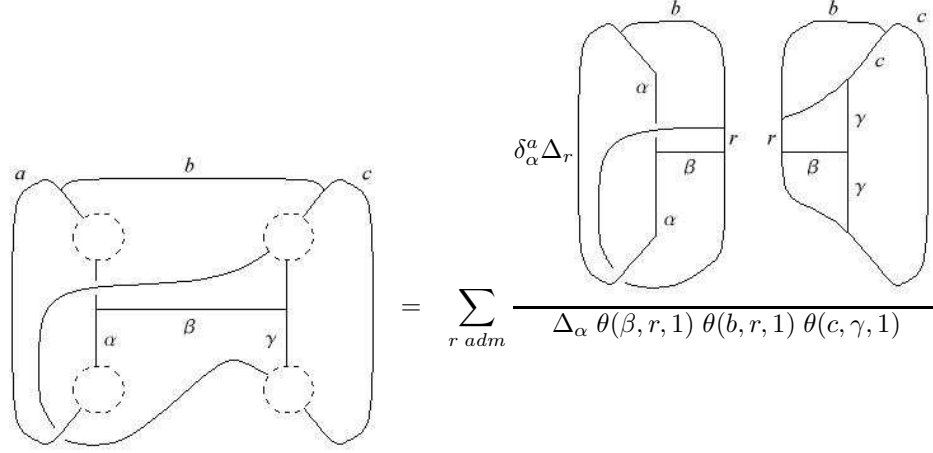
(a, b, c) can only appear in r_3 if $a = \alpha \pm 1, b \in \{0, 2\}$, and $c = \gamma \pm 1$. See Figure 15.

(a, b, c) can only appear in r_4 if $a = \alpha \pm 1, b \in \{0, 2, 4\}$, and $c \in \{0, 2\}$. See Figure 16.

(a, b, c) can only appear in r_5 if $a \in \{0, 2\}, b \in \{0, 2, 4\}$, and $c = \gamma \pm 1$.
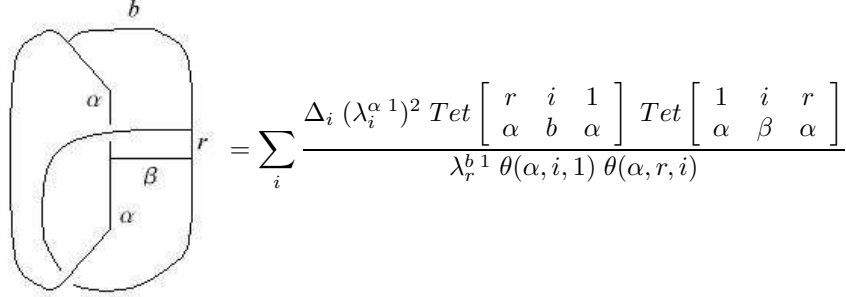
(a, b, c) can only appear in r_6 if $a \in \{\alpha - 2, \alpha, \alpha + 2\}, b \in \{0, 2, 4\}$, and $c \in \{\alpha - 2, \alpha, \alpha + 2\}$. See Figure 17.

Proposition 9. *$S(M)$ is spanned by* $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, *and* $(0, 0, 2)$.
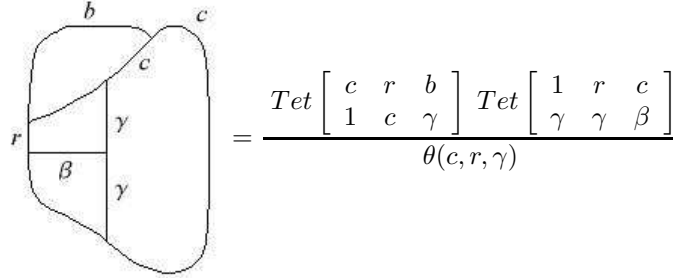
$$= \sum_{r\ adm} \frac{\delta^a_\alpha \Delta_r \cdots}{\Delta_\alpha\ \theta(\beta,r,1)\ \theta(b,r,1)\ \theta(c,\gamma,1)}$$

where

$$= \sum_i \frac{\Delta_i\ (\lambda_i^{\alpha\,1})^2\ Tet\begin{bmatrix} r & i & 1 \\ \alpha & b & \alpha \end{bmatrix} Tet\begin{bmatrix} 1 & i & r \\ \alpha & \beta & \alpha \end{bmatrix}}{\lambda_r^{b\,1}\ \theta(\alpha,i,1)\ \theta(\alpha,r,i)}$$

and

$$= \frac{Tet\begin{bmatrix} c & r & b \\ 1 & c & \gamma \end{bmatrix} Tet\begin{bmatrix} 1 & r & c \\ \gamma & \gamma & \beta \end{bmatrix}}{\theta(c,r,\gamma)}$$

