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Optimization Approach to the Treatment of Open Boundary-Conditions

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ABSTRACT

A solution to an optimization problem is developed that deals with minimizing a measure of difference between the values of observed and predicted variables at an open ocean boundary. Minimization is based on the change of the flux of energy through the open boundary. It is shown that many of the longwave radiation conditions that are commonly used in ocean modeling can be derived using this optimization criteria. However, the minimization process is seen to produce a modification of these radiation conditions in that they are multiplied by a coefficient, which allows the conditions to adapt to a change in the flux of energy penetrating the boundary. An example of the numerical implementation is presented for the Reid and Bodine boundary formulation. For a standing wave problem with an analytical solution, use of the modified Reid and Bodine formulation is seen to eliminate almost entirely errors in the predicted amplitudes and phases. Overall, this approach is seen to allow a modeler to generate different types of boundary conditions based on observations as well as the inclinations of the modeler.

1. Introduction

The treatment of open boundaries is one of the most interesting problems to be solved while modeling oceanic phenomena, especially in finite ocean coastal areas. In most ocean models, open boundary conditions are chosen locally, that is, depending on the solution of the governing equations near the boundary. Many approaches of the local type have been developed (Reid and Bodine 1968; Orlanski 1976; Chapman 1985; Blumberg and Kantha 1985; Oey and Chen 1992). The results of numerical studies show that the application of many local-type boundary conditions reproduce the modeled physical phenomenon and work for most practical purposes. However, it is known (Bennett 1992, Oliger and Sundstrom 1978) that the local treatment of open boundaries for primitive equations models is an ill-posed problem in that it is difficult to prove that a unique solution exists that is continuously dependent on available observations.

Some researchers use the inverse approach to the modeling of open boundary conditions. The open boundary conditions are chosen in such way as to simultaneously provide the "best" fit to the governing equations and observations. The best fit means the minimization of the norm of the deviation between model results and observations. Thus, the interior solution and available observations are used to choose the open boundary conditions. This approach has been applied to atmospheric and oceanic circulation problems (e.g., Bennett 1992; Zou et al. 1993; Seiler 1993). The most popular algorithm for solving an inverse problem is an adjoint method in which the initial problem of circulation with open boundary conditions is reduced to integrating the governing equations and an equation for the adjoint variable forward and backward in time (Zou et al. 1993). Although the inverse approach leads to a well posedness, it suffers from a few drawbacks that may restrict its use: requirements of large amounts of computer time and memory and the problem of stable integration of the adjoint equation.

We propose methods for modeling open boundary conditions that are based on the integration of the governing equations forward in time and choosing open boundary conditions via a specific inverse problem that provides the "best" fit to available observations on the open boundary and to the energy flux through the open boundary. In this way, we avoid the local treatment of the open boundary conditions. All observations on the open boundary and its interior vicinity are used in determining the open boundary conditions for each particular point on the boundary. Numerically, the proposed methods consist of the integration of the governing equations and solving the optimization problem

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for each time step. We show that some of the well-known "local" longwave radiation boundary conditions, commonly used in ocean modeling, are special cases of boundary conditions derived using this approach. The derivations presented here suggest methods for the generation of new boundary conditions based on the requirements of the modeled phenomena.

2. Derivation of the optimization problem

We first derive the equations for the optimization problem based on the surface height at the open boundary. Consider the system of equations for shallow water:

$$\frac{\partial \mathbf{u}}{\partial t} = -g\nabla\eta + \phi, \qquad (1)$$

$$\frac{\partial \eta}{\partial t} = -\nabla\cdot(H\mathbf{u}), \qquad (2)$$

where $\mathbf{u}$ is the horizontal velocity vector, η is the sea surface deviation, H is the depth, and ϕ represents the forcing. The equation for the total energy within the model domain D is