FIGURE 13. Computing the coefficients for the right side of Relation 1

Proof. We proceed by induction on (a,b,c). Using the six sets of relations, we can rewrite (x,y,z) as a linear combination of terms appearing earlier in the following ordering:

$(a',b',c') > (a,b,c)$

- if $m' = \max(a',c') > m = \max(a',c')$, or
- if $m' = m$ and $a' > a$, or
- if $m' = m$, $a' = a$, and $c' > c$, or
- if $m' = m$, $a' = a$, $c' = c$, and $b' > b$.

$$\sum_{a,b,c\ adm} \frac{\delta^a_\alpha\ \delta^b_\beta\ \theta(\alpha,\alpha,\beta)\ Tet\begin{bmatrix} c & \gamma & \beta \\ \gamma & c & 1 \end{bmatrix}}{\Delta_\alpha\ \Delta_\beta\ \theta(c,\gamma,1)<<a,b,c>>}(a,b,c)$$

$$= \sum_{a,b,c\ adm}\sum_r \frac{\delta^a_\alpha\Delta_r\ Tet\begin{bmatrix} c & r & b \\ 1 & c & \gamma \end{bmatrix}\ Tet\begin{bmatrix} 1 & r & c \\ \gamma & \gamma & \beta \end{bmatrix}\sum_i Di}{\Delta_\alpha\ \theta(\beta,r,1)\ \theta(b,r,1)\ \theta(c,\gamma,1)\theta(c,r,\gamma)<<a,b,c>>}(a,b,c)$$

$$\text{where } D_i = \frac{\Delta_i\ (\lambda^\alpha_i\ ^1)^2\ Tet\begin{bmatrix} r & i & 1 \\ \alpha & b & \alpha \end{bmatrix}\ Tet\begin{bmatrix} 1 & i & r \\ \alpha & \beta & \alpha \end{bmatrix}}{\lambda^b_r\ ^1\ \theta(\alpha,i,1)\ \theta(\alpha,r,i)}.$$

FIGURE 14. Relation 1

$$\sum_{a,b,c\ adm} \frac{Tet\begin{bmatrix} 1 & a & b \\ a & 1 & \alpha \end{bmatrix}\ Tet\begin{bmatrix} c & 1 & b \\ 1 & c & \gamma \end{bmatrix}}{\theta(a,\alpha,1)\ \theta(b,1,1)\ \theta(c,\gamma,1)<<a,b,c>>}(a,b,c)$$

$$= \sum_{a,b,c\ adm} \frac{-A^3\ Tet\begin{bmatrix} c & 1 & b \\ 1 & c & \gamma \end{bmatrix}\sum_i D_i}{\lambda^\alpha_a\ ^1\theta(a,\alpha,1)\ \theta(b,1,1)\ \theta(c,\gamma,1)<<a,b,c>>}(a,b,c),$$

$$\text{where } D_i = \frac{\Delta_i\ \lambda^a_i\ ^1\ Tet\begin{bmatrix} i & a & b \\ a & \alpha & 1 \end{bmatrix}\ Tet\begin{bmatrix} \alpha & 1 & b \\ 1 & i & a \end{bmatrix}}{\theta(a,i,1)\ \theta(b,\alpha,i)}.$$

FIGURE 15. Relation 3

$$\sum_{a,b,c\ adm} \frac{\delta(a,\alpha,1)\ \delta^b_0\ \delta^c_0}{<<a,b,c>>}\ (a,b,c)$$

$$= \sum_{a,b,c\ adm}\sum_r \frac{-A^{-3}\ \Delta_r\ Tet\begin{bmatrix} c & r & b \\ 1 & c & 1 \end{bmatrix}\ Tet\begin{bmatrix} a & a & \alpha \\ r & 1 & b \end{bmatrix}\sum_i D_i}{\theta(a,\alpha,1)\ \theta(c,1,1)\ \theta(b,r,1)\ \theta(r,c,1)\ \theta(a,r,\alpha)<<a,b,c>>}(a,b,c),$$

$$\text{where } D_i = \frac{\Delta_i\ (\lambda^a_i\ ^1)^2\ Tet\begin{bmatrix} 1 & i & c \\ \alpha & 1 & a \end{bmatrix}\ Tet\begin{bmatrix} c & i & 1 \\ a & r & \alpha \end{bmatrix}}{\lambda^r_c\ ^1\ \theta(a,i,1)\ \theta(c,\alpha,i)}.$$