$$K(t) = \frac{1}{2}\int_D [Hu^2 + g\eta^2]d\tau, \qquad (3)$$

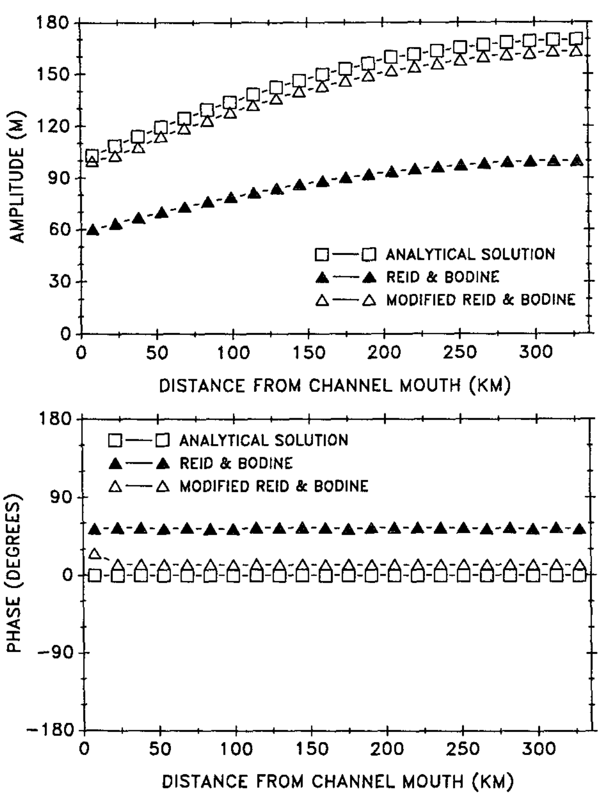

FIG. 1. Amplitude (top) and phase (bottom) of the standing wave for a channel 255 m deep.

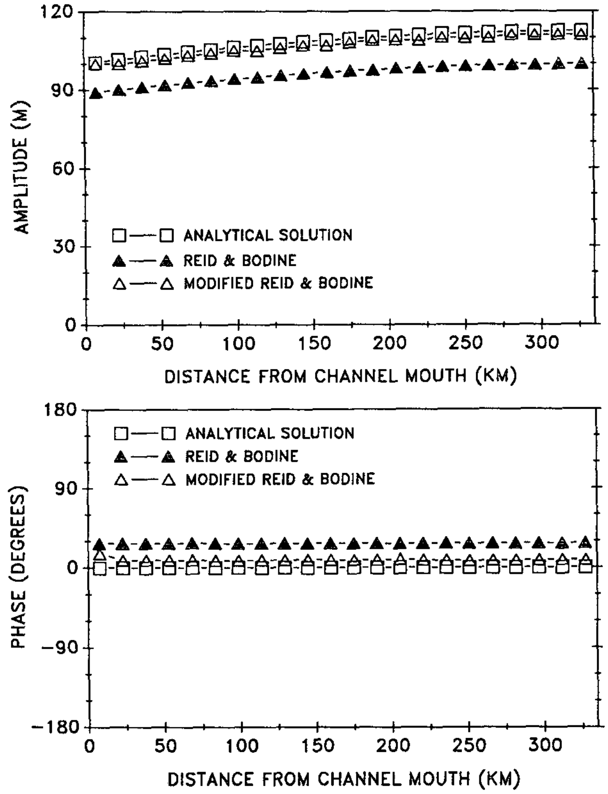

FIG. 2. Same as Fig. 1 but for a channel 1000-m deep.

and its time rate of change using (1) and (2) is

$$d_tK = \int_D [Hu\partial_t u + g\eta\partial_t\eta]d\tau$$

$$= \int_D -g\nabla\cdot(H\eta\mathbf{u})d\tau + \int_D H\mathbf{u}\phi d\tau. \qquad (4)$$

By using the Gauss divergence theorem, (4) can be rewritten as

$$d_tK = -g\int_S H\eta\mathbf{u}_\mathbf{n}ds + \int_D H\mathbf{u}\phi d\tau,$$

where u_n is the outward normal velocity and S is the open boundary of D. Thus, the temporal variations of the total energy can be expressed as a sum of energy changes occurring along the open boundary plus changes related to ϕ inside the model domain.

Now consider the following optimization problem:

$$\min_\eta J(\eta), \qquad (5)$$

constrained by the flux of energy through the open boundary

$$P_t = -g\int_S H\eta\mathbf{u}_\mathbf{n}ds. \qquad (6)$$

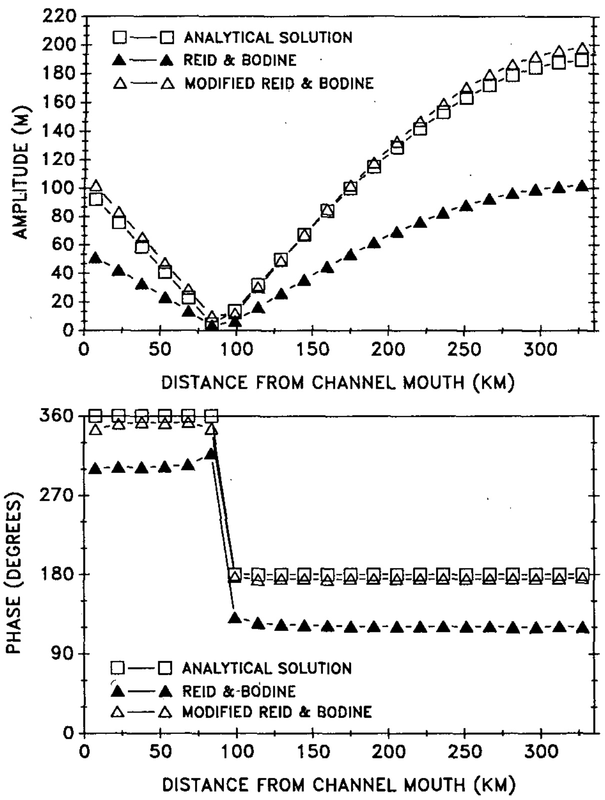

FIG. 3. Same as Fig. 1 but for a channel 50 m deep.

Here $J(\eta)$ is an objective functional depending on η, such that $J(\eta) \geq 0$. The term P_t can be interpreted as the flux of total energy penetrating the open boundary S. Using the Lagrangian method (Fletcher 1987) to solve the problem (5)–(6), we minimize:

$$\min_{\eta}\left[J(\eta) - \lambda_t\left(P_t + g\int_S H\eta u_n ds\right)\right],$$

where λ_t is a constant (the Lagrange multiplier). Thus, the solution of the problem satisfies the following condition:

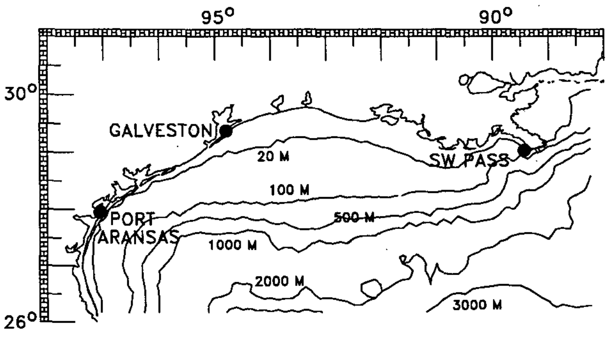

FIG. 4. Model domain and bathymetry (m) for the Texas–Louisiana shelf.

TABLE 1. Observed and model-predicted amplitudes (cm) and phases (degrees, relative to UTC) for the M_2 tide at Galveston, Texas; Port Aransas, Texas; and South West Pass, Louisiana. The model forcing used the standard Reid and Bodine boundary formulation.

	Model		Observed	
Station	Amplitude	Phase	Amplitude	Phase
Galveston	9.1	259.8	13.5	275.1
Port Aransas	6.2	238.0	7.7	262.3
South West Pass	0.2	333.5	1.7	127.2

$$\frac{dJ}{d\eta} - \lambda_t gHu_n = 0. \qquad (7)$$

It is important to note that the constant λ_t measures the rate of change in the function $J(\eta)$ due to changes in P_t (the flux of mechanical energy across the open boundary S). Let ϵ_t be the perturbation of P_t: $P_t^* = P_t + \epsilon_t$. It can be shown that (Fletcher 1987)

$$\lambda_t = -\frac{dJ}{d\epsilon_t}, \qquad (8)$$

which provides us with a direct relationship between the various terms of the problem.

3. Some examples

In the following we will show that several of the open boundary conditions used in numerical ocean modeling can be related to the optimization approach of the last section. In each case a specific objective functional J is minimized.

a. Simple longwave radiation

Let us consider the following problem:

$$\min_{\eta}\left(J_1 = \frac{g}{2}\int_S \sqrt{gH}\eta^2 ds\right), \qquad (9)$$

with condition (6). In this case, (7) becomes

$$u_n = \frac{\eta}{\lambda_t}\left(\frac{g}{H}\right)^{1/2}. \qquad (10)$$

TABLE 2. Observed and model-predicted amplitudes (cm) and phases (degrees, relative to UTC) for the M_2 tide at Galveston, Texas; Port Aransas, Texas; and South West Pass, Louisiana. The model forcing used the optimized Reid and Bodine boundary formulation.