FIGURE 16. Relation 4

The proof splits into five cases:

(1) $x \geq 1$, $y \geq 2$, and $z \geq 1$,
(2) $x \geq 1$, $y = 0$, $z \geq 1$, and $z \neq x$,
(3) $x \geq 2$ and $y = z = 0$,
(4) $x = y = 0$ and $z > 2$, and
(5) $x \geq 2$, $y = 0$, and $z = x$.

After these have been rewritten, only $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, $(1,0,1)$, and $(0,0,2)$ remain.

Case 1: $x \geq 1$, $y \geq 2$, and $z \geq 1$.

If we let $(\alpha, \beta, \gamma) = (x, y-2, z-1)$, then (x, y, z) is the highest term appearing in relator r_1. Using Mathematica, we calculate that the coefficient of (x, y, z) is

$$A^{-2-2x-y}(A^{2+2x} - A^y)(A^{2+2x} + A^y).$$

This is invertible, so (x, y, z) is a linear combination of lesser terms.

$$\sum_{a,b,c\ adm} \frac{\delta_\alpha^a\ \delta_0^b\ \delta_\alpha^c\ \Delta_1}{<< a,b,c >>}\ (a,b,c) = \sum_{a,b,c\ adm} \sum_{p,q,q',r} \frac{C_{p,q,q',r}}{<< a,b,c >>}\ (a,b,c),$$

for

$$C_{p,q,q',r} = \frac{\Delta_p\ \Delta_q\ \Delta_{q'}\ \Delta_r\ Tet\begin{bmatrix} 1 & c & q' \\ \alpha & 1 & r \end{bmatrix} Tet\begin{bmatrix} q' & c & q \\ r & 1 & \alpha \end{bmatrix} Tet\begin{bmatrix} q & b & c \\ c & r & 1 \end{bmatrix}}{\theta(\alpha,1,p)\ \theta(a,1,p)\ \theta(q',1,1)\ \theta(q,q',1)\ \theta(b,q,1)\ \theta(\alpha,r,1)\theta(c,r,1)\ \theta(c,\alpha,q')\ \theta(c,q,r)} \sum_i D_i,$$

where $D_i = \dfrac{\Delta_i\ Tet\begin{bmatrix} p & 1 & a \\ i & 1 & \alpha \end{bmatrix}}{\lambda_i^{\alpha\ 1}\ \theta(\alpha,i,1)\ \theta(a,i,1)} \sum_j E_j,$

and

$$E_j = \frac{\Delta_j\ \lambda_j^{a\ 1}\ Tet\begin{bmatrix} 1 & q' & j \\ i & a & 1 \end{bmatrix} Tet\begin{bmatrix} q' & q & j \\ \alpha & i & 1 \end{bmatrix} Tet\begin{bmatrix} q & b & j \\ p & \alpha & 1 \end{bmatrix} Tet\begin{bmatrix} j & a & b \\ a & p & 1 \end{bmatrix}}{\theta(a,j,1)\ \theta(i,j,q')\ \theta(\alpha,j,q)\ \theta(p,b,j)}.$$

FIGURE 17. Relation 6

Case 2: $x \geq 1$, $y = 0$, $z \geq 1$, and $z \neq x$.

Let $(\alpha, \beta, \gamma) = (x, 0, z-1)$ in r_1, and let $(\alpha, \beta, \gamma) = (x-1, 0, z)$ in r_2. Then $(x, 2, z)$ and $(x, 0, z)$ are the two highest terms appearing in the relators.

Rearranging terms, we have

$$a_1(x, 2, z) + a_2(x, 0, z) = \text{lesser terms}$$

and

$$b_1(x, 2, z) + b_2(x, 0, z) = \text{lesser terms},$$

where a_i is the coefficient of the ith-highest term appearing in r_1 and b_i is the coefficient of the ith-highest term appearing in r_2.