	Model		Observed	
Station	Amplitude	Phase	Amplitude	Phase
Galveston	13.2	262.1	13.5	275.1
Port Aransas	7.9	242.0	7.7	262.3
South West Pass	3.1	136.1	1.7	127.2

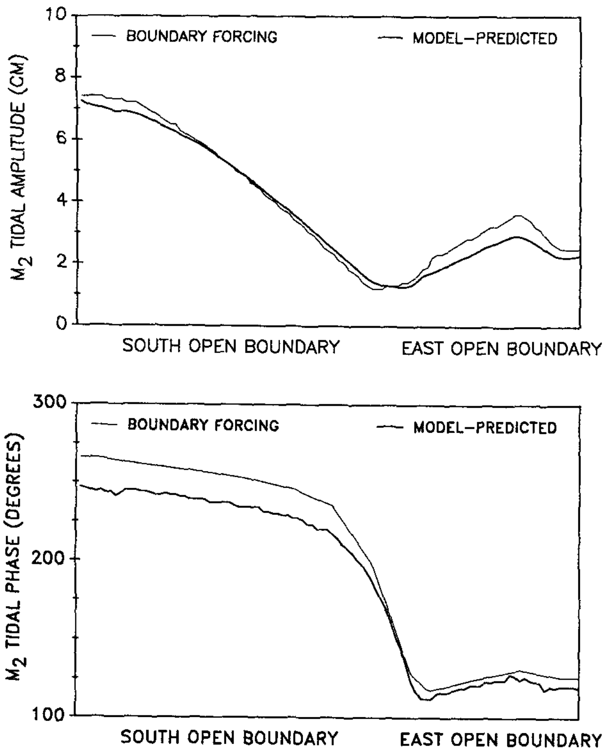

FIG. 5. Amplitude (top) and phase (bottom) of the M_2 tide along the open boundaries of the LATEX shelf model using the modified Reid and Bodine open boundary formulation.

Note that (10) is the commonly used longwave radiation condition with a "tuning" coefficient λ_t, which is time dependent. With this coefficient, the relationship given by (10) minimizes the functional $J(\eta)$ given in (9) (in essence, the square of the sea surface height on the open boundary). If $\lambda_t = 1$ (the common radiation condition), then according to (8) we have $dJ/d\epsilon_t = 1$. Under such conditions, we are specifying that the change in the flux of energy penetrating the open boundary will be equal to the change of the functional specified in (9).

b. The Reid and Bodine boundary condition

There are situations when some extraneous information on the open boundary in the form of sea surface height η_T is available. This information could be direct observations or output of another numerical model. In this case, we specify the objective functional to be

$$\min_{\eta}\left[J_2 = \frac{g}{2}\int_S \sqrt{gH}(\eta - \eta_T)^2 ds\right], \quad (11)$$

where the model height is fitted to the known function in the least squares sense. In this case, (7) becomes

$$u_n = \frac{(\eta - \eta_T)}{\lambda_t}\left(\frac{g}{H}\right)^{1/2}. \quad (12)$$

Note that condition (12) is a modified version of that used by Reid and Bodine (1968), which is often used to force regional tidal models while still allowing for the radiation of longwave energy from the interior domain of the model (Lewis et al. 1992). In this case, the relationship given by (12) minimizes the deviation of the sea surface height from the known function η_T (e.g., observed tidal variations). When $\lambda_t = 1$, the change in the flux of the total mechanical energy penetrating the boundary S is equal to the change of the deviation η from η_T.

Oey and Chen (1992) incorporated current velocity information $u_T(s, t)$ into the open boundary condition. To do so in our formulation, we rewrite the condition (12) in the following form:

$$\sqrt{g/H}[\eta - (\eta_T + \lambda_t\sqrt{H/g}u_{n,T})] = \lambda_t(u_n - u_{n,T}), \quad (13)$$

where $u_{n,T}$ is the outer normal component of the known velocity $u_T(s, t)$. Let us introduce the following notation:

$$\eta_T^* = \eta_T + \lambda_t\sqrt{H/g}u_{n,T}. \quad (14)$$

By substituting (14) into (13), we can get the following:

$$u_n - u_{n,T} = \frac{(\eta - \eta_T^*)}{\lambda_t}\left(\frac{g}{H}\right)^{1/2}. \quad (15)$$

Condition (15) is a modification of the boundary condition employed by Oey and Chen (1992) with η_T^* calculated from (14). The boundary conditions given by (12) and (14)–(15) both minimize the difference between the modeled and observed surface heights [Eq. (11)]. The difference is that (14)–(15) allows for the inclusion of surface heights and currents observed at the boundary.

c. The Blumberg and Kantha boundary condition

We specify the objective functional to be

$$\min_{\eta}\left\{J_3 = \frac{g}{2}\int_S \sqrt{gH}\left[\eta^2 + \frac{1}{T}\left(\int_0^t (\eta - \eta_T)dt\right)^2\right]ds\right\}, \quad (16)$$

TABLE 3. Observed and model-predicted amplitudes (cm) and phases (degrees, relative to UTC) for the M_2 tide at Galveston, Texas; Port Aransas, Texas; and South West Pass, Louisiana. The model forcing used the optimized Reid and Bodine boundary formulation for the eastern open boundary and again for the southern open boundary.

	Model		Observed	
Station	Amplitude	Phase	Amplitude	Phase
Galveston	15.4	251.0	13.5	275.1
Port Aransas	9.6	234.2	7.7	262.3
South West Pass	8.1	120.5	1.7	127.2

where $\bar{T}$ is a known constant. The minimization of J_3 subject to the energy flux constraint leads to

$$\eta - \lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\frac{1}{\bar{T}}\int_0^t (\eta - \eta_T)dt. \quad (17)$$

By differentiation of Eq. (17) with respect to t, we can get the following:

$$\frac{\partial\eta}{\partial t} - \lambda_t\left(\frac{H}{g}\right)^{1/2}\frac{\partial u_n}{\partial t} - d_t\lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\frac{1}{\bar{T}}(\eta - \eta_T). \quad (18)$$

According to (1), (19) can be rewritten in the following form:

$$\frac{\partial\eta}{\partial t} - \lambda_t\left(\frac{H}{g}\right)^{1/2}\left(-g\frac{\partial\eta}{\partial n} + \frac{\partial\phi}{\partial n}\right) - d_t\lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\frac{1}{\bar{T}}(\eta - \eta_T). \quad (19)$$

Suppose that $\partial\phi/\partial n|_S = 0$. Thus,

$$\frac{\partial\eta}{\partial t} + \lambda_t\sqrt{gH}\frac{\partial\eta}{\partial n} - d_t\lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\frac{1}{\bar{T}}(\eta - \eta_T). \quad (20)$$

The condition specified in (20) is a variation of the boundary condition developed by Blumberg and Kantha (1985) (if $\lambda_t = 1$ for all t). According to (16), condition (20) minimizes the square of η and the deviation of η from η_T over time along the open boundary. If $\bar{T} = t^2$, according to (16) we are minimizing the square of η and the average deviation of η from η_T over time.

d. Some enhancements

A few simple enhancements of the objective functionals can be considered to derive different open boundary conditions. Consider

$$\min_\eta\left\{J_4 = \beta_1 J_1 + \beta_2 J_2 = \frac{g}{2}\int_S \sqrt{gH}[\beta_1\eta^2 + \beta_2(\eta - \eta_T)^2]ds\right\}, \quad (21)$$

again with constraint (6). For this case, (7) becomes

$$\beta_1\eta - \lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\beta_2(\eta - \eta_T).$$

If $\beta_1 = 1$ and $\beta_2 = 1/\bar{T}$, we have

$$\eta - \lambda_t\left(\frac{H}{g}\right)^{1/2} u_n = -\frac{1}{\bar{T}}(\eta - \eta_T). \quad (22)$$

This formulation provides for the direct specification of u_n at the boundary.

4. Discussion

There are many possible numerical approaches to implementing the proposed boundary conditions. These approaches depend on the numerics of the hydrodynamical model. Here we discuss one of these approaches for the Reid and Bodine modified boundary condition (12) implemented for a vertically averaged model. Suppose η_t and $u_{n,t}$ are the sea surface elevation and velocity at time t. The model uses the two previous time steps for calculating variables for the $t + 1$ time step. As always, the sea surface elevations η_{t+1} are calculated from the continuity equation, and the velocities u_{t+1} are calculated from the momentum equation. On the open boundary, the following numerical scheme was implemented based on our optimization approach. By using values of η_t and $u_{n,t}$, we can find an estimate of P_t from (6). We have the following optimization problem (11), (6) for time t:

$$\min_\eta\left[J = \frac{g}{2}\int_S \sqrt{gH}(\eta - \eta_{T,t})^2 ds\right], \quad (23)$$