So, we can rewrite $(x, 0, z)$ if

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

is invertible.

Using Mathematica, we can see that this determinant is $-A^{-2-2x-2z}(-1 + A^x)(1 + A^x)(A^x - A^z)(-1 + A^z)(1 + A^z)(A^x + A^z)$, which is invertible when $x \geq 1$, $z \geq 1$, and $x \neq z$.

Case 3: $x \geq 2$, $y = z = 0$.

Using r_1 with $(\alpha, \beta, \gamma) = (x, 2, 1)$, r_2 with $(\alpha, \beta, \gamma) = (x-1, 0, 2)$, r_3 with $(\alpha, 0, \gamma) = (x-1, 0, 1)$, and r_4 with $\alpha = x-1$, we obtain four relations with $(x, 4, 2), (x, 2, 2), (x, 0, 2)$, and $(x, 0, 0)$ appearing as the four highest terms.

Thus, we can rewrite $(x, 0, 0)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & 0 \\ 0 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is invertible, where $a_i, b_i, c_i,$ and d_i are the coefficients of the ith-highest terms appearing in $r_1, r_2, r_3,$ and r_4, respectively.

This determinant is

$$-A^{-10-2x}(-1+A)(1+A)(1+A^2)(-A+A^x)^2(A+A^x)^2(A^2+A^{2x})^2,$$

which is invertible when $x \geq 2$.

Case 4: $x = y = 0$, $z > 2$.

Using r_1 with $(\alpha, \beta, \gamma) = (2, 2, z - 1)$, r_2 with $(\alpha, \beta, \gamma) = (1, 0, z)$, r_3 with $(\alpha, 0, \gamma) = (1, 0, z - 1)$, and r_5 with $\gamma = z - 1$, we obtain four relations with $(2, 4, z), (2, 2, z), (2, 0, z)$, and $(0, 0, z)$ appearing as the four highest terms. Note that this only holds for $z > 2$: for $z = 2$, $(0, 0, z)$ is no longer the fourth-highest term.

Thus, we can rewrite $(0, 0, z)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is invertible, where a_i, b_i, c_i, and d_i are the coefficients of the ith-highest terms appearing in r_1, r_2, r_3, and r_5, respectively.

This determinant is $-A^{-6-2z}(-1+A)(1+A)(1+A^2)(-1+A^z)(1+A^z)(-A+A^z)(A+A^z)(A^2+A^{2z})$, which is invertible when $z > 2$.

Case 5: $x \geq 2$, $y = 0$, and $z = x$.

Using r_2 with $(\alpha, \beta, \gamma) = (x - 1, 2, x)$, r_3 with $(\alpha, 0, \gamma) = (x - 1, 0, x - 1)$, and r_6 with $\alpha = x - 2$, we obtain three relations with $(x, 4, x), (x, 2, x)$, and $(x, 0, x)$ appearing as the three highest terms.

Thus, we can rewrite $(x, 0, x)$ if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is invertible, where a_i, b_i, and c_i are the coefficients of the ith-highest terms appearing in r_2, r_3, and r_6, respectively.

This determinant is $A^{-4+2x}(-1+A^x)(1+A^x)$, which is invertible for $x \geq 2$.

For all five cases, the determinants are indeed invertible, though their complexity compels us to use Mathematica for their evaluations. See [H] for the code and the output. $\square$

Remark Note that all the determinants that we have computed are up to sign and powers of A products of cyclotomic polynomials in A. There is no obvious reason that these determinants should even be polynomials; the entries of the matrices are generally quite complicated elements of $\mathcal{R}$. It is this unexpected fact which allows us to make our computation over $\mathcal{R}$.

4. Linear Independence

Proposition 10. $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, and $(0, 0, 2)$ are linearly independent in $S(M)$.

Proof. First, we recall that $S(M)$ is a direct sum of four submodules $S_1(M)$, $S_i(M)$, $S_j(M)$, and $S_k(M)$, and that the latter three are isomorphic.

Hence, our task is greatly simplified. We only have to show that $(0, 0, 0)$ and $(0, 0, 2)$ are linearly independent, and that $(0, 0, 1)$ is nonzero.