$$P_t = -g\int_S H\eta u_{n,t} ds. \quad (24)$$

We assume that P_t and $u_{n,t}$ contain some errors in the estimates of the energy flux and the normal velocity. We employ the regularization method (Sabatier 1987; Parker 1994) for solving the problem

$$\min_\eta\left[\frac{1}{2}\left(P_t + g\int_S H\eta u_{n,t} ds\right)^2 + \gamma J\right], \quad (25)$$

where γ is a parameter of regularization. The solution of (25) gives the following expression for λ_t in (12):

$$\lambda_t = -\frac{P_t + g\int_S H\eta_{t,T} u_{n,t} ds}{g^{1/2}\int_S H^{3/2} u_{n,t}^2 ds + \gamma}. \quad (26)$$

After the calculation of λ_t, we can find a corrected value of the sea surface elevation from (12). This corrected value of the sea surface elevation at time t is used for calculating the value of $u_{n,t+1}$:

$$u_{n,t+1} = (\eta_t - \eta_{T,t})\sqrt{g/H}. \quad (27)$$

Because we do not know the norms of the errors in the estimates of P_t and $u_{n,t}$, the parameter γ is chosen in such a way that the two terms in (25) are equal.

We performed simulations for a flat bottom, frictionless channel that is closed at one end. The channel was forced at the other end by surface oscillations at the M_2 tidal frequency with an amplitude of 1 m. The length of the channel was 355 km. There are 23 grid points along the channel (the 23rd is a wall). We performed simulations for three different values of channel depth: 255 m, 1000 m, and 50 m. Figures 1–3 show the amplitudes and phases of the model-predicted standing waves using the Reid and Bodine boundary condition and the modified Reid and Bodine as well as the analytical solution. The use of the modified Reid and Bodine formulation is seen to eliminate, almost entirely, errors in the predicted amplitudes while more than halving the errors in the predicted phases.

We also performed simulations for the M_2 tide over the Texas–Louisiana shelf (LATEX) area (Fig. 4). The model domain has southern and eastern open boundaries. Tables 1 and 2 contain the results of the simulations performed with both the Reid and Bodine and modified Reid and Bodine conditions [calculating λ_t and η_t from (26) and (12)]. The use of the optimization approach gives much better predictions of the amplitudes and phases at Galveston and Port Aransas, Texas, and the weak tide at South West Pass, Louisiana. Figure 5 shows a comparison between the tidal forcing at the open boundaries and the tides predicted at the model grid cells next to the open boundaries. These results indicate that the modified Reid and Bodine formulation provides a good estimate of the tidal amplitudes, with the largest deviations occurring on the eastern open boundary along the shelf slope. However, the minimization process for this simulation shows a clear bias in overestimating the phases, with the largest overestimates occurring along the southern open boundary.

We also conducted an experiment when the southern and eastern open boundaries were treated separately in our optimization approach. Different values of λ_t (26) were calculated for each open boundary. Thus, the technique would minimize differences along the eastern open boundary independently of those differences along the longer southern open boundary. The results of the simulation are presented in Table 3. In this case, the amplitudes were overestimated and the phases were underestimated. As mentioned in the Introduction, the original Reid and Bodine condition is a local treatment approach to the specification of the open boundary condition: only neighborhood data are used in determining the boundary condition for a grid cell on the boundary. In our approach, all data along and near the boundary are used in calculating the boundary values for each grid point on the boundary, and this approach results in a substantial improvement in predicting amplitudes and phases. However, the separate treatment of the southern and eastern boundaries gives inferior results, likely due to its more local treatment of the boundaries.

5. Conclusions

As we have shown, many of the familiar formulations used as open boundary conditions in numerical simulations can have certain optimization properties based on the coefficient λ_t. This coefficient allows the adaptation of the boundary condition to the change in the flux of total mechanical energy penetrating the open boundary, as well as the minimization of differences between observations and predictions. By choosing the appropriate functional to minimize along the open boundary, the proposed approach allows a modeler to generate different types of boundary conditions based on a priori information and the inclinations of the modeler. Conditions (12), (16), (21), and (23) can be considered as data assimilation schemes, where functions η_T and $u_T(s, t)$ include a priori information of the phenomena being modeled. For future research we plan to investigate the percentage of reflected versus absorbed energy on the open boundary. We plan to extend our approach to the modeling of three-dimensional, open boundary conditions and develop techniques for coupling basin and coastal models by using estimated energy flux through the interface of the basin–coastal models.

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