Definition 11. *The triple a, b, c is said to be r-admissible if it is admissible and $a + b + c \leq 2r - 4$.*

The recoupling theory we have used in previous chapters works when A is replaced by a primitive $2r$th root of unity, for odd $r > 1$. But we must replace "admissible" with "r-admissible" and restrict our labels to the range 0, ..., r-2: our insistence on r-admissibility ensures that the fusion formula still makes sense.

Let A_r be the $2r$th root of unity $e^{\pi i/r}$ with $r > 1$ odd, let e_i denote the core of the solid torus, labelled i. Let $\Omega_r = \sum_{i=0}^{\lfloor (r-3)/2 \rfloor} \Delta_i e_i$, choose η such that $\eta^2 < \Omega_r >= 1$, and let κ be a root of unity such that $\kappa^6 = A_r^{-6-r(r+1)/2}$.

Definition 12. *For a framed link K in a closed, connected, oriented 3-manifold M described by surgery on a framed link $L \subset S^3$, we define the quantum invariant*

$$I_r(M, K) = \kappa^{3(b_-(L)-b_+(L))} < L(\eta\Omega_r) \cup K >,$$

where $b_+(L)$ and $b_-(L)$ are the numbers of the positive and negative eigenvalues of the linking matrix of L.

We follow the notation of Masbaum and Roberts in [MR]. See [BHMV1], [Lic], [KM], [RT], and [W] for the origins of this formula.

Note that for the quaternionic manifold M, with the surgery description L presented in the introduction, $b_+(L) = b_-(L)$, and so $I_r(M, K) = \eta^2 < L(\Omega_r) \cup K >$.

Proposition 13. *For odd $r > 1$,*

$$(1 - A_r^4)I_r(M) = \sum_{k=1}^{r-1}(-1)^k A_r^{2k^2+2k},$$

$$I_r(M, (0, 0, 1)) = (-1)^{\frac{r-1}{2}} \frac{A_r^{-2}}{A_r^2 + 1},$$

and

$$A_r^4 I_r(M, (0, 0, 2)) = I_r(M) - 1.$$

Proof.

$$= \sum_{i=0}^{(r-3)/2} (-A_r)^{i(i+2)} \Delta_i \sum_{\alpha,\beta \ adm} \frac{\Delta_\alpha \Delta_\beta}{\theta(i,i,\alpha)\theta(\alpha,\beta,c)}$$

$$= \sum_{i=0}^{(r-3)/2} (-A_r)^{i(i+2)} \Delta_i \sum_{\alpha,\beta \ as \ above} \frac{\Delta_\beta \lambda_\alpha^{i\,i}}{\theta(\alpha,\beta,c)}$$

$$= \sum_{i=0}^{(r-3)/2} (-A_r)^{i(i+2)} \Delta_i \sum_{\alpha,\beta \ as \ above} \lambda_\alpha^{i\,i} \ \beta$$

By the encirclement lemma, β $\mathbf{\Omega}_r = \eta^{-2}\Delta_\beta$ if $\beta = 0$ or $\beta = r-2$,

and 0 otherwise. See, for example, [Lic].

Furthermore, the sums are restricted to r-admissible labellings, and so, $0 \le \alpha \le 2i$, $|\alpha - c| \le \beta \le \alpha + c$, and $\beta \equiv \alpha + c \bmod 2$.

Hence, a contribution to the sum can only occur when $\alpha = 0, \beta = 0$ for $c = 0$, $\alpha = r-3, \beta = r-2, i = \frac{r-3}{2}$ for $c = 1$, and $\alpha = 2, \beta = 0$ for $c = 2$.

Hence,

$$I_r(M) = \sum_{i=0}^{(r-3)/2} (-A_r)^{i(i+2)} \Delta_i \lambda_0^{i\,i},$$

$$I_r(M,(0,0,2)) = \sum_{i=1}^{(r-3)/2} (-A_r)^{i(i+2)} \Delta_i \lambda_2^{i\,i},$$

and

$$I_r(M,(0,0,1)) = (-A_r)^{\frac{r-3}{2}(\frac{r-3}{2}+2)} \Delta_{\frac{r-3}{2}} \Delta_{r-2} \lambda_{r-3}^{\frac{r-3}{2}\,\frac{r-3}{2}} = (-1)^{\frac{r-1}{2}} \frac{A_r^{-2}}{A_r^2+1}.$$

Hence, $A_r^4 I_r(M,(0,0,2)) = I_r(M) - 1$, and letting $q = A_r^2$ and $k = i + 1$, we obtain

$$-q(q - q^{-1})I_r(M) = \sum_{k=1}^{(r-1)/2} (-1)^k (q^{k^2+k} - q^{k^2-k}).$$

Since $q^r = 1$, $(-1)^{r-k} q^{(r-k)^2+(r-k)} = -(-1)^k q^{k^2-k}$. Hence,

$$(1 - q^2)I_r(M) = \sum_{k=1}^{r-1} (-1)^k q^{k^2+k}.$$

□

Before proceeding, we need more notation and a useful lemma:

Notation 14. *Let* $\zeta_N = e^{2\pi i/N}$. *(Hence,* $A_r = \zeta_{2r}$.*) Also, let* $Q(A)$ *denote the field of fractions of* $\mathbb{Z}[A]$. *Elements of* $Q(A)$ *are called rational functions.*

Lemma 15. *If* $F(A)$ *is a rational function, then the imaginary part of* $F(\zeta_{2r})$ *cannot change sign infinitely often as* $r \to \infty$.

Proof. We have that $F(A^{-1}) \in Q(A)$ and so $G(A) = F(A) - F(A^{-1}) \in Q(A)$. For z on the unit circle, Im $F(z) = \frac{1}{2i}(F(z) - \overline{F(z)}) = \frac{1}{2i}(F(z) - F(\bar{z})) = \frac{1}{2i}G(z)$. As $G(z)$ has at most finitely many zeros and poles, $F(z)$ is zero or undefined for at most finitely many z on the unit circle. Thus, we can choose N such that $F(z)$ has no poles or zeros for $|z| = 1$ with Arg$z \in (0, \frac{2\pi}{N})$.

Let $h(x) = $ Im $F(e^{2\pi i x})$, with domain $(0, \frac{1}{N})$. Then h is a real-valued continuous function defined on an interval with no zeros . Thus, h cannot change sign. So Im $F(\zeta_{2r})$ cannot change sign infinitely often as $r \to \infty$. $\qquad\square$

Proposition 16. $(0, 0, 0)$ *and* $(0, 0, 2)$ *are linearly independent.*

Proof. Suppose that $(0, 0, 0)$ and $(0, 0, 2)$ are linearly dependent. Then we have g_0 and g_2 in $\mathcal{R}$ such that

$$g_0(A)(0, 0, 0) + g_2(A)(0, 0, 2) = 0 \in S(M)$$

Hence, there are elementary Kauffman relations among framed links, R_i, such that

$$g_0(A)(0, 0, 0) + g_2(A)(0, 0, 2) = \sum_i h_i(A) R_i$$

in the free $\mathcal{R}$-module generated by all isotopy classes of framed links in M. Here, all the $h_i(A)$ are in $\mathcal{R}$. Taking quantum invariants on both sides, we have, for r sufficiently large,

$$g_0(A_r) I_r(M) + g_2(A_r) I_r(M, (0, 0, 2)) = 0.$$

Since $A_r^4 I_r(M, (0, 0, 2)) = I_r(M) - 1$ for odd $r > 1$,

$$g_0(A_r) I_r(M) + A_r^{-4} g_2(A_r)(I_r(M) - 1) = 0,$$

for r odd and sufficiently large.

Hence,

$$I_r(M) = \frac{g_2(A_r)}{A_r^4 g_0(A_r) + g_2(A_r)},$$

for r odd and sufficiently large.

Thus, it suffices to show that there is no $f(A)$ in $Q(A)$ such that $f(\zeta_{2r}) = I_r(M)$ for r odd and sufficiently large.

By Proposition 13, $(1 - A_r^4) I_r(M) = \sum_{k=1}^{r-1} (-1)^k A_r^{2k^2 + 2k}$.

Notation 17. $g_N(m) = \frac{2}{\sqrt{N}} \sum_{k=0}^m \zeta_N^{k^2}$.

The following lemma was provided by Paul van Wamelen. See [H, Appendix C].

Lemma 18. *(van Wamelen)*

$$\sum_{k=1}^{r-1} \zeta_{2r}^{rk} \zeta_{2r}^{2k^2 + 2k} + 1 = \zeta_{16r}^{-(r+2)^2} (\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))).$$

Hence, $(1 - A_r^4) I_r(M) + 1 = \zeta_{16r}^{-(r+2)^2} (\zeta_{16r}^{9r^2} - 2\sqrt{r}(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})))$.

Now, we can apply a result in [Le] by D.H. Lehmer to estimate each of the new sums.
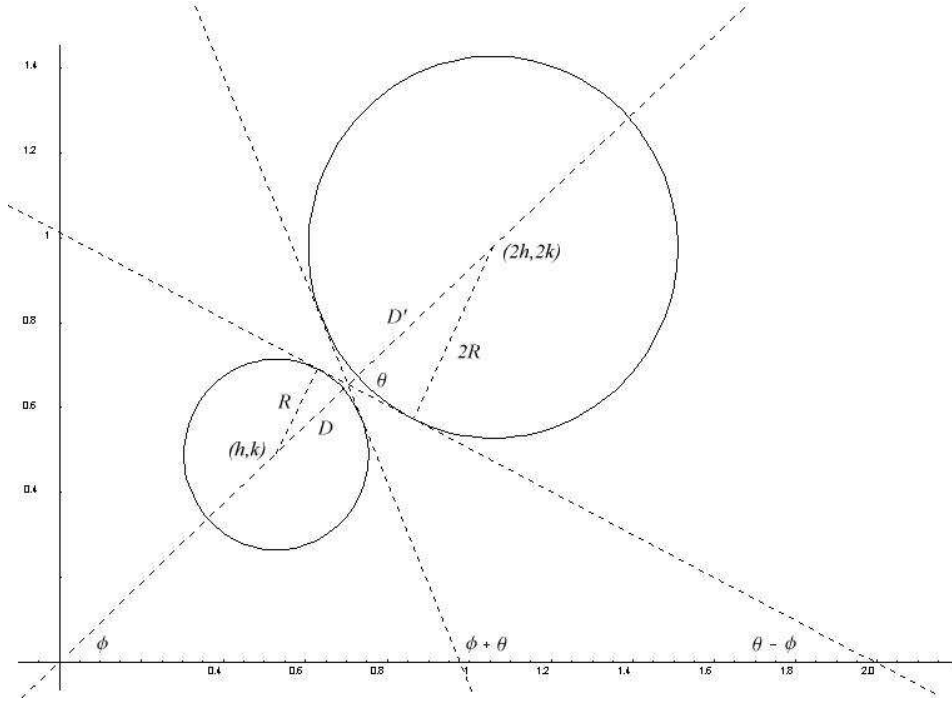
FIGURE 18. Estimating $g_{4r}(\frac{r-1}{2})$ and $2g_{16r}(r-1)$

Theorem 19. *(Lehmer) For $N \geq 100$ and $\sqrt{\frac{N}{2}} \leq m \leq \frac{N}{4}$, $g_N(m)$ lies within the circle with center $(h,k) = (C(\sqrt{(2)}), S(\sqrt{2}) - \frac{1}{\sqrt{2\pi}}) \approx (0.529, 0.489)$ and radius $\frac{1}{\sqrt{2\pi}} + \frac{101}{40\sqrt{N}}$, where $C(u) = \int_0^u \cos(\frac{1}{2}\pi x^2)dx$ and $S(u) = \int_0^u \sin(\frac{1}{2}\pi x^2)dx$ are the Fresnel integrals.*

The proof of this result in Lehmer's paper is only sketched, and moreover, there are some minor errors. However, Litherland has given a detailed proof [Lit] along the lines indicated by Lehmer.

Let $R = \frac{1}{\sqrt{2\pi}} + 0.0001$. Then, for sufficiently large r, $g_{4r}(\frac{r-1}{2})$ lies inside a circle with center (h,k) and radius R, and $2g_{16r}(r-1)$ lies inside a circle with center $(2h, 2k)$ and radius $2R$.

Let D be the distance from (h,k) to the intersection of the tangent lines between the circles depicted in Figure 18, let D' be the distance from $(2h, 2k)$ to the point of intersection, let L be the distance between the two centers, let θ be the angle between one of the tangent lines and the line joining the centers, and let ϕ be the angle between the line joining the centers and the x-axis.

$$\phi - \theta < \arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) < \phi + \theta,$$

where arg takes values in $[-90º, 270º)$.

We will now show that multiplication by $\zeta_{16r}^{-(r+2)^2}$ rotates the difference into the upper half-plane for certain values of r and into the lower half-plane for other values of r.
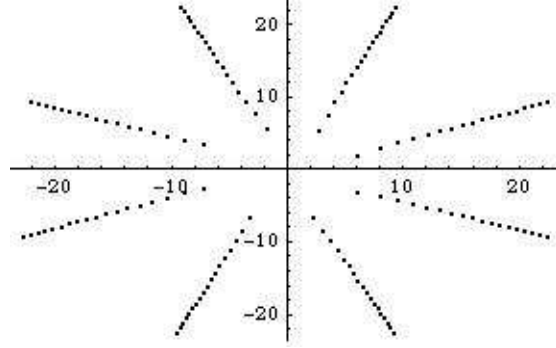
FIGURE 19. Values of $(1 - A_r^4)I_r(M)$ for $17 \leq r \leq 301$, and r odd.

Since $\frac{R}{D} = \sin(\theta) = \frac{2R}{D'}$, $D' = 2D$. $L = D + D' = 3D$, so $\sin(\theta) = \frac{3R}{3D} = \frac{3R}{L} = \frac{3R}{\sqrt{h^2+k^2}}$, and so $\theta = \sin^{-1}(\frac{3R}{\sqrt{h^2+k^2}}) \approx 69.7078°$.

Also, $\tan(\phi) = \frac{k}{h}$, so $\phi \approx 42.7495°$.

As $r \to \infty$, $\zeta_{16r}^{-(r+2)^2} \to \zeta_{16}^3$, for $r \equiv 9$ mod 16 and $\zeta_{16r}^{-(r+2)^2} \to \zeta_{16}^{11}$, for $r \equiv 1$ mod 16.

$$0° < 40.541° < \phi - \theta + 67.5° < \arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) + 67.5°$$

and

$$\arg(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) + 67.5° < \phi + \theta + 67.5° < 179.96° < 180°.$$

Hence, for sufficiently large r, multiplication by $\zeta_{16r}^{-(r+2)^2}$ rotates $(2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2}))$ into the upper half-plane for $r \equiv 9$ mod 16 and into the lower half-plane for $r \equiv 1$ mod 16.

Hence, for sufficiently large r, the imaginary part of

$$\frac{(1 - A_r^4)I_r(M) + 1}{2\sqrt{r}} = \zeta_{16r}^{-(r+2)^2} \left(\frac{\zeta_{16r}^{9r^2}}{2\sqrt{r}} - (2g_{16r}(r-1) - g_{4r}(\frac{r-1}{2})) \right)$$

is positive for $r \equiv 1$ mod 16 and is negative for $r \equiv 9$ mod 16, and the same must hold for $(1 - A_r^4)I_r(M) + 1$.

Hence, the imaginary part of a function F such that $F(\zeta_{2r}) = (1 - \zeta_{2r}^4)I_r(M) + 1$ for all but finitely many odd r changes sign infinitely often, and so, by Lemma 15, F cannot be rational. Hence, there can be no rational function f such that $f(\zeta_{2r}) = I_r(M)$ for r odd and sufficiently large, as required. $\square$

Proposition 20. $(0, 0, 1) \neq 0$

Proof. If $(0, 0, 1) = 0$, then there exists a linear combination of Kauffman bracket skein relators R_i such that $(0, 0, 1) = \sum_i a_i(\mathcal{A})R_i$ in the free module over $\mathcal{R}$, and hence as in the previous proof, there is a product of cyclotomic polynomials $k(A)$ such that $k(A_r)I_r(M, (0, 0, 1)) = 0$ for all r.

As shown in Proposition 13, $I_r(M, (0, 0, 1)) = (-1)^{\frac{r-1}{2}} \frac{A^{-2}}{A^2+1} \neq 0$ for all odd $r > 1$, and so, no such $k(A)$ can exist.

Hence, $(0, 0, 1) \neq 0$. $\square$

Thus, $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, and $(0, 0, 2)$ are all linearly independent in $S(M)$, and so, $S(M) = \mathcal{R}^5$. □

Experiments run in Mathematica suggest that, restricted to each odd congruence class of r mod 16, $I_r(M)$ takes on the values of some well-behaved function of r: see Figure 19.

